

Algebra Booster

with Problems & Solutions

for

JEE

Main and Advanced

Fully Solved

Rejaul Makshud

Mc Graw Hill Education

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About the Author

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Rejaul Makshud
M. Sc. (Calcutta University, Kolkata)

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Dedicated to

light of my life

(Rushda Manha, Daughter)

Preface

ALGEBRA BOOSTER with Problems & Solutions for JEE Main and Advanced is meant for aspirants preparing for the entrance examinations of different technical institutions, especially NIT/IIT/BITSAT/IISc. In writing this book, I have drawn heavily from my long experience of teaching at National Level Institutes. After many years of teaching, I have realised the need of designing a book that will help the readers to build their base, improve their level of mathematical concepts and enjoy the subject.

This book is designed keeping in view the new pattern of questions asked in JEE Main and Advanced Exams. It has eight chapters. Each chapter has the concept booster followed by a large number of exercises with the exact solutions to the problems as given below:

Level - I	:	Questions based on Fundamentals
Level - II	:	Mixed Problems (Objective Type Questions)
Level - III	:	Problems for JEE Advanced Exam
Level - IV	:	Tougher problems for JEE Advanced Exams
(0.......9)	:	Integer type Questions
Passages	:	Comprehensive link passages
Matching	:	Matrix Match
Reasoning	:	Assertion and Reasoning
Previous years' papers	:	Questions asked in past IIT-JEE Exams

Remember friends, no problem in mathematics is difficult. Once you understand the concept, they will become easy. So please don't jump to exercise problems before you go through the Concept Booster and the Objectives. Once you are confident in the theory part, attempt the exercises. The exercise problems are arranged in a manner that they gradually require advanced thinking.

I hope this book will help you to build your base, enjoy the subject and improve your confidence to tackle any type of problem easily and skilfully.

My special thanks goes to Mr. M.P. Singh (IISc Bangalore), Mr. Manoj Kumar (IIT, Delhi), Mr. Nazre Hussain (B. Tech.), Dr. Syed Kashan Ali (MBBS) and Mr. Shahid Iqbal, who have helped, inspired and motivated me to accomplish this task. As a matter of fact, teaching being the best learning process, I must thank all my students who inspired me most for writing this book.

I would like to convey my affectionate thanks to my wife, who helped me immensely and my children who bore with patience my neglect during the period I remained devoted to this book.

I also convey my sincere thanks to Mr. Biswajit Das of McGraw Hill Education for publishing this book in such a beautiful format.

I owe a special debt of gratitude to my father and elder brother, who taught me the first lesson of Mathematics and to all my learned teachers—Mr. Swapan Halder, Mr. Jadunandan Mishra, Mr. Mahadev Roy and Mr. Dilip Bhattacharya, who instilled the value of quality teaching in me.

I have tried my best to keep this book error-free. I shall be grateful to the readers for their constructive suggestions toward the improvement of the book.

Rejaul Makshud
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Contents

CHAPTER

1

Sequence and Series

CONCEPT BOOSTER

1. Introduction

It is formally defined as, a sequence is a set of numbers arranged in a definite order so that all the numbers after the first will maintain a definite rule.

For example, 1, 3, 5, ... is a list of odd natural numbers. It can be written as $2n - 1$, where $n \in N$.

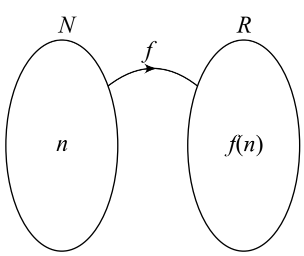
So a sequence may be defined like a function by a variable expression.

Sequence

It is a function whose domain is the set of natural numbers. It can also be defined as a function whose range is the set of real numbers.

Let $f(n) = n + 2$, where $n \in N$

Thus, {3, 4, 5, 6, 7, …} is a sequence.



1.1 Series

If a sequence is connected by either +ve sign or –ve sign or both, it is called a series.

For example, 1 + 2 + 3 + 4 + … is a series.

1.2 Finite and Infinite Sequences

A sequence is said to be finite or infinite according as the number of elements is finite or infinite.

For example, {1, 3, 5, 7, …, 99} is a finite sequence.

2. Progression

It is a sequence in which each term is followed by a certain pattern.

There are four types of progression, namely, AP, GP, HP and AGP.

2.1 Arithmetic Progression (AP)

If the algebraic difference between any two consecutive terms is same throughout the sequence, then it is called an A.P sequence.

i.e. $t_{n+1} - t_n$ = constant = d, which is known as the common difference.

2.2 *n*th Term from the Beginning of an AP Sequence

It is given as $t_n = a + (n - 1)d$,

where a is the first term and d the common difference of the AP.

Which is also known as the general term of A.P

2.3 *n*th Term from the end of an AP Sequence

Let the first term of a sequence is a, common difference d and the number of terms is m of an AP.

Thus, $T_n = l - nd = a + (m - n)d$,

where l is the last term

2.4 *n*th Term of an AP Sequence is a Linear Expression of *n*.

Let the first term is a and the common difference is d of an AP.

Then $t_n = a + (n - 1)d$

$= (a - d) + dn = A + Bn$

where $A = a - d$ and $B = d$.

which is a linear expression of n.

2.5 Sum of *n*-terms of an AP Sequence

Let first term is a and the common difference is d of an AP.

$\therefore$ The sum, $S_n = \frac{n}{2}[2a + (n - 1)d]$

2.6 Properties of AP Sequences

Property I

Let $a_1, a_2, a_3, \ldots, a_n$ be in AP and k be a non-zero real number, then

(i) $a_1 + k, a_2 + k, a_3 + k, \ldots, a_n + k$ are also in AP.

(ii) $a_1 - k, a_2 - k, a_3 - k, \ldots, a_n - k$ are also in AP.
(iii) $a_1.k, a_2.k, a_3.k, \ldots, a_n.k$ are also in AP.
(iv) $\frac{a_1}{k}, \frac{a_2}{k}, \frac{a_3}{k}, \ldots, \frac{a_n}{k}$ are also in AP.

Property II

If $a_1, a_2, a_3, \ldots, a_n$ and $b_1, b_2, b_3, \ldots, b_n$ are two APs, then
(i) $a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots, a_n + b_n$ are also in AP.
(ii) $a_1 - b_1, a_2 - b_2, a_3 - b_3, \ldots, a_n - b_n$ are also in AP.
(iii) $a_1 \times b_1, a_2 \times b_2, a_3 \times b_3, \ldots, a_n \times b_n$ are not in AP.
(iv) $a_1/b_1, a_2/b_2, a_3/b_3, \ldots, a_n/b_n$ are not in AP.

Property III

The sum of the terms of an AP is equidistant from the beginning and from end is constant and is equal to the sum of the first and the last terms, i.e.

if $a_1, a_2, a_3, \ldots, a_n$ are in AP, then

$a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \ldots$

Property IV

Any term of an AP (other than the first term) is equal to half the sum of terms which are equidistant from it, i.e.

if $a_1, a_2, a_3, \ldots, a_n$ are in AP, then

$$a_r = \frac{a_{r-k} + a_{r+k}}{2},$$

where $0 \le k \le n - r$.

Property V

If a, b, c are in AP, then $b = \frac{a+c}{2}$

Property VI

Terms of an AP sequence

(i) Three numbers of an AP can be taken as $a - d, a, a + d$.
(ii) Four numbers of an AP can be taken as $a - 3d, a - d, a + d, a - 3d$
(iii) Five numbers of an AP can be taken as $a - 2d, a - d, a, a + d, a + 2d$.
(iv) Six numbers of an AP can be taken as $a - 5d, a - 3d, a - d, a + d, a + 3d, a + 5d$.

Property VII

If log a, log b and log c are in AP, then a, b, c are in G. P and conversely.

3. Geometric Progression (GP)

If the algebraic ratio between any two consecutive terms is the same throughout the sequence, the sequence is called a GP sequence.

i.e. $\frac{t_{r+1}}{t_r} = \text{constant} = r$, and r is known as the common ratio.

Note

1. No term of a GP sequence can be zero.
2. All terms of a GP sequence can never be negative.
3. $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$ is holds good only when either a or b is non-negative.

3.1 *n*th Term of a GP Sequence

Let the first term be a and the common ratio be r of a GP.

Then the sequence is $\{a, ar, ar^2, ar^3, \ldots, ar^{n-1}\}$

Thus $t_n = ar^{n-1}$

which is also known as general term of the GP.

3.2 *n*th Term of a GP Sequence from the End

Let the first term be a, the common ratio r and the numbers of terms be m of a GP.

The sequence is $\left\{a, ar, ar^2, \ldots, \frac{1}{r^2}, \frac{1}{\text{r}}, 1\right\}$.

Thus $t_n = \frac{l}{r^{n-1}} = \frac{ar^{m-1}}{r^{n-1}} = ar^{m-n}$, where a is the first term, m is the number of terms and n is the required term, which we want to find out.

3.3 Sum of *n* Terms of a GP

Let the first term be a and the common ratio be r of a GP.

Then the sequence is $\{a, ar, ar^2, ar^3, \ldots, ar^{n-1}\}$.

The sum of n terms of a GP, $S_n = \begin{cases} \left(\frac{1-r^n}{1-r}\right), & r < 1 \\ \left(\frac{r^n - 1}{r-1}\right), & r > 1 \\ n, & r = 1 \end{cases}$

Note: If $r = 1$, the sum is $S_n = N$

3.4 Basic Concepts of Infinity

1. $\lim_{n\to\infty} x^n = \infty$ when $x > 1$
2. $\lim_{n\to\infty} x^n = 0$ when $0 < x < 1$
3. The value of $2^\infty = \infty, 3^\infty = \infty, 4^\infty = \infty, 5^\infty = \infty$
4. The value of

$$\left(\frac{1}{2}\right)^\infty = 0, \left(\frac{1}{3}\right)^\infty = 0, \left(\frac{2}{3}\right)^\infty = 0, \left(\frac{3}{4}\right)^\infty = 0$$

3.5 Sum of Infinite Terms of a GP

As we know that,

$$S_n = a\left(\frac{1-r^n}{1-r}\right),$$

where a = first term, r = common ratio.

When $n \to \infty$, $r^n \to 0$

Thus $S_\infty = \frac{a}{1-r}$.

3.6 Properties of a GP Sequence

Property I

If $a_1, a_2, a_3, \ldots, a_n$ are in GP and k be a non-zero real number, then

(i) $a_1k, a_2k, a_3k, \ldots, a_nk$ are also in GP

(ii) $a_1/k, a_2/k, a_3/k, \ldots, a_n/k$ are also in GP
(iii) $a_1 + k, a_2 + k, a_3 + k, \ldots, a_n + k$ are not in G.P
(iv) $a_1 - k, a_2 - k, a_3 - k, \ldots, a_n - k$ are not in GP.

Property II
If $a_1, a_2, a_3, \ldots, a_n$ and $b_1, b_2, b_3, \ldots, b_n$ are two GPs, then
(i) $a_1b_1, a_2b_2, a_3b_3, \ldots, a_nb_n$ are also in GP
(ii) $a_1/b_1, a_2/b_2, a_3/b_3, \ldots, a_n/b_n$ are also in GP
(iii) $a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots, a_n + b_n$ are not in GP.
(iv) $a_1 - b_1, a_2 - b_2, a_3 - b_3, \ldots, a_n - b_n$ are not in GP

Property III
If $a_1, a_2, a_3, \ldots, a_n$ are in GP, then
(i) $a_1^r, a_2^r, a_3^r, \ldots, a_n^r$ are also in GP.
(ii) $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots, \frac{1}{a_n}$ are also in GP

Property IV
If $a_1, a_2, a_3, \ldots, a_n$ are in GP, then
$\log a_1, \log a_2, \log a_3, \ldots, \log a_n$ are in AP and vice-versa.

Property V
If $a_1, a_2, a_3, \ldots, a_n$ are in GP, then
$$a_1a_n = a_2a_{n-1} = a_3a_{n-2}$$

Property VI
If $a_1, a_2, a_3, \ldots, a_n$ are in GP, then
$$a_r = \sqrt{a_{r-k} \times a_{r+k}}$$
where $0 \le k \le n - r$.

Property VII
If a, b and c are in GP, then
$$b^2 = ac$$

Property VIII
If $a_1, a_2, a_3, \ldots, a_n$ are in G.P, then
$$a_2^2 = a_1a_3,\ a_3^2 = a_2a_4,\ a_4^2 = a_3a_5$$
and so on.

Property IX
Number of terms and terms of GP sequence
(i) Three numbers of a GP can be considered as $\frac{a}{r}, a, ar$.
(ii) Four numbers of a GP can be considered as $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3$
(iii) Five numbers of a GP can be considered as $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$.
(iv) Six numbers of a GP can be considered as $\frac{a}{r^5}, \frac{a}{r^3}, \frac{a}{r}, ar, ar^3, ar^5$.

4. Harmonic Progression (HP)

A sequence is said to be in HP if the sequence formed by the reciprocals of each terms are in AP.

If the sequence $a_1, a_2, a_3, \ldots, a_n$ are in HP, then $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots \frac{1}{a_n}$ are in AP.

Note
1. No term of HP can be zero.
2. The general term of an HP is
$$t_n = \frac{1}{a + (n-1)d} \quad [\text{as in AP, } t_n = a + (n-1)d]$$
3. There is no general formula to find out the sum of n-terms of an HP.
4. Questions of HP are generally solved by inverting the terms and making use of the properties of the corresponding AP sequence.

5. Recognisation of AP, GP and HP

Let a, b and c be three non-zero real numbers. Then
(i) a, b, c are in AP if and only if
$$\frac{a-b}{b-c} = \frac{a}{a} \text{ or } b = \frac{a+c}{2}$$
(ii) a, b and c are in GP if and only if
$$\frac{a-b}{b-c} = \frac{a}{b} \text{ or } b^2 = ac$$
(iii) a, b and c are in HP if and only if
$$\frac{a-b}{b-c} = \frac{a}{c} \text{ or } b = \frac{2ac}{a+c}.$$

6. Means (AM, GM and HM)

6.1 Arithmetic Mean (AM)

When three quantities are in AP, the middle one is said to be the arithmetic mean (AM) of the other two.
If a, b and c be in AP, b is said to be the AM between a and c.
(i) If $a, b \in R^+$, then AM = $(A) = \left(\frac{a+\text{b}}{2}\right)$
(ii) If $a, b, c \in R^+$, then
$$A = \left(\frac{a+b+c}{3}\right)$$
(iii) If $a_1, a_2, a_3, \ldots, a_n \in R^+$, then
$$A = \left(\frac{a_1 + a_2 + a_3 + \ldots + a_n}{n}\right)$$

Insertion of n arithmetic means between two positive real numbers.

Let two positive real numbers are a and b.
Let $A_1, A_2, \ldots, A_n$ be n arithmetic means inserted between two numbers a and b.
Then $a, A_1, A_2, \ldots, A_n, b$ must be in AP.
Let the common difference be d.
Thus $\quad t_{n-2} = b$

$\Rightarrow \quad a + (n+1)d = b$

$\Rightarrow \quad d = \dfrac{(b-a)}{n+1}$

Hence, $A_1 = a + d = a + \left(\dfrac{b-a}{n+1}\right)$.

Similarly,

$$A_2 = a + 2d = a + 2\left(\dfrac{b-a}{n+1}\right).$$

$$\vdots$$

$$A_n = a + nd = a + n\left(\dfrac{b-a}{n+1}\right)$$

Property

If n arithmetic means inserted between two positive real numbers, say a and b, the sum of the n-arithmetic means is equal to n-times the single arithmetic mean between two positive real numbers, i.e.

$$A_1 + A_2 + A_3 + \ldots + A_n = n \times \left(\dfrac{a+b}{2}\right).$$

6.2 Geometric Mean

When three quantities are in GP, the middle one is called the geometric mean (GM) between the other two.

If a, b and c are in GP, then b is the geometric mean between a and c.

(i) If $a, b \in R^+$, then GM = $(G) = \sqrt{ab}$.

(ii) If $a, b, c \in R^+$, then GM = $(G) = \sqrt[3]{abc}$.

(iii) If $a_1, a_2, a_3, \ldots, a_n \in R^+$, then GM = $(G) = \sqrt[n]{a_1 a_2 \ldots a_n}$.

Insertion of n geometric means between two positive real numbers

Let two positive real numbers be a and b.

Let $G_1, G_2, \ldots, G_n$ are n geometric means inserted between the two given positive numbers, a and b.

Then $a, G_1, G_2, \ldots, G_n, b$ must be in G.P.

Thus, $t_{n-2} = b$

$\Rightarrow \quad ar^{n+1} = b$

$\Rightarrow \quad r = \left(\dfrac{b}{a}\right)^{\frac{1}{n+1}}$

So, $G_1 = ar = a\left(\dfrac{b}{a}\right)^{\frac{1}{n+1}}$,

Similarly,

$$G_2 = ar^2 = a\left(\dfrac{b}{a}\right)^{\frac{2}{n+1}}$$

$$\vdots$$

$$G_n = ar^n = a\left(\dfrac{b}{a}\right)^{\frac{n}{n+1}}$$

Property

The product of n geometric means inserted between two positive numbers, say a and b, is equal to the nth power of the single geometric mean between the two given positive numbers, i.e.

$$G_1, G_2, \ldots, G_n = (ar)(ar^2) \ldots (ar^n)$$

$$= a^n \times r^{1+2+3+\ldots+n}$$

$$= a^n \times (r)^{\frac{n(n+1)}{2}}$$

$$\left[\because 1 + 2 + 3 + \cdots + n = \dfrac{n(n+1)}{2}\right]$$

$$= a^n \times \left(\left(\dfrac{b}{a}\right)^{\frac{1}{n+1}}\right)^{\frac{n(n+1)}{2}}$$

$$= a^n \times \left(\dfrac{b}{a}\right)^{\frac{n}{2}}$$

$$\left[\because \left(\dfrac{b}{a}\right)^m = 6^m \times a^{-m}\right]$$

$$= (ab)^{\frac{n}{2}}$$

$$= (\sqrt{ab})^n.$$

6.3 Harmonic Mean (HM)

When three non-zero quantities are in HP, the middle one is called the harmonic mean (HM) between the other two.

If a, b and c are in HP, then b is called the harmonic mean between a and c.

(i) If $a, b \in R^+$, then HM = $(H) = \dfrac{2}{\frac{1}{a} + \frac{1}{b}}$

(ii) If $a, b, c \in R^+$, then $H = \dfrac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}$.

(iii) If $a_1, a_2, a_3, \ldots, a_n \in R^+$, then

$$H = \dfrac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_n}}$$

Insertion of n harmonic means between two positive real numbers

Let two positive numbers be a and b respectively.

Let $H_1, H_2, \ldots, H_n$ are n harmonic means inserted between two positive real numbers a and b.

Then $a, H_1, H_2, \ldots, H_n, b$ are in HP.

Thus, $\dfrac{1}{a}, \dfrac{1}{H_1}, \dfrac{1}{H_2}, \ldots, \dfrac{1}{H_n}, \dfrac{1}{b}$ are in AP.

Now $\dfrac{1}{b} = \dfrac{1}{a} + (n+1)d$

$$\Rightarrow \quad \frac{1}{b} - \frac{1}{a} = (n+1)d$$

$$\Rightarrow \quad d = \frac{a-b}{(n+1)ab}$$

Therefore, $\dfrac{1}{H_1} = \dfrac{1}{a} + d = \dfrac{1}{a} + \left(\dfrac{a-b}{(n+1)ab}\right)$

Similarly, $\dfrac{1}{H_2} = \dfrac{1}{a} + d = \dfrac{1}{a} + 2\left(\dfrac{a-b}{(n+1)ab}\right)$

$$\frac{1}{H_3} = \frac{1}{a} + d = \frac{1}{a} + 3\left(\frac{a-b}{(n+1)ab}\right)$$

$$\vdots$$

$$\frac{1}{H_n} = \frac{1}{a} + d = \frac{1}{a} + n\left(\frac{a-b}{(n+1)ab}\right)$$

Property

The sum of the reciprocal of n harmonic means is equal to n times the single harmonic mean between the two given positive real numbers.

Let $H_1, H_2, \ldots, H_n$ are n harmonic means inserted between two positive real numbers a and b.

Since $a, H_1, H_2, \ldots, H_n, b$ are in HP

$\dfrac{1}{a}, \dfrac{1}{H_1}, \dfrac{1}{H_2}, \ldots, \dfrac{1}{H_n}, \dfrac{1}{b}$ are in AP

Therefore, $\dfrac{1}{H_1} + \dfrac{1}{H_2} + \ldots + \dfrac{1}{H_n} = n \times \left(\dfrac{\frac{1}{a} + \frac{1}{b}}{2}\right)$

$$= n \times \frac{1}{\left(\dfrac{2}{\frac{1}{a} + \frac{1}{b}}\right)}$$

Relation amongst the AM, GM and HM

Let $a, b \in R^+$.

Then AM = $(A) = \left(\dfrac{a+b}{2}\right)$,

GM = $(G) = \sqrt{ab}$

and HM = $(H) = \left(\dfrac{2ab}{a+b}\right)$

Now, $A \times H = \left(\dfrac{a+b}{2}\right) \times \left(\dfrac{2ab}{a+b}\right)$

$$= ab = G^2$$

Thus, A, G, H are in GP.

$$\Rightarrow \quad G^2 = AH$$

$$\Rightarrow \quad \frac{A}{G} = \frac{G}{H} \qquad \ldots\text{(i)}$$

Also, $A - G = \dfrac{a+b}{2} - \sqrt{ab}$

$$= \frac{1}{2}(\sqrt{a} - \sqrt{b})^2 \ge 0$$

$$\Rightarrow \quad A \ge G \Rightarrow \frac{A}{G} \ge 1 \qquad \ldots\text{(ii)}$$

From Relations (i) and (ii), we get

$$\frac{G}{H} = \frac{A}{G} \ge 1$$

$$\Rightarrow \quad G \ge H$$

Hence, $A \ge G \ge H$

$$\Rightarrow \quad \text{AM} \ge \text{GM} \ge \text{HM}$$

> ***Notes***
>
> (i) If $\dfrac{a^{n+1} + b^{n+1}}{a^n + b^n}$ be the AM between two numbers a and b, then $n = 0$
>
> (ii) If $\dfrac{a^{n+1} + b^{n+1}}{a^n + b^n}$ be GM between two positive numbers a and b, then $n = -1/2$
>
> (iii) If $\dfrac{a^{n+1} + b^{n+1}}{a^n + b^n}$ be the HM between two positive numbers a and b, then $n = -1$.

7. *m*th Powers Theorem

If $a_1, a_2, \ldots, a_n$ be a set of positive numbers and all the a's are not equal,

$$\frac{\left(\sum_{i=1}^{n} a_i^m\right)}{n} > \left(\frac{\sum_{i=1}^{n} a_i}{n}\right)^m,$$

when $0 < m < 1$

and $$\frac{\left(\sum_{i=1}^{n} a_i^m\right)}{n} < \left(\frac{\sum_{i=1}^{n} a_i}{n}\right)^m$$

when $m \in R - (0, 1)$.

8. Cauchy-Schwartz Inequality

If a, b, c and x, y, z are any real numbers (positive, negative or zero), then

$$(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) \ge (ax + by + cz)^2$$

9. Maximum and Minimum Values of Positive Real Numbers

Let us suppose that $x, y, z, \ldots, w$ and are n positive variables and c is constant.

(i) **Maximum Value**

Let $x + y + z + \ldots + w = c$

As we know that, AM ≥ GM

Thus, $\left(\frac{x+y+z+\ldots+w}{n}\right) \geq \sqrt[n]{x \cdot y \cdot z \ldots w}$

Hence, the greatest value of

$\sqrt[n]{x \cdot y \cdot z \cdots w}$ is $\left(\frac{c}{n}\right)^{1/n}$

(ii) **Minimum Value**

Let $x + y + z + \ldots + w = c$.

As we know that, AM ≥ GM

Thus, $\left(\frac{x+y+z+...+w}{n}\right) \geq \sqrt[n]{x \cdot y \cdot z \ldots w}$

Hence, the least value of $x + y + z + \ldots + w$ is $n(c)^{1/n}$.

10. Arithmetico Geometric Progression

The combine form of AP and GP will form an AGP.

Let the first term of an AP be a, d the common difference and r the common ratio of a GP.

Then the AP Sequence is $\{a, a+d, a+2d, \ldots, a+(n-1)d\}$ and the GP sequence is $\{1, r, r^2, \ldots, r^{n-1}\}$.

Thus the AGP sequence is

$$\{a, (a + d)r, (a + 2d)r^2, \ldots, (a + (n - 1)d)r^{n-1}\}$$

10.1 *n*th term of an AGP Sequence

The nth term of AGP sequence is

$$t_n = (a + (n - 1)d)r^{n-1},$$

where a = first term, d = common difference of AP, r = the common ratio of GP.

10.2 Sum of *n* terms of AGP

Let the first term be a, common difference d and the common ratio be r of an AGP sequence.

Then the AGP sequence is

$$\{a, (a + d)r, (a + 2d)r^2, \ldots, (a + (n - 1)d)r^{n-1}\}$$

Let $S_n = a + (a + d)r + (a + 2d)r^2 + \ldots + (a + (n - 1)d)r^{n-1}$

$\therefore \quad rS_n = ar + (a + d)r^2 + (a + 2d)r^3 + \ldots$
$\ldots + (a + (n - 2)d)r^{n-1} + (a + (n - 1)d)r^n$

Now, $(1 - r)S_n$

$= a + (dr + dr^2 + dr^3 + \ldots + dr^{n-1}) + [a + (n - 1)d]r^n$

$= a + (1 + r + r^2 + \ldots + r^{n-2})dr\ [a + (n - 1)d]r^n$

$$\Rightarrow \quad S_n = \left(\frac{a}{1-r}\right) + \left(\frac{1+r+r^2+\ldots+r^{n-2}}{1-\mathrm{r}}\right)dr + \left(\frac{(a+(n-1)d)r^n}{r-1}\right)$$

$$\Rightarrow \quad S_n = \left(\frac{a}{1-r}\right) + \left(\frac{1-r^{n-1}}{(1-r)^2}\right)dr + \left(\frac{[a+(n-1)d]r^n}{r-1}\right)$$

10.3 Sum of Infinite Terms of AGP

When n is infinite and $-1 < r < 1$, then

$r^n, r^{n-1} \to 0$.

Thus $S_\infty = \left(\frac{a}{1-r}\right) + \left(\frac{dr}{(1-r)^2}\right)$

11. Summation of Series

(i) Sum of the first n natural numbers

$$= 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$$

(ii) Sum of the first $2n$ even natural numbers

$$= 2 + 4 + 6 + \ldots + 2n = n(n + 1)$$

(iii) Sum of the first $2n - 1$ odd natural numbers

$$= 1 + 3 + 5 + \ldots + (2n - 1) = n^2$$

(iv) Sum of the squares of first n natural numbers

$$= 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

(v) Sum of the cubes of first n natural numbers

$$= 1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

(vi) Sum of the fourth power of first n natural numbers

$$= 1^4 + 2^4 + \ldots + n^4$$
$$= \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

(vii) $a + a + a + a + \ + a(n \text{ times}) = na$.

(viii) $\sin\alpha + \sin(\alpha + \beta) + \sin(\alpha + 2\beta) + \ldots + \sin[\alpha + (n - 1)\beta]$

$$= \frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)} \times \sin\left(\alpha + (n-1)\frac{\beta}{2}\right)$$

(ix) $\cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \ldots + \cos(\alpha + (n - 1)\beta)$

$$= \frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)} \times \cos\left(\alpha + (n-1)\frac{\beta}{2}\right)$$

(x) Sum of the product of numbers taken two at a time

Let $a_1, a_2, a_3, \ldots, a_n \in R$, then

$$(a_1 + a_2 + a_3 + \ldots + a_n)^2 = \sum_{i=1}^{n} a_i^2 + 2\sum_{1 \leq i < j \leq n} a_i a_j$$

$$\Rightarrow \quad \left(\sum_{i=1}^{n} \mathrm{a_i}\right)^2 = \sum_{i=1}^{n} a_i^2 + 2\sum_{1 \leq i < j \leq n} a_i a_j$$

$$\Rightarrow \quad \sum_{1 \leq i < j \leq n} a_i a_j = \frac{1}{2}\left(\left(\sum_{i=1}^{n} a_i\right)^2 - \sum_{i=1}^{n} a_i^2\right)$$

12. Method of Differences

(i) Consider the series
$3 + 7 + 14 + 24 + 37 + ...$
Here, the differences between the successive terms are 4, 7, 10, 13, …, which are in AP. Whenever the successive differences are in AP, we consider its nth term as $t_n = an^2 + bn + c$.
In order to find the value of a, b and c, let us put $n = 1, 2, 3$
Thus, $t_1 = 3 = a + b + c$,
$t_2 = 7 = 4a + 2b + c$
and $t_3 = 14 = 9a + 3b + c$.

On solving, we get,

$$a = \frac{3}{2}, b = -\frac{1}{2} \text{ and } c = 2.$$

Thus, $t_n = \frac{1}{2}(3n^2 - n + 4)$.

$$\text{Hence, } S_n = \frac{1}{2}[3\Sigma n^2 - \Sigma n + \Sigma 4]$$
$$= \frac{1}{2}\left[3 \cdot \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + 4n\right]$$
$$= \frac{n}{2}(n^2 + n + 4)$$

Exercises

Level I

(Questions Based on Fundamentals)

nth TERM OF AN AP

1. In an AP, if its $t_n = 3n + 5$, find its common difference.
2. If the pth term of an AP is q and the qth term is p, prove that its nth term is $(p + q - n)$.
3. In an AP, if $mt_m = nt_n$, show that $t_{m+n} = 0$
4. If the mth term of an AP be $\frac{1}{n}$ and nth term be $\frac{1}{m}$, then show that its (mn)th term is 1.
5. If $(m + 1)$th term of an AP is twice the $(n + 1)$th term, prove that $(3m + 1)$th term is twice the $(m + n + 1)$th term.
6. In an AP, if $T_p = q$, $T_{p+q} = 0$, find T_q.
7. A man starts repaying a loan as first instalment of ₹100. If he increases the instalment by ₹5 every month, what amount he will pay in the 30th instalment?
8. In an AP, prove that $t_{m+n} + t_{m-n} = 2t_m$
9. If pth, qth, rth terms of an AP be a, b, c respectively, show that:
   (i) $a(q - r) + b(r - p) + c(p - q) = 0$
   (ii) $(a - b)r + (b - c)p + (c - a)q = 0$
10. If $<a_n>$ is an AP such that $\frac{a_4}{a_7} = \frac{2}{3}$, find $\frac{a_6}{a_8}$.

SELECTIONS OF TERMS IN AN AP

11. If a, b, c, d, e are in AP, find the value of
    $a - 4b + 6c - 4d + e$.
12. Divide 69 into three parts to form an AP and the product of the two smaller parts is 483.
13. The angles of a quadrilateral are in AP whose common difference is 10°. Find the angles.
14. If the roots of the equation $x^3 - 12x^2 + 39x - 28 = 0$ are in AP, find the common difference.
15. Divide 20 into four parts to form an AP such that the ratio of the product of the first and the fourth to the product of the second and third is 2 : 3.
16. If $a_1, a_2, \ldots, a_n$ are in AP, prove that
$$\frac{1}{a_1a_2} + \frac{1}{a_2a_3} + \ldots + \frac{1}{a_{n-1}a_n} = \frac{n-1}{a_1a_n}$$
17. If $a_1, a_2, \ldots, a_n$ are in AP, prove that
$$\frac{1}{\sqrt{a_1} + \sqrt{a_2}} + \frac{1}{\sqrt{a_2} + \sqrt{a_3}} + \ldots + \frac{1}{\sqrt{a_{n-1}} + \sqrt{a_n}} = \frac{n-1}{\sqrt{a_1} + \sqrt{a_n}}$$
    $a_i > 0$ for all i
18. If $a_1, a_2, \ldots, a_n$ are in AP, prove that
$$\frac{1}{a_1a_n} + \frac{1}{a_2a_{n-1}} + \frac{1}{a_3a_{n-2}} + \ldots + \frac{1}{a_na_1} = \frac{2}{a_1 + a_n}\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_n}\right)$$
19. In an AP, $t_7 = 15$, find the value of common difference, d that would make the value of $t_2\, t_7\, t_{12}$ maximum.
20. If the non-zero numbers a, b, c are in AP and $\tan^{-1} a$, $\tan^{-1} b$, $\tan^{-1} c$ are also in AP, prove that
    (i) $b^2 = ac$
    (ii) $a = b = c$

SUM TO N-TERMS OF AN AP

21. Find the sum of the series
    $5 + 13 + 21 + \ldots + 181$.
22. Find the sum of all three-digit natural numbers which are divisible by 7.
23. Find the sum of all natural numbers between 250 and 1000 which are exactly divisible by 3.
24. Find the sum of first 24 terms of the AP $a_1, a_2, a_3, \ldots, a_n$ if it is known that $a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$.

25. In an AP,
 (i) if $S_n = 3n^2 + 4n$, find its 10th term
 (ii) if $S_n = 2n^2 + 5n$, find its common difference.
26. If the mth term of an AP is $\frac{1}{n}$ and the nth term is $\frac{1}{m}$, show that the sum of mnth term is $\frac{1}{2}$ $(mn + 1)$.
27. Solve for x $1 + 6 + 11 + 16 + \ldots + x = 148$.
28. The sum of the first p, q, r terms of an AP are a, b, c respectively. Show that
$$\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)=0$$
29. The sum of n, $2n$, $3n$ terms of an AP are S_1, S_2, S_3 respectively. Prove that
$$S_3 = 3(S_2 - S_1).$$
30. The pth term of an AP is a and qth term is b. Prove that the sum of its $(p + q)$ terms is
$$\left(\frac{p+q}{2}\right)\left\{a+b+\frac{a-b}{p-q}\right\}.$$
31. If in an AP, the sum of m terms is equal to n and the sum of n terms is equal to m, prove that the sum of $(m + n)$ terms is $-(m + n)$.
32. The ratio of the sum of n terms of two APs is $\frac{(7n+1)}{(4n+27)}$. Find the ratio of their 11th terms.
33. The ratio of the sum of m and n terms of an AP is $m^2 : n^2$. Show that the ratio of the mth and nth terms is $(2m - 1) : (2n - 1)$.
34. The interior angles of a polygon are in AP. The smallest angle is 120° and the common difference is 5°. Find the number of sides of the polygon.
35. If $S_1, S_2, S_3, \ldots, S_m$ are the sum of n terms of m APs whose first terms are $1, 2, 3, \ldots, m$ and the common differences are $1, 3, 5, \ldots, (2m - 1)$, respectively. Show that
$$S_1 + S_2 + \ldots + S_m = \frac{mn}{2}(mn+1).$$
36. If $S_n = n^2p$ and $S_m = m^2p$, $m \neq n$, in an AP, prove that $S_p = p^3$.
37. If S_1 be the sum of $(2n + 1)$ terms of an AP and S_2 be the sum of its odd terms, prove that
$$S_1 : S_2 = (2n + 1) : (n + 1).$$
38. If S_1 be the sum of odd terms and S_2 be the sum of even terms of an AP consists of $(2n + 1)$ terms, prove that $S_1 : S_2 = (n + 1) : n$.
39. If the sum of n terms of an AP is $nP+\frac{1}{2}n(n-1)Q$, where P and Q are constants, find the common difference.

PROPERTIES OF AP

40. If a, b, c are in AP, prove that the following are also in AP.
 (i) $b + c, c + a, a + b$
 (ii) $a^2(b + c), b^2(c + a), c^2(a + b)$
 (iii) $\frac{1}{\sqrt{b}+\sqrt{c}}, \frac{1}{\sqrt{c}+\sqrt{a}}, \frac{1}{\sqrt{a}+\sqrt{b}}$
 (iv) $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$
 (v) $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$
 (vi) $[(b+c)^2-a^2], [(a+c)^2-b^2], [(b+a)^2-c^2]$
41. If a^2, b^2, c^2 are in AP, prove that the following terms are also in AP.
 (i) $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$
 (ii) $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$
42. If a, b, c are in AP, show that
 (i) $a^2(b + c), b^2(c + a), c^2(a + b)$ are in AP.
 (ii) $b + c - a, c + a - b, a + b - c$ are in AP.
 (iii) $bc - a^2, ca - b^2, ab - c^2$ are in AP.
43. If $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in AP, prove that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are also in AP.

INSERTION OF AM

44. Insert three arithmetic means between 3 and 19.
45. For what value of n, $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is the arithmetic mean of a and b?
46. If n arithmetic means are inserted between 20 and 80 such that the ratio of first mean to the last mean is 1: 3, find the value of n.
47. Prove that the sum of n arithmetic means between two numbers is n times the single AM between them.
48. If the AM between pth and qth terms of an AP be equal to the AM between rth and sth terms of the AP, show that $p + q = r + s$.
49. There are n AMs between 3 and 17. If the ratio of the last mean to the first mean is 3: 1, find the value of n.
50. If x, y, z are in AP and A_1 is the AM of x and y; and A_2 is the AM of y and z, prove that the AM of A_1 and A_2 is y.

GEOMETRIC PROGRESSION

*n*th term of the GP

51. Find the 10th term of the GP $\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}, \ldots$
52. Which term of the GP $\sqrt{2}, \frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{4\sqrt{2}}, \ldots$ is $\frac{1}{512\sqrt{2}}$?

53. Which term of the GP 2, 1, $\frac{1}{2}, \frac{1}{4}, \ldots$ is $\frac{1}{128}$?
54. If the 2nd and 5th terms of a GP are 24 and 81, respectively, find the GP.
55. If the 3rd term of a GP is 3, then find the product of its first five terms.
56. The sum of three numbers in GP is 21 and the sum of their squares is 189, find the numbers.
57. If the first term of a GP is 1 and the sum of the 5th and 1st term is 82, find the common ratio.
58. If the pth, qth and rth terms of a GP are a, b and c, respectively, prove that
$$a^{q-r} \cdot b^{r-p} \cdot c^{p-q} = 1$$
59. If the first and the nth terms of a GP are a and b, respectively and if P is the product of the first n terms, prove that $P^2 = (ab)^n$.
60. The $(m + n)$th and $(m - n)$th terms of a GP are p and q, respectively. Show that the mth and nth terms are $\sqrt{pq}$ and $p\left(\frac{q}{p}\right)^{m/2n}$, respectively.
61. If a, b, c, d and p are different real numbers such that $(a^2 + b^2 + c^2)p^2 - 2(ab + bc + cd)p + (b^2 + c^2 + d^2) = 0$, show that a, b, c and d are in GP.
62. If a, b, c are respectively the pth, qth and rth terms of a GP, show that
$$(q - r) \log a + (r - p) \log b + (p - q) \log c = 0$$
63. If $(1 - k)(1 + 2x + 4x^2 + 8x^3 + 16x^4 + 32x^5) = 1 - k^6$, where $k \neq 1$, find the value of $\frac{k}{x}$.
64. If α and β be the roots of $x^2 - 3x + a = 0$ and γ and δ be the roots of $x^2 - 12x + b = 0$ and numbers $\alpha, \beta, \gamma, \delta$ (in order) form an increasing GP, prove that the value of $a = 2$ and $b = 32$.
65. If three distinct real numbers x, y, z are in GP such that $x + y + z = ax$, find the value of a.

SUM OF *N* TERMS OF GP

66. Find the sum of $1 + 3 + 9 + 27 + \ldots$ to n terms.
67. Find the sum of $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ to n terms.
68. How many terms of the series $1 + 3 + 3^2 + 3^3 + \ldots$ must be taken to make 3280?
69. Find the sum of the geometric series $(x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + \ldots$ to n-terms.
70. Find the sum of the geometric series
$$\frac{a}{(1+i)} + \frac{a}{(1+i)^2} + \frac{a}{(1+i)^3} + \ldots + \frac{a}{(1+i)^n}.$$
71. Find the value of $\sum_{k=1}^{10}(2 + 3^k)$.
72. Evaluate $\sum_{n=1}^{n}(2^{n-1} + 3^n)$.
73. Evaluate
 (i) $\sum_{k=1}^{n}(4^{k-1} + 5^{k+1})$
 (ii) $\sum_{n=1}^{n}\left(\left(\frac{1}{3}\right)^{n-1} + \left(\frac{1}{5}\right)^{n+1}\right)$
74. Find the sum of the following series
 (i) $5 + 55 + 555 + \ldots$ to n terms
 (ii) $7 + 77 + 777 + \ldots$ to n terms
 (iii) $9 + 99 + 999 + \ldots$ to n terms
75. Find the sum of $(6666 \ldots 6)^2 + (8888 \ldots 8)^2$ (upto n digits).
76. Find the sum $\sum_{n=1}^{10}\left[\left(\frac{1}{2}\right)^{n-1} + \left(\frac{1}{5}\right)^{n+1}\right]$.
77. Prove that the sum to n terms of the series $11 + 103 + 1005 + \ldots$ is $\frac{10}{9}(10^n - 1) + n^2$.
78. Find the least value of n for which the sum $1 + 3 + 3^2 + \ldots$ to n terms is greater than 7000.
79. Find the sum of n terms of the series $\frac{1}{3} + \frac{5}{9} + \frac{19}{27} + \frac{65}{81} + \ldots$ to n terms.
80. If $S = \frac{2}{3} + \frac{8}{9} + \frac{26}{27} + \frac{30}{81} + \ldots$ to n terms, find the value of S.
81. If S be the sum, P the product and R the sum of the reciprocals of n terms of a GP, prove that $\left(\frac{S}{R}\right)^n = P^2$.
82. If f is a function satisfying $f(x + y) = f(x) + f(y)$ for all $x, y \in N$ such that $f(1) = 3$ and $\sum_{x=1}^{n} f(x) = 120$, find the value of n.
83. Let a_n be the nth term of the GP of positive numbers. Let $\sum_{n=1}^{100} a_{2n} = \alpha$ and $\sum_{n=1}^{100} a_{2n-1} = \beta$ such that $\alpha \neq \beta$. Prove that the common ratio of the GP is $\frac{\alpha}{\beta}$.

SUM OF AN INFINITE GP

84. Find the sum of
$$(\sqrt{2} + 1) + 1 + (\sqrt{2} - 1) + \ldots \text{to } \infty$$
85. Find the sum of
$$\frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^4} + \frac{1}{2^5} + \frac{1}{3^6} + \ldots \infty$$

86. If $b = a + a^2 + a^3 + \ldots \infty$, prove that $a = \frac{b}{1+b}$.

87 If $x = a + \frac{a}{r} + \frac{a}{r^2} + \ldots \infty$, $y = b - \frac{b}{r} + \frac{b}{r^2} - \ldots \infty$ and, $z = c + \frac{c}{r^2} + \frac{c}{r^4}$ … prove that $\frac{xy}{z} = \frac{ab}{c}$.

88. If $x = 1 + a + a^2 + \ldots \infty$, where $|a| < 1$ and $y = 1 + b + b^2 + \ldots \infty$, where $|b| < 1$. Prove that

$$1 + ab + (ab)^2 + \ldots \infty = \frac{xy}{x+y-1}$$

89. If $A = 1 + r^a + r^{2a} + \ldots$ to ∞ and $B = 1 + r^b + r^{2b} + \ldots \infty$, prove that

$$r = \left(\frac{A-1}{A}\right)^{1/a} = \left(\frac{B-1}{B}\right)^{1/b}$$

90. If $x = \sum_{n=0}^{\infty} \cos^{2n}\theta,\ y = \sum_{n=0}^{\infty} \sin^{2n}\varphi\ \ z = \sum_{n=0}^{\infty} \cos^{2n}\theta \sin^{2n}\varphi$, where $0 < \theta, \varphi < \frac{\pi}{2}$, prove that $xz + yz - z = xy$.

91. If $|x| < 1$ and $|y| < 1$, find the sum $(x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + \ldots$ to ∞

92. If S_p denotes the sum of the series $1 + r^p + r^{2p} + \ldots$ to ∞ and s_p the sum of the series $1 - r_p + r^{2p} - \ldots$ to ∞ prove that $S_p + s_p = 2.\ S_{2p}$.

93. Let $S \subset (-\pi, \pi)$ denotes the set of values of x satisfying the equation $8^{1+|\cos x|+|\cos x|^2+|\cos x|^3+\ldots \text{ to } \infty} = 4^3$, prove that $S = \left(-\frac{\pi}{3}, -\frac{2\pi}{3}\right)$.

94. If $\exp\{(\sin^2 x + \sin^4 x + \sin^6 x + \ldots \infty)$ satisfies the equation $x^2 - 9x + 8 = 0$, prove that the value of $\left(\frac{\cos x}{\cos x + \sin x}\right) = \frac{1}{2}(\sqrt{3} - 1)$.

95. Find the value of $(0.2)^{\log_{\sqrt{2}}\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots\text{inf}\right)}$.

PROPERTIES OF GP

96. If p, q, r are in an AP, show that the pth, qth and rth terms of a GP are in GP.

97. If a, b, c are in AP and x, y, z are in G.P., show that $x^{b-c}.\ y^{c-a}.\ z^{a-b} = 1$.

98. If a, b, c are in GP, prove that $\log(a^n), \log(b^n), \log(c^n)$ are in AP.

99. If a, b, c are in GP and x, y are the arithmetic means of a, b and b, c respectively, prove that $\frac{a}{x} + \frac{c}{y} = 2$ and $\frac{1}{x} + \frac{1}{y} = \frac{2}{b}$.

100. If $a, b, 3c$ are in AP and $a, b, 4c$ are in GP, find the value of a/b.

101. If $x^a = x^{\frac{b}{2}} z^{\frac{b}{2}} = z^c$, prove that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in AP.

102. If a, b, c are three distinct real numbers in GP and $a + b + c = xb$, prove that $x < -1$ or $x > 3$.

103. If a, b, c in GP, prove that

(i) $a(b^2 + c^2) = c(a^2 + b^2)$

(ii) $a^2b^2c^2\left(\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3}\right) = a^3 + b^3 + c^3$

(iii) $\frac{(a+b+c)^2}{a^2+b^2+c^2} = \frac{a+b+c}{a-b+c}$

(iv) $\frac{1}{a^2-b^2} + \frac{1}{b^2} = \frac{1}{b^2-c^2}$

(v) $(a + 2b + 2c)(a - 2b + 2c) = a^2 + 4c^2$

104. Suppose a, b, c are in AP and a^2, b^2, c^2 are in GP. If $a < b < c$ and $a + b + c = \frac{3}{2}$, find the value of a.

INSERTION OF GM

105. Find the value of n so that $\frac{a^{n+1} + b^{n+1}}{a^n + b^n}$ may be the geometric mean between a and b.

106. Find two positive numbers whose difference is 12 and whose AM exceeds the GM by 2.

107. Let x be the arithmetic mean, and y and z be two geometric means between any two positive numbers. Prove that $\frac{y^3 + z^3}{xyz} = 2$.

108. If a be the AM of b and c and the two geometric means are G_1 and G_2, prove that $G_1^3 + G_2^3 = 2abc$.

109. If one geometric mean G and two arithmetic means A_1 and A_2 be inserted between two given quantities, prove that $G^2 = (2A_1 - A_2)(2A_2 - A_1)$.

110. If a, b, c are in GP and the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root, show that $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in AP.

111. The sum of two numbers is 6 times their geometric means, show that the numbers are in the ratio $(3 + 2\sqrt{2}) : (3 - 2\sqrt{2})$.

HARMONIC PROGRESSION

112. If mth term of an HP is n and nth term is m, prove that (mn)th term is 1.

113. If mth term of an HP is n and nth term is m, prove that $(m + n)$th term is $\frac{mn}{m+n}$.

114. If pth term of an HP is $1/q$ and qth term is $1/p$, prove that nth term is $\frac{1}{p+q-n}$.

115. If a_1, a_2, a_3 are in AP; a_2, a_3, a_4 are in GP and a_3, a_4, a_5 are in HP, prove that a_1, a_3, a_5 are in GP.
116. If $a^x = b^y = c^z = d^w$ and a, b, c, d are in GP, prove that x, y, z, w are in HP
117. If $x > 1$, $y > 1$ and $z > 1$ are in GP, prove that $\frac{1}{1+\ln x}, \frac{1}{1+\ln y}, \frac{1}{1+\ln z}$ are in HP.
118. If pth, qth and rth terms of an HP are a, b, c, respectively, prove that $\frac{q-r}{a}+\frac{r-p}{b}+\frac{p-q}{c}=0$.
119. If $x_1, x_2, x_3, \ldots, x_n$ are in HP, prove that
$$x_1x_2 + x_2x_3 + \ldots + x_{n-1}x_n = (n-1)x_1x_n$$
120. Let a, b, c be three distinct real numbers in AP such that $|a| < 1, |b| < 1, |c| < 1$, if
$$x=\sum_{n=0}^{\infty} a^n,\ y=\sum_{n=0}^{\infty} b^n,\ z=\sum_{n=0}^{\infty} c^n,$$
then prove that x, y, z are in HP.
121. Let $a_1, a_2, a_3, \ldots, a_{10}$ are in AP and $h_1, h_2, h_3, \ldots, h_{10}$ are in HP. If $a_1 = 2 = h_1$ and $a_{10} = 3 = h_{10}$, find the value of $a_4h_7 + 2007$.
122. If $a^2+9b^2+25c^2=abc\left(\frac{15}{a}+\frac{5}{b}+\frac{3}{c}\right)$, prove that a, b, c are in HP
123. If a, x, b are in AP, a, y, b are in GP and a, z, b are in HP such that $x = 9z$, where $a, b > 0$, prove that $y = 3|z|$ and $x = 3|y|$.
124. If A_1A_2; G_1G_2; H_1H_2 are arithmetic, geometric and harmonic means, respectively between two quantities a and b, prove that
    (i) $\frac{G_1G_2}{H_1H_2}=\frac{A_1+A_2}{H_1+H_2}$
    (ii) $A_1H_2 = A_2H_1 = G_1G_2 = ab$.
125. If $4a^2 + 9b^2 + 16c^2 = 2(3ab + 6bc + 4ac)$, where a, b, c are non-zero real numbers, prove that a, b, c are in HP.
126. If a, b, c are in HP such that $\left(\frac{a+b}{2a-b}\right)+\left(\frac{c+b}{2c-b}\right)>\lambda$, find the value of λ.
127. If a, b, c are three positive real numbers, prove that $\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\ge 3$.
128. If a, b, c are in HP, find the value of $\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{c}+\frac{1}{a}-\frac{1}{b}\right)$.
129. If a, b, c, d are in HP, prove that $ab + bc + cd = 3ad$.
130. If a, b, c are in AP; x, y, z are in HP; ax, by, cz are in GP, prove that $\frac{x}{z}+\frac{z}{x}=\frac{a}{c}+\frac{c}{a}$.
131. If $\frac{a-x}{px}=\frac{a-y}{qy}=\frac{a-z}{rz}$ and p, q, r are in AP, prove that x, y, z are in HP.

ARITHMETICO GEOMETRIC PROGRESSION

132. Find the nth term of the following arithmetico geometric series.
    (i) $1 + 5x + 9x^2 + 13x^3 + \ldots$
    (ii) $1 - 3x + 5x^2 - 7x^3 + \ldots$
    (iii) $1+\frac{2}{3}+\frac{3}{3^2}+\frac{4}{3^3}+\ldots$
133. Find the sum of n terms of the following series.
    (i) $1 + 2x + 3x^2 + 4x^3 + \ldots$
    (ii) $1 + 4x + 7x^2 + 10x^3 + \ldots$
    (iii) $1+\frac{3}{2}+\frac{5}{4}+\frac{7}{8}+\ldots$
134. Find the sum to infinity of the following series when $|x| < 1$.
    (i) $1 + 2x + 3x^2 + 4x^3 + \ldots$
    (ii) $1 + 3x + 5x^2 + 7x^3 + \ldots$
    (iii) $1 + 4x + 7x^2 + 10x^3 + \ldots$
135. Find the sum to n terms of the series
    (i) $1+\frac{3}{2}+\frac{5}{4}+\frac{7}{8}+\frac{9}{16}+\ldots$
    (ii) $\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+\ldots$
    (iii) $\frac{2}{3}+\frac{8}{9}+\frac{26}{27}+\frac{80}{81}+\ldots$
    (iv) $2+\frac{7}{4}+\frac{10}{8}+\frac{13}{16}+\frac{16}{32}+\ldots$
136. If the sum to infinity of the series
    (i) $3+(3+d)\frac{1}{4}+(3+2d)\frac{1}{4^2}+\ldots$ to ∞ is $\frac{44}{9}$, find d.
    (ii) $1 + 2r + 3r^2 + 4r^3 + \ldots$ is 9/4, find r.

SOME SPECIAL SEQUENCES

137. Find the sum to n terms of
    1.2 + 2.3 + 3.4 + 5.6 + …
138. Find the sum to n terms of
    1.4 + 3.7 + 5.10 + 7.13 + …
139. Find the sum to n terms of
    1.2.3 + 2.3.4 + 3.4.5 + …
140. Find the sum to n terms of
    1.3.5 + 2.5.7 + 3.7.9 + …
141. Find the sum to n terms of
    $1 + (1 + 2) + (1 + 2 + 3) + (1 + 2 + 3 + 4) + \ldots$
142. Find the sum to n terms of
    $1 + (1 + 3) + (1 + 3 + 5) + (1 + 3 + 5 + 7) + \ldots$
143. Find the sum to n terms of
    $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + (1^2 + 2^2 + 3^2 + 4^2) + \ldots$
144. Find the sum to n terms of
$$\frac{1^2}{1}+\left(\frac{1^2+2^2}{1+2}\right)+\left(\frac{1^2+2^2+3^2}{1+2+3}\right)+\ldots$$

145. Find the sum to n terms of

$$\frac{1^2}{1}+\left(\frac{1^3+2^3}{1+2}\right)+\left(\frac{1^3+2^3+3^3}{1+2+3}\right)+\ldots$$

146. Find the sum to n terms of

$$\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\ldots$$

147. Find the sum to n terms of

$$\frac{1}{1.3}+\frac{1}{4.5}+\frac{1}{7.7}+\frac{1}{10.9}+\ldots$$

148. Find the sum to n terms of

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\ldots$$

149. Find the sum to n terms of

$$\frac{1}{1.3.5}+\frac{1}{3.5.7}+\frac{1}{5.7.9}+\ldots$$

150. Find the sum to n terms of

$$\frac{1}{1.4.7}+\frac{1}{4.7.10}+\frac{1}{7.10.13}+\ldots$$

151. Prove that $\dfrac{1\times 2^2+2\times 3^2+\ldots+n\times(n+1)^2}{1^2\times 2+2^2\times 3+\ldots+n^2\times(n+1)}=\dfrac{3n+5}{3n+1}$.

152. Find the sum to n terms of the series

$$\frac{4}{2.3.4}+\frac{7}{3.4.5}+\frac{10}{4.5.6}+\ldots$$

153. Find the sum of the series

$$\frac{1}{\log_2 4}+\frac{1}{\log_{2^2} 4}+\frac{1}{\log_{2^3} 4}+\ldots+\frac{1}{\log_{2^n} 4}$$

154. Find the value of the expression $\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j}1$.

155. Find the sum of n terms of the series
$3+7+13+21+\ldots$

156. Prove that the sum of the series
$1+3+7+15+31+\ldots$ to n terms $=2^{n+1}-n=2$.

157. Find the sum to n terms of the series
$1+4+10+22+\ldots$

A.M/G.M/H.M

158. If $x, y, z > 0$ and $x+y+z=1$, find the maximum value of $\dfrac{xyz}{(1-x)(1-y)(1-z)}$.

159. If p and q are positive real numbers such that $p+q=1$, prove that

$$\left(p+\frac{1}{p}\right)^2+\left(q+\frac{1}{q}\right)^2\geq\frac{25}{2}.$$

160. If a, b, c are positive real numbers such that $a+b+c=1$, prove that

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right)\geq 64.$$

161. If a, b, c, d are positive real numbers, prove that $(a+b+c+d)\left(\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}+\dfrac{1}{d}\right)\geq\dfrac{1}{16}$.

162. If a, b, c are positive real numbers, prove that $(a+b+c)(ab+bc+ca)\geq 9abc$.

163. If $(1+x)(1+x^2)(1+x^4)\ldots(1+x^{128})=\sum_{r=0}^{n}x^r$, find the value of n.

164. Let x be arithmetic mean, and y, and z be two geometric means between any two positive numbers, find the value of $\dfrac{y^3+z^3}{xyz}$.

165. The sides of a right-angled triangle form a GP. The shortest side has length 2. The length of the hypotenuse is of the form $a+\sqrt{b}$, where $a, b \in N$, find the value of a^2+b^2+10.

166. If a, b, c are the sides of a triangle, prove that $\dfrac{1}{2}<\dfrac{ab+bc+ca}{a^2+b^2+c^2}<1$.

167. If a, b, c, d are four positive real numbers such that $abcd=1$, prove that
$(1+a)(1+b)(1+c)(1+d)\geq 16$.

168. If $a>1$, $b>1$, $c>1$ and $d>1$, prove that
$(1+a)(1+b)(1+c)(1+d)<8(abcd+1)$.

169. If $x+y+z=1$, prove that

$$(1-x)(1-y)(1-z)<\frac{8}{27}.$$

170. If a, b, c are positive real numbers such that $a+b+c=1$, prove that

$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\geq 27$$

171. If $a^2+b^2=1$ and $x^2+y^2=1$, prove that
$(ax+by)<1$.

172. If p, q, x, y are positive, prove that
$(px+qy)(pq+xy)\geq 4pqxy$.

173. If $\log_2 x+\log_2 y\geq 6$, prove that the least value of $x+y$ is 16.

174. If $a>1$, $b>1$, prove that $\log_b a+\log_a b\geq 2$.

175. If $x>1$, find the least value of $2\log_{10}x-\log_x(.01)$.

176. If $a^2+b^2=1$, prove that

$$\left(a^2-\frac{1}{a^2}\right)^2+\left(b^2-\frac{1}{b^2}\right)^2\geq\frac{9}{2}.$$

177. Let $a, b, c, d>0$ and $a\neq b\neq c\neq d$ such that $m=a+b+c+d$, prove that
$(m-a)(m-b)(m-c)(m-d)>81abcd$

178. If $a+b=1$, find the least value of $\left(a+\dfrac{1}{a}\right)^2+\left(b+\dfrac{1}{b}\right)^2$.

179. If $x^2+y^2=2$ and $a^2+b^2=8$, find the greatest value of $ax+by$.

180. If $a > 0$, $b > 0$, $c > 0$ and $a + b + c = 1$, find the least value of
$$\left(a+\frac{1}{a}\right)^2+\left(b+\frac{1}{b}\right)^2+\left(c+\frac{1}{c}\right)^2.$$
181. If $a + 2b + 3c = 12$ and $a > 0, b > 0, c > 0$, find the greatest value of ab^2c^3.
182. If $4a + 3b + 2c = 45$ and $a > 0$, $b > 0$, $c > 0$, find the greatest value of $a^2b^3c^4$.
183. If $2x + 3y = 10$, find the maximum value of x^2y^3.
184. If $x + 5y = 18$, find the greatest value of xy^5.
185. If three positive real numbers a, b, c are in AP with $abc = 64$, find the minimum value of b.
186. Find the minimum value of
$$f(x) = x^2 + 2 + \frac{1}{x^2+1}, x > 0.$$
187. Find the minimum value of
$$f(x)=x^{10}+\frac{10}{x}, x>0.$$
188. Find the minimum value of
$$f(x)=x^{2012}+\frac{2012}{x}, x>0$$
189. Find the minimum value of
$$f(a) = a^9 + a^7 + a^5 + a^3 + 1 + a^{-3} + a^{-5} + a^{-7} + a^{-9}$$
where $a > 0$.
190. Find the minimum value of
$$f(x)=2^{x^2+2x+2}+2^{2-2x-x^2}, x>0.$$
191. Find the minimum value of
$f(x) = 2^x + 3^x + 4^x + 5^x + 2^{-x} + 3^{-x} + 4^{-x} + 5^{-x} + 10$
192. Let a, b, c, d are all positive real numbers such that $a + 2b + 3c + 4d = 10$. If M is the maximum value of $(a + 2b)(3c + 4d)$ and N is the maximum value of $(a + c + 2d)(b + c + d)$ then find the value of $(M + 2N + 10)$.
193. Let $a + b + c = 1$ such that $a > 0, b > 0, c > 0$, find the minimum value of $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$.
194. Let $x + y + z = 1$ such that $x > 0, y > 0, z > 0$, find the least value of $(x^3 + y^3 + z^3)(x^6 + y^6 + z^6)$.
195. If $a > 0$, $b > 0$, $c > 0$ and $abc = 8$, find the minimum value of $(1 + a + a^2)(1 + b + b^2)(1 + c + c^2)$.
196. Let $x > 0, y > 0, z > 0$ with $xyz = 2$, find the minimum value of $(1 + x^3)(1 + y^3)(1 + z^3)$.
197. If a, b are positive real numbers, prove that $\{(1+a)(1+b)\}^3 \ge \frac{3^6}{16}(a^2b^2)$.
198. If a, b, c are positive real numbers, prove that
$$(1 + a)(1 + b)(1 + c)^7 > 7^7.a^4b^4c^4.$$
199. If a, b, c are positive real numbers such that $a + b + c = 1$, find the minimum value of $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}$.
200. If a, b, c, d are positive real numbers such that $a + b + c + d + e = 8$ and $a^2 + b^2 + c^2 + d^2 + e^2 = 16$, find the range of e.
201. Let a, b, c are positive real numbers with $ab + bc + ca = 8$. Find the maximum value of abc.
202. Let $\alpha, \beta, \gamma, \delta$ are four positive roots of $x^4 + bx^3 + cx^2 + dx + 6 = 0$, find the minimum value of (bd).
203. If a, b, c are real numbers such that $a + 2b + c = 4$, find the maximum value of $ab + bc + ca$.
204. If a, b, c, d are positive real numbers such that $a + b + c + d = 2$, find the maximum value $(a + b)(c + d)$.
205. If $a + b + c = 1$, find the minimum value of $\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right)$.
206. If a, b, c are all positive, prove that
$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\ge\frac{3}{2}.$$
207. If x, y, z, w are all greater than 1, prove that $(1 + x)(1 + y)(1 + z)(1 + w) < 4(xy + 1)(zw + 1)$.
208. If x, y, z and w are all greater than 1, prove that $(1 + x)(1 + y)(1 + z)(1 + w) < 8(xyzw + 1)$.
209. If a, b, c are any three real numbers, show that $a^4+b^4+c^2\ge 2\sqrt{2}\,abc$.
210. If a, b, c are positive real numbers, prove that
$$\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b}\ge a+b+c$$
211. If a, b, c are positive real numbers, prove that $\left(\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}+\frac{a^2+b^2}{a+b}\right)\ge(a+b+c)$.
212. Find the least value of $\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)$ where $x^2 + y^2 + z^2 = 1$.
213. Find the minimum value of the sum of real numbers $a^{-5}, a^{-4}, 3a^{-3}, 1, a^8$ and a^{10} with $a > 0$.

Level II

(Mixed Problems)

1. If a, b, c are in GP, the equations $ax^2 + 2bx + c - 0$ and $dx^2 + 2ex + f = 0$ have a common root if $\frac{d}{a},\frac{e}{b},\frac{f}{c}$ are in
(a) AP (b) GP (c) HP (d) None
2. The sum of the first n terms of the series $\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+\ldots$ is equal to
(a) $2^n - n - 1$ (b) $1 - 2^{-n}$
(c) $n + 2^{-n} - 1$ (d) $2^n - 1$
3. For $0<\varphi<\frac{\pi}{2}$, if
$$x=\sum_{n=0}^{\infty}\cos^{2n}\varphi,\ y=\sum_{n=0}^{\infty}\sin^{2n}\varphi;$$

and $z = \sum_{n=0}^{\infty} \cos^{2n}\varphi \sin^{2n}\varphi$, then

(a) $xyz = xz + y$ (b) $xyz = xy + z$
(c) $xyz = x + y + z$ (d) $xyz = yz + x$.

4. If in a triangle PQR, $\sin P$, $\sin Q$, $\sin R$ are in AP,
(a) the altitudes are in AP.
(b) the altitudes are in HP.
(c) the medians are in GP.
(d) the medians are in AP.

5. Let T_r be the rth term of an AP, for $r = 1, 2, 3, \ldots$ if for some positive integers m and n, we have $T_m = 1/n$ and $T_n = 1/m$, then T_{mn} equals
(a) $1/mn$ (b) $1/m + 1/n$
(c) 1 (d) 0

6. If $x > 1, y > 1, z > 1$ are in GP, $\frac{1}{1+\ln x}, \frac{1}{1+\ln y}, \frac{1}{1+\ln z}$ are in
(a) AP (b) HP
(c) GP (d) None of these

7. Let $a_1, a_2, \ldots, a_{10}$ be in AP and $h_1, h_2, \ldots h_{10}$ be in HP. If $a_1 = h_1 = 2$ and $a_{10} = h_{10} = 3$, a_4h_7 is
(a) 12 (b) 3 (c) 5 (d) 6

8. If the sum of the first $2n$ terms of the AP 2, 5, 8, … is equal to the sum of the first n terms of the AP 57, 59, 61, …, n equals
(a) 10 (b) 12 (c) 11 (d) 13

9. Let the positive numbers a, b, c, d be in AP. Then abc, abd, acd, bcd are
(a) not in AP/GP/HP (b) in AP
(c) in GP (d) in HP

10. If $a_1, a_2, \ldots a_n$ are positive real numbers whose product is a fixed number c, the minimum value of $a_1 + a_2 + \ldots + a_{n-1} + 2a_n$ is
(a) $n(2c)^{1/n}$ (b) $(n+1)(c)^{1/n}$
(c) $2n(c)^{1/n}$ (d) $(n+1)(2n)^{1/n}$

11. Suppose a, b, c are in AP and a^2, b^2, c^2 are in GP. If $a < b < c$ and $a + b + c = \frac{3}{2}$, the value of a is
(a) $\frac{1}{2\sqrt{2}}$ (b) $\frac{1}{2\sqrt{3}}$
(c) $\frac{1}{2} - \frac{1}{\sqrt{3}}$ (d) $\frac{1}{2} - \frac{1}{\sqrt{2}}$

12. If x, y, z are real numbers satisfying the expression $25(9x^2 + y^2) + 9z^2 - 15(5xy + yz + 3zx) = 0$, x, y and z are in
(a) AP (b) GP (c) HP (d) None

13. Let a_n be the nth term of an AP. If $\sum_{r=1}^{100} a_{2r} = \alpha$ and $\sum_{r=1}^{100} a_{2r-1} = \beta$, the common ratio is
(a) $\frac{\alpha}{\beta}$ (b) $\frac{\beta}{\alpha}$ (c) $\frac{\beta^2}{\alpha^2}$ (d) $\frac{\alpha^2}{\beta^2}$

14. Let a_n be the nth term of an AP. If $\sum_{r=1}^{100} a_{2r} = \alpha$ and $\sum_{r=1}^{100} a_{2r-1} = \beta$, the common difference is
(a) $\alpha - \alpha$ (b) $\beta - \alpha$
(c) $2(\beta - \alpha)$ (d) $\frac{(\alpha - \beta)}{100}$

15. If $(r)_n = \left(\underset{(n \text{ times})}{r \cdot r \cdot r \ldots r}\right)$ such that $a = (6)_n$, $b = (8)_n$, $c = (4)_n$, then
(a) $a^2 + b + c = 0$ (b) $a^2 + b - c = 0$
(c) $a^2 + b + 2c = 0$ (d) $a^2 + b + ac = 0$

16. The sum of $1 + \frac{1}{5} + \frac{3}{5^2} + \frac{5}{5^3} + \frac{7}{5^4} + \ldots$ to ∞ is
(a) 3/8 (b) 8/3 (c) 11/8 (d) None

17. The co-efficient of x^n in the expansion of $(1 + x + 2x^2 + 3x^3 + \ldots + nx^n)^2$ where $-1 < x < 1$, is
(a) $\frac{n(n+1)}{2}$ (b) $\frac{n(n+1)(2n+1)}{6}$
(c) $\frac{n(n^2+11)}{6}$ (d) None

18. If a, b, c are in GP, x and y be the AMs between a and b; and b and c respectively, the value of $\left(\frac{a}{x} + \frac{c}{y}\right)\left(\frac{b}{x} + \frac{c}{y}\right)$ is
(a) 2 (b) –4 (c) –2 (d) 4

19. If $0 < x < \frac{\pi}{4}$, the minimum value of $(\sin x + \cos x + \operatorname{cosec} 2x)^3$ is
(a) 27 (b) 13.5 (c) 6.75 (d) 40.5

20. If x, y, z are positive, the minimum value of $(a^{\log y - \log z} + a^{\log z - \log x} + a^{\log x - \log y})$ is
(a) 3 (b) 1 (c) 9 (d) 16

21. The sum of all the product of the first n positive integers, taken two at a time, is
(a) $\frac{1}{24} \times n(n-1)(n+1)(3n+2)$
(b) $\frac{1}{48} \times n^2(n-1)(n-2)$
(c) $\frac{1}{6} \times n(n+1)(n+2)(n+5)$
(d) None

22. If $t_n = 2^{\frac{n}{2}} + 2^{-\frac{n}{2}}$, $\sum_{n=1}^{15} t_n^2$ is
(a) $\frac{2^{21}-1}{2^{10}} + 20$ (b) $\frac{2^{21}-1}{2^{10}} + 19$
(c) $\frac{2^{21}-1}{2^{10}} + 1$ (d) None

23. If $u_n = \sum_{n=0}^{n}\left(\frac{1}{2^n}\right)$, the sum of $\sum_{n=1}^{n} u_n$ is
 (a) $2^{-n} + 2n - 1$ (b) $2^n + 2n - 1$
 (c) $2^{-n} - 2n - 1$ (d) None
24. If a, b, c are in AP and a^2, b^2, c^2 are in HP, then
 (a) $a = b = c$ (b) a, b, c in GP
 (c) $-a/2, b, c$ in GP (d) $a^2 = b^2 = \frac{c^2}{2}$
25. If $\sum_{k=1}^{n}\left(\sum_{m=1}^{n} m^2\right) = an^4 + bn^3 + cn^2 + dn + e$, then
 (a) $a = \frac{1}{12}$ (b) $b = \frac{1}{6}$
 (c) $d = \frac{1}{6}$ (d) $e = 0$
26. The real roots of $2x^3 - 19x^2 + 57x + k = 0$ are in GP, the value of k is
 (a) 216 (b) –108 (c) –54 (d) 27
27. If the roots of $x^3 + px^2 + qx - 1 = 0$ form an increasing GP, where p and q are real numbers, then
 (a) $p + q = 0$ (b) $p - q = 0$
 (c) $2p + 3q = 0$ (d) $2p - 3q = 0$
28. Let α, β, γ be the roots of $x^3 + px^2 + qx - q = 0$, where p and q are non-zero real numbers, the minimum value of $\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2}$ is
 (a) 1/3 (b) 1 (c) 4/3 (d) 3
29. The sum of infinite series
 $1 + \frac{3}{4} + \frac{7}{16} + \frac{15}{64} + \frac{31}{256} + \ldots$ is
 (a) 3/8 (b) 8/3 (c) 3/16 (d) 16/3.
30. The equation $x^{x\sqrt{x}} = (x\sqrt{x})^x$ has two solutions in positive real numbers. If one solution is $x = 1$, the other one is
 (a) 4/9 (b) 16/25 (c) 9/4 (d) 81/16
31. If $x^2 + y^2 + z^2 = 9$, the least value of $\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}$ is
 (a) 4 (b) 9 (c) 16 (d) 1
32. If $a + b + c + d = 8$ such that a, b, c, d are all positive real numbers, the maximum value of $(a + b)(c + d)$ is
 (a) 9 (b) 4 (c) 16 (d) 25
33. If a, b, c are all positive real numbers such that $a + b + c = 1$, find the minimum value of $\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}$ is
 (a) 9 (b) 27 (c) 81 (d) 243
34. If the sum of 3 + 7 + 13 + 21 + … is of the form $\frac{n}{a}(bn^2 + cn + d)$, the value of $(a + b + c + d)$ is
 (a) 10 (b) 11 (c) 12 (d) 13
35. If the sum of the series 1 + 4 + 10 + 22 + … is of the form $a.2^n + b.n + c$, the value of $(a + b + c + 10)$ is
 (a) 2 (b) 4 (c) 6 (d) 8
36. If x^3, y^3, z^3 are in AP, $\log_x y, \log_z x, \log_y z$ are in
 (a) AP (b) GP (c) HP (d) AGP
37. The third term of a geometric progression is 4, the product of the first 5 terms is
 (a) 4^3 (b) 4^5 (c) 4^4 (d) None
38. If a, b, c are in GP, the equation $ax^2 + 2bx + c = 0$, $dx^2 + 2ex + f = 0$ have a common root if $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in
 (a) AP (b) GP (c) HP (d) None
39. If a, b, c, d and p are distinct real numbers such that $(a^2 + b^2 + c^2)p^2 - 2(ab + bc + ca)p + (b^2 + c^2 + d^2) \le 0$, then a, b, c, d are in
 (a) AP (b) GP (c) HP (d) $ab = cd$
40. The sum of the first n terms of the series $\frac{1}{2} + \frac{3}{4} + \frac{7}{8} + \frac{15}{16} + \ldots$ is equal to
 (a) $2^n - n - 1$ (b) $1 - 2^{-n}$
 (c) $n - 1 + 2^{-n}$ (d) $2^n + 1$
41. If the first and $(2n - 1)$th terms of an AP, a GP and an HP are equal and their nth terms are a, b, c respectively, then
 (a) $a = b = c$ (b) $a \ge b \ge c$
 (c) $a + c = b$ (d) $ac - b^2 = 0$
42. The third term of a GP is 4, the product of its first five terms is
 (a) 2^6 (b) 2^{10} (c) 2^8 (d) None
43. If $\ln(a + c), \ln(a - c), \ln(a - 2b + c)$ are in AP, then
 (a) a, b, c are in AP (b) a^2, b^2, c^2 are in AP
 (c) a, b, c are in GP (d) a, b, c are in HP
44. If $a, b, c \in R^+$ and are in AP such that $abc = 4$, the minimum value of b is
 (a) $2^{2/3}$ (b) $2^{1/3}$ (c) $4^{2/3}$ (d) None
45. If $4x^2 + 9y^2 + 16z^2 - 6xy - 12yx - 8zx = 0$, then x, y, z are in
 (a) AP (b) GP (c) HP (d) AGP
46. If $\frac{1}{a} + \frac{1}{c} + \frac{1}{a-b} + \frac{1}{b-c} = 0$ and $a + c - b \ne 0$, then a, b, c are in
 (a) AP (b) GP (c) HP (d) AGP
47. The harmonic mean of the roots of the equation $(5 + \sqrt{3})x^2 - (6 + \sqrt{5})x + (18 + 3\sqrt{5}) = 0$ is
 (a) 2 (b) 4 (c) 6 (d) 8
48. If a, b, c are in AP, a, b, d are in GP, then $a, a - b, d - c$ are in
 (a) AP (b) GP (c) HP (d) None
49. If a, b, c are in HP, the value of $\frac{b+a}{b-a} + \frac{b+c}{b-c}$ is
 (a) 1 (b) 2 (c) 3 (d) 4
50. If $a, b, c \in R^+$ are in AP, then
 $a + \frac{1}{bc}, b + \frac{1}{ca}, c + \frac{1}{ab}$ are in
 (a) AP (b) GP (c) HP (d) AGP

51. In a series, if $t_n = \frac{n}{(n+1)!}$, the sum of $\sum_{n=1}^{20} t_n$ is

(a) $\left(1 - \frac{1}{20!}\right)$ (b) $\left(1 - \frac{1}{21!}\right)$

(c) $\left(1 + \frac{1}{21!}\right)$ (d) $\left(1 + \frac{1}{20!}\right)$

52. If $1 + \lambda + \lambda^2 + \lambda^3 + \ldots + \lambda^n$
$= (1 + \lambda)(1 + \lambda^2)(1 + \lambda^4)(1 + \lambda^8)(1 + \lambda^{16})$,
the value of n is
(a) 15 (b) 31 (c) 32 (d) 16

53. If $\log_2(a + b) + \log_2(c + d) \geq 4$, the minimum value of $(a + b + c + d)$ is
(a) 2 (b) 4 (c) 8 (d) 16

54. An infinite GP has first term x and sum '5' then
(a) $x < -10$ (b) $-10 < x < 0$
(c) $0 < x < 10$ (d) $x > 10$

55. If α and β are the roots of $ax^2 + bx + c = 0$ and $\alpha + \beta$, $\alpha^2 + \beta^2$, $\alpha^3 + \beta^3$ are in GP and $\Delta = b^2 - 4ac$, then
(a) $c\Delta = 0$ (b) $cb \neq 0$ (c) $b\Delta = 0$ (d) $\Delta \neq 0$

56. If $a > 0$, $b > 0$, $c > 0$ and $s = a + b + c$, then $\left[\left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c}\right) - \frac{9}{2s}\right]$ is
(a) less than 0 (b) equal to 0
(c) greater than 0 (d) less than equal to 0

57. If $I_n = \int_0^{\pi/2} \left(\frac{\sin^2 nx}{\sin^2 x}\right) dx$, then $I_1, I_2, I_3, \ldots$ are in
(a) AP (b) GP (c) HP (d) AGP

58. If $U_n = \sum_{n=0}^{n} \left(\frac{1}{2^n}\right)$, then $\sum_{n=1}^{n} U_n$ is
(a) $2^{-n} + 2n - 1$ (b) $2^n - 2n - 1$
(c) $2^{-n} - 2n + 1$ (d) $2^{-n} - 2n + 2$

59. If a, b, c are in GP, x, y be the AMs between a and b; and b and c, respectively, the value of $\left(\frac{a}{x} + \frac{c}{y}\right)\left(\frac{b}{x} + \frac{b}{y}\right)$ is
(a) 2 (b) 4 (c) 3 (d) None

60. The sum of n terms of the series
$1 + 5\left(\frac{4n+1}{4n-3}\right) + 9\left(\frac{4n+1}{4n-3}\right)^2 + 13\left(\frac{4n+1}{4n-3}\right)^3 + \ldots$ is
(a) $4n^2 - 3n$ (b) $4n^2 + 3n$
(c) $n^2 + 2$ (d) $n^2 + 3n$

61. The sum of all the product of the first n positive integers, taken two at a time, is

(a) $\frac{1}{24} n(n-1)(n+1)(3n+2)$

(b) $\frac{1}{48} n^2(n-1)(n-2)$

(c) $\frac{1}{6} n(n+1)(n+2)(n+5)$

(d) $\frac{1}{12} n(n-1)(n+2)$

62. The nth term of the series $2 + 5 + 12 + 31 + 86 + \ldots$ is
(a) $n + 3^{n-1}$ (b) $(n - 1) + 3^{n-2}$
(c) $(n + 1) + 3^n$ (d) $(n - 2) + 3^n$

63. The sum to n terms of the series $2 + 5 + 14 + 41 + \ldots$ is

(a) $\frac{1}{4}(3^n - 1)$ (b) $\frac{1}{4}(3^{n+1} - 3) + \frac{n}{2}$

(c) $\frac{1}{4}(3^{n+1} - 2) + \frac{n}{3}$ (d) $\frac{1}{4}(3^{n+1} - 2) + \frac{n}{4}$

64. If the numbers a, b, c, d and e are in AP, the value of $a - 4b + 6c - 4d + e$ is
(a) 1 (b) 2 (c) 0 (d) -2

65. If $a^3 + b^3 + 6abc = 8c^3$ and ω is a cube root of unity, then
(a) a, c, b are in AP
(b) a, c, b are in GP
(c) $a + b\omega - 2c\omega^2 = 0$
(d) $a + b\omega - 2c\omega = 0$

Level III

Problems for JEE-Advanced

1. Prove that the value of the expression

$$\left(1 + \frac{1}{\omega}\right)\left(1 + \frac{1}{\omega^2}\right) + \left(2 + \frac{1}{\omega}\right)\left(2 + \frac{1}{\omega^2}\right) + \left(3 + \frac{1}{\omega}\right)\left(3 + \frac{1}{\omega^2}\right) + \ldots + \left(n + \frac{1}{\omega}\right)\left(n + \frac{1}{\omega^2}\right) = \frac{n(n^2+2)}{3}$$

2. Prove that the value of the expression
$1 \cdot (2 - \omega)(2 - \omega^2) + 2 \cdot (3 - \omega)(3 - \omega^2) + \ldots + (n - 1)(n - \omega)(n - \omega^2)$
$$= \left(\frac{n(n+1)}{2}\right)^2 - n$$

3. Prove that the sum of the series
$n \cdot 1 + (n - 1) \cdot 2 + (n - 2) \cdot 3 + \ldots + 1 \cdot n$
$$= \frac{1}{6} n(n+1)(n+2)$$

4. If a, b, c are three distinct real numbers in GP and $a + b + c = xb$, prove that $x \leq -1$ or $x \geq 3$.

5. If a, b, c are in GP and the equations $ax^2 + 2bx + c = 0$ and $dx^2 + 2ex + f = 0$ have a common root, show that $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in AP.

6. If $a_1, a_2, a_3, a_4, \ldots, a_n$ are in HP, prove that $a_1a_2 + a_2a_3 + a_3a_4 + \ldots a_{n-1} = (n - 1)a_1a_n$.

7. If $\exp\{(\sin^2 x + \sin^4 x + \sin^6 x + \dots \infty)\log_e 2\}$ satisfies the equation $x^2 - 9x + 8 = 0$, prove that the value of $\frac{\cos x}{\cos x + \sin x} = \frac{1}{2}(\sqrt{3} - 1)$.
8. If a, b, c are the sides of a triangle, prove that $\frac{1}{2} < \frac{ab + bc + ca}{a^2 + b^2 + c^2} < 1$.
9. If a, b, c be any three positive real numbers, prove that $\frac{b + c}{b^2 + c^2} + \frac{c + a}{c^2 + a^2} + \frac{a + b}{a^2 + b^2} \le \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$
10. Let $f(x) = 4x^4 - ax^3 + bx^2 - cx + 5$ be a polynomial has four positive roots m_1, m_2, m_3 and m_4 such that $\frac{m_1}{2} + \frac{m_2}{4} + \frac{m_3}{5} + \frac{m_4}{8} = 1$, find the value of $(a + 1995)$.
11. If p and q are two positive real numbers such that $p + q = 2$, find the least value of $\left(p + \frac{1}{p}\right)^2 + \left(q + \frac{1}{q}\right)^2$.
12. Find the co-efficient of x^n in the expansion of
$$(1 + x + 2x^2 + 3x^3 + \dots + nx^n)^2, |x| < 1.$$
13. Find the value of $\sum_{r=1}^{n} r^2 - \sum_{m=1}^{n}\sum_{r=1}^{m} r$.
14. Let a_n be the nth terms of an AP such that $\sum_{r=1}^{2014} a_{2r} = \alpha$ and $\sum_{r=1}^{2014} a_{2r-1} = \beta$, find the common difference of the AP.
15. If x, y, z are real numbers satisfying the expression
$$25(9x^2 + y^2) + 9z^2 - 15(5xy + yz + 3zx) = 0,$$
then x, y, z are in HP.
16. Find the sum of all the product of the first n positive real integers, taken two at a time.
17. If $\sum_{k=1}^{n}\left(\sum_{m=1}^{n} m^2\right) = an^4 + bn^3 + cn^2 + dn + e$, find the value of a, b, c, d and e.
18. Let $a_1, a_2, \dots, a_{10}$ be in AP and $b_1, b_2, \dots, b_{10}$ be in HP. If $a_1 = b_1 = 2$ and $a_{10} = b_{10} = 3$, prove that the value of $a_4 b_7$ is 6.
19. Let $x + y + z = 1$ such that $x > 0$, $y > 0$, $z > 0$, find the least value of $(x^3 + y^3 + z^3)(x^6 + y^6 + z^6)$.
20. If $a > 0$, $b > 0$, $c > 0$ and $abc = 8$, find the minimum value of $(1 + a + a^2)(1 + b + b^2)(1 + c + c^2)$.
21. Let $x > 0$, $y > 0$, $z > 0$ with $xyz = 2$, find the minimum value of $(1 + x^3)(1 + y^3)(1 + z^3)$.
22. If a, b are positive real numbers, prove that
$$((1 + a)(1 + b))^3 > 33a^2b^2.$$
23. If a, b, c are positive real numbers such that $a + b + c = 1$, find the minimum value of $\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca}$.
24. If $t_1 = 1$, $t_r - t_{r-1} = 2^{r-1}$, $r \ge 2$, find the value of $\sum_{r=1}^{n} t_r$.
25. If $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \infty = \frac{\pi^2}{6}$, find the value of $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \infty$.
26. If between any two quantities there be inserted two arithmetic means A_1 and A_2, two geometric means G_1 and G_2 and two harmonic means H_1 and H_2, prove that $G_1G_2 : H_1H_2 = (A_1 + A_2) : (H_1 + H_2)$ **[Roorkee, 1983]**
27. Given that $a^x = b^y = c^z = d^u$ and a, b, c, d are in geometric progression, show that x, y, z, u are in Harmonic Progression. **[Roorkee, 1984]**
28. Balls are arranged in rows to form an equilateral triangle. The first row consists of one ball, the second of 2 balls and so on. If 669 more balls are added, all the balls can be arranged in the shape of a square, and each of its sides then contains 8 balls less than each side of the triangle did. Determine the initial number of balls. **[Roorkee, 1985]**
29. The sum of n terms of two arithmetic series are in the ratio of $(7n + 1) : (4n + 27)$. Find the ratio of their nth terms. **[Roorkee, 1986]**
30. Solve for x and y, given
$y = \log_{10} x + \log_{10} x^{1/2} + \log_{10} x^{1/4} + \dots$
$$\frac{1 + 3 + 5 + \dots + (2y - 1)}{4 + 7 + 10 + \dots + (3y + 1)} = \frac{20}{7 \log_{10} x}$$
[Roorkee, 1987]
31 The sum of the fist ten terms of an AP is 155 and the sum of the first two terms of a GP is 9. Find these progressions, if the first term of the AP equals the common ratio of the GP and the first term of GP equals the common difference of the AP. **[Roorkee, 1993]**
Note *No question asked in 1988.*
32. The sum of an infinite geometric progression is 2 and the sum of the geometric progression made from the cubes of the terms of this infinite series is 24. Find the series. **[Roorkee, 1989]**
33. Find the sum S_n of the cubes of the first n terms of an AP and show that the sum of the first n terms of the AP is a factor of S_n. **[Roorkee, 1992]**
34. The sum of the first ten terms of an AP is 155 and the sum of first two terms of a G.P is 9, the first term of A.P is equal to the common ratio of the G.P and the first term of G.P is equal to the common difference of the A.P . Find the two progressions. **[Roorkee, 1993]**
35. If the $(m + 1)$th, $(n + 1)$th and $(r + 1)$th terms of an AP are in GP and m, n, r are in HP, find the ratio of the first term of the AP. to its common difference in terms of n. **[Roorkee, 1994]**
Note *No question asked in 1995.*
36. Let $x = 1 + 3a + 6a^2 + 10^3 + \dots, |a| < 1$
$y = 1 + 4b + 10b^2 + 20b^3 + \dots |b| < 1$
Find $S = 1 + 3(ab) + 5(ab)^2 + \dots$ in terms of x and y. **[Roorkee, 1996]**

37. Observing that $1^3 = 1$, $2^3 = 3 + 5$, $3^3 = 7 + 9 + 1$, $4^3 = 13 + 15 + 17 + 19$,
find a general formula for the cube of natural numbers. **[Roorkee, 1997]**

38. Let a, b, c are the first three terms of a geometric series. If the harmonic mean of a and b is 12 and that of b and c is 36, find the first five terms of the series. **[Roorkee, 1998]**

39. The sum of the infinite series is 162 and the sum of the first n terms is 160. If the inverse of its common ratio is an integer, find all possible values of the common ratio, n and the first term of the series. **[Roorkee, 1999]**
Note *No question asked in 2000.*

40. The sum of three numbers in a GP is 42. If the first two numbers are increased by 2 and third is decreased by 4, the resulting numbers form an AP. Find the numbers of the GP. **[Roorkee, 2001]**

41 If the sum of
$$\sqrt{1+\frac{1}{1^2}+\frac{1}{2^2}}+\sqrt{1+\frac{1}{2^2}+\frac{1}{3^2}}+\sqrt{1+\frac{1}{3^2}+\frac{1}{4^2}}+\ldots+\sqrt{1+\frac{1}{1999^2}+\frac{1}{2000^2}}$$
is equal to
$\left(n-\frac{1}{n}\right), n \in N$, find n.

42. Find the value of $\sum_{k=0}^{359} k \cdot \cos(k°)$.

43 Find the sum to n terms of
$$1+2\left(1+\frac{1}{n}\right)+3\left(1+\frac{1}{n}\right)^2+4\left(1+\frac{1}{n}\right)^3+\ldots$$

44. Find the sum of $f(x)=\sum_{n=1}^{\infty}\sin\left(\frac{2x}{3^n}\right)\sin\left(\frac{x}{3^n}\right)$.

45. If the roots of $10x^3 - cx^2 - 54x - 27 = 0$ are in HP, find the value of c.

Level IV

(Tougher Problems For JEE-Advanced)

1. In a triangle ABC, if $\cot A$, $\cot B$, $\cot C$ are in AP, prove that a^2, b^2, c^2 are in AP.

2. The sum of the sequence of three distinct real numbers, which are in GP is S^2. If their sum is αS, show that $\alpha^2 \in \left(\frac{1}{3}, 1\right) \cup (1, 3)$.

3. The sum of n terms of two arithmetic progressions are in the ratio $(7n + 1) : (4n + 17)$. Find the ratio of their nth terms.

4. Solve the following equations for x and y:
$$\log_{10}x + \log_{10}x^{1/2} + \log_{10}x^{1/4} + \ldots = y$$
and $$\frac{1+3+5+\ldots+(2y-1)}{4+7+10+\ldots+(3y+1)}=\frac{20}{7\log_{10}x}.$$

5. If $a_i > 0$ for every i in N such that $\prod_{i=1}^{n} a_i = 1$, prove that $(1 + a_1)(1 + a_2)(1 + a_3)\ldots(1 + a_n) \geq 2^n$.

6. Obtain the sum of
$$\frac{1}{(x+1)}+\frac{2}{(x^2+1)}+\frac{4}{(x^4+1)}+\ldots+\frac{2^n}{(x^{2n}+1)}.$$

7. If $S_1, S_2, S_3, \ldots, S_n$ are the sums of infinite geometric series whose first terms are $1, 2, 3, \ldots, n$ and common ratios $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n+1}$, respectively, find the values of $S_1^2 + S_2^2 + S_3^2 + S_{2n-1}^2$.

8. If $\exp(\sin^2 x + \sin^4 x + \sin^6 x + \ldots \infty) \ln 2$ satisfies the equation $x^2 - 9x + 8 = 0$, find the value of $\frac{\cos x}{\cos x + \sin x}, 0 < x < \frac{\pi}{2}$.

9. Find the sum of the following infinite series
$$1+\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)+\left(\frac{3-2\sqrt{2}}{8}\right)+\left(\frac{5\sqrt{2}-7}{16\sqrt{2}}\right)\frac{17-12\sqrt{2}}{64}+\ldots \text{ to } \infty$$

10. Find the natural number a for which $\sum_{k=1}^{n} f(a+k) = 16(2^n - 1)$, where the function f satisfies the relation $f(x + y) = f(x) \cdot f(y)$ for all natural numbers x and y and further $f(1) = 2$.

11. The real numbers x_1, x_2 and x_3 satisfying the equation $x^3 - x^2 + \beta x + \gamma = 0$ are in AP, find the intervals in which β and γ lie.

12. Let $x = 1 + 3a + 6a^2 + 10a^3 + \ldots, |A| < 1, y = 1 + 4b + 10b^2 + 20b^3 + \ldots, |b| < 1$.
Find $S = 1 + 3(ab) + 5(ab)^2 + \ldots$ in terms of x and y.

13. Let $(1+x^2)^2(1+x)^n = \sum_{k=0}^{n+4} a_k x^k$. If a_1, a_2 and a_3 are in arithmetic progression, find n.

14. Let $\cos(x - y)$, $\cos x$ and $\cos(x + y)$ are in HP, and a, b and c are positive real numbers. If m is the value of $\left|\cos x \sec\left(\frac{y}{2}\right)\right|$ and n is the minimum value of $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$, find the value of $(n + m^2 - 4)$.

15. Let x be the arithmetic mean, and y and z be the two geometric means between any two positive numbers, find $\frac{y^3+z^3}{xyz}$.

16. Let a, b and c are the first three terms of a geometric series. If the harmonic mean of a and b is 12 and that of b and c is 36, find the first five terms of the series.
17. Given that α and γ are roots of the equation $Ax^2 - 4x + 1 = 0$, and β and δ be the roots of the equation $Bx^2 - 6x + 1 = 0$, find the values of A and B such that α, β, γ and δ are in HP.
18. The sum of three number in GP is 42. If the first two number are increased by 2 and third is decreased by 4, the resulting form an AP, find the numbers of GP.
19. If a, b and c are in AP and a^2, b^2 and c^2 are in HP, prove that either $a = b = c$ or a, b and $-\frac{c}{2}$ are in GP.
20. If p and q are positive real numbers such that $p + q = 1$, prove that $\left(p+\frac{1}{p}\right)^2+\left(q+\frac{1}{q}\right)^2 \geq \frac{25}{2}$.
21. If a, b and c are positive real numbers such that $a + b + c = 2$, prove that $\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right) \geq \frac{49}{4}$
22. Find the co-efficients of x^{99} in $(x - 1)(x - 2)(x - 3)\ldots(x - 100)$.
23. Find the co-efficients of x^{99} in $(2x - 1)(2x - 3)(2x - 5)\ldots(2x - 199)$.
24. If the sum of first n terms of an AP is cn^2, find the sum of squares of these n terms.
25. If a, b and c are positive real numbers such that $\{(1 + a)(1 + b)(1 + c)\}^7 > k^m(a^4b^4c^4)$, find the value of $k + m$.
26. Find the sum of $\sum_{r=1}^{\infty}\left(\frac{r+3}{r(r+1)(r+2)}\right)$.
27. A cricket player plays n (> 1) matches and scores $k(2^{n-k+1})$ runs in his kth match $(1 \leq k \leq n)$. If the total runs scored by him in the n matches is $\frac{1}{4}(n+1)(2^{n+1}-n-2)$, find the value of n.
28. If $t_r = 1$, $t_r - t_{r-1} = 2^{r-1}$, $r \geq 2$, find the sum of $\sum_{r=1}^{n} t_r$.
29. Find the sum of $\sum_{q=1}^{n}\sum_{p=1}^{q}\sum_{k=1}^{p} 1$.
30. One of the roots of the equation $2000x^6 + 100x^5 + 10cx^3 + x - 2 = 0$ is of the form $\frac{m+\sqrt{n}}{r}$, where m is non-zero integer and n and r are relatively prime natural numbers. Find the value of $m + n + r$.
31. If $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots \text{to } \infty = \frac{\pi^4}{90}$, find the value of $\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\ldots \text{to } \infty$.
32 Find the value of $\sum_{m=1}^{n}\left(\sum_{k=1}^{m}(2^k+3k)\right)$.
33. Evaluate $\lim_{n\to\infty}\left\{\tan\left(\sum_{r=1}^{n}\tan^{-1}\left(\frac{1}{2r^2}\right)\right)\right\}$
34 Find the sum to n terms of the series $\frac{1}{x+1}+\frac{2x}{(x+1)(x+2)}+\frac{3x^2}{(x+1)(x+2)(x+3)}$ to n terms
35 Find $\sum_{n=1}^{n} U_n$ if $U_n = \sum_{k=0}^{n}\left(\frac{1}{2^k}\right)$.
36. If $a + b = 2$, find the minimum value of $\left(a^2-\frac{1}{a^2}\right)^2+\left(b^2-\frac{1}{b^2}\right)^2$.
37. Let $f(x)=\left[\frac{1}{2}+\frac{x}{100}\right]$, where [] = G.I.F. Find the value of $\sum_{x=1}^{100} f(x)$.
38. Find the sum of the series upto n terms of $\frac{1}{\log_2 4}+\frac{1}{\log_4 4}+\frac{1}{\log_8 4}+\ldots+\frac{1}{\log_{2^n} 4}$.
39. If $S_n = 1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}$ such that $2 - S_n < \frac{1}{100}$, find the least value of n.
40. If a, b and c are the sides of a triangle, prove that $\Delta \leq \frac{s^2}{4}$, where s is the semi-perimeter and Δ be the area of the triangle.

Integer Type Questions

1. If $x, y, z > 0$ and $x + y + z = 1$, find the maximum value of $\frac{16xyz}{(1-x)(1-y)(1-z)}$.
2. If $0 < x < \frac{\pi}{4}$ such that the minimum value of $(\sin x + \cos x + \operatorname{cosec} 2x)^3$ is $\frac{m^n}{2}$, where $m, n \in N$, find the value of $(m + n + 2)$.
3. If a, b and c are positive real numbers such that $a + b + c = 1$ and $\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right) \geq p^q$, find the value of $(p + q)$.
4. If $a + b + c = 1$ such that $(1 - a)(1 - b)(1 - c) \geq kabc$, find the value of $(\sqrt[3]{k}+1)$.
5. If a, b, c and d are four positive real numbers and $abcd = 1$ such that the minimum value of $(1 + a)(1 + b)(1 + c)(1 + d)$ is λ, find the value of $(\sqrt[4]{\lambda}+1)$.
6. Let $a_1, a_2, \ldots, a_n$ are in GP. If $a_n > a_m$, where $n > m$ and $a_1 + a_n = 66$ while $a_2 \cdot a_{n-1} = 128$ and $\sum_{i=1}^{n} a_i = 126$, find the value of n.

7. If a, b and c are real numbers such that $a + b + c = 3$ and $\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}=\frac{10}{3}$, find the value of $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$.
8. If $b_n=\left(\frac{1}{5}\right)^{\log_{\sqrt{5}}\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots+\frac{1}{4.2^{n-1}}\right)}$, find $([\lim_{n\to\infty}(b_n)]+3)$.
9. If a, x, y, z and b are in AP, the value of $(x + y + z)$ is 15 while $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ is $\frac{5}{3}$ if a, x, y, z and b are in HP, find the value $(a - b - 2)$.
10. If $(1 - p)(1 + 3x + 9x^2 + 27x^3 + 8ax^4 + 243x^5) = (1 - p^6)$, $p \neq 1$, find the value of $\left(\frac{p}{x}+2\right)$.
11. If $\sum_{x=5}^{n+5} 4(x-3) = An^2 + Bn + C$, find the value of $(A + B - C - 4)$.
12. If $\sum_{r=1}^{n} r(r+1)(2r+3) = an^4 + bn^3 + cn^2 + dn + e$, find the value of $(a + c + 1)$.
13. If x and y be two positive real numbers such that $x^2 + y^2 = 8$, find the maximum value of $(x + y)$.
14. If $\sum_{k=1}^{n}\left(\sum_{m=1}^{k} m^2\right) = an^4 + bn^3 + cn^2 + dn + e$, find the value of $12(a + d)$.
15. Let a, b and c are positive real numbers such that $ab + bc + ca = 12$, then find the greatest value of abc.

Comprehensive Link Passage (For JEE-Advanced Examination Only)

Passage I

Let α, β and γ be the roots of $\frac{x-a}{b}+\frac{x-b}{a}=\frac{a}{x-b}+\frac{b}{x-a}$, where $a, b > 0$ and $\alpha > \beta > \gamma$.

(i) Then β is
(a) $a + b$ (b) $\frac{a^2+b^2}{a+b}$
(c) 0 (d) $\frac{a+b}{a^2+b^2}$

(ii) If $\alpha = 2\beta$, the maximum value of the area of a triangle whose perimeter 3α, is
(a) α^2 (b) $\frac{3}{4}a^2$ (c) $a^2\sqrt{3}$ (d) $2\sqrt{3}\,a^2$

(iii) If $\alpha - \beta - \gamma = c$, then
(a) a, b, c are in AP (b) a, c, b are in AP
(c) a, b, c are in HP (d) a, c, b are in HP

Passage II

Suppose $A_1, A_2, \ldots, A_n$ be AMs; $G_1, G_2, \ldots, G_n$ be GMs; $H_1, H_2, \ldots, H_n$ be HMs between two positive real numbers a and b.

(i) A_n, G_n, H_n are in
(a) AP (b) GP (c) HP (d) AGP

(ii) H_1 is
(a) $\frac{a+(2n-1)b}{2n}$ (b) $\frac{a(2n+1)+b}{2n}$
(c) $\frac{a(2n+1)-b}{2n}$ (d) $\frac{2nab}{a+(2n-1)b}$

(iii) $\frac{H_1+a}{H_1-a}+\frac{H_{2n-1}+b}{H_{2n-1}-b}$ is
(a) $2(n - 1)$ (b) $4n$ (c) $4n - 2$ (d) $4n + 2$

Passage III

Suppose a, b and c are the sides of a triangle, which are in GP.

(i) If the common ratio, r of the series is less than unity, then r is
(a) $\left(n, \frac{1}{\sqrt{2}+1}\right)$ (b) $\left(0, \frac{\sqrt{5}-1}{2}\right)$
(c) $\left(0, \frac{\sqrt{5}+1}{4}\right)$ (d) $\left(\frac{\sqrt{5}-1}{2}, 1\right)$

(ii) If $\log a - \log 2b$, $\log 2b - \log 3c$ and $\log 3c - \log a$ are in AP, the least side of the triangle is
(a) a (b) b (c) c (d) $a = b = c$

(iii) The greatest angle of the triangle is
(a) 135° (b) 90°
(c) 120° (d) $\pi - \cos^{-1}\left(\frac{1}{3}\right)$

Passage IV

Let α, β and γ are the real roots of $ax^3 + 3bx^2 + 3cx + d = 0$ such that $\alpha \neq \beta \neq \gamma$.

(i) If roots are in AP, the value of $2a^3$ is
(a) $3abc + 2a^2c$ (b) $3abc - ac^2$
(c) $3abc + ac^2$ (d) $3abc - a^2c$

(ii) If roots are in GP, the value of $\frac{a}{c}$ is
(a) $\left(\frac{a}{b}\right)^3$ (b) $\left(\frac{b}{d}\right)^3$ (c) $\left(\frac{d}{b}\right)$ (d) $\left(\frac{b}{d}\right)$

(iii) If roots are in HP and $a = b = c = 1$, the value of d is
(a) 1, 2 (b) 1 (c) 2 (d) –2

Passage V

Let $x_1, x_2, x_3, \ldots, x_n$ are distinct real numbers in HP.

(i) Which of the following is true?
(a) $x_1x_4 > x_2x_3$ (b) $x_1x_4 < x_2x_3$
(c) $x_1x_2 > x_3x_4$ (d) $x_1x_2 < x_3x_4$

(ii) Which of the following is true?
(a) $x_5 + x_8 > x_6 + x_7$ (b) $x_5 + x_8 < x_6 + x_7$
(c) $x_5 + x_6 > x_8 + x_7$ (d) $x_5 + x_6 < x_8 + x_7$

(iii) $x_1x_2 + x_2x_3 + x_3x_4 + x_1x_5$ is
(a) $4x_1x_5$ (b) $2x_1x_5$ (c) $3x_1x_5$ (d) $5x_1x_5$

Passage VI

Suppose $s_1, s_2, \ldots, s_n$ are the sum of n geometric series of infinite terms, whose first terms are 1, 2, 3, …, n and the common ratios $\frac{1}{2}, \frac{1}{3}, \ldots \frac{1}{n+1}$, respectively.

(i) If $s_1 + s_2 + \ldots + s_n = 7$, the value of n is
(a) 12 (b)* 13 (c) 11 (d) 14

(ii) The value of $s_1^2 + s_2^2 + \ldots + s_{2n-1}^2$ is
(a) $n(2n + 1)(4n + 1) - 1$
(b) $\frac{1}{3}n(2n+1)(4n+1)-1$
(c) $\frac{1}{3}n(2n+1)(4n+1)$
(d) $\frac{1}{3}n(2n+1)(4n+1)-2$

(iii) If $s_1^3 + s_2^3 + \ldots + s_{2k-1}^3 = 1800$, the value of k is
(a) 15 (b) 16 (c) 5 (d) 6

Passage VII

Let $A_1, A_2, A_3, \ldots, A_n$ be the arithmetic means between –2 and 1027 and $G_1, G_2, G_3, \ldots, G_n$ be geometric means between 1 and 1024. The product of geometric means is 2^{45} and the sum of the arithmetic means is 1025×171.

On the basis of above information, answer the following questions:

(i) The value of n is
(a) 7 (b) 9 (c) 11 (d) None

(ii) The value of m is
(a) 340 (b) 342 (c) 344 (d) 346

(iii) The value of $G_1 + G_2 + G_3 + \ldots + G_n$ is
(a) 1022 (b) 2044 (c) 512 (d) None

(iv) The common difference of the progression $A_1, A_3, A_5, \ldots, A_{m-1}$ is
(a) 6 (b) 3 (c) 2 (d) 1

(v) The numbers $2A_{171}$, $(G_5)^2 + 1$ and $2A_{172}$ are in
(a) AP (b) GP (c) HP (d) AGP

Passage VIII

There are two sets A and B each of which consists of three numbers in AP, whose sum is 15 where D and d are the common differences such that $D - d = 1$. If $\frac{p}{q} = \frac{7}{8}$, where p and q are the products of the numbers, respectively, and $d > 0$, in two sets.

On the basis of the above information, answer the following questions:

(i) The value of p is
(a) 100 (b) 120 (c) 105 (d) 110.

(ii) The value of q is
(a) 100 (b) 120 (c) 105 (d) 110

(iii) The value of $D + d$ is
(a) 1 (b) 2 (c) 3 (d) 4

Passage IX

Four different integers form an increasing AP. One of these numbers is equal to the sum of the squares of the other three numbers.

On the basis of the above information, answer the following questions:

(i) The smallest number is
(a) –2 (b) 0 (c) –1 (d) 2

(ii) The common difference of the four numbers is
(a) 2 (b) 1 (c) 3 (d) 4

(iii) The sum of the four numbers is
(a) 10 (b) 8 (c) 2 (d) 6

Matrix Match
(For JEE-Advanced Examination Only)

1. Match the following columns:

	Column I		Column II
(A)	If a^2, b^2, c^2 are in AP, $\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ are in	(P)	AP
(B)	If a, b, c are in HP, $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in	(Q)	GP
(C)	If a, b, c are in AP as well as in GP, a^2, b^2, c^2 are in	(R)	HP
(D)	If $b + c$, $c + a$, $a + b$ are in HP, $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in	(S)	AGP

2. Match the following columns:

	Column I		Column II
(A)	If 2, A_1, A_2, A_3, A_4, 8 are in AP, the value of $A_1 + A_2 + A_3 + A_4$ is	(P)	2
(B)	If 2, G_1, G_2, G_3, G_4, G_5, 16 are in GP, the value of $G_1 \cdot G_2 \cdot G_3 \cdot G_4 \cdot G_5$ is	(Q)	5/2
(C)	If 2, $H_1, H_2, H_3, H_4, H_5, H_6$, 3 are in HP, the value of $\frac{1}{H_1}+\frac{1}{H_2}+\frac{1}{H_3}+\frac{1}{H_4}+\frac{1}{H_5}+\frac{1}{H_6}$ is	(R)	20

3. Match the following columns:

	Column I		Column II
(A)	If $x_1, x_2, x_3, \ldots, x_{10}$ are in HP, $\frac{x_1x_2 + x_2x_3 + x_3x_4 + \ldots + x_1x_{10}}{x_1x_{10}}$ is	(P)	60

(B)	If 2, x_1, x_2, …, x_{10}, 10 are in AP then $x_1 + x_2 + \ldots + x_{10}$ is	(Q)	1
(C)	If a, b, c are in AP and x, y, z are in GP, $x^{b-c} \cdot y^{c-a} \cdot z^{a-b}$ is	(R)	9
(D)	If $a^x = b^y = c^z$ and a, b, c are in GP, $y\left(\frac{1}{x}+\frac{1}{z}\right)$ is	(S)	2

4. Match the following columns:

Column I		Column II	
(A)	The sum of n terms of an AP is $5n^2 + 4n$, the common difference of the series is	(P)	4
(B)	The sum of n terms of a GP is $\left(\frac{A^n - B}{2}\right)$, the value of $A + B$ is	(Q)	10
(C)	The sum of n terms of an AP is $4n^2 + 3n$, then t_0 is	(R)	3
(D)	The sum of $\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+\ldots$ to 100 terms is $100+\left(1-\left(\frac{A}{B}\right)^{100}\right)$, then $A + B$ is	(S)	79

Matching List Type (Only One Option is Correct)

This section contains four questions, each having two matching list. Choices for the correct combination of elements from List I and List II are given as options (A), (B), (C) and (D), out of which only ONE is correct.

5. Match the following lists:

List I		List II	
(P)	Suppose that $F(n+1) = \frac{2F(n)+1}{2}$ for n = 1, 2, 3, … and $F(1) = 2$. Then $F(101)$ is	(1)	42
(Q)	If $a_1, a_2, a_3, \ldots, a_{21}$ are in AP and $a_1 + a_5 + a_{11} + a_{17} + a_{19} = 10$, the value of $\sum_{i=1}^{21} a_i$ is	(2)	1620
(R)	10th term of the sequence $S = 1 + 5 + 13 + 29 + \ldots$ is	(3)	52
(S)	The sum of all two-digit numbers which are not divisible by 2 or 3 is	(4)	2045

Codes

	P	Q	R	S
(A)	2	3	4	1
(B)	3	2	1	4
(C)	3	1	4	2
(D)	3	1	2	4

6. Match the following columns:

List I		List II	
(P)	The arithmetic mean of two numbers is 6 and their geometric mean G and harmonic H satisfy the relation $G^2 + 3H = 48$. The two numbers are	(1)	240 and 77
(Q)	The sum of the series $\frac{5}{1^2 \cdot 4^2}+\frac{11}{4^2 \cdot 7^2}+\frac{17}{7^2 \cdot 10^2}+\ldots$ is	(2)	(4, 8)
(R)	If the first two terms of an harmonic progression be 1/2 and 1/3, the harmonic mean of the first four terms is	(3)	16
(S)	If a, b, c and d are four positive real numbers such that $abcd = 1$, the minimum value of $(1 + a)(1 + b)(1 + c)(1 + d)$ is	(4)	1/2

Codes

	P	Q	R	S
(A)	2	3	1	4
(B)	3	1	4	2
(C)	1	3	4	2
(D)	3	1	2	4

7. Match the following lists:

List I		List II	
(P)	The co-efficient of x^{49} in $(x - 1)(x - 2)(x - 3) \ldots (x - 100)$ is	(1)	5050
(Q)	The co-efficient of x^{99} in $(x + 1)(x + 2)(x + 3) \ldots (x + 100)$ is	(2)	2500
(R)	The co-efficient of x^{99} in $(x - 1)(x - 3)(x - 5) \ldots (x - 99)$ is	(3)	–2550
(S)	The co-efficient of x^{49} in $(x - 2)(x - 4) \ldots (x - 100)$ is	(4)	–1225

Codes

	P	Q	R	S
(A)	2	3	1	4
(B)	1	4	2	3
(C)	2	3	4	1
(D)	4	1	2	3

8. Match the following columns:

List I		List II	
(P)	If $a + 2b + 3c = 12$, the maximum value of ab^2c^3 is	(1)	1024
(Q)	If $x + 4y + 5z = 20$, the max value of xy^4z^5 is	(2)	16
(R)	If $a + b + c + d = 8$, the minimum value of $(a + b)(c + d)$ is	(3)	32
(S)	If $abc = 4$, the least value of $4(1 + a)(1 + b)(1 + c)$ is	(4)	64

Codes

	P	Q	R	S
(A)	1	3	2	4
(B)	3	1	2	4
(C)	2	3	4	1
(D)	3	2	4	1

Questions asked in Previous Years' JEE-Advanced Examinations

1. The value of $x + y + z$ is 15 . If a, x, y, z, b are in AP while the value of $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ is $\frac{5}{3}$. If a, x, y, z, b are in HP, find a and b. **[IIT-JEE, 1978]**
2. If x, y and z are in HP, show that
$\log (x + z) + \log (x + z - y) = 2 \log (x - z)$ **[IIT-JEE, 1978]**
3. If the mth, nth and pth terms of an AP and GP are equal and x, y, z, then prove that $x^{y-z} \cdot y^{z-x} \cdot z^{x-y} = 1$. **[ITT-JEE, 1979]**
4. The interior angles of a polygon are in AP, the smallest angle is 120° and the common difference is 5°. Find the number of sides of the polygon. **[IIT-JEE, 1980]**
5. Let the angles of a triangle ABC be in AP and let $b : c = \sqrt{3} : \sqrt{2}$. Find the angle A. **[IIT-JEE, 1981]**
6. If $a_1, a_2, a_3, \ldots, a_n$ are in AP where $a_i > 0$, $i = 1, 2, \ldots, n$ for all i, show that
$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\ldots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}} = \left(\frac{(n-1)}{\sqrt{a_n}+\sqrt{a_1}}\right)$$ **[IIT-JEE, 1982]**
7. Does there exist a GP containing 27, 8 and 12 as three of its terms? If it exists, how many such progressions are possible? **[IIT–JEE, 1982]**
8. The third term of a geometric progression is 4, the product of the first 5 terms is
(a) 4^3 (b) 4^5 (c) 4^4 (d) None **[IIT-JEE,1982]**
9. Find three numbers a, b and c between 2 and 18 such that
(i) their sum is 25.
(ii) 2, a and b are consecutive terms of an AP
(iii) the numbers b, c and 18 are consecutive terms of a GP. **[IIT-JEE, 1983]**
10. The rational number, which equals the number $2.\overline{357}$ with recurring decimal is
(a) $\frac{2355}{1001}$ (b) $\frac{2379}{997}$
(c) $\frac{2355}{999}$ (d) None **[IIT-JEE, 1983]**
11. If n is a natural number such that $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots p_k^{\alpha_k}$ and $p_1, p_2, \ldots, p_k$ are distinct primes, show that $\log n \geq k \log 2$ **[IIT-JEE, 1984]**
12. The sum of the integers from 1 to 100 that are divisible by 2 or 5 is … **[IIT-JEE, 1984]**
13. In a triangle ABC if $\cot A$, $\cot B$, $\cot C$ are in AP, a, b, c are in … progression. **[IIT-JEE, 1985]**
14. If three complex numbers are in AP, they lie on a circle in the complex plane. **[IIT-JEE, 1985]**
15. If a, b and c are in GP, the equation $ax^2 + 2bx + c = 0$, $dx^2 + 2ex + f = 0$ have a common root if $\frac{d}{a}, \frac{e}{b}$ and $\frac{f}{c}$ are in
(a) AP (b) GP (c) HP (d) None **[IIT-JEE, 1985]**
16. The sum of the squares of three distinct real numbers, which are in GP is s^2. If their sum is αS, show that $\alpha^2 \in \left(\frac{1}{3}, 1\right) \cup (1, 3)$. **[IIT-JEE, 1986]**
17. If a, b, c, d and p are distinct real numbers such that $(a^2 + b^2 + c^2)p^2 - 2(ab + bc + cd)p + (b^2 + c^2 + d^2) \leq 0$ then a, b, c and d are in
(a) AP (b) GP (c) HP (d) $ab = cd$ **[IIT-JEE, 1987]**
18. The sides of a triangle inscribed in a given circle subtend angles α, β and γ at the centre. The minimum value of the arithmetic mean of
$$\cos\left(\alpha+\frac{\pi}{2}\right)\cdot\cos\left(\beta+\frac{\pi}{2}\right)\cdot\cos\left(\gamma+\frac{\pi}{2}\right)$$
is… **[IIT-JEE, 1987]**
19. The sum of the first n terms of the series $\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+\ldots$ is equal to
(a) $2^n - n - 1$ (b) $1 - 2^{-n}$
(c) $n - 1 + 2^{-n}$ (d) $2^n + 1$ **[IIT-JEE, 1988]**
20. The sum of the first n terms of the series $1^2 + 2.2^2 + 3^2 + 2.4^2 + 5^2 + 2.6^2 + \ldots$ is $\frac{n(n+1)^2}{2}$, where n is even. Then the sum, when n is odd, is **[IIT-JEE, 1988]**
21. If the first and $(2n - 1)$th term of an AP, a GP and an HP are equal and their nth terms are a, b and c respectively, then
(a) $a = b = c$ (b) $a \geq b \geq c$
(c) $a + c = b$ (d) $ac - b^2 = 0$ **[IIT-JEE, 1988]**

22. No questions asked in 1989.
23. If $\log_3 2, \log_3(2^x - 5), \log_3\left(2^x - \frac{7}{2}\right)$ are in AP, determine the value of x. **[IIT-JEE, 1990]**
24. Let p be the first of the n arithmetic means between two numbers and q the first of the n harmonic means between the same numbers. Show that q does not lie between p and $\left(\frac{n+1}{n-1}\right)^2 p$. **[IIT-JEE, 1991]**
25. If $S_1, S_2, S_3, \ldots, S_n$ are the sums of infinite geometric series whose first terms are $\{1, 2, 3, \ldots, n\}$ and whose common ratios are $\left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n+1}\right\}$, respectively, find the value of $S_1^2 + S_2^2 + \ldots + S_{2n-1}^2$. **[IIT-JEE, 1991]**
26. Let the harmonic mean and geometric mean of two positive numbers be in the ratio 4 : 5. Then the two numbers in the ratio… **[IIT-JEE, 1992]**
27. For $0 < \varphi < \frac{\pi}{2}$, if $x = \sum_{n=0}^{\infty} \cos^{2n}\varphi$, $y = \sum_{n=0}^{\infty} \sin^{2n}\varphi$ and $z = \sum_{n=0}^{\infty} \cos^{2n}\varphi \sin^{2n}\varphi$, then
    (a) $xyz = xz + y$ (b) $xyz = xy + z$
    (c) $xyz = x + y + z$ (d) $xyz = yz + x$
    [IIT-JEE, 1993]
28. If ln $(a + c)$, ln $(a - c)$ and ln $(a - 2b + c)$ are in AP, then
    (a) a, b, c are in AP (b) a^2, b^2, c^2 are in AP
    (c) a, b, c are in GP (d) a, b, c are in HP
    [IIT-JEE, 1994]
29. No questions asked in 1995.
30. For any odd integer $n \geq 1$,
    $n^3 - (n-1)^3 + \ldots + (-1)^{n-1} \cdot 1^3 = \ldots$
    [IIT-JEE, 1996]
31. The real number x_1, x_2 and x_3 satisfying the equation $x^3 - x^2 + \beta x + \gamma = 0$ are in AP. Find the intervals in which β and γ lie. **[IIT-JEE, 1996]**
32. Let p and q be the roots of the equation $x^2 - 2x + A = 0$ and let r and s be the roots of $x^2 - 18x + B = 0$. If $p < q < r < s$ are in arithmetic progression, then $A = \ldots$ and $B = \ldots$ **[IIT-JEE, 1997]**
33. Let x be the arithmetic mean and y and z be the two geometric means between any two positive numbers. Then $\frac{y^3 + z^3}{xyz} = \ldots.$ **[IIT-JEE, 1997]**
34. Let T_r be the rth term of an AP for $r = 1, 2, 3$. If for some positive integers m and n, we have $T_m = 1/n$ and $T_n = 1/m$, then T_{mn} equals
    (a) $\frac{1}{mn}$ (b) $\frac{1}{m} + \frac{1}{n}$
    (c) 1 (d) 0 **[IIT-JEE, 1998]**
35. If $x > 1, y > 1$ and $z > 1$ are in GP, then $\frac{1}{1 + \ln x}, \frac{1}{1 + \ln y}, \frac{1}{1 + \ln z}$ are in
    (a) AP (b) GP (c) HP (d) None
    [IIT-JEE, 1998]
36. Let $a_1, a_2, \ldots, a_{10}$ be in AP and $h_1, h_2, \ldots, h_{10}$ be in HP. If $a_1 = 2 = h_1$ and $a_{10} = 3 = h_{10}$, the value of $a_4 h_7$ is **[IIT-JEE, 1999]**
    (a) 2 (b) 3 (c) 5 (d) 6
37. The harmonic mean of the roots of the equation $(5 + \sqrt{2})x^2 - (4 + \sqrt{5})x + (8 + 2\sqrt{5}) = 0$ is
    (a) 2 (b) 4 (c) 6 (d) 8
    [IIT-JEE, 1999]
38. Consider an infinite series with first term a and common ratio r. If its sum is 4 and the second term is 3/4, then
    (a) $a = 4/7, r = 3/7$ (b) $a = 2, r = 3/8$
    (c) $a = 3/2, r = 1/2$ (d) $a = 3, r = 1/4$
    [IIT-JEE, 2000]
39. The fourth power of the common difference of an AP with integer entries in added to the product of any four consecutive terms of it, prove that the resulting sum is square of an integer. **[IIT-JEE, 2000]**
40. Let T_n denotes the number of triangles which can be formed using the vertices of a regular polygon of n-sides. If $T_{n+1} - T_n = 21$, then n equals
    (a) 5 (b) 7 (c) 6 (d) 4
    [IIT-JEE, 2001]
41. Let the positive numbers a, b, c, d be in AP, then abc, abd, acd, bcd are
    (a) not in AP/GP/HP (b) in AP
    (c) in GP (d) in HP
    [IIT-JEE, 2001]
42. If the sum of the first $2n$ terms of the AP = $\{2, 5, 8, \ldots\}$ is equal to the sum of the first n terms of the AP = $\{57, 59, 61, \ldots\}$, then n equals
    (a) 100 (b) 12 (c) 11 (d) 13
    [IIT-JEE, 2001]
43. Let $a_1, a_2, a_3, \ldots, a_n$ be positive real number in GP for each a, let A_n, G_n, H_n, be the arithmetic mean, geometric mean, harmonic mean respectively. Find an expression for the geometric mean $G_1, G_2, G_3, \ldots, G_n$ in terms of $A_1, A_2, A_3, \ldots, A_n$ and $H_1, H_2, \ldots, H_n$ **[IIT-JEE, 2001]**
44. If $a_1, a_2, a_3, \ldots, a_n$ are positive real numbers whose product is a fixed number c, the minimum value of $a_1 + a_2 + \ldots + a_{n-1} + 2a_n$ is **[IIT-JEE, 2002]**
    (a) $n(2c)^{1/n}$ (b) $(n + 1)c^{1/n}$
    (c) $2n.c^{1/n}$ (d) $(n + 1)(2c)^{1/n}$
45. Suppose a, b and c are in AP and a^2, b^2 and c^2 are in GP. If $a < b < c$ and $a + b + c = \frac{3}{2}$, the value of a is
    (a) $\frac{1}{2\sqrt{2}}$ (b) $\frac{1}{2\sqrt{3}}$ (c) $\frac{1}{2} + \frac{1}{\sqrt{2}}$ (d) $\frac{1}{2} - \frac{1}{\sqrt{2}}$
    [IIT-JEE, 2002]

46. Let a and b are positive real numbers. If a, A_1, A_2, b are in AP and a, G_1, G_2, b are in GP and a, H_1, H_2, b are in HP. Show that
$$\frac{G_1G_2}{H_1H_2} = \frac{A_1 + A_2}{H_1 + H_2} = \frac{(2a+b)(a+2b)}{9ab}$$
[IIT-JEE, 2002]

47. If a, b and c are in AP and a^2, b^2 and c^2 are in HP, prove that either $a = b = c$ or a, b and $-c/2$ are in GP. **[IIT-JEE, 2003]**

48. An infinite GP has first term x and sum 5, then
(a) $x < -10$ (b) $-10 < x < 0$
(c) $0 < x < 10$ (d) $x > 10$
[IIT-JEE, 2004]

49. If a, b and c are positive numbers, prove that $(1 + a)^7 (1 + b)^7 (1 + c)^7 < 7^7 a^4 b^4 c^4$. **[IIT-JEE, 2004]**

50. If α and β are the roots of $ax^2 + bx + c = 0$ and $\alpha + \beta$, $\alpha^2 + \beta^2$ and $\alpha^3 + \beta^3$ are in GP and $\Delta = b^2 - 4ac$, then
(a) $c\Delta = 0$ (b) $cb \neq 0$ (c) $b\Delta = 0$ (d) $\Delta \neq 0$
[IIT-JEE, 2005]

51. If total number of runs scored by a cricketer in n matches is $\left(\frac{n+1}{4}\right)(2^{n+1} - n - 2)$ where $n > 1$ and the runs scored in the kth match are given by $k \cdot 2^{n+1-k}$, where $1 \leq k \leq n$, find n. **[IIT-JEE, 2005]**

52. If $a_n = \frac{3}{4} - \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 - \ldots + (-1)^{n-1}\left(\frac{3}{4}\right)^n$ and $b_n = 1 - a_n$, find the least natural number n_0 such that $b_n > a_n$, for every $n \geq n_0$. **[IIT-JEE, 2006]**

53. If $\theta = \sum_{k=1}^{\infty} \tan^{-1}\left(\frac{1}{2k^2}\right)$, find $\tan\theta$. **[IIT-JEE, 2006]**

54. Suppose 4 distinct positive numbers a_1, a_2, a_3 and a_4 are in GP. Let
$b_1 = a_1, b_2 = b_1 + a_2$
$b_3 = b_2 + a_3, b_4 = b_3 + a_4$
Statement I: The numbers b_1, b_2, b_3, b_4 are neither in AP nor in GP.
Statement II: The numbers b_1, b_2, b_3, b_4 are in HP.
[IIT-JEE, 2008]

55. If the sum of first n terms of an AP is cn^2, the sum of the squares of these n terms is
(a) $\frac{n(4n^2-1)c^2}{6}$ (b) $\frac{n(4n^2+1)c^2}{3}$
(c) $\frac{n(4n^2-1)c^2}{3}$ (d) $\frac{n(4n^2+1)c^2}{6}$
[IIT-JEE, 2009]

56. Let $a_1, a_2, \ldots, a_{11}$ be real numbers satisfying $a_1 = 15$, $27 - 2a_2 > 0$ and $a_k = 2a_{k-1} - a_{k-2}$ for $k = 3, 4, 5, \ldots, 11$. If $\frac{a_1^2 + a_2^2 + \ldots + a_{11}^2}{11} = 90$, find the value of $\frac{a_1 + a_2 + \ldots + a_{11}}{11}$. **[IIT-JEE, 2010]**

57. Let $a_1, a_2, \ldots, a_{100}$ be an AP with $a_1 = 3$ and $S_p = \sum_{i=1}^{p}(a_i), 1 \leq p \leq 100$. For any integer n with $1 \leq n \leq 20$, let $m = 5n$. If $\frac{S_m}{S_n}$ does not depend on n, find a_2. **[IIT-JEE, 2011]**

58. Let $a_1, a_2, \ldots$ be in HP with $a_1 = 5$ and $a_{20} = 25$. The least positive integer n for which $a_n < 0$ is
(a) 22 (b) 23 (c) 24 (d) 25
[IIT-JEE, 2012]

59. Let $S_n = \sum_{k=1}^{4n} (-1)^{\frac{k(k+1)}{2}} \cdot k^2$. Then S_n can take values
(a) 1056 (b) 1088 (c) 1120 (d) 1132
[IIT-JEE, 2013]

60. Let a, b and c be positive integers such that $\frac{b}{a}$ is an integer. If a, b and c are in GP and the arithmetic mean of a, b, c is $b + 2$, find the value of $\left(\frac{a^2+a-14}{a+1}\right)$.
[IIT-JEE, 2014]

Answers

Level I

1. 3
6. p
7. ₹245
10. 4/5
11. 0
12. 21, 23, 25
13. 75°, 85°, 95° and 105°
14. ±3
15. 2, 4, 6, 8
21. 2139
22. 70336
24. 900
25. (i) 61
(ii) 10
27. 36
32. (148/111)
34. 9
39. $2Q$
45. $n = 0$
46. $n = 11$
49. $n = 6$

50. $\left(\frac{1}{\sqrt{2}}\right)^{17}$
52. 11th term
53. 9th term
54. (16, 24, 36, …)
55. 243
56. (3, 6, 12 or 12, 6, 3)
57. ($r = \pm 3$)
63. 2
66. $\left(\frac{3^n - 1}{2}\right)$
67. $2\left(1 - \frac{1}{2^n}\right)$
68. $n = 8$
69. $\frac{1}{(x-y)}\left[x^2\left(\frac{x^n - 1}{x - 1}\right) - y^2\left(\frac{y^n - 1}{y - 1}\right)\right]$
70. $a\left(\frac{(1+i)^n - 1}{i - 1}\right)$
71. $\frac{1}{2}(3^{11} + 37)$
72. $\frac{1}{2}(2^{n+1} + 3^{n+1} - 5)$
73. (i) $\left(\frac{4^n - 1}{3}\right) + \frac{25}{4}(5^n - 1)$
    (ii) $\frac{3}{2}\left(1 - \frac{1}{3^n}\right) + \frac{1}{20}\left(1 - \frac{1}{5^n}\right)$
74. (i) $\frac{5}{81}(10^{n+1} - 9n - 10)$
    (ii) $\frac{7}{81}(10^{n+1} - 9n - 10)$
    (iii) $\frac{1}{9}(10^{n+1} - 9n - 10)$
75. $\frac{4}{3}(10^n - 1)^2$
76. $\frac{2^{10} - 1}{2^9} + \frac{5^{10} - 1}{4 \times 5^{11}}$
78. 9
79. $n + 2\left(1 - \left(\frac{2}{3}\right)^n\right)$
80. $n + \frac{\left(1 - \left(\frac{1}{3}\right)^n\right)}{2}$
82. $n = 4$
84. $\left(\frac{3 + 2\sqrt{2}}{\sqrt{2}}\right)$
85. $\left(\frac{19}{24}\right)$
95. 4
105. –1/2
106. 16, 4
136. (i) $n - 1 + 2^{-n}$
    (ii) $\frac{1}{2}(2n - 1 + 3^{-n})$
    (iii) $\left(7 - \frac{7 + 3n}{2^n}\right)$
137. (i) 2 (ii) 1/3
153. $\log_4 (2^{n+1} - 2)$
158. 20
159. $\leq \left(\frac{1}{8}\right)$
164. $n = 255$
165. 2
176. 4
179. 21/2
180. 4
181. 118/9
182. 64
183. 5^9
184. 32
185. 729
186. 4
187. 3
188. 11
189. 2013
190. 9
191. 8
192. 18
193. 3141/4
194. 9
195. $(1/3)^9$
196. 8
197. 27
167. 27
201. $\left[0, \frac{16}{5}\right]$
202. 8
203. 96

Level II

1. (a)	2. (c)	3. (c)	4. (b)	5. (c)
6. (b)	7. (d)	8. (c)	9. (d)	10. (a)
11. (d)	12. ()	13. (a)	14. (d)	15. (b)
16. (c)	17. (c)	18. (c)	19. (b)	20. (a)

21. (a)	22. (d)	23. (a)	24. (a)	25. (a, c)
26. (c)	27. (a)	28. (a)	29. (b)	30. (c)
31. (d)	32. (c)	33. (b)	34. (c)	35. (d)
36. (b)	37. (b)	38. (a)	39. (b)	40. (c)
41. (b, d)	42. (b)	43. (d)	44. (a)	45. (c)
46. (c)	47. (a)	48. (b)	49. (b)	50. (a)
51. (b)	52. (b)	53. (c)	54. (c)	55. (a)
56. (c)	57. (b)	58. (a)	59. (a)	60. (a)
61. (a)	62. (a)	63. (b)	64. (c)	65. (a)

Level III

1. $\left(\dfrac{n(n^2+2)}{3}\right)$
2. $\left(\dfrac{n(n+1)}{2}\right)^2 - n$
3. $\dfrac{n(n+1)(n+2)}{6}$
4. $x \le -1$ or $x \ge 3$
5. $\dfrac{d}{a}, \dfrac{e}{b}, \dfrac{f}{c}$ are in AP
6. $(n-1)a_1 a_n$
7. $\left(\dfrac{\sqrt{3}-1}{2}\right)$
8. $\dfrac{1}{2} < \left(\dfrac{(ab+bc+ca)}{(a^2+b^2+c^2)}\right) < 1$
10. $a = 19$
11. $\left(p+\dfrac{1}{p}\right)^2 + \left(q+\dfrac{1}{q}\right)^2 \ge 8$
12. $\dfrac{n(n+1)(n+2)}{6}$
13. $\dfrac{n(n+1)(n-2)}{6}$
14. $\left(\dfrac{\alpha-\beta}{1007}\right)$
15. x, y and z are in HP
16. $\left(\dfrac{n(n+1)(n-1)(3n+2)}{12}\right)$
17. $a = \dfrac{1}{24}, b = \dfrac{1}{4}, c = \dfrac{3}{8}, d = \dfrac{1}{6}, e = 0$
18. 2013
19. $\dfrac{1}{2187}$
20. 27
21. $16\sqrt{2}$
22. $((1+a)(1+b))^3 \ge 3^3 \cdot a^2 b^2$
23. 27
24. $(2^{n+1} - n - 2)$
25. $\left(\dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\ldots\right) = \dfrac{\pi^2}{8}$
26. $\dfrac{(2a+b)(a+2b)}{9ab}$
27. $x, y\, z, u \in$ HP
29. $\dfrac{13n-6}{23-19n}$
30. $x = 10^{35},\ x = 10^{\left(-\frac{5}{7}\right)}$
31. $\dfrac{25}{2}, \dfrac{79}{6}, \dfrac{83}{6}, \ldots$ or $\dfrac{2}{3}, \dfrac{25}{3}, \dfrac{625}{6}, \ldots$
32. $\begin{cases} 3, -\dfrac{3}{2}, \dfrac{3}{4}, \ldots \\ \text{or} \\ 6, -12, 24, \ldots \end{cases}$
33. $\dfrac{n}{2}[2a^3 + 3a^2 d(n-1) + (n-1)(2n-1) + d^3\left(\dfrac{n(n-1)^2}{2}\right)$
34. $\dfrac{25}{2}, \dfrac{79}{6}, \dfrac{83}{6}, \ldots$ or $\dfrac{2}{3}, \dfrac{25}{3}, \dfrac{625}{6}, \ldots$
35. $-\dfrac{n}{2}$
36. $S = \dfrac{1+(1-x^{-1/3})(1-y^{-1/4})}{[1-(1-x^{-1/3})(1-y^{-1/4})]^2}$
37. $n^2 - n + 1$
38. 8, 24, 72, 216, 648
39. 108, 144, 160
40. 6, 12, 24 or 24, 12, 6
41. 2000
42. –180
43. $S = \left(\dfrac{1-x^n}{(1-x)^2}\right) - \dfrac{nx^n}{(1-x)}$, where $x = \left(1+\dfrac{1}{n}\right)$
44. $\dfrac{1}{2}(1-\cos x)$
45. $\left(-\dfrac{3}{5}, -\dfrac{3}{2}, 3\right)$

Level IV

1. Arithmetic
3. $\left(\dfrac{14n-6}{8n+23}\right)$
4 $x = 10^5, y = 10$
5. $x = 3$
6. $\dfrac{2^{n+1}}{1-x^{2n+1}} - \dfrac{1}{1-x}$
7. $\left[\dfrac{(n+1)(n+2)(2n+3)}{6} - 1\right] + S_{n+1}^2 + S_{n+2}^2 + S_{2n-1}^2$

8. $\dfrac{\sqrt{3}-1}{2}$
9. $2\sqrt{2}(\sqrt{2}-1)$
10. $a = 3$
11. $\beta \in \left(-\alpha, \dfrac{1}{3}\right]$ and $r \in \left[-\dfrac{1}{27}, \alpha\right)$
12. $\dfrac{(2x^{1/3}y^{1/4} - x^{1/3} - y^{1/4} + 1)\, x^{1/3} y^{1/4}}{(x^{1/3} + y^{1/4} - 1)^2}$
13. $n = 2, 3, 4$
14. $\pm\sqrt{2}$
15. 1/201
16. 8, 24, 72, 216, 648
17. $A = 3, B = 8$
18. 24, 12, 6
22. –5050
23. $-(2)^{99} \times 10000$
24. $\dfrac{n(4n^2-1)c^2}{3}$
25. 14
26. 3/2
27. 7
28. $2^{n-1} - n - 2$
29. $\dfrac{n(n+1)(n+2)}{6}$
30. 200
31. $\dfrac{\pi^4}{96}$
32. $2(2^{n+1} - n - 2) + \dfrac{3n(n+1)(n+2)}{6}$
33. $\dfrac{\pi}{4}$
34. $\left(1 - \dfrac{x^n}{(x+1)(x+2)\ldots(x+n)}\right)$
35. $(2^{-n} + 2n - 1)$
36. 0
37. 51
38. $\dfrac{n(n+1)}{4}$

INTEGER TYPE QUESTIONS

1. 2	2. 4	3. 7	4. 3	5. 3
6. 6	7. 7	8. 5	9. 6	10. 5
11. 6	12. 6	13. 4	14. 3	15. 8

COMPREHENSIVE LINK PASSAGES

Passage I: (i) → b, (ii) → c, (iii) → d;
Passage II: (i) → b, (ii) → a, (iii) → c;
Passage III: (i) → d, (ii) → a, (iii) → c;
Passage IV: (i) → d, (ii) → b, (iii) → c;
Passage V: (i) → a, (ii) → a, (iii) → d;
Passage VI: (i) → b, (ii) → b, (iii) → c;
Passage VII: (i) → a, (ii) → b, (iii) → a, (iv) → a, (v) → a;
Passage VIII: (i) → c, (ii) → b, (iii) → c;
Passage IX: (i) → c, (ii) → b, (iii) → c;

MATRIX MATCH

1. (A) → R; (B) → P ;(C) → P, Q, R; (D) → P
2. (A) → R; (B) → P; (C) → Q;
3. (A) → R; (B) → P; (C) → Q; (D) → S
4. (A) → Q; (B) → P; (C) → S; (D) → R

MATCHING LISTS

5. (C)
6. (B)
7. (D)
8. (B)

Hints and Solutions

Level I

1. We have $t_{n+1} - t_n$
$= (3(n+1) + 5) - (3n + 5)$
$= 3.$
Hence, its common difference is 3.

2. Let the first term = a and the common difference = d
$t_p = q \Rightarrow a + (p-1)d = q$
$t_q = p \Rightarrow a + (q-1)d = p$
On subtracting, we get
$(p-q)d = (p-q)$
$\Rightarrow \quad d = \dfrac{(p-q)}{(q-p)} = -1$
Also, $a + (p-1)\cdot(-1) = q$
$\Rightarrow \quad a - (p-1) = q$
$\Rightarrow \quad a = q + p - 1$
Thus, $t_n = a + (n-1)d$
$= q + p - 1 - (n-1)$
$= p + q + n$

3. Let the first term = a and the common difference = d
We have
$mt_m = nt_n$
$\Rightarrow \quad m(a + (m-1)d) = n(a + (n-1)d)$
$\Rightarrow \quad (m-n)a = -(m^2 - m - n^2 + n)d$
$\Rightarrow \quad (m-n)a = -[m^2 - n^2 - (m-n)]d$
$\Rightarrow \quad a = -(m+n-1)d$
Now, $t_{m+n} = a + (m+n-1)d$
$= -(m+n-1)d + (m+n-1)d$
$= 0.$
Hence, the result.

4. Let the first term = a and the common difference = d

$$t_m = \frac{1}{n} \Rightarrow a + (m-1)d = \frac{1}{n}$$

$$t_n = \frac{1}{m} \Rightarrow a + (n-1)d = \frac{1}{m}$$

On subtracting, we get

$$(m-n)d = \frac{1}{n} - \frac{1}{m} = \frac{m-n}{mn}$$

$$\Rightarrow \quad d = \frac{1}{mn}$$

Also, $a + (m-1)\frac{1}{mn} = \frac{1}{n}$

$$\Rightarrow \quad a = \frac{1}{mn}$$

Thus, $t_{mn} = a + (mn-1)d$

$$= \frac{1}{mn} + (mn-1)\frac{1}{mn}$$

$$= \frac{1 + (mn-1)}{mn}$$

$$= 1$$

5. Let the first term = a and the common difference = d

We have $t_{m-1} + 2t_{n+1}$

$\Rightarrow \quad a + md = 2(a + nd)$

$\Rightarrow \quad a = (m - 2n)d$

Now, $\quad t_{3m+1} = a + 3md$

$= (m - 2n)d + 3md = 2(2m - n)d$

and $\quad t_{m+n+1} = a + (m+n)d$

$= (m - 2n)d + (m + n)d$

$= (2m - n)d$

Hence, the result.

6. Let the first term = a and the common difference = d.

We have $T_p = q \Rightarrow a + (p-1)d = q$

and $\quad T_{p+q} = 0 \Rightarrow a + (p + q - 1)d = 0 \qquad$ (i)

On subtracting, we get

$\{(p-1) - (p+q-1)\}d = q$

$\Rightarrow \quad d = -1$

Put $d = -1$ in eq.(i), we get

$a - (p-1) = q$

$\Rightarrow \quad a = p + q - 1$

Now, $T_q = a + (q-1)d$

$= p + q - 1 - (q - 1)$

$= p$

7. Do yourself.

8. Let the first term = a and the common difference = d

We have

$t_{m-n} + t_{m+n} = a + (m - n - 1)d + a + (m + n - 1)d$

$= 2a + (m - n - 1 + m + n - 1)d$

$= 2a + (2m - 1)d$

$= 2(a + (m-1)d)$

$= 2t_m$

9. (i) Let the first term = A and the common difference = D

It is given that,

$t_p = A + (p-1)D = a$

$t_q = A + (q-1)D = b$

$t_r = A + (r-1)D = c$

Now, $a(q-r) + b(r-p) + c(p-q)$

$= \{A + (p-1)D\}(q-r)$

$+ \{A + (q-1)D\}(r-p)$

$+ \{A + (r-1)D\}(p-q)$

$= A(q - r + r - p + p - q)$

$- D(q - r + r - p + p - q)$

$+ \{p(q-r) + q(r-p) + r(p-q)\}$

$= 0 + 0 + 0$

$= 0.$

10. We have

$$\frac{a_4}{a_7} = \frac{2}{3}$$

$$\Rightarrow \quad \frac{a+3d}{a+6d} = \frac{2}{3}$$

$\Rightarrow \quad 3a + 9d = 2a + 12d$

$\Rightarrow \quad a = 3d$

Now, $\dfrac{a_6}{a_8} = \dfrac{a+5d}{a+7d}$

$$= \frac{3d+5d}{3d+7d}$$

$$= \frac{8d}{10d}$$

$$= \frac{4}{5}$$

11. Since a, b, c, d and e are in AP, so

$a + e = b + d = 2c$

We have

$a - 4b + 6c - 4d + e = (a + e) - 4(b + d) + 6c$

$= 2c - 4(2c) + 6c$

$= 8c - 8c$

$= 0$

12 Do yourself

13. Let $ABCD$ be a quadrilateral in which $\angle A$, $\angle B$, $\angle C$, $\angle D$ are in AP.

Let $\angle A = a - 3d$, $\angle B = a - d$, $\angle C = a + d$, $\angle D = a + 3c$ where $2d$ is the common difference.

It is given that $2d = 10$, $\Rightarrow d = 5$

Clearly, $a = 90°$

Thus, the angles are

$\angle A = 90° - 15° = 75°$

$\angle B = 90° - 5° = 85°$

$\angle C = 90° + 5° = 95°$

$\angle D = 90° + 15° = 105°$

Hence, the angles are

75°, 85°, 95° and 105°

14. Let the roots be $a - d$, a, $a + d$

So, $\quad a - d + a + a + d = 12$

$\Rightarrow \quad 3a = 12$

$\Rightarrow \quad a = 4$

Also, $(a - d) \cdot a \cdot (a + d) = 28$

$\Rightarrow \quad a(a^2 - d^2) = 28$

$\Rightarrow \quad 4(16 - d^2) = 28$

$\Rightarrow \quad (16 - d^2) = 7$

$\Rightarrow \quad d^2 = 9$

$\Rightarrow \quad d = 3$

15 Do yourself.

16. We have

$$\frac{1}{a_1a_2}+\frac{1}{a_2a_3}+\ldots+\frac{1}{a_{n-1}a_n}$$
$$=\frac{1}{d}\left(\frac{d}{a_1a_2}+\frac{d}{a_2a_3}+\ldots+\frac{d}{a_{n-1}a_n}\right)$$
$$=\frac{1}{d}\left(\frac{a_2-a_1}{a_1a_2}+\frac{a_3-a_2}{a_2a_3}+\ldots+\frac{a_n-a_{n-1}}{a_{n-1}a_n}\right)$$
$$=\frac{1}{d}\left(\frac{1}{a_1}-\frac{1}{a_2}+\frac{1}{a_2}-\frac{1}{a_3}+\ldots+\frac{1}{a_{n-1}}-\frac{1}{a_n}\right)$$
$$=\frac{1}{d}\left(\frac{1}{a_1}-\frac{1}{a_n}\right)$$
$$=\frac{1}{d}\left(\frac{a_n-a_1}{a_1a_n}\right)$$
$$=\frac{1}{d}\left(\frac{\{a_1+(n-1)d\}-a_1}{a_1a_n}\right)$$
$$=\frac{(n-1)}{a_1a_n}$$

Hence, the result.

17. Given $a_1, a_2, \ldots, a_n$ are in AP. So,
$a_2 - a_1 = a_3 - a_2 = \ldots = a_{n-1} - a_n = d(\text{say})$
We have

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\ldots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}$$
$$=\left(\frac{\sqrt{a_2}-\sqrt{a_1}}{a_2-a_1}+\frac{\sqrt{a_3}-\sqrt{a_2}}{a_3-a_2}+\ldots+\frac{\sqrt{a_n}-\sqrt{a_{n-1}}}{a_n-a_{n-1}}\right)$$
$$=\left(\frac{\sqrt{a_2}-\sqrt{a_1}}{d}+\frac{\sqrt{a_3}-\sqrt{a_2}}{d}+\ldots+\frac{\sqrt{a_n}-\sqrt{a_{n-1}}}{d}\right)$$
$$=\frac{1}{d}(\sqrt{a_n}-\sqrt{a_1})$$
$$=\frac{1}{d}\left(\frac{a_n-a_1}{\sqrt{a_n}+\sqrt{a_1}}\right)$$
$$=\frac{1}{d}\left(\frac{a_1+(n-1)d-a_1}{\sqrt{a_n}+\sqrt{a_1}}\right)$$
$$=\left(\frac{(n-1)}{\sqrt{a_n}+\sqrt{a_1}}\right)$$

18. Given $a_1, a_2, \ldots, a_n$ are in AP. So,
$a_2 - a_1 = a_3 - a_2 = \ldots = a_{n-1} - a_n = d(\text{say})$

We have

$$\frac{1}{a_1a_n}+\frac{1}{a_2a_{n-1}}+\frac{1}{a_3a_{n-2}}+\ldots+\frac{1}{a_na_1}$$
$$=\frac{1}{a_1+a_n}\left(\frac{a_1+a_n}{a_1a_n}+\frac{a_1+a_n}{a_2a_{n-1}}+\frac{a_1+a_n}{a_3a_{n-2}}+\ldots+\frac{a_1+a_n}{a_na_1}\right)$$
$$=\frac{1}{a_1+a_n}\left(\frac{a_1+a_n}{a_1a_n}+\frac{a_2+a_{n-1}}{a_2a_{n-1}}+\frac{a_3+a_{n-2}}{a_3a_{n-2}}+\ldots+\frac{a_1+a_n}{a_na_1}\right)$$
$$=\frac{1}{a_1+a_n}\left(\left(\frac{1}{a_1}+\frac{1}{a_n}\right)+\left(\frac{1}{a_2}+\frac{1}{a_{n-1}}\right)+\left(\frac{1}{a_3}+\frac{1}{a_{n-2}}\right)+\left(\frac{1}{a_1}+\frac{1}{a_n}\right)\right)$$
$$=\frac{2}{a_1+a_n}\left(\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\ldots+\frac{1}{a_n}\right)$$

Hence, the result.

19. Let the first term be a and the common difference be d.
It is given that,

$t_7 = 15 \Rightarrow a + 6d = 15$

Let $P = t_2t_7t_{12}$

$= (a + d)(a + 6d)(a + 11d)$

$= 15(15 - 5d)(15 + 5d)$

$= 15(225 - 25d^2)$

$$\Rightarrow \quad \frac{dP}{dd} = -30\times 25d$$
$$\Rightarrow \quad \frac{d^2P}{dd^2} = -30\times 25 < 0$$

So, the product is maximum.

$\therefore \quad d = 0.$

20. (i) Given a, b and c are in AP.

$\therefore \quad 2b = a + c$

Also, $\tan^{-1} a$, $\tan^{-1} b$ and $\tan^{-1} c$ are in AP.

$\Rightarrow \quad 2\tan^{-1} b = \tan^{-1} a + \tan^{-1} c$

$$\Rightarrow \quad \tan^{-1}\left(\frac{2b}{1-b^2}\right) = \tan^{-1}\left(\frac{a+c}{1-ac}\right)$$
$$\Rightarrow \quad \left(\frac{2b}{1-b^2}\right) = \left(\frac{a+c}{1-ac}\right)$$
$$\Rightarrow \quad \left(\frac{2b}{1-b^2}\right) = \left(\frac{2b}{1-ac}\right)$$

$\Rightarrow \quad 1 - b^2 = 1 - ac$

$\Rightarrow \quad b^2 = ac$

$\Rightarrow \quad a$, b and c are in GP.

(ii) Also, $b^2 = ac$

$\Rightarrow \left(\frac{a+c}{2}\right)^2 = ac$

$\Rightarrow (a+c)^2 = 4ac$

$\Rightarrow (a+c)^2 - 4ac = 0$

$\Rightarrow (a-c)^2 = 0$

$\Rightarrow (a-c) = 0$

$\Rightarrow a = c$

Again, $2b = a + c = a + a = 2a$

$\Rightarrow b = a$

Thus, $a = b = c$

21. Do yourself
22. Do yourself
23. Do yourself
24. Given $a_1 + a_5 + a_{10} + a_{15} + a_{20} + a_{24} = 225$

$\Rightarrow (a_1 + a_{24}) + (a_5 + a_{20}) + (a_{10} + a_{15}) = 225$

$\Rightarrow (a_1 + a_{24}) + (a_1 + a_{24}) + (a_1 + a_{24}) = 225$

$\Rightarrow 3(a_1 + a_{24}) = 225$

$\Rightarrow (a_1 + a_{24}) = 75$

Thus, $s_{24} = \frac{24}{2} \times (a_1 + a_{24})$

$= 12 \times 75 = 900$

25. (i) Given $S_n = 3n^2 + 4n$

$\Rightarrow S_{n+1} = 3(n+1)^2 + 4(n+1)$

Thus, $t_n = S_{n+1} - S_n$

$= \{3(n+1)^2 + 4(n+1)\} - (3n^2 + 4n)$

$= 6n + 7$

Therefore, $t_{10} = 60 + 7 = 67$

(ii) Do yourself.

26. Let the first term = a and the common difference = d.

Clearly, $a = \frac{1}{mn} = d$

Now, $S_{mn} = \frac{mn}{2}[2a + (mn-1)d]$

$= \frac{mn}{2}\left(\frac{2}{mn} + \frac{(mn-1)}{mn}\right)$

$= \frac{mn}{2}\left(\frac{2 + (mn-1)}{mn}\right)$

$= \frac{(mn+1)}{2}$

27. Let the nth term, $t_n = x$.

So, $x = a + (n-1)d$

$= 1 + (n-1)5$

$= 5n - 4$

Thus, $\frac{n}{2}(1 + 5n - 4) = 148$

$\Rightarrow \frac{n}{2}(5n-3) = 148$

$\Rightarrow n(5n-3) = 296$

$\Rightarrow 5n^2 - 3n - 296 = 0$

$\Rightarrow 5n^2 - 40n + 37n - 296 = 0$

$\Rightarrow 5n(n-8) + 37(n-8) = 0$

$\Rightarrow (5n+37)(n-8) = 0$

$\Rightarrow n = 8, -\frac{37}{5}$

Since n is a natural number, so $n = 8$

Hence, the value of x is

$= 5n - 4$

$= 5 \times 8 - 4$

$= 36$

28. Do yourself.
29. Let the first term = a and the common difference = d.

Here, $S_1 = \frac{n}{2}[2a + (n-1)d]$

$S_2 = \frac{2n}{2}[2a + (2n-1)d]$

$S_3 = \frac{3n}{2}[2a + (3n-1)d]$

Now, $3(S_2 - S_1)$

$= 3\left\{\left(\frac{2n}{2}[2a + (2n-1)d]\right) - \left(\frac{n}{2}[2a + (n-1)d]\right)\right\}$

$= \frac{3n}{2}\{2[2a + (2n-1)d] - [a + (3n-1)d]\}$

$= \frac{3n}{2}[2a + (3n-1)d]$

$= S_3$

Hence, the result.

30. Let the first term = A and the common difference = d.

It is given that,

$t_p = a \Rightarrow A + (p-1)d = a$

$t_q = b \Rightarrow A + (q-1)d = b$

On subtraction, we get

$(p-q)d = (a-b)$

$\Rightarrow d = \frac{(a-b)}{(p-q)}$

Now,

$S_{p+q} = \left(\frac{p+q}{2}\right)\{2A + (p+q-1)d\}$

$= \left(\frac{p+q}{2}\right)\{A + (p-1)d + A + (q-1)d + d\}$

$= \left(\frac{p+q}{2}\right)\left(a + b + \left(\frac{a-b}{p-q}\right)\right)$

Hence, the result.

31. Let the first term = a and the common difference = d.

It is given that,

$S_m = n \Rightarrow \frac{m}{2}\{2a + (m-1)d\} = n$

$S_n = m \Rightarrow \frac{n}{2}\{2a + (n-1)d\} = m$

On subtraction, we get

$$(m-n)d = \frac{2n}{m} - \frac{2m}{n}$$

$$\Rightarrow \quad (m-n)d = \frac{2(n^2-m^2)}{mn}$$

$$\Rightarrow \quad d = -\frac{2(m+n)}{mn}$$

Now, $\frac{m}{2}\left(2a-(n-1)\frac{2(m+n)}{mn}\right) = n$

$$\Rightarrow \quad \left(2a-(n-1)\frac{2(m+n)}{mn}\right) = \frac{2n}{m}$$

$$\Rightarrow \quad a = \frac{n}{m} + (n-1)\frac{(m+n)}{mn}$$

$$\Rightarrow \quad a = \frac{n^2+mn+n^2-m-\text{n}}{mn}$$

Thus, $S_{(m+n)} = \left(\frac{m+n}{2}\right)[2a+(m+n-1)d]$

$$= \left(\frac{m+n}{2}\right)\left(\frac{2(n^2+mn+n^2-m-n)}{mn} - \frac{2(m+n-1)(m+n)}{mn}\right)$$

$$= \left(\frac{m+n}{mn}\right)\{(n^2+mn+n^2-m-n) - (m+n-1)(m+n)\}$$

$$= \left(\frac{m+n}{mn}\right)(-mn)$$

$$= -(m+n)$$

Hence, the result.

32. It is given that,

$$\frac{S_n}{S_{n'}} = \frac{7n+1}{4n+27}$$

$$= \frac{7n^2+n}{4n^2+27n}$$

Now, $\frac{t_n}{t_{n'}} = \frac{S_n - S_{n-1}}{S_{n'} - S_{n-1'}}$

$$= \frac{(7n^2+n)-\{7(n-1)^2+(n-1)\}}{(4n^2+27n)-\{4(n-1)^2+27(n-1)\}}$$

$$= \frac{14n-6}{8n+33}$$

Thus, $\frac{t_{11}}{t_{11'}} = \frac{14.11-6}{8.11+23} = \frac{154-6}{88+23} = \frac{148}{111}$

33. Clearly, $\frac{t_m}{t_n} = \frac{S_m - S_{m-1}}{S_n - S_{n-1}}$

$$= \frac{m^2-(m-1)^2}{n^2-(n-1)^2}$$

$$= \frac{(2m-1)}{(2n-1)}$$

Hence, the result.

34. Let the first term be a and the common difference is d and the number of sides be n.
It is given that, $a = 120°$, $d = 5°$

Also, $\frac{n}{2}\{2a+(n-1)d\} = (2n-4)\times 90°$

$$\Rightarrow \quad \frac{n}{2}\{240+5(n-1)\} = (2n-4)\times 90$$

$\Rightarrow \quad n(5n + 235) = (2n - 4) \times 180$

$\Rightarrow \quad 5n^2 + 235n = 360n - 720$

$\Rightarrow \quad 5n^2 - 125n + 720 = 0$

$\Rightarrow \quad n^2 - 25n + 144 = 0$

$\Rightarrow \quad (n-9)(n-16) = 0$

$\Rightarrow \quad n = 9, 16$

But $n = 16$ does not satisfy the angles property of a regular polygon.

So, $n = 9$

Hence, the number of sides is 9.

35. We have

$$S_1 = \frac{n}{2}\{2.1+(n-1)\cdot 1\}$$

$$S_2 = \frac{n}{2}\{2.2+(n-1)\cdot 3\}$$

$$S_3 = \frac{n}{2}\{2.3+(n-1)\cdot 5\}$$

$$\vdots$$

$$S_m = \frac{n}{2}\{2\cdot m+(n-1)\cdot(2m-1)\}$$

Thus, $S_1 + S_2 + S_3 + \ldots + S_m$

$$= \frac{n}{2}[2(1+2+3+\ldots+m) + (n-1)\{1+3+5+\ldots+(2m-1)\}]$$

$$= \frac{n}{2}\left[\frac{2m(m+1)}{2} + (n-1)m^2\right]$$

$$= \frac{mn}{2}[(m+1)+m(n-1)]$$

$$= \frac{mn(mn+1)}{2}$$

Hence, the result.

36. Let the first term be a and the common difference be d.
It is given that

$$S_n = n^2 p \Rightarrow \frac{n}{2}(2a+(n-1)d) = n^2 p$$

$\Rightarrow \quad 2a + (n-1)d = 2np$ ...(i)

Similarly, $2a + (m-1)d = 2mp$ ...(ii)

On subtraction, we get

$(n-m)d = 2(n-m)p$

$\Rightarrow \quad d = 2p$

put $d = 2p$ in Eq. (i), we get

$\Rightarrow \quad 2a + 2(n-1)p = 2np$

$\Rightarrow \quad a = p$

Thus, $S_p = \frac{p}{2}\left(2a + (p-1)d\right)$

$= \frac{p}{2}\left(2p + 2(p-1)p\right)$

$= \frac{p}{2}(2p + 2p^2 - 2p)$

$= p^3$

38. It is given that

$S_1 = a_1 + a_3 + a_5 + \ldots + a_{2n+1}$
$= a + (a + 2d) + (a + 4d) + \ldots + (a + 2nd)$
$= (n+1)a + 2d(1 + 2 + 3 + \ldots + n)$
$= (n+1)a + n(n+1)d$
$= (n+1)(a+d)$

Also, $S_2 = a_2 + a_4 + a_6 + \ldots + a_{2n}$
$= (a+d) + (a+3d) + (a+5d) + \ldots + \{a + (2n-1)d\}$
$= na + \{1 + 3 + 5 + 7 + \ldots + (2n-1)\}d$
$= na + n^2 d$
$= n(a + nd)$

Hence, $\frac{S_1}{S_2} = \frac{(n+1)(a+nd)}{n(a+nd)} = \frac{(n+1)}{n}$

39. We have $S_n = nP + \frac{1}{2}n(n-1)Q$

$= \frac{n}{2}\{2p + (n-1)Q\}$

Clearly, the common difference = Q.

40. (i) It is given that $a, b, c \in$ AP

$\Rightarrow \quad a - (a+b+c),\ b - (a+b+c),\ c - (a+b+c) \in$ AP

$\Rightarrow \quad -(b+c), -(a+c), -(a+b) \in$ AP

$\Rightarrow \quad (b+c), (a+c), (a+b) \in$ AP

Hence, the result.

(iii) It is given that a, b and c are in AP.

$\Rightarrow \quad b - a = c - b$

$\Rightarrow \quad (\sqrt{b} - \sqrt{a})(\sqrt{b} + \sqrt{a}) = (\sqrt{c} - \sqrt{b})(\sqrt{c} + \sqrt{b})$

$\Rightarrow \quad \frac{(\sqrt{b} - \sqrt{a})}{(\sqrt{c} + \sqrt{b})} = \frac{(\sqrt{c} - \sqrt{b})}{(\sqrt{b} + \sqrt{a})}$

$\Rightarrow \quad \frac{(\sqrt{b} - \sqrt{a})}{(\sqrt{c} + \sqrt{b})(\sqrt{c} + \sqrt{a})} = \frac{(\sqrt{c} - \sqrt{b})}{(\sqrt{b} + \sqrt{a})(\sqrt{c} + \sqrt{a})}$

$\Rightarrow \quad \frac{(\sqrt{b} + \sqrt{c}) - (\sqrt{c} + \sqrt{a})}{(\sqrt{c} + \sqrt{b})(\sqrt{c} + \sqrt{a})}$

$= \frac{(\sqrt{c} + \sqrt{a}) - (\sqrt{b} + \sqrt{c})}{(\sqrt{b} + \sqrt{a})(\sqrt{c} + \sqrt{a})}$

$\Rightarrow \quad \frac{1}{(\sqrt{c} + \sqrt{a})} - \frac{1}{(\sqrt{c} + \sqrt{b})}$

$= \frac{1}{(\sqrt{b} + \sqrt{a})} - \frac{1}{(\sqrt{c} + \sqrt{a})}$

$\Rightarrow \quad \frac{1}{(\sqrt{b} + \sqrt{a})}, \frac{1}{(\sqrt{c} + \sqrt{a})}, \frac{1}{(\sqrt{c} + \sqrt{b})} \in$ AP

Hence, the result.

(iv) It is given that a, b and $c \in$ AP

$\Rightarrow \quad \frac{a}{abc}, \frac{b}{abc}, \frac{c}{abc} \in$ AP

$\Rightarrow \quad \frac{1}{bc}, \frac{1}{ac}, \frac{1}{ab} \in$ AP

Hence, the result.

(v) It is given that a, b and $c \in$ AP

$a\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right), b\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right),$

$c\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \in$ AP

$\Rightarrow \quad 1 + a\left(\frac{1}{b} + \frac{1}{c}\right), 1 + b\left(\frac{1}{a} + \frac{1}{c}\right),$

$1 + c\left(\frac{1}{a} + \frac{1}{b}\right) \in$ AP

$\Rightarrow \quad a\left(\frac{1}{b} + \frac{1}{c}\right), b\left(\frac{1}{a} + \frac{1}{c}\right), c\left(\frac{1}{a} + \frac{1}{b}\right) \in$ AP

(vi) It is given that a, b and c are in AP.

Now, $\{(a+c)^2 - b^2\} - \{(b+c)^2 - a^2\}$
$= (a^2 - b^2) + \{(a+c)^2 - (b+c)^2\}$
$= (a^2 - b^2) + \{(a^2 - b^2) + 2c(a-b)\}$
$= 2(a^2 - b^2) + 2c(a-b)$
$= 2(a-b)(a+b+c)$

Also, $[(a+b)^2 - c^2] - [(a+c)^2 - b^2]$
$= (b^2 - c^2) + [(a+b)^2 - (a+c)^2]$
$= 2(b^2 - c^2) + 2a(b-c)$
$= 2(b-c)(a+b+c)$
$= 2(a-b)(a+b+c)$

Hence, the result.

41. (i) It is given that a^2, b^2 and c^2 are in AP.

$\Rightarrow \quad b^2 - a^2 = c^2 - b^2$

$\Rightarrow \quad (b+a)(b-a) = (c+b)(c-b)$

$\Rightarrow \quad \frac{(b+a)}{(c-b)} = \frac{(c+b)}{(b-a)}$

$\Rightarrow \quad \frac{(c-b)}{(a+b)} = \frac{(b-a)}{(b+c)}$

$\Rightarrow \quad \frac{(c-b)}{(a+\mathrm{b})(c+\mathrm{a})} = \frac{(b-a)}{(b+c)(c+\mathrm{a})}$

$\Rightarrow \quad \frac{((a+c)-(b+a))}{(a+b)(c+a)} = \frac{((b+c)-(a+c))}{(b+c)(c+a)}$

$\Rightarrow \quad \dfrac{1}{(a+b)} - \dfrac{1}{(a+c)} = \dfrac{1}{(c+a)} - \dfrac{1}{(b+c)}$

$\Rightarrow \quad \dfrac{1}{(a+b)}, \dfrac{1}{(a+c)}, \dfrac{1}{(b+c)} \in \text{AP}$

Hence, the result.

(ii) $\dfrac{1}{(a+b)}, \dfrac{1}{(a+c)}$ and $\dfrac{1}{(b+c)} \in \text{AP}$

$\Rightarrow \quad \dfrac{(a+b+c)}{(a+b)}, \dfrac{(a+b+c)}{(a+c)}, \dfrac{(a+b+c)}{(b+c)} \in \text{AP}$

$\Rightarrow \quad 1+\dfrac{c}{(a+b)}, 1+\dfrac{b}{(a+c)}, 1+\dfrac{c}{(b+c)} \in \text{AP}$

$\Rightarrow \quad \dfrac{c}{(a+b)}, \dfrac{b}{(a+c)}, \dfrac{c}{(b+c)} \in \text{AP}$

Hence, the result.

42. (i) It is given that a, b and c are in AP.

Now, $b^2(c+a) - a^2(b+c)$
$= b^2c + b^2a - a^2b - a^2c$
$= (b^2 - a^2)c + ba(b-a)$
$= [(b+a)c + ba](b-a)$
$= (ab+bc+ca)(b-a)$

Also, $c^2(a+b) - b^2(c+a)$
$= (c^2 - b^2)a + bc(c-b)$
$= [(c+b)a + bc](c-b)$
$= (ab+bc+ca)(c-b)$
$= (ab+bc+ca)(b-a)$

Hence, the result.

(ii) It is given that a, b and c are in AP.

$\Rightarrow \quad 2a, 2b, 2c \in \text{AP}$

$\Rightarrow \quad -2a, -2b, -2c \in \text{AP}$

$\Rightarrow \quad (a+b+c)-2a, (a+b+c)-2b, (a+b+c)-2c \in \text{AP}$

$\Rightarrow \quad (b+c)-a, (a+c)-b, (a+b)-c \in \text{AP}$

(iii) It is given that a, b and c are in AP.

Now, $(ca-b^2)-(bc-a^2)$
$= c(a-b) + (a^2-b^2)$
$= (a+b+c)(a-b)$

Also, $(ab-c^2)-(ac-b^2)$
$= a(b-c) + (b^2-c^2)$
$= (a+b+c)(b-c)$
$= (a+b+c)(a-b)$,

Hence $bc-a^2$, $ca-b^2$, $ab-c^2$ are also in AP.

43. Given $\dfrac{b+c-a}{a}, \dfrac{c+a-b}{b}, \dfrac{a+b-c}{c}$ are in AP

$\Rightarrow \quad \dfrac{b+c-a}{a}+2, \dfrac{c+a-b}{b}+2, \dfrac{a+b-c}{c}+2 \in \text{AP}$

$\Rightarrow \quad \dfrac{b+c+a}{a}, \dfrac{c+a+b}{b}, \dfrac{a+b+c}{c} \in \text{AP}$

$\Rightarrow \quad \dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c} \in \text{AP}$

44. Do yourself

45. We have $\dfrac{a^{n+1}+b^{n+1}}{a^n+b^n} = \dfrac{a+b}{2}$

$\Rightarrow \quad 2a^{n+1} + 2b^{n+1} = a^{n+1} + a^nb + ab^n + b^{n+1}$

$\Rightarrow \quad a^{n+1} + b^{n+1} = a^nb + ab^n$

$\Rightarrow \quad a^n(a-b) = b^n(a-b)$

$\Rightarrow \quad a^n = b^n$

$\Rightarrow \quad \left(\dfrac{a}{b}\right)^n = 1 = \left(\dfrac{a}{b}\right)^0$

$\Rightarrow \quad n = 0$

46. Let $A_1, A_2, A_3, \ldots, A_n$ be n arithmetic means are inserted between 20 and 80. So,

$20, A_1, A_2, A_3, \ldots, A_n, 80 \in \text{AP}$

Now, $t_{n+2} = 80$

$\Rightarrow \quad 20 + (n+1)d = 80$

$\Rightarrow \quad d = \dfrac{60}{(n+1)}$ …(i)

Also, it is given that,

$\dfrac{A_1}{A_n} = \dfrac{1}{3}$

$\Rightarrow \quad \dfrac{A_n}{A_1} = 3$ (ii)

$\Rightarrow \quad \dfrac{A_n}{A_1} + 1 = 3 + 1 = 4$

$\Rightarrow \quad \dfrac{A_n + A_1}{A_1} = 4$

$\Rightarrow \quad 4A_1 = 100$ from (ii)

$\therefore \quad A_1 = 25$

$\Rightarrow \quad 20 + d = 25$

$\Rightarrow \quad d = 5$ $\because a = 20$

Putting the value of d in Eg. (i), we get

$\dfrac{60}{(n+1)} = 5$

$\Rightarrow \quad (n+1) = 12$

$\Rightarrow \quad n = 1$

Hence, the value of n is 11.

47. Let $A_1, A_2, A_3, \ldots, A_n$ be n arithmetic means are inserted between two positive numbers, say a and b, respectively. Thus, $a, A_1, A_2, A_3, \ldots, A_n, b \in \text{AP}$

Let d be the common difference.

Now,

$$A_1 + A_2 + A_3 + \ldots + A_n = \frac{n}{2}(A_1 + A_n)$$
$$= \frac{n}{2}(a+d+b-d)$$
$$= \frac{n}{2}(a+b)$$
$$= n\left(\frac{a+b}{2}\right)$$

Hence, the result.

48. Do yourself.
49. Do yourself.
50. Do yourself.
51. Do yourself.
52. Let $t_n = \dfrac{1}{512\sqrt{2}}$

$\Rightarrow \quad ar^{n-1} = \dfrac{1}{512\sqrt{2}}$

$\Rightarrow \quad \sqrt{2}\left(\dfrac{1}{2}\right)^{n-1} = \dfrac{1}{512\sqrt{2}}$

$\Rightarrow \quad \left(\dfrac{1}{2}\right)^{n-1} = \dfrac{1}{512\times 2} = \dfrac{1}{1024} = \dfrac{1}{2^{10}}$

$\Rightarrow \quad \left(\dfrac{1}{2}\right)^{n-1} = \left(\dfrac{1}{2}\right)^{10}$

$\Rightarrow \quad n - 1 = 10$

$\therefore \quad n = 11$

Hence, the 11th term is $\dfrac{1}{512\sqrt{2}}$.

53. Let $t_n = \dfrac{1}{128}$

$\Rightarrow \quad ar^{n-1} = \dfrac{1}{128}$

$\Rightarrow \quad 2\cdot\left(\dfrac{1}{2}\right)^{n-1} = \dfrac{1}{128}$

$\Rightarrow \quad \left(\dfrac{1}{2}\right)^{n-1} = \dfrac{1}{256} = \left(\dfrac{1}{2}\right)^{8}$

$\Rightarrow \quad n - 1 = 8$

$\therefore \quad n = 9$

Hence, the 9th term is $\dfrac{1}{128}$.

54. Do yourself
55. It is given that $t_3 = 3$.

$\Rightarrow \quad ar^2 = 3$

Now,

$t_1t_2t_3t_4t_5 = a.ar.ar^2.ar^3.ar^4$
$= a^5r^{10}$
$= (ar^2)^5$
$= (3)^5 = 243$

56. Do yourself.
57. It is given that, $a = 1$ and

$t_5 + t_n = 82$

$\Rightarrow \quad ar^4 + a = 82$

$\Rightarrow \quad r^4 + 1 = 82$

$\Rightarrow \quad r^4 = 81$

$\Rightarrow \quad r = \pm 3$

58. Clearly, $t_p = a \Rightarrow AR^{p-1} = a$

$t_q = b \Rightarrow AR^{q-1} = b$

$t_r = c \Rightarrow AR^{r-1} = c$

Now, $a^{q-r}\cdot b^{r-p}, c^{p-q}$

$= (AR^{p-1})^{q-r}\cdot(AR^{q-1})^{r-p}\cdot(AR^{r-1})p - q$

$= A^{q-r+r-p+p-q}\cdot R^{(p-1)(q-r)+(q-1)(r-p)+(r-1)(p-q)}$

$= A^0\cdot R^{p(q-r)+q(r-p)+r(p-q)}\cdot R^{-(q-r+r-p+p-q)}$

$= A^0R^0.R^0$

$= 1.$

59. It is given that, $t_1 = a \Rightarrow t_1 = a$, and $t_n = b \Rightarrow ar^{n-1} = b$

Now, $P = t_1t_2t_3 \ldots t_n$

$= (a)(ar)(ar^2)(ar^3)\ldots(ar^{n-1})$

$= (a)^n r^{1+2+3+\ldots+(n-1)}$

$= (a)^n r^{\frac{n-1}{2}\times(1+n-1)}$

$= (a)^n (r^{n-1})^{\frac{n}{2}}$

$= (a)^{\frac{n}{2}}(ar^{n-1})^{\frac{n}{2}}$

$= (a)^{\frac{n}{2}}(b)^{\frac{n}{2}}$

$= (ab)^{\frac{n}{2}}$

$P^2 = (ab)^n$

Hence, the result.

60. It is given that, $t_{m+n} = p \Rightarrow ar^{m+n-1} = p$ and $t_{m-n} = q \Rightarrow ar^{m-n-1} = q$

Multiplying both, we get

$ar^{m+n-1}\times ar^{m-n-1} = pq$

$\Rightarrow \quad a^2r^{m+n-1+m-n-1} = pq$

$\Rightarrow \quad a^2r^{2m-2} = pq$

$\Rightarrow \quad (ar^{m-1})^2 = pq$

$\Rightarrow \quad (ar^{m-1}) = \sqrt{pq}$

$\therefore \quad t_m = \sqrt{pq}$

Now, $\left(\dfrac{q}{p}\right) = \dfrac{ar^{m-n-1}}{ar^{m+n-1}} = r^{-2n}$

$\Rightarrow \quad r = \left(\dfrac{p}{q}\right)^{\frac{1}{2n}}$

Also, $ar^{m+n-1} = p$

$\Rightarrow \quad a\left(\dfrac{p}{q}\right)^{\frac{m=n-1}{2n}} = p$

$\Rightarrow \quad a = q(p)^{\frac{2n}{m+n-1}-1} = qp^{\frac{n-m+1}{m+n-1}}$

Thus, $t_n = ar^{n-1} = qp^{\frac{n-m+1}{m+n-1}}\times\left(\dfrac{p}{q}\right)^{\frac{n-1}{2n}}$

$= p\left(\dfrac{q}{p}\right)^{m/2n}$

Hence, the result.

61. We have

$$(a^2 + b^2 + c^2)p^2 - 2(ab + bc + cd)p + (b^2 + c^2 + d^2) \le 0$$

$$\Rightarrow (a^2p^2 - 2abp + b^2) + (b^2p^2 - 2bcp + c^2) + (c^2p^2 - 2cdp + d^2) \le 0$$

$$\Rightarrow (ap - b)^2 + (bp - c)^2 + (cp - d)^2 \le 0$$

$$\Rightarrow (ap - b)^2 + (bp - c)^2 + (cp - d)^2 = 0$$

$$\Rightarrow (ap - b) = (bp - c) = (cp - d) = 0$$

$$\Rightarrow \frac{b}{a} = \frac{1}{p} = \frac{c}{b} = \frac{d}{c}$$

$$\Rightarrow \frac{b}{a} = \frac{c}{b} = \frac{d}{c}$$

$\Rightarrow$ a, b, c, d are in G.P.

62. It is given that,

$$t_p = AR^{p-1} = a$$
$$t_q = AR^{q-1} = b$$
$$t_r = AR^{r-1} = c$$

Now, $(q - r)\log a + (r - p)\log b + (p - q)\log c$

$$= (q - r)\{\log A + (p - 1)\log R\} + (r - p)\{\log A + (q - 1)\log R\} + (p - q)\{\log A + (r - 1)\log R\}$$

$$= \log A\{q - r + r - p + p - q\} - \log R\,\{q - r + r - p + p - q\} + \log R\,\{p(q - r) + q(r - p) + r(p - q)\}$$

$$= 0 + 0 + 0$$

$$= 0.$$

63. We have

$$(1 - k)(1 + 2x + 4x^2 + 8x^3 + 16x^4 + 32x^5) = 1 - k^6$$

$$\Rightarrow (1 + 2x + 4x^2 + 8x^3 + 16x^4 + 32x^5) = \frac{1 - k^6}{(1 - k)}$$

$$\Rightarrow (1 + 2x + (2x)^2 + (2x)^3 + (2x)^4 + (2x)^5) = \frac{1 - k^6}{(1 - k)}$$

$$\Rightarrow \frac{1 - (2x)^6}{1 - 2x} = \frac{1 - k^6}{(1 - k)}$$

$$\Rightarrow k = 2x$$

$$\therefore \left(\frac{k}{x}\right) = 2$$

64. Clearly, $\alpha + \beta = 3$, $\alpha\beta = a$

and $\gamma + \delta = 12$, $\gamma\delta = b$

Also, it is given that,

$\alpha, \beta, \gamma, \delta \in$ GP

Let A be the first term and R be the common ratio.

Now, $\dfrac{\gamma + \delta}{\alpha + \beta} = \dfrac{12}{3} = 4$

$$\Rightarrow \frac{AR^2 + AR^3}{A + AR} = 4$$

$$\Rightarrow \frac{AR^2(1 + R)}{A(1 + R)} = 4$$

$$\Rightarrow R^2 = 4$$

$$\therefore R = 2$$

and $A = 1$

Now, $a = \alpha\beta = A \cdot AR = A^2R = 2$

$b = \gamma\delta = AR^2 \cdot AR^3 = A^2R^5 = (2)^5 = 32$

65. Since x, y and z are in GP, so $y = xr$ and $z = xr^2$

We have, $x + y + z = ax$

$$\Rightarrow x + xr + xr^2 = ax$$

$$\Rightarrow r^2 + r + (1 - a) = 0$$

Since r is real, so $D \ge 0$

$$\therefore 1 - 4(1 - a) \ge 0$$

$$\Rightarrow 1 - 4 + 4a \ge 0$$

$$\Rightarrow 4a \ge 3$$

$$\Rightarrow a \ge \frac{3}{4}$$

Hence, the value of a is $\left[\dfrac{3}{4}, \infty\right)$.

66. Do yourself.
67. Do yourself.
68. Do yourself.
69. We have,

$$\text{Sum} = (x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + \ldots \text{ to } n \text{ terms}$$

$$= \frac{1}{(x - y)}[(x^2 - y^2) + (x^3 - y^3) + \ldots + (x^n - y^n)]$$

$$= \frac{1}{(x - y)}[(x^2 + x^3 + \ldots + x^n) - (y^2 + y^3 + \ldots + y^n)]$$

$$= \frac{1}{(x - y)}\left[\frac{x^2(1 - x^n)}{(1 - x)} - \frac{y^2(1 - y^n)}{(1 - y)}\right]$$

70. We have,

$$\text{Sum} = \frac{a}{(1 + i)} + \frac{a}{(1 + i)^2} + \frac{a}{(1 + i)^3} + \ldots + \frac{a}{(1 + i)^n}$$

$$= \frac{a}{(a + i)}\left[1 + \frac{1}{(1 + i)} + \frac{1}{(1 + i)^2} + \ldots + \frac{1}{(1 + i)^{n-1}}\right]$$

$$= \frac{a}{(a + i)}\left(\frac{1 - \frac{1}{(1 + i)^n}}{1 - \frac{1}{(1 + i)}}\right)$$

71. We have $\displaystyle\sum_{k=1}^{10}(2 + 3^k)$

$$= (2 + 3) + (2 + 3^2) + (2 + 3^3) + \ldots + (2 + 3^{10})$$

$$= 10 \times 2 + 3(1 + 3 + 3^2 + \ldots + 3^9)$$

$$= 20 + 3\left(\frac{3^{10} - 1}{3 - 1}\right)$$

$$= 20 + \frac{3}{2}(3^{10} - 1)$$

$$= \left(\frac{37 + 3^{11}}{2}\right)$$

72. We have,

$$\sum_{n=1}^{n}(2^{n-1}+3^n)$$
$$=\sum_{n=1}^{n}2^{n-1}+\sum_{n=1}^{n}3^n$$
$$=(1+2+2^2+\ldots+2^{n-1})+(3+3^2+3^3+\ldots+3^n)$$
$$=\left(\frac{2^n-1}{2-1}\right)+3\left(\frac{3^n-1}{3-1}\right)$$
$$=(2^n-1)+\frac{3}{2}(3^n-1)$$

73. (i) We have

$$\sum_{k=1}^{n}(4^{k-1}+5^{k+1})$$
$$=\sum_{k=1}^{n}4^{k-1}+\sum_{k=1}^{n}5^{k+1}$$
$$=(1+4+4^2+\ldots+4^{n-1})+(5^2+5^3+5^4+\ldots+5^{n+1})$$
$$=\left(\frac{4^n-1}{4-1}\right)+5^2\left(\frac{5^n-1}{5-1}\right)$$
$$=\left(\frac{4^n-1}{3}\right)+\frac{25}{4}(5^n-1)$$

(ii) Do yourself.

74. (i) We have,

5 + 55 + 555 + ... to n terms

$$=5(1+11+111+\ldots \text{ to } n \text{ terms})$$
$$=\frac{5}{9}(9+99+999+\ldots \text{ to } n \text{ terms})$$
$$=\frac{5}{9}[(10-1)+(10^2-1)+(10^3-1)+\ldots+(10^n-1)]$$
$$=\frac{5}{9}[10(1+10+10^2+\ldots+10^{n-1})-n]$$
$$=\frac{5}{9}\left[10\left(\frac{10^n-1}{10-1}\right)-n\right]$$
$$=\frac{5}{9}\left[\left(\frac{10^{n+1}-10}{9}\right)-n\right]$$
$$=\frac{5}{81}(10^{n+1}-9n-10)$$

(ii) Do yourself.

(iii) Do yourself.

75. We have,

$$\text{Sum}=(6666\ldots6)^2+(888\ldots8)$$
$$=36(111\ldots11)^2+8(111\ldots11)$$
$$=36(1+10+10^2+\ldots+10^{n-1})^2+8(1+10+10^2+\ldots+10^{n-1})$$
$$=36\left(\frac{10^n-1}{10-1}\right)^2+8\left(\frac{10^n-1}{10-1}\right)$$
$$=36\left(\frac{10^n-1}{9}\right)^2+8\left(\frac{10^n-1}{9}\right)$$
$$=\left(\frac{10^n-1}{9}\right)\left(36\left(\frac{10^n-1}{9}\right)+8\right)$$
$$=\left(\frac{10^n-1}{9}\right)[4(10^n-1)+8]$$
$$=\left(\frac{10^n-1}{9}\right)[4(10^n-1+2)]$$
$$=\frac{4}{9}(10^n-1)(10^n+1)$$
$$=\frac{4}{9}(10^{2n}-1)$$

76. We have,

$$\text{Sum}=\sum_{n=1}^{10}\left[\left(\frac{1}{2}\right)^{n-1}+\left(\frac{1}{5}\right)^{n+1}\right]$$
$$=\sum_{n=1}^{10}\left[\left(\frac{1}{2}\right)^{n-1}\right]+\sum_{n=1}^{10}\left[\left(\frac{1}{5}\right)^{n+1}\right]$$
$$=\left(1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\ldots+\left(\frac{1}{2}\right)^9\right)+\left(\left(\frac{1}{5}\right)^2+\left(\frac{1}{5}\right)^3+\ldots+\left(\frac{1}{5}\right)^{11}\right)$$
$$=\left(\frac{1-\left(\frac{1}{2}\right)^{10}}{1-\left(\frac{1}{2}\right)}\right)+\left(\frac{1}{5}\right)^2\left(\frac{1-\left(\frac{1}{5}\right)^{10}}{1-\left(\frac{1}{5}\right)}\right)$$
$$=2\left(1-\left(\frac{1}{2}\right)^{10}\right)+\left(\frac{1}{20}\right)\left(1-\left(\frac{1}{5}\right)^{10}\right)$$

77. We have,

$$\text{Sum}=11+103+1005+\ldots$$
$$=(10+1)+(10^2+3)+\ldots+[10^n+(2n-1)]$$
$$=(10+10^2+\ldots+10^n)+[1+3+5+\ldots+(2n-1)]$$
$$=10(1+10+\ldots+10^{n-1})+[1+3+5+\ldots+(2n-1)]$$
$$=10\left(\frac{10^n-1}{10-1}\right)+n^2$$
$$=\frac{10}{9}(10^n-1)+n^2$$

78. It is given that

$$1+3+3^2+3^3+\ldots+3^{n-1}>7000$$
$$\Rightarrow \left(\frac{3^n-1}{3-1}\right)>7000$$
$$\Rightarrow (3^n-1)>14000$$

$\Rightarrow \quad 3^n > 14001$

$\Rightarrow \quad 3^n > 3^9$

$\therefore \quad n > 9$

Hence, the least value of n is 9.

79. We have,

$$\text{Sum } = \frac{1}{3} + \frac{5}{9} + \frac{19}{27} + \frac{65}{81} + \ldots \text{ to } n \text{ terms}$$

$$= \left(1 - \frac{2}{3}\right) + \left(1 - \left(\frac{2}{3}\right)^2\right) + \left(1 - \left(\frac{2}{3}\right)^3\right) + \ldots + \left(1 - \left(\frac{2}{3}\right)^n\right)$$

$$= n - \frac{2}{3}\left(1 + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \ldots + \left(\frac{2}{3}\right)^{n-1}\right)$$

$$= n - \frac{2}{3}\left(\frac{1 - \left(\frac{2}{3}\right)^n}{1 - \frac{2}{3}}\right)$$

$$= n - 2\left(1 - \left(\frac{2}{3}\right)^n\right)$$

80. We have,

$$S = \frac{2}{3} + \frac{8}{9} + \frac{26}{27} + \frac{30}{81} + \ldots \text{ to } n\text{-terms}$$

$$= \left(1 - \frac{1}{3}\right) + \left(1 - \frac{1}{9}\right) + \left(1 - \frac{1}{27}\right) + \ldots$$

$$= \left(1 - \frac{1}{3}\right) + \left(1 - \left(\frac{1}{3}\right)^2\right) + \left(1 - \left(\frac{1}{3}\right)^3\right) + \ldots + \left(1 - \left(\frac{1}{3}\right)^n\right)$$

$$= n - \frac{1}{3}\left(1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \ldots + \left(\frac{1}{3}\right)^{n-1}\right)$$

$$= n - \frac{1}{3}\left(\frac{1 - \left(\frac{1}{3}\right)^n}{1 - \left(\frac{1}{3}\right)}\right)$$

$$= n - \frac{1}{2}\left(1 - \left(\frac{1}{3}\right)^n\right)$$

81. We have,

$$S = a + ar + ar^2 + \ldots + ar^{n-1} = a\left(\frac{r^n - 1}{r - 1}\right)$$

$$P = a.ar.ar^2 \ldots ar^{n-1}$$

$$= an.r^{1+2+3+\ldots+(n-1)}$$

$$= a^n r^{n\left(\frac{n-1}{2}\right)}$$

and $R = \dfrac{1}{a} + \dfrac{1}{ar} + \dfrac{1}{ar^2} + \ldots + \dfrac{1}{ar^{n-1}}$

$$= \frac{1}{a}\left(1 + \frac{1}{r} + \frac{1}{r^2} + \ldots + \frac{1}{r^{n-1}}\right)$$

$$= \frac{1}{a}\left(\frac{1 - \left(\frac{1}{r}\right)^n}{1 - \frac{1}{r}}\right)$$

$$= \frac{1}{a}\left(\frac{r^n - 1}{r^{n-1}(r - 1)}\right)$$

$$= \frac{1}{ar^{n-1}}\left(\frac{r^n - 1}{(r - 1)}\right)$$

Now, $\left(\dfrac{S}{R}\right) = \dfrac{a(r^n - 1)}{(r - 1)} \div \dfrac{1}{ar^{n-1}}\left(\dfrac{r^n - 1}{(r - 1)}\right)$

$$= a^2 r^{n-1}$$

$$\left(\frac{S}{R}\right)^n = a^{2n} r^{n(n-1)} = \left(a^n r^{\frac{n(n-1)}{2}}\right)^2 = P^2$$

Hence, the result.

82. Clearly, $\quad f(1) = 3, f(2) = 3^2, f(3) = 3^3, \ldots, f(n) = 3^n$

Now, $\displaystyle\sum_{x=1}^{n} f(x) = 120$

$\Rightarrow \quad f(1) + f(2) + f(3) + \ldots + f(n) = 120$

$\Rightarrow \quad 3 + 3^2 + 3^3 + \ldots + 3^n = 120$

$\Rightarrow \quad 3\left(\dfrac{3^n - 1}{3 - 1}\right) = 120$

$\Rightarrow \quad \left(\dfrac{3^n - 1}{3 - 1}\right) = 40$

$\Rightarrow \quad 3^n - 1 = 80$

$\Rightarrow \quad 3^n = 81 = 3^4$

$\therefore \quad n = 4$

Hence, the value of n is 4.

83. We have,

$$\sum_{n=1}^{100} a_{2n} = \alpha$$

$\Rightarrow \quad \alpha = a_2 + a_4 + a_6 + \ldots + a_{200}$

$= ar + ar^3 + ar^5 + \ldots + ar^{199}$

$= ar(1 + r^2 + r^4 + \ldots + r^{198}) \qquad \ldots\text{(i)}$

Also, $\displaystyle\sum_{n=1}^{100} a_{2n-1} = \beta$,

$\Rightarrow \quad \beta = a_1 + a_3 + a_5 + \ldots + a_{199}$

$= a + ar^2 + ar^4 + \ldots + ar^{198}$

$= a(1 + r^2 + r^4 + \ldots + r^{198}$

Now, $\dfrac{\alpha}{\beta}=\dfrac{ar(1+r^2+r^4+\ldots+r^{198})}{(1+r^2+r^4+\ldots+r^{198})}$

$\Rightarrow \quad r=\dfrac{\alpha}{\beta}$

Hence, the result.

84. We have,

$$\text{Sum}=\frac{(\sqrt{2}+1)}{1-\left(\frac{1}{\sqrt{2}+1}\right)}$$
$$=\frac{(\sqrt{2}+1)}{1-(\sqrt{2}-1)}$$
$$=\frac{(\sqrt{2}+1)}{2-\sqrt{2}}$$
$$=\frac{(\sqrt{2}+1)}{\sqrt{2}(\sqrt{2}-1)}$$
$$=\frac{(\sqrt{2}+1)}{\sqrt{2}(\sqrt{2}-1)}=\frac{(\sqrt{2}+1)^2}{\sqrt{2}}=\left(\frac{3+2\sqrt{2}}{\sqrt{2}}\right)$$

85. We have,

$$\text{Sum}=\frac{1}{2}+\frac{1}{3^2}+\frac{1}{2^3}+\frac{1}{3^4}+\frac{1}{2^5}+\frac{1}{3^6}+\ldots\infty$$
$$=\left(\frac{1}{2}+\frac{1}{2^3}+\frac{1}{2^5}+\ldots\right)+\left(\frac{1}{3^2}+\frac{1}{3^4}+\frac{1}{3^6}+\ldots\right)$$
$$=\left(\frac{\frac{1}{2}}{1-\frac{1}{2^2}}\right)+\left(\frac{\frac{1}{3^2}}{1-\frac{1}{3^2}}\right)$$
$$=\frac{2}{3}+\frac{1}{8}=\frac{19}{24}$$

86. We have,

$b=a+a^2+a^3+\ldots\infty$

$\Rightarrow \quad b=\dfrac{a}{1-a}$

$\Rightarrow \quad b-ab=a$

$\Rightarrow \quad a(1+b)=b$

$\Rightarrow \quad a=\dfrac{b}{(1+b)}$

Hence, the result.

87. We have,

$$x=a+\frac{a}{r}+\frac{a}{r^2}+\ldots\infty$$

$\Rightarrow \quad x=\dfrac{a}{1-\frac{1}{r}}=\dfrac{ar}{r-1}$

Similarly,

$$y=\frac{b}{1+\frac{1}{r}}=\frac{br}{r+1}$$

and $\quad z=\dfrac{c}{1-\frac{1}{r^2}}=\dfrac{cr^2}{r^2-1}$

Now, $\dfrac{xy}{z}=\dfrac{\frac{abr^2}{r^2-1}}{\frac{cr^2}{r^2-1}}=\dfrac{ab}{c}$

88. We have,

$x=1+a+a^2+\ldots\infty$

$\Rightarrow \quad x=\dfrac{1}{1-a}$

$\Rightarrow \quad 1-a=\dfrac{1}{x}$

$\Rightarrow \quad a=\dfrac{x-1}{x}$

Similarly, $b=\dfrac{y-1}{y}$

Now,

$1+ab+(ab)^2+(ab)^3+\ldots$

$$=\frac{1}{1-ab}$$
$$=\frac{1}{1-\left(\frac{x-1}{x}\right)\left(\frac{y-1}{y}\right)}$$
$$=\frac{xy}{xy-(xy-x-y+1)}$$
$$=\frac{xy}{x+y-1}$$

89. We have,

$A=1+r^a+r^{2a}+\ldots$ to ∞

$\Rightarrow \quad A=\dfrac{1}{1-r^a}$

$\Rightarrow \quad (1-r^a)=\dfrac{1}{A}$

$\Rightarrow \quad r^a=\left(1-\dfrac{1}{A}\right)$

$\Rightarrow \quad r=\left(\dfrac{A-1}{A}\right)^{1/a}$

Similarly, $r=\left(\dfrac{B-1}{B}\right)^{1/b}$

Hence, $\left(\dfrac{A-1}{A}\right)^{1/a}=r=\left(\dfrac{B-1}{B}\right)^{1/b}$

90. We have,

$$x=\sum_{n=0}^{\infty}\cos^{2n}\theta$$
$$=1+\cos^2\theta+\cos^4\theta+\cos^6\theta+\ldots$$
$$=\frac{1}{1-\cos^2\theta}=\frac{1}{\sin^2\theta}$$

$$y = \sum_{n=0}^{\infty} \sin^{2n}\varphi$$
$$= 1 + \sin^2\varphi + \sin^4\varphi + \sin^6\varphi + \ldots$$
$$= \frac{1}{1-\sin^2\varphi} = \frac{1}{\cos^2\varphi}$$

and $z = \sum_{n=0}^{\infty} \cos^{2n}\theta \sin^{2n}\varphi$

$$= 1 + \cos^2\theta \sin^2\varphi + \cos^4\theta \sin^4\varphi + \cos^6\theta \sin^6\varphi + \ldots$$
$$= \frac{1}{1-\cos^2\theta \sin^2\varphi} = \frac{1}{1-(1 \sin^2\theta)(1-\cos^2\varphi)}$$
$$= \frac{1}{1-\left(1-\frac{1}{x}\right)\left(1-\frac{1}{y}\right)}$$
$$= \frac{xy}{xy-(x-1)(y-1)}$$
$$= \frac{xy}{xy-(xy-x-y+1)} = \frac{xy}{x+y-1}$$
$$xz + yz - z = xy$$

91. We have,

$$\text{Sum} = (x+y) + (x^2+xy+y^2) + (x^3+x^2y+xy^2+y^3) + \ldots \text{ to } \infty$$
$$= \frac{1}{x-y}\left[(x^2-y^2)+(x^3-y^3)+(x^4-y^4)+\ldots\right]$$
$$= \frac{1}{x-y}[(x^2+x^3+x^4+\ldots)-(y^2+y^3+y^4+\ldots)]$$
$$= \frac{1}{(x-y)}\left(\frac{x^2}{1-x} - \frac{y^2}{1-y}\right)$$

92. We have,

$$S_p = 1 + r^p + r^{2p} + \ldots$$
$$= \frac{1}{1-r^p}$$

Also, $s_p = 1 - r^p - r^{2p} - \ldots$

$$= \frac{1}{1-(-r^p)} = \frac{1}{1+r^p}$$

Now,

$$S_p + s_p = \frac{1}{1-r^p} + \frac{1}{1+r^p}$$
$$= \frac{1+r^p+1-r^p}{(1-r^p)(1+r^p)}$$
$$= \frac{2}{(1-r^{2p})}$$
$$= 2S_{2p}$$

93. We have,

$$8^{1+|\cos x|+|\cos x|^2+|\cos x|^3+\ldots \text{ to } \infty} = 4^3$$
$$\Rightarrow 8^{\frac{1}{1-|\cos x|}} = 4^3 = 64 = 8^2$$
$$\Rightarrow \frac{1}{1-|\cos x|} = 2$$
$$\Rightarrow |1-|\cos x|| = \frac{1}{2}$$
$$\Rightarrow |\cos x| = \frac{1}{2}$$
$$\Rightarrow x = -\frac{2\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{2\pi}{3}$$

Hence, the solutions are

$$\left\{\pm\frac{2\pi}{3}, \pm\frac{\pi}{3}\right\}$$

94. Given equation is

$$x^2 - 9x + 8 = 0$$
$$\Rightarrow (x-1)(x-8) = 0$$
$$\Rightarrow x = 1 \text{ or } 8$$

Now, given expression is

$$\exp\{(\sin^2 x + \sin^4 x + \sin^6 x + \ldots \infty)\log_e 2\}$$
$$= 2^{\sin^2 x + \sin^4 x + \sin^6 x + \ldots}$$
$$= 2^{\frac{\sin^2 x}{1-\sin^2}} = 2^{\tan^2 x}$$

When, $2^{\tan^2 x} = 1 = 2^0$

$$\Rightarrow \tan^2 x = 0$$
$$\Rightarrow \tan x = 0$$

When $2^{\tan^2 x} = 8 = 2^3$

$$\Rightarrow \tan^2 x = 3 = (\sqrt{3})^2$$
$$\Rightarrow \tan x = \sqrt{3}$$

Now, $\dfrac{\cos x}{\cos x + \sin x} = \dfrac{1}{1+\tan x} = \dfrac{1}{\sqrt{3}+1}$

$$= \frac{(\sqrt{3}-1)}{2}$$

95. Let $S = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots$

$$= \frac{1}{4}\left(1 + \frac{1}{2} + \frac{1}{4} + \ldots\right)$$
$$= \frac{1}{4} \times \frac{1}{1-\frac{1}{2}} = \frac{1}{4} \times 2 = \frac{1}{2}$$

We have,

$$(0.2)^{\log\sqrt{2}\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots\text{inf}\right)} = (0.2)^{\log_{\sqrt{2}}\left(\frac{1}{2}\right)}$$
$$= (0.2)^{\log_{2^{1/2}}(2^{-1})}$$
$$= (0.2)^{-2\log_2 2}$$
$$= (0.2)^{-2} = \left(\frac{1}{5}\right)^{-2} = 5^2 = 25$$

96. Given p, q and r are in AP,

$\Rightarrow \quad 2q = p + r$

Let a be the first term and R be the common ratio of the GP.

Now, $t_p = aR^{p-1}$, $t_q = aR^{q-1}$, $t_r = aR^{r-1}$

Here, $\dfrac{t_q}{t_p} = \dfrac{aR^{q-1}}{aR^{p-1}} = R^{q-p}$

and $\dfrac{t_r}{t_q} = \dfrac{aR^{r-1}}{aR^{q-1}} = R^{r-q} = R^{q-p}$

Thus, t_p, t_q, t_r are in GP.

97. It is given that a, b and c are in AP.

$2b = a + c$

Also, x, y and z are in GP.

$y^2 = xz$

$y = \sqrt{xz}$

Now,

$$\begin{aligned} x^{b-c} \cdot y^{c-a} \cdot z^{a-b} &= x^{b-c} \cdot (\sqrt{xz})^{c-a} \cdot z^{a-b} \\ &= x^{b-c} \cdot (xz)^{\frac{c-a}{2}} \cdot z^{a-b} \\ &= x^{b-c+\frac{c-a}{2}} \cdot z^{a-b+\frac{c-a}{2}} \\ &= x^{\frac{2b-2c+c-a}{2}} \cdot z^{\frac{2a-2b+c-a}{2}} \\ &= x^{\frac{2b-c-a}{2}} \cdot z^{\frac{a+c-2b}{2}} \\ &= x^{\frac{2b-2b}{2}} \cdot z^{\frac{2b-2b}{2}} \\ &= x^0 . z^0 = 1 \end{aligned}$$

98. It is given that a, b and c are in GP.

$\therefore \quad b^2 = ac$

$\Rightarrow \quad \log_n(b^2) = \log_n(ac)$

$\Rightarrow \quad 2\log_n(b) = \log_n(a) + \log_n(c)$

$\Rightarrow \quad \log_n(a), \log_n(b), \log_n(c)$ are in AP.

99. It is given that a, b and c are in GP.

$\Rightarrow \quad b^2 = ac$

Also, $x = \dfrac{a+b}{2}$, $y = \dfrac{b+c}{2}$

Now,

$$\begin{aligned} \frac{a}{x} + \frac{c}{y} &= \frac{2a}{a+b} + \frac{2c}{b+\text{c}} \\ &= \frac{2(ab+ac+ac+bc)}{(a+b)(b+c)} \\ &= \frac{2(ab+2b^2+bc)}{(ab+ac+b^2+bc)} \\ &= \frac{2(ab+2b^2+bc)}{(ab+2b^2+bc)} \\ &= 2 \end{aligned}$$

$$\begin{aligned} \text{Also, } \frac{1}{x} + \frac{1}{y} &= \frac{2}{(a+b)} + \frac{2}{(b+c)} \\ &= \frac{2(a+b+b+c)}{(a+b)(b+c)} \\ &= \frac{2(a+2b+c)}{(ab+ac+b^2+bc)} \\ &= \frac{2(a+2b+\text{c})}{(ab+b^2+b^2+bc)} \\ &= \frac{2(a+2b+c)}{(ab+2b^2+bc)} \\ &= \frac{2(a+2b+c)}{b(a+2b+c)} = \frac{2}{b} \end{aligned}$$

Hence, the result.

100. We have a, b and $3c$ are in AP and a, b and $4c$ are in GP.

$\therefore \quad 2b = a + 3c$ and $b^2 = 4ac$

Squaring $2b = a + 3e$, we get

$\Rightarrow \quad 4b^2 = a^2 + 9c^2 + 6ac$

$\Rightarrow \quad 16ac = a^2 + 9c^2 + 6ac \qquad (\because b^2 = 4ac)$

$\Rightarrow \quad a^2 + 9c^2 - 10ac = 0$

$\Rightarrow \quad a^2 - 10ac + 9c^2 = 0$

$\Rightarrow \quad (a - c)(a - 9c) = 0$

$\Rightarrow \quad a = c$ or $9c$

When $a = c$,

$$b = \left(\frac{a+3c}{2}\right) = \left(\frac{c+3c}{2}\right) = 2c$$

$\Rightarrow \quad \dfrac{a}{b} = \dfrac{c}{2c} = \dfrac{1}{2}$

When $a = 9c$

$\Rightarrow \quad b = \left(\dfrac{a+3c}{2}\right) = \left(\dfrac{9c+3c}{2}\right) = 6c$

$\Rightarrow \quad \dfrac{a}{b} = \dfrac{9c}{6c} = \dfrac{3}{2}$

101. Given $x^a = x^{\frac{b}{2}} z^{\frac{b}{2}} = z^c$

$\Rightarrow \quad x^a = x^{\frac{b}{2}} z^{\frac{b}{2}} = z^c = k$ (say)

$\Rightarrow \quad x = k^{\frac{1}{a}}, z = k^{\frac{1}{c}}, (xz) = k^{\frac{2}{b}}$

Now, $(xz) = k^{\frac{2}{b}}$

$\Rightarrow \quad (k^{\frac{1}{a}} k^{\frac{1}{c}}) = k^{\frac{2}{b}}$

$\Rightarrow \quad k^{\frac{1}{a}+\frac{1}{c}} = k^{\frac{2}{b}}$

$\Rightarrow \quad \dfrac{1}{a} + \dfrac{1}{c} = \dfrac{2}{b}$

$\Rightarrow \quad \dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c} \in \text{AP}$

Hence, the result.

102. We have,

$a+b+c=xb$

$\Rightarrow \quad a+ar+ar^2=arx$

$\Rightarrow \quad 1+r+r^2=rx$

$\Rightarrow \quad r^2+(1-x)r+1=0$

Since a, b and c are three distinct real numbers, so its $D>0$

$\Rightarrow \quad (1-x)^2-4>0$

$\Rightarrow \quad (x-1)^2-(2)^2>0$

$\Rightarrow \quad (x-1+2)(x-1-2)>0$

$\Rightarrow \quad (x+1)(x-3)>0$

$\Rightarrow \quad x<-1$ or $x>3$

103. Do yourself.

104. Given a, b and c are in AP.

$\Rightarrow \quad 2b=a+c \qquad \ldots$(i)

Also, a^2, b^2 and c^2 are in GP.

$\Rightarrow \quad b^4=a^2c^2 \qquad \ldots$(ii)

$\Rightarrow \quad a+b+c=\frac{3}{2}$

$\Rightarrow \quad 3b=\frac{3}{2}$

$\Rightarrow \quad b=\frac{1}{2}$

From Eg. (ii), we get,

$a^2c^2=\left(\frac{1}{2}\right)^4=\frac{1}{16}$

$\Rightarrow \quad ac=\pm\frac{1}{4}$

Also $a+c=2b=1$

Thus, a and c are the roots of $x^2-x\pm\frac{1}{4}=0$.

$\Rightarrow \quad x^2-x+\frac{1}{4}=0$ and $x^2-x-\frac{1}{4}=0$

$\Rightarrow \quad \left(x-\frac{1}{2}\right)^2=0$ and $\left(x-\frac{1}{2}\right)^2=\frac{1}{2}$

$\Rightarrow \quad x=\frac{1}{2}$ and $x=\frac{1}{2}\pm\frac{1}{\sqrt{2}}$

$\Rightarrow \quad x=\frac{1}{2}\pm\frac{1}{\sqrt{2}}$, since $a<b<c$.

Thus, $a=\frac{1}{2}-\frac{1}{\sqrt{2}}$, as $a<b<c$.

105. We have

$\frac{a^{n+1}+b^{n+1}}{a^n+b^n}=\sqrt{ab}$

$\Rightarrow \quad \frac{a^{n+1}+b^{n+1}}{a^n+b^n}=(ab)^{1/2}$

$\Rightarrow \quad a^{n+1}+b^{n+1}=a^{n+\frac{1}{2}}b^{\frac{1}{2}}+b^{n+\frac{1}{2}}a^{\frac{1}{2}}$

$\Rightarrow \quad a^{n+\frac{1}{2}}(a^{\frac{1}{2}}-b^{\frac{1}{2}})=b^{n+\frac{1}{2}}(a^{\frac{1}{2}}-b^{\frac{1}{2}})$

$\Rightarrow \quad a^{n+\frac{1}{2}}=b^{n+\frac{1}{2}}$

$\Rightarrow \quad \left(\frac{a}{b}\right)^{n+\frac{1}{2}}=1=\left(\frac{a}{b}\right)^0$

$\Rightarrow \quad n+\frac{1}{2}=0$

$\Rightarrow \quad n=-\frac{1}{2}$

106. Do yourself.

107. Let a and b be two positive numbers.

$\therefore \quad x=\frac{a+b}{2}$

and $\quad a, y, z, b \in$ GP

$\Rightarrow \quad t_4=b$

$\Rightarrow \quad ar^3=b$

$\Rightarrow \quad r^3=\frac{b}{a}$

$\Rightarrow \quad r=\left(\frac{b}{a}\right)^{1/3}$

Now, $y=ar=a\left(\frac{b}{a}\right)^{1/3}=a^{2/3}b^{1/3}$

and $\quad z=ar^2=a\left(\frac{b}{a}\right)^{2/3}=a^{1/3}b^{2/3}$

Therefore, $y^3+z^3=a^2b+ab^2=ab(a+b)$

$\Rightarrow \quad y^3+z^3=2xyz$

$\Rightarrow \quad \frac{y^3+z^3}{xyz}=2.$.

Hence, the result.

108. Do yourself

109. If one geometric mean G and two arithmetic means A_1 and A_2 be inserted between two given quantities, prove that $G^2=(2A_1-A_2)(2A_2-A_1)$.

110. Given a, b, c are in GP

$\Rightarrow \quad b^2=ac$

$\Rightarrow \quad b=\sqrt{ac}$

Given equation is $ax^2+2bx+c=0$

$\Rightarrow \quad ax^2+2\sqrt{ac}\,x+c=0$

$\Rightarrow \quad (\sqrt{a}x+\sqrt{c})^2=0$

$\Rightarrow \quad (\sqrt{a}x+\sqrt{c})=0$

$\Rightarrow \quad x=-\frac{\sqrt{c}}{\sqrt{a}}$

Since the equations have a common root, so

$d\left(\frac{c}{a}\right)-2e\left(\frac{\sqrt{c}}{\sqrt{a}}\right)+f=0$

$\Rightarrow \quad d\left(\frac{c}{a}\right)-2e\left(\frac{\sqrt{c^2}}{\sqrt{ac}}\right)+f=0$

$$\Rightarrow \quad d\left(\frac{c}{a}\right) - 2e\left(\frac{c}{b}\right) + f = 0$$

$$\Rightarrow \quad \left(\frac{d}{a}\right) - 2\left(\frac{e}{b}\right) + \frac{f}{c} = 0$$

$$\Rightarrow \quad \left(\frac{d}{a}\right) + \frac{f}{c} = 2\left(\frac{e}{b}\right)$$

$$\Rightarrow \quad \frac{d}{a}, \frac{e}{b}, \frac{f}{c} \text{ are in AP}$$

111. Let two numbers be a and b respectively. It is given that,

$$a + b = 6\sqrt{ab}$$

$$\Rightarrow \quad (a + b)^2 = 36ab$$

$$\Rightarrow \quad (a - b)^2 = (a + b)^2 - 4ab = 32ab$$

Now, $\dfrac{(a+b)^2}{(a-b)^2} = \dfrac{36ab}{32ab}$

$$\Rightarrow \quad \frac{(a+b)^2}{(a-b)^2} = \frac{9}{8}$$

$$\Rightarrow \quad \frac{(a+b)}{(a-b)} = \frac{3}{2\sqrt{2}}$$

$$\Rightarrow \quad \frac{(a+b)+(a-b)}{(a+b)-(a-b)} = \frac{3+2\sqrt{2}}{3-2\sqrt{2}}$$

$$\Rightarrow \quad \frac{2a}{2b} = \frac{3+2\sqrt{2}}{3-2\sqrt{2}}$$

$$\Rightarrow \quad \frac{a}{b} = \frac{3+2\sqrt{2}}{3-2\sqrt{2}}$$

112. In AP, $t_m = \dfrac{1}{n} \Rightarrow a + (m-1)d = \dfrac{1}{n}$

and $\quad t_n = \dfrac{1}{m} \Rightarrow a + (n-1)d = \dfrac{1}{m}$

On subtraction, we get,

$$(m-n)d = \frac{1}{n} - \frac{1}{m} = \frac{(m-n)}{mn}$$

$$\Rightarrow \quad d = \frac{1}{mn}$$

Put $d = \dfrac{1}{mn}$ in (i), we get,

$$a + (m-1)\frac{1}{mn} = \frac{1}{n}$$

$$\Rightarrow \quad a + \frac{1}{n} - \frac{1}{mn} = \frac{1}{n}$$

$$\Rightarrow \quad a = \frac{1}{mn}$$

Thus, $t_{mn} = a + (mn - 1)d$

$$= \frac{1}{mn} + (mn-1)\frac{1}{mn}$$

$$= 1$$

Hence, the mnth term of HP is

$$= \frac{1}{t_{mn}} = \frac{1}{1} = 1$$

113. In an AP,

$$t_m = \frac{1}{n} \Rightarrow a + (m-1)d = \frac{1}{n}$$

and $\quad t_n = \dfrac{1}{m} \Rightarrow a + (n-1)d = \dfrac{1}{m}$

On subtraction, we get,

$$(m-n)d = \frac{1}{n} - \frac{1}{m} = \frac{(m-n)}{mn}$$

$$\Rightarrow \quad d = \frac{1}{mn}$$

Put $d = \dfrac{1}{mn}$ in Eg. (i), we get

$$a + (m-1)\frac{1}{mn} = \frac{1}{n}$$

$$\Rightarrow \quad a + \frac{1}{n} - \frac{1}{mn} = \frac{1}{n}$$

$$\Rightarrow \quad a = \frac{1}{mn}$$

Thus, $t_{m+n} = a + (m + n - 1)d$

$$= \frac{1}{mn} + \frac{(m+n-1)}{mn}$$

$$= \frac{1}{mn} + \frac{(m+n)}{mn} - \frac{1}{mn}$$

$$= \left(\frac{m+n}{mn}\right)$$

Hence, the $(m + n)$th term is $\left(\dfrac{mn}{m+n}\right)$.

114. In an AP,

$t_p = q \Rightarrow a + (p - 1)d = q$ (i)

and $\quad t_q = p \Rightarrow a + (q - 1)d = p$

On subtraction, we get

$$(p - q)d = (q - p)$$

$$\Rightarrow \quad d = -1$$

Put $d = -1$, in Eg. (i), we get

$$\Rightarrow \quad a - (p - 1) = q$$

$$\Rightarrow \quad a = p + q - 1$$

Now,

$$t_n = a + (n - 1)d$$

$$= p + q - 1 - (n - 1)$$

$$= p + q - n$$

Hence, the nth term of the HP is

$$\frac{1}{t_n} = \frac{1}{p+q-n}$$

115. It is given that,

$a_1, a_2, a_3 \in$ AP $\Rightarrow 2a_2 = a_1 + a_3$

Also $a_2, a_3, a_4 \in$ GP $\Rightarrow a_3^2 = a_2 \cdot a_4$

and, $a_3, a_4, a_5 \in$ HP $\Rightarrow a_4 = \dfrac{2a_3a_5}{a_3 + a_5}$

Now,

$$a_3^2 = a_2 \cdot a_4$$

$$\Rightarrow a_3^2 = \left(\frac{a_1 + a_3}{2}\right) \times \left(\frac{2a_3a_5}{a_3 + a_5}\right)$$

$$\Rightarrow a_3^2 = (a_1 + a_3) \times \left(\frac{a_3a_5}{a_3 + a_5}\right)$$

$$\Rightarrow a_3^2(a_3 + a_5) = a_3a_5(a_1 + a_3)$$

$$\Rightarrow a_3^3 + a_3^2a_5 = a_1a_3a_5 + a_3^2a_5$$

$$\Rightarrow a_3^3 = a_1a_3a_5$$

$$\Rightarrow a_3^2 = a_1a_5$$

$\Rightarrow$ $a_1, a_3, a_5 \in$ GP

116. $a^x = b^y = c^z = d^w = k$ (say)

It is given that a, b, c and d are in GP.

$$\therefore \frac{a}{b} = \frac{b}{c} = \frac{c}{d}$$

$$\Rightarrow \frac{k^{\frac{1}{x}}}{k^{\frac{1}{y}}} = \frac{k^{\frac{1}{y}}}{k^{\frac{1}{z}}} = \frac{k^{\frac{1}{z}}}{k^{\frac{1}{w}}}$$

$$\Rightarrow k^{\frac{1}{x}-\frac{1}{y}} = k^{\frac{1}{y}-\frac{1}{z}} = k^{\frac{1}{z}-\frac{1}{w}}$$

$$\Rightarrow \frac{1}{x} - \frac{1}{y} = \frac{1}{y} - \frac{1}{z} = \frac{1}{z} - \frac{1}{w}$$

$$\Rightarrow \frac{1}{x}, \frac{1}{y}, \frac{1}{z}, \frac{1}{w} \in \text{AP}$$

$\Rightarrow$ x, y, z and $w \in$ HP

117. It is given that x, y and z are in GP.

$\therefore$ $y^2 = xz$

$\Rightarrow$ $\log(y^2) = \log(xz)$

$\Rightarrow$ $2\log(y) = \log(x) + \log(z)$

$\Rightarrow$ $\log(x), \log(y), \log(z) \in$ AP

$\Rightarrow$ $1 + \log(x), 1 + \log(y), 1 + \log(z) \in$ AP

$$\Rightarrow \frac{1}{1 + \log(x)}, \frac{1}{1 + \log(y)}, \frac{1}{1 + \log(z)} \in \text{HP}$$

118. In AP, $t_p = \dfrac{1}{a} \Rightarrow A + (p-1)d = \dfrac{1}{a}$ (i)

$t_q = \dfrac{1}{b} \Rightarrow A + (q-1)d = \dfrac{1}{b}$ (ii)

and $t_r = \dfrac{1}{c} \Rightarrow A + (r-1)d = \dfrac{1}{c}$ (iii)

Solving Eqs (i), (ii) and (iii), we get

$$(p - q) = \frac{b - a}{abd}$$

$$(q - r) = \frac{(b - c)}{bcd}$$

and $(r - p) = \dfrac{(c - a)}{acd}$

Thus,

$$\frac{q - \text{r}}{a} + \frac{r - p}{b} + \frac{p - q}{c}$$

$$= \frac{(b - \text{c})}{bcd} + \frac{(c - a)}{acd} + \frac{(a - b)}{abd}$$

$$= \frac{1}{d}\left(\frac{1}{c} - \frac{1}{b} + \frac{1}{a} - \frac{1}{c} + \frac{1}{b} - \frac{1}{a}\right)$$

$$= \frac{1}{d} \times 0$$

$= 0.$

119. It is given that

$x_1, x_2, x_3, \ldots, x_n$ are in HP.

$$\therefore \frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}, \ldots, \frac{1}{x_n} \in \text{AP}$$

$$\Rightarrow \frac{1}{x_2} - \frac{1}{x_1} = \frac{1}{x_3} - \frac{1}{x_2} = \ldots = \frac{1}{x_n} - \frac{1}{x_{n-1}} = d$$

$$\Rightarrow \frac{x_1 - x_2}{x_1x_2} = \frac{x_2 - x_3}{x_2x_3} = \ldots = \frac{x_{n-1} - x_n}{x_{n-1}x_n} = \text{d}$$

Now,

$$x_1x_2 + x_2x_3 + \ldots + x_{n-1}x_n$$

$$= \frac{x_1 - x_2}{d} + \frac{x_2 - x_3}{d} + \ldots + \frac{x_{n-1} - x_n}{d}$$

$$= \frac{1}{d}(x_1 - x_2 + x_2 - x_3 + x_{n-1} - x_n)$$

$$= \frac{1}{d}(x_1 - x_n)$$

$$= (n - 1)x_1x_n$$

120. It is given that a, b and c are in AP.

$\therefore$ $2b = a + c$

It is given that

$$x = \sum_{n=0}^{\infty} a^n, \; y = \sum_{n=0}^{\infty} b^n, \; z = \sum_{n=0}^{\infty} c^n$$

$$\Rightarrow x = \frac{1}{1 - a}, \; y = \frac{1}{1 - b}, \; z = \frac{1}{1 - c}$$

$$\Rightarrow (1 - a) = \frac{1}{x}, \; (1 - b) = \frac{1}{y}, \; (1 - c) = \frac{1}{z}$$

$$\Rightarrow a = \left(1 - \frac{1}{x}\right), b = \left(1 - \frac{1}{y}\right), c = \left(1 - \frac{1}{z}\right)$$

$\Rightarrow \quad a = \left(\frac{x-1}{x}\right), b = \left(\frac{y-1}{y}\right), c = \left(\frac{z-1}{z}\right)$

Now, putting these values in

$2b = a + c$, we have,

$$2\left(\frac{y-1}{y}\right) = \left(\frac{x-1}{x}\right) + \left(\frac{z-1}{z}\right)$$

$\Rightarrow \quad 2\left(1-\frac{1}{y}\right) = \left(1-\frac{1}{x}\right) + \left(1-\frac{1}{z}\right)$

$\Rightarrow \quad \frac{2}{y} = \frac{1}{x} + \frac{1}{z}$

$\Rightarrow \quad \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \in \text{AP}$

$x, y, z \in$ HP

Hence, the result.

121. Given $a_1 = 2$. So,

$a_{10} = a_1 + 9d$

$\Rightarrow \quad 9d = a_{10} - a_1 = 3 - 2 = 1$

$\Rightarrow \quad d = \frac{1}{9}$

Now, $a_4 = a_1 + 3d = 2 + \frac{1}{3} = \frac{7}{3}$

Also, $h_{10} = 3$

$\Rightarrow \quad \frac{1}{h_{10}} = \frac{1}{3}$

$\Rightarrow \quad \frac{1}{h_1} + 9D = \frac{1}{3}$

$\Rightarrow \quad 9D = \frac{1}{3} - \frac{1}{h_1}$

$\Rightarrow \quad 9D = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$

$\Rightarrow \quad D = -\frac{1}{54}$

Now, $\frac{1}{h_7} = \frac{1}{h_1} + 6D$

$\Rightarrow \quad \frac{1}{h_7} = \frac{1}{2} - \frac{1}{9} = \frac{9-2}{18} = \frac{7}{18}$

$\Rightarrow \quad h_7 = \frac{18}{7}$

Thus, $a_4 h_7 = \frac{7}{3} \times \frac{18}{7} = 6$

$\Rightarrow \quad a_4 h_7 + 2007 = 6 + 2007 = 2013.$

122. It is given that

$$a^2 + 9b^2 + 25c^2 = abc\left(\frac{15}{a} + \frac{5}{b} + \frac{3}{c}\right)$$

$\Rightarrow \quad (a)^2 + (3b)^2 + (5c)^2 = 15bc + 5ac + 3ab$

$\Rightarrow \quad (a)^2 + (3b)^2 + (5c)^2 = (3b)(5c) + a(5c) + a(3b)$

$\Rightarrow \quad (a)^2 + (3b)^2 + (5c)^2 - (3b)(5c) - a(5c) - a(3b) = 0$

$\Rightarrow \quad \frac{1}{2}[(a-3b)^2 + (3b-5c)^2 + (5c-a)^2] = 0$

$\Rightarrow \quad |(a-3b)^2 + (3b-5c)^2 + (5c-a)^2| = 0$

$\Rightarrow \quad (a-3b)^2 = 0, (3b-5c)^2 = 0, (5c-a)^2 = 0$

$\Rightarrow \quad (a-3b) = 0, (3b-5c) = 0, (5c-a = 0$

$\Rightarrow \quad a = 3b = 5c$

$\Rightarrow \quad \frac{a}{\frac{1}{1}} = \frac{b}{\frac{1}{3}} = \frac{c}{\frac{1}{5}}$

Clearly, a, b and $c \in$ HP.

123. It is given that,

a, x and b are in AP $\Rightarrow 2x = a + b$

a, y and $b \in$ GP $\Rightarrow y^2 = ab$

a, z and $b \in$ HP $\Rightarrow z = \frac{2ab}{a+b}$

Also, $x = 9z$

$\Rightarrow \quad \frac{a+b}{2} = 9\left(\frac{2ab}{a+b}\right)$

$\Rightarrow \quad (a+b)(a+b) = 36ab$

$\Rightarrow \quad (a+b) = 6\sqrt{ab} = 6|y|$

Clearly, $2x = 6|y|$

$\Rightarrow \quad x = 3|y|$

Also, $z = \frac{2ab}{a+\text{b}} = \frac{2y^2}{6|y|} = \frac{|y|}{3}$

$\Rightarrow \quad |y| = 3z$

Hence, the result.

124. Do yourself.

125. It is given that

$4a^2 + 9b^2 + 16c^2 = 2(3ab + 6bc + 4ac)$

$\Rightarrow \quad (2a)^2 + (3b)^2 + (4c)^2 - (2a)(3b)$
$\qquad -(3b)(4c) - (2a)(4c) = 0$

$\Rightarrow \quad (2a-3b)^2 + 3b - 4c)^2 + (4c-2a)^2 = 0$

$\Rightarrow \quad (2a-3b)^2 = 0, (3b-4c)^2 = 0, (4c-2a)^2 = 0$

$\Rightarrow \quad (2a-3b) = 0, (3b-4c) = 0, (4c-2a) = 0$

$\Rightarrow \quad 2a = 3b = 4c$

$\Rightarrow \quad \frac{a}{\frac{1}{2}} = \frac{b}{\frac{1}{3}} = \frac{c}{\frac{1}{4}}$

Thus, $a, b, c \in$ HP

126. It is given that a, b and c are in HP.

$$b = \frac{2ac}{a+c}$$

Now, $\left(\frac{a+b}{2a-b}\right)+\left(\frac{c+b}{2c-b}\right)$

$$=\left(\frac{1+\frac{b}{a}}{2-\frac{b}{a}}\right)+\left(\frac{1+\frac{b}{c}}{2-\frac{b}{c}}\right)$$

$$=\left(\frac{1+\frac{2c}{a+c}}{2-\frac{2c}{a+c}}\right)+\left(\frac{1+\frac{2a}{a+c}}{2-\frac{2a}{a+c}}\right)$$

$$=\left(\frac{a+c+2c}{2a+2c-2c}\right)+\left(\frac{a+c+2a}{2a+2c-2a}\right)$$

$$=\left(\frac{a+3c}{2a}\right)+\left(\frac{3a+c}{2c}\right)$$

$$=\frac{ac+3c^2+3a^2+ac}{2ac}$$

$$=\frac{3c^2+3a^2+2ac}{2ac}$$

$$=\frac{3(a^2+c^2)}{2ac}+1$$

$$=\frac{3}{2}\left(\frac{a}{c}+\frac{c}{a}\right)+1$$

$$\geq\frac{3}{2}\times 2+1=4$$

Hence, the value of λ is 4.

127. As we know that,

AM $\geq$ GM

$$\Rightarrow \frac{\frac{a}{b}+\frac{b}{c}+\frac{c}{a}}{3}\geq\sqrt[3]{\frac{a}{b}\cdot\frac{b}{a}\cdot\frac{c}{a}}=1$$

$$\Rightarrow \left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\geq 3$$

Hence, the result.

128. It is given that a, b and c are in HP, so we can write,

$\frac{1}{a},\frac{1}{b},\frac{1}{c}$ are in AP.

Thus, $\frac{2}{b}=\frac{1}{a}+\frac{1}{c}$

Now, $\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{c}+\frac{1}{a}-\frac{1}{b}\right)$

$$=\left(\frac{3}{b}-\frac{1}{a}\right)\left(\frac{2}{b}-\frac{1}{b}\right)$$

$$=\frac{1}{b}\left(\frac{3}{b}-\frac{1}{a}\right)=\frac{3}{b^2}-\frac{1}{ab}.$$

129. Given a, b, c and d are in HP.

$\Rightarrow \frac{1}{a},\frac{1}{b},\frac{1}{c},\frac{1}{d}$ are in AP

Let D be the common difference.

Thus, $\frac{1}{b}-\frac{1}{a}=D$

$$\Rightarrow ab=\frac{1}{D}(a-b)$$

Similarly, $bc=\frac{1}{D}(b-c)$

and $cd=\frac{1}{D}(c-d)$

Now, $ab+bc+cd$

$$=\frac{1}{D}(a-b)+\frac{1}{D}(b-c)+\frac{1}{D}(c-d)$$

$$=\frac{1}{D}(a-b+b-c+c-d)$$

$$=\frac{1}{D}(a-d)$$

$$=\left(\frac{3ad}{a-d}\right)\times(a-d)$$

$$=3ad$$

130. Since a, b and c are in AP

$\Rightarrow \quad 2b=a+c \quad$ …(i)

Also, x, y and z are in HP

$\Rightarrow \quad y=\frac{2xz}{x+z} \quad$ …(ii)

And, ax, by and cz are in GP.

$\therefore \quad b^2y^2=axcz \quad$ …(iii)

$\Rightarrow \left(\frac{a+c}{2}\right)^2\times\frac{4x^2z^2}{(x+z)^2}=acxz$ [from Eqs. (i) and (ii)]

$$\Rightarrow \frac{(a+c)^2}{ac}=\frac{(x+z)^2}{xz}$$

$$\Rightarrow \frac{a^2+c^2+2ac}{ac}=\frac{x^2+z^2+2xz}{xz}$$

$$\Rightarrow \frac{a}{c}+\frac{c}{a}+2=\frac{x}{z}+\frac{z}{x}+2$$

$$\Rightarrow \frac{a}{c}+\frac{c}{a}=\frac{x}{z}+\frac{z}{x}$$

Hence, the result.

131. We have,

$$\frac{a-x}{px}=\frac{a-y}{qy}=\frac{a-z}{rz}$$

$$\Rightarrow \quad \frac{\frac{a}{x}-1}{p}=\frac{\frac{a}{y}-1}{q}=\frac{\frac{a}{z}-1}{r}$$

Since, p, q, r are in AP, so we can write,

$2q = p + r$

$$\Rightarrow \quad 2\left(\frac{a}{y}-1\right)=\left(\frac{a}{x}-1\right)+\left(\frac{a}{z}-1\right)$$

$$\Rightarrow \quad \frac{2a}{y}-2=\frac{a}{x}+\frac{a}{z}-2$$

$$\Rightarrow \quad \frac{2a}{y}=\frac{a}{x}+\frac{a}{z}$$

$$\Rightarrow \quad \frac{1}{x}, \frac{1}{y}, \frac{1}{z} \text{ are in AP}$$

$\Rightarrow$ x, y, z are in HP.

132. Do yourself.

133. (i) Let $S_n = 1 + 2x + 3x^2 + 4x^3 + \ldots + nx^{n-1}$

$\Rightarrow \quad x \cdot S_n = x + 2x^2 + 3x^3 + 4x^4 + \ldots + nx^n$

Subtracting, we get

$\Rightarrow \quad (1-x)S_n = 1 + x + x^2 + \ldots + x^{n-1} - nx^n$

$$=\left(\frac{1-x^n}{1-x}\right)-nx^n$$

$$\Rightarrow \quad S_n=\left(\frac{1-x^n}{(1-x)^2}\right)-\frac{nx^n}{(1-x)}$$

134. Do yourself.

135. (i) Let $S_n = 1+\frac{3}{2}+\frac{5}{4}+\frac{7}{8}+\ldots+\frac{2n-1}{2^n}$

$$\Rightarrow \quad \frac{S_n}{2}=\frac{1}{2}+\frac{3}{4}+\frac{5}{8}+\ldots+\frac{2n-3}{2^n}+\frac{2n-1}{2^{n+1}}$$

Subtracting, we get

$$\left(1-\frac{1}{2}\right)S_n=\left(1+\frac{2}{2}+\frac{2}{4}+\frac{2}{8}+\ldots+\frac{2}{2^n}-\frac{2n-1}{2^{n+1}}\right)$$

$$=-1+2\left(1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots+\frac{1}{2^n}\right)-\frac{2n-1}{2^{n+1}}$$

$$=-1+2\left(\frac{1-\left(\frac{1}{2}\right)^{n+1}}{1-\frac{1}{2}}\right)-\frac{2n-1}{2^{n+1}}$$

$$=-1+4\left(1-\left(\frac{1}{2}\right)^{n+1}\right)-\frac{2n-1}{2^{n+1}}$$

$$=\left(3-\left(\frac{1}{2}\right)^{n-1}\right)-\frac{2n-1}{2^{n+1}}$$

$$\Rightarrow \quad S_n=\left(\frac{3}{2}-\left(\frac{1}{2}\right)^n\right)-\frac{2n-1}{2^{n+2}}$$

(ii) We have,

$$\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+\ldots$$

$$=\left(1-\frac{1}{2}\right)+\left(1-\frac{1}{2^2}\right)+\left(1-\frac{1}{2^3}\right)+\ldots+\left(1-\frac{1}{2^n}\right)$$

$$=n-\left(\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^n}\right)$$

$$=n-\frac{1}{2}\left(\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}\right)$$

$$=n-\left(1-\left(\frac{1}{2}\right)^n\right)$$

$$=(n+2^{-n}-1)$$

(iii) We have,

$$\frac{2}{3}+\frac{8}{9}+\frac{26}{27}+\frac{80}{81}+\ldots$$

$$=\left(1-\frac{1}{3}\right)+\left(1-\frac{1}{3^2}\right)+\left(1-\frac{1}{3^3}\right)+\ldots+\left(1-\frac{1}{3^n}\right)$$

$$=n-\left(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots+\frac{1}{3^n}\right)$$

$$=n-\frac{1}{3}\left(\frac{1-\frac{1}{3^n}}{1-\frac{1}{3}}\right)$$

$$=n-\frac{1}{2}\left(1-\frac{1}{3^n}\right)$$

(iv) We have

$$2+\frac{7}{4}+\frac{10}{8}+\frac{13}{16}+\frac{16}{32}+\ldots$$

$$=\frac{4}{2}+\frac{7}{4}+\frac{10}{8}+\frac{13}{16}+\frac{16}{32}+\ldots+\frac{3n+1}{2^n}$$

Let $S=\frac{4}{2}+\frac{7}{4}+\frac{10}{8}+\frac{13}{16}+\frac{16}{32}+\ldots+\frac{3n+1}{2^n}$

$$\therefore \quad \frac{S}{2}=\frac{4}{4}+\frac{7}{8}+\frac{10}{16}+\frac{13}{32}+\frac{16}{64}+\ldots+\frac{3n+1}{2^{n+1}}$$

Subtracting, we get

$$\left(S-\frac{S}{2}\right)=2+\frac{3}{4}+\frac{3}{8}+\frac{3}{16}+\ldots+\frac{3}{2^n}-\frac{3n+1}{2^{n+1}}$$

$$\Rightarrow \frac{S}{2} = 2 + \frac{3}{4} + \frac{3}{8} + \frac{3}{16} + \ldots + \frac{3}{2^n} - \frac{3n+1}{2^{n+1}}$$

$$\Rightarrow \frac{S}{2} = 2 + \frac{3}{4}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-2}}\right) - \frac{3n+1}{2^{n+1}}$$

$$\Rightarrow \frac{S}{2} = 2 + \frac{3}{2}\left(1 - \frac{1}{2^{n-1}}\right) - \frac{3n+1}{2^{n+1}}$$

$$\Rightarrow S = 4 + 3\left(1 - \frac{1}{2^{n-1}}\right) - \frac{3n+1}{2^n}$$

136. (i) We have

$$3 + (3+d)\frac{1}{4} + (3+2d)\frac{1}{4^2} + \ldots \text{to } \infty = \frac{44}{9}$$

$$\Rightarrow \frac{a}{1-r} + \frac{dr}{(1-r)^2} = \frac{44}{9}$$

$$\Rightarrow \frac{3}{1-\frac{1}{4}} + \frac{d\left(\frac{1}{4}\right)}{\left(1-\frac{1}{4}\right)^2} = \frac{44}{9}$$

$$\Rightarrow 4 + \frac{4}{9}d = \frac{44}{9}$$

$$\Rightarrow \frac{4}{9}d = \frac{44}{9} - 4 = \frac{8}{9}$$

$$\Rightarrow d = 2$$

137 Let $t_r = r(r+1) = r^2 + r$.

Then $S_n = \sum_{r=1}^{n} t_r$

$$= \sum_{r=1}^{n} (r^2 + r)$$

$$= \sum_{r=1}^{n} r^2 + \sum_{r=1}^{n} r$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left(\frac{2n+1}{3} + 1\right)$$

$$= \frac{n(n+1)(n+2)}{3}$$

138. Do yourself.
139. Do yourself.
140. Do yourself.
141. Let $t_n = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$.

Then $S_n = \sum_{n=1}^{n} t_n$

$$= \frac{1}{2}\sum_{n=1}^{n} (n^2 + n)$$

$$= \frac{1}{2}\left(\sum_{n=1}^{n} n^2 + \sum_{n=1}^{n} n\right)$$

$$= \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right)$$

$$= \frac{n(n+1)}{4}\left(\frac{(2n+1)}{3} + 1\right)$$

$$= \frac{n(n+1)(n+2)}{6}$$

142. Do yourself.
143. Let $t_n = (1^2 + 2^2 + 3^2 + \ldots + n^2)$

$$= \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{(2n^3 + 3n^2 + n)}{6}$$

$$= \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}$$

Then $S_n = \sum_{n=1}^{n}\left(\frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}\right)$

$$= \frac{1}{3}\sum_{n=1}^{n} n^3 + \frac{1}{2}\sum_{n=1}^{n} n^2 + \frac{1}{6}\sum_{n=1}^{n} 1.$$

$$= \frac{1}{3} \times \left(\frac{n(n+1)}{2}\right)^2 + \frac{1}{2} \times \frac{n(n+1)(2n+1)}{6} + \frac{1}{6} \times n$$

$$= \frac{n(n+1)}{12} \times (1 + 2n + 1) + \frac{1}{6} \times n$$

$$= \frac{n(n+1)^2}{6} + \frac{n}{6}$$

$$= \frac{n}{6}(n^2 + 2n + 2).$$

144. Let $t_n = \frac{1^2 + 2^2 + \ldots + n^2}{1 + 2 + 3 + \ldots + n}$

$$= \frac{\frac{n(n+1)(2n+1)}{6}}{\frac{n(n+1)}{2}}$$

$$= \frac{(2n+1)}{3}$$

Thus, $S_n = \sum t_n = \sum \frac{(2n+1)}{3}$

$$= \frac{1}{3}\left(\sum 2n + \sum 1\right)$$

$$= \frac{1}{3}(n(n+1)+n)$$

$$= \frac{(n^2+2n)}{3}$$

145. Let $t_n = \dfrac{1^3+2^3+3^3+\ldots+n^3}{1+3+5+\ldots+(2n-1)}$

$$= \frac{\left(\frac{n(n+1)}{2}\right)^2}{n^2}$$

$$= \frac{1}{4}(n^2+2n+1)$$

Then $S_n = \sum_{n=1}^{n} \frac{1}{4}(n^2+2n+1)$

$$= \frac{1}{4}\sum_{n=1}^{n} n^2 + \frac{1}{2}\sum_{n=1}^{n} n + \frac{1}{4}\sum_{n=1}^{n} 1$$

$$= \frac{1}{4} \times \frac{n(n+1)(2n+1)}{6} + \frac{1}{2} \times \frac{n(n+1)}{2} + \frac{1}{4} \times n$$

$$= \frac{n}{4}\left(\frac{n^2+3n+1}{6} + n^2+n+1\right)$$

$$= \left(\frac{n(7n^2+10n+7)}{24}\right)$$

146. Let $S_n = \dfrac{1}{1.2} + \dfrac{1}{2.3} + \dfrac{1}{3.4} + \ldots$

$$= \frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \ldots + \frac{1}{n \cdot (n+1)}$$

$$= \frac{(2-1)}{1.2} + \frac{(3-2)}{2.3} + \frac{(4-3)}{3.4} + \ldots + \frac{(n-1-n)}{n \cdot (n+1)}$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right)$$

$$= \left(1 - \frac{1}{n}\right)$$

$$= \left(\frac{n-1}{n}\right)$$

> ***Note*** If $S_\infty = \dfrac{1}{1.2} + \dfrac{1}{2.3} + \dfrac{1}{3.4} + \ldots \infty$, then $S_\infty = 1$.

147. Do yourself.

148. Let

$$S_n = \frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \ldots$$

$$= \frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \ldots + \frac{1}{n(n+1)(n+2)}$$

$$= \frac{1}{2}\left[\frac{(3-1)}{1.2.3} + \frac{(4-2)}{2.3.4} + \frac{(5-3)}{3.4.5} + \ldots + \frac{(n+2-n)}{n(n+1)(n+2)}\right]$$

$$= \frac{1}{2}\left[\left(\frac{1}{1.2} - \frac{1}{2.3}\right) + \left(\frac{1}{2.3} - \frac{1}{3.4}\right) + \ldots + \ldots + \left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right)\right]$$

$$= \frac{1}{2}\left[\frac{1}{2} - \frac{1}{(n+1)(n+2)}\right]$$

> ***Note*** If $S_\infty = \dfrac{1}{1.2.3} + \dfrac{1}{2.3.4} + \dfrac{1}{3.4.5} + \ldots$, then $S_\infty = 1/4$.

149. Do yourself.

150. Do yourself.

151. Let $t_n = n(n+1)^2$

$$= n(n^2+2n+1)$$

$$= n^3+2n^2+n$$

Then $S_n = \Sigma n^3 + 2\Sigma n^2 + \Sigma n$

$$= \left(\frac{n(n+1)}{2}\right)^2 + 2\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{n(n+1)}{2}$$

$$= \frac{n(n+1)}{2}\left(\frac{n(n+1)}{2} + \frac{2(2n+1)}{3} + 1\right)$$

$$= \frac{n(n+1)}{12}(3n^2+3n+8n+10)$$

$$= \frac{n(n+1)(3n^2+11n+10)}{12}$$

$$= \frac{n(n+1)(n+2)(3n+5)}{12}$$

Also, let $T_n = n^2(n+1) = n^3+n^2$

Then $S'_n = \Sigma n^3 + \Sigma n^2$

$$= \left(\frac{n(n+1)}{2}\right)^2 + \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(n+1)}{2}\left(\frac{n(n+1)}{2} + \frac{2n+1}{3}\right)$$

$$= \frac{n(n+1)}{12}(3n^2+3n+4n+2)$$

$$= \frac{n(n+1)}{12}(3n^2+7n+2)$$

$$= \frac{n(n+1)(n+2)(3n+1)}{12}$$

Thus, $\dfrac{S_n}{S'_n} = \dfrac{n(n+1)(n+2)(3n+5)}{n(n+1)(n+2)(3n+1)}$

$$= \frac{3n+5}{3n+1}$$

152. Let $t_n = \dfrac{3n+1}{(n+1)(n+2)(n+3)}$

$$= \frac{3n+3-2}{(n+1)(n+2)(n+3)}$$

$$= \frac{3}{(n+2)(n+3)} - \frac{2}{(n+1)(n+2)(n+3)}$$

$$= 3\left(\frac{1}{(n+2)} - \frac{1}{(n+3)}\right) - \frac{(n+3)-(n+1)}{(n+1)(n+2)(n+3)}$$

$$= 3\left(\frac{1}{(n+2)} - \frac{1}{(n+3)}\right) - \frac{1}{(n+1)(n+2)} + \frac{1}{(n+2)(n+3)}$$

$$= 3\left(\frac{1}{n+2} - \frac{1}{n+3}\right) - \left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \left(\left(\frac{1}{n+2} - \frac{1}{n+3}\right)\right)$$

$$= 4\left(\frac{1}{n+2} - \frac{1}{n+3}\right) - \left(\frac{1}{n+1} - \frac{1}{n+2}\right)$$

Thus,

$$S_n = 4\left(\left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \ldots + \left(\frac{1}{n+2} - \frac{1}{n+3}\right)\right) - \left(\left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{n+1} - \frac{1}{n+2}\right)\right)$$

$$= 4\left(\frac{1}{3} - \frac{1}{n+3}\right) - \left(\frac{1}{2} - \frac{1}{n+2}\right)$$

$$= \left(\frac{2}{3} - \frac{(3n+5)}{(n+2)(n+3)}\right)$$

153. We have,

$$\text{Sum} = \frac{1}{\log_2 4} + \frac{1}{\log_{2^2} 4} + \frac{1}{\log_{2^3} 4} + \ldots + \frac{1}{\log_{2^n} 4}$$

$$= \log_4 2 + \log_4(2^2) + \log_4(2^3) + \ldots + \log_4(2^n)$$

$$= \log_4(2 \cdot 2^2 \cdot 2^3 \cdot \ldots \cdot 2^n)$$

$$= \log_4(2^{1+2+3+\ldots+n})$$

$$= \log_{2^2}(2^{1+2+3+\ldots+n})$$

$$= \left(\frac{1+2+3+\ldots+n}{2}\right)\log_2(2)$$

$$= \left(\frac{1+2+3+\ldots+n}{2}\right)$$

$$= \frac{n(n+1)}{4}$$

154. We have,

$$\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{j} 1$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{i}(j)$$

$$= \sum_{i=1}^{n}\left(\frac{n(n+1)}{2}\right)$$

$$= \frac{1}{2}\sum_{i=1}^{n}(n^2+n)$$

$$= \frac{1}{2}\left(\sum_{i=1}^{n} n^2 + \sum_{i=1}^{n} n\right)$$

$$= \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right)$$

$$= \frac{1}{2} \times \frac{n(n+1)}{2}\left(\frac{(2n+1)}{3} + 1\right)$$

$$= \frac{n(n+1)(n+2)}{6}$$

155. Let $S_n = 3 + 7 + 13 + 21 + \ldots + T_{n-1} + T_n$ …(i)

Also, $S_n = 3 + 7 + 13 + 21 + \ldots + T_{n-1} + T_n$ …(ii)

Subtracting

(i) – (ii), we get

$0 = 3 + 4 + + 6 + \ldots + (T_n - T_{n-1}) - T_n$

$\Rightarrow \quad T_n = 3 + 4 + + 6 + \ldots + (T_n - T_{n-1})$

$\Rightarrow \quad T_n = 3 + (4 + + 6 + \ldots + (T_n - T_{n-1}))$

$$= 3 + \frac{n-1}{2}(2.4 + (n-2)2)$$

$= 3 + (n-1)(n+2)$

$= n^2 + n + 1$

Therefore, $S_n = \sum(n^2 + n + 1)$

$$= \sum n^2 + \sum n + \sum 1$$

$$= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} + n$$

$$= \frac{n(n^2+3n+5)}{3}$$

156. Given series is

$1 + 3 + 7 + 15 + 31 + \ldots$

Here, the differences between the successive terms are 2, 4, 8, 16, …, which are in GP. Whenever the successive differences are in GP, we consider its nth term as

$t_n = ar^n + br^{n-1} + c$

In order to find the value of a, b and c, we put $n = 1, 2, 3$.

Thus, $t_1 = 1 = a + b + c$

$t_2 = 3 = ar^2 + br + c$,

and $t_3 = 7 = ar^2 + br^2 + c$

On solving, we get

$r = 2, a = 0, b = 1, c = -1$

Therefore, $t_n = 2^n - 1$.

Hence, $S_n = \sum 2^n - \sum 1$

$$= \left(\frac{2^n - 1}{2 - 1}\right) - n$$

$$= 2^n - n - 1$$

157. Let $S_n = 1 + 4 + 10 + 22 + \ldots + T_{n-1} + T_n$ …(i)

Also $S_n = 1 + 4 + 10 + 22 + \ldots + T_{n-1} + T_n$ …(ii)

Subtracting

(i) – (ii), we get

$0 = 1 + 3 + 6 + 12 + \ldots + (T_n - T_{n-1}) - T_n$

$\Rightarrow \quad T_n = 1 + 3 + 6 + 12 + \ldots + (T_n - T_{n-1})$

$\Rightarrow \quad T_n = 1 + (3 + 6 + 12 + \ldots + (T_n - T_{n-1}))$

$\Rightarrow \quad T_n = 1 + 3(1 + 2 + 2^2 + \ldots + 2^{n-1})$

$$\Rightarrow \quad T_n = 1 + 3\left(\frac{2^{n-1} - 1}{2 - 1}\right) = 3.2^{n-1} - 2$$

Therefore, $S_n = \sum T_n$

$$= \sum (3.2^{n-1} - 2)$$

$$= 3\sum 2^{n-1} - \sum 2$$

$$= 3\left(\frac{2^n - 1}{2 - 1}\right) - 2n$$

$$= 3.2^n - 2n - 3$$

158. It is given that $x + y + z = 1$

$\therefore \quad (1 - x) = y + z, (1 - y) = x + z, (1 - z) = x + y$

Solving, we have $x + y = \frac{1}{2}, y + z = \frac{1}{2}$ and $z + x = \frac{1}{2}$

We know that for any two numbers, $\frac{y + z}{2} \geq \sqrt{yz}$

$\Rightarrow \quad (y + z) \geq 2\sqrt{yz}$

Similarly, $(x + z) \geq 2\sqrt{xz}$

and $(x + y) \geq 2\sqrt{xy}$

On multiplication, we get,

$(y + z)(z + x)(x + y) \geq 8xyz$

$(1 - x)(1 - y)(1 - z) \geq 8xyz$

$$\frac{xyz}{(1 - x)(1 - y)(1 - z)} \leq \frac{1}{8}$$

Hence, the maximum value is $\frac{1}{8}$.

159. We have

$$\left(p + \frac{1}{p}\right)^2 + \left(q + \frac{1}{q}\right)^2$$

$$= \left(p^2 + \frac{1}{p^2}\right) + \left(q^2 + \frac{1}{q^2}\right) + 4$$

$$= \left(p^2 + q^2 + \frac{1}{p^2} + \frac{1}{q^2}\right) + 4$$

Now, $\left(\frac{p^2 + q^2}{2}\right) \geq \left(\frac{p + q}{2}\right)^2$

$\Rightarrow \quad \left(\frac{p^2 + q^2}{2}\right) \geq \frac{1}{4}$

$\Rightarrow \quad (p^2 + q^2) \geq \frac{1}{2}$ …(i)

Also, $\left(\frac{p^{-2} + q^{-2}}{2}\right) \geq \left(\frac{p + q}{2}\right)^{-2}$

$\Rightarrow \quad \left(\frac{p^{-2} + q^{-2}}{2}\right) \geq 4$

$\Rightarrow \quad \left(\frac{1}{p^2} + \frac{1}{q^2}\right) \geq 8$

Thus, $\left(p^2 + q^2 + \frac{1}{p^2} + \frac{1}{q^2}\right) + 4 \geq \frac{1}{2} + 8 + 4 = \frac{25}{2}$

$\Rightarrow \quad \left(p + \frac{1}{p}\right)^2 + \left(q + \frac{1}{q}\right)^2 \geq \frac{25}{2}$

Hence, the result.

160. We have,

$$\frac{\left(1 + \frac{1}{a}\right) + \left(1 + \frac{1}{b}\right) + \left(1 + \frac{1}{c}\right)}{3} \geq \sqrt[3]{\left(1 + \frac{1}{a}\right) \cdot \left(1 + \frac{1}{b}\right) \cdot \left(1 + \frac{1}{c}\right)}$$

$$\Rightarrow \quad \sqrt[3]{\left(1 + \frac{1}{a}\right) \cdot \left(1 + \frac{1}{b}\right) \cdot \left(1 + \frac{1}{c}\right)} \leq \left(\frac{3 + \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)}{3}\right)$$

$$= \left(1 + \left(\frac{a^{-1} + b^{-1} + c^{-1}}{3}\right)\right)$$

$$\geq \left(1 + \left(\frac{a + b + c}{3}\right)^{-1}\right) = (1 + 3) = 4$$

$$\Rightarrow \quad \sqrt[3]{\left(1 + \frac{1}{a}\right) \cdot \left(1 + \frac{1}{b}\right) \cdot \left(1 + \frac{1}{c}\right)} \geq 4$$

$$\Rightarrow \quad \left(1+\frac{1}{a}\right)\cdot\left(1+\frac{1}{b}\right)\cdot\left(1+\frac{1}{c}\right) \ge 4^3 = 64$$

161. As we know that,

AM ≥ HM

$$\therefore \quad \left(\frac{a+b+c+d}{4}\right) \ge \frac{4}{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)}$$

$$\Rightarrow \quad \left(\frac{a+b+c+d}{4}\right)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \ge 4$$

$$\Rightarrow \quad (a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \ge 16$$

163. As we know that,

AM ≥ GM

$$\therefore \quad \left(\frac{a+b+c}{3}\right)\left(\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3}\right) \ge \sqrt[3]{(abc)\times\frac{1}{(abc)}}$$

$$\Rightarrow \quad \left(\frac{a+b+c}{3}\right)\left(\frac{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}{3}\right) \ge 1$$

$$\Rightarrow \quad (a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \ge 9$$

$$\Rightarrow \quad (a+b+c)\left(\frac{ab+bc+ca}{abc}\right) \ge 9$$

$\Rightarrow \quad (a+b+c)(ab+bc+ca) \ge 9abc$

Hence, the result.

164. We have,

$$\sum_{r=0}^{n}(x^r) = (1+x)(1+x^2)(1+x^4)(1+x^8)\ldots(1+x^{128})$$

$$= \left(\frac{1-x^{256}}{1-x}\right)$$

$$= \left(\frac{1-x^{256}}{1-x}\right)$$

$$\Rightarrow \quad (1+x+x^2+x^3+\ldots+x^n) = \left(\frac{1-x^{256}}{1-x}\right)$$

$$\Rightarrow \quad \left(\frac{1-x^{256}}{1-x}\right) = \left(\frac{1-x^{n+1}}{1-x}\right)$$

$\Rightarrow \quad n+1 = 256$

$\Rightarrow \quad n = 255$

164 Do yourself.

165. Let the sides of a triangle be $2, c, (a+\sqrt{b})$.

Since the sides of a triangle are in GP. So,

$c^2 = 2(a+\sqrt{b})$ …(i)

Also, $c^2 + 4 = (a+\sqrt{b})^2$ …(ii)

From Eqs (i) and (ii), we get

$(a+\sqrt{b})^2 = 2(a+\sqrt{b}) + 4$

$\Rightarrow \quad (a+\sqrt{b})^2 - 2(a+\sqrt{b}) = 4$

$\Rightarrow \quad (a+\sqrt{b})^2 - 2(a+\sqrt{b}) + 1 = 4+1$

$\Rightarrow \quad (a+\sqrt{b}-1)^2 = (\sqrt{5})^2$

$\Rightarrow \quad (a+\sqrt{b}-1) = (\sqrt{5})$

$\Rightarrow \quad (a+\sqrt{b}) = (1+\sqrt{5})$

Thus, $a = 1$ and $b = 5$

Hence, the value of $a^2 + b^2 + 10 = 1 + 25 + 10 = 36$

166. Since a, b and c are the sides of a triangle, so,

$a^2 + b^2 + c^2 > ab + bc + ca$

$$\Rightarrow \quad \frac{ab+bc+ca}{a^2+b^2+c^2} < 1 \qquad \ldots\text{(i)}$$

Also, $a < b + c$

$\Rightarrow \quad a^2 < a(b+c)$

Similarly, $b^2 < b(a+c)$

and $c^2 < c(a+b)$

Thus, $a^2 + b^2 + c^2 < 2(ab + bc + ca)$

$\Rightarrow \quad 2(ab + bc + ca) > a^2 + b^2 + c^2$

$$\Rightarrow \quad \frac{(ab+bc+ca)}{(a^2+b^2+c^2)} > \frac{1}{2} \qquad \ldots\text{(ii)}$$

From Relations (i) and (ii), we get

$$\frac{1}{2} < \frac{(ab+bc+ca)}{(a^2+b^2+c^2)} < 1$$

167. Applying, AM ≥ GM, we have

$$\left(\frac{1+a}{2}\right) \ge \sqrt{a}$$

$\Rightarrow \quad (1+a) \ge 2\sqrt{a}$

Similarly, $(1+b) \ge 2\sqrt{b}$

$(1+c) \ge 2\sqrt{c}$

and $(1+d) \ge 2\sqrt{d}$

Multiplying, we get

$(1+a)(1+b)(1+c)(1+d) \ge 16\sqrt{abcd} = 16$

Hence, $(1+a)(1+b)(1+c)(1+d) \ge 16$

168. Now,

$(1+a)(1+b) - 2(ab+1)$

$= 1 + a + b + ab - 2ab - 2$

$= -1 + a + b - ab$

$= -(1-a)(1-b)$

$= -\text{ve}$

Thus, $(1+a)(1+b) < 2(ab+1)$

Similarly, $(1+c)(1+d) < 2(cd+1)$

Hence,

$(1+a)(1+b)(1+c)(1+d) < 4(ab+1)(cd+1)$

$< 8(abcd+1)$

Hence, the result.

169. Applying AM ≥ GM, we get

$$\frac{(1-x)+(1-y)+(1-z)}{3} \geq \sqrt[3]{(1-x)(1-y)(1-z)}$$

$$\Rightarrow \quad \frac{3-(x+y+z)}{3} \geq \sqrt[3]{(1-x)(1-y)(1-z)}$$

$$\Rightarrow \quad \frac{3-1}{3} \geq \sqrt[3]{(1-x)(1-y)(1-z)}$$

$$\Rightarrow \quad \frac{2}{3} \geq \sqrt[3]{(1-x)(1-y)(1-z)}$$

$$\Rightarrow \quad (1-x)(1-y)(1-z) \leq \frac{8}{27}$$

Hence, the result.

170. $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}=\frac{c+a+b}{abc}=\frac{1}{abc}$ (given)

We know that

$$\left(\frac{a+b+c}{3}\right) \geq \sqrt[3]{abc}$$

$$\Rightarrow \quad \left(\frac{a+b+c}{3}\right)^3 \geq abc$$

$$\Rightarrow \quad abc \leq \frac{1}{27}$$

$$\Rightarrow \quad \frac{1}{abc} \geq 27$$

$$\Rightarrow \quad \left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right) \geq 27$$

Hence, the result.

171. We have

$(ax+by)^2-(a^2+b^2)(x^2+y^2)$

$= a^2x^2+b^2y^2+2abxy-a^2x^2-a^2y^2-b^2x^2-b^2y^2$

$= -(a^2y^2+b^2x^2-2abxy)$

$= -(bc-ay)^2$

$= -\text{ve}$

Thus, $(ax+by)^2 < (a^2+b^2)(x^2+y^2)$

$\Rightarrow \quad (ax+by)^2 < 1 \cdot 1 = 1$

$\Rightarrow \quad (ax+by) < 1$

Hence, the result.

172. Applying AM ≥ GM, we get

$$\left(\frac{px+qy}{2}\right) \geq \sqrt{pqxy}$$

$$\Rightarrow \quad (px+qy) \geq 2\sqrt{pqxy}$$

Similarly, $(pq+xy) \geq 2\sqrt{pqxy}$

Thus, $(px+qy)(pq+xy) \geq 4pqxy$

Hence, the result.

173. We have,

$\log_2 x + \log_2 y \geq 6$

$\Rightarrow \quad \log_2(xy) \geq 6$

$\Rightarrow \quad (xy) \geq 2^6 = 64$

Applying AM ≥ GM, we have

$$\left(\frac{x+y}{2}\right) \geq \sqrt{xy}$$

$$\Rightarrow \quad \left(\frac{x+y}{2}\right) \geq \sqrt{64} = 8$$

$\Rightarrow \quad (x+y) \geq 16$

Hence, the minimum value of $(x+y)$ is 16.

174. Applying AM ≥ GM, we have

$$\left(\frac{\log_a b+\log_b a}{2}\right) \geq \sqrt{\log_a b+\log_b a}$$

$$\Rightarrow \quad \left(\frac{\log_a b+\log_b a}{2}\right) \geq 1$$

$\Rightarrow \quad \log_a b + \log_b a \geq 2$

Hence, the minimum value is 2.

175. We have,

$$2\log_{10} x - \log_x(.01) = 2\log_{10} x - \log_x(10)^{-2}$$
$$= 2\log_{10} x + 2\log_x(10)$$
$$= 2(\log_{10} x + \log_x(10))$$
$$\geq 2.2 - 4$$

Hence, the least value of $2\log_{10} x - \log_x(.01)$ is 4.

176. We know that,

$$\left(a^2-\frac{1}{a^2}\right)^2+\left(b^2-\frac{1}{b^2}\right)^2$$
$$=\left(a^4+\frac{1}{a^4}\right)+\left(b^4+\frac{1}{b^4}\right)-4$$

Also, $\left(\frac{(a^2)^2+(b^2)^2}{2}\right) \geq \left(\frac{a^2+b^2}{2}\right)^2$

$\Rightarrow \quad \left(\frac{(a^2)^2+(b^2)^2}{2}\right) \geq \frac{1}{4}$ (given)

$\Rightarrow \quad (a^4+b^4) \geq \frac{1}{2}$

Also, $\frac{(a^2)^{-2}+(b^2)^{-2}}{2} \geq \left(\frac{a^2+b^2}{2}\right)^{-2}$

$\Rightarrow \quad \frac{(a^2)^{-2}+(b^2)^{-2}}{2} \geq 4$

$\Rightarrow \quad \left(\frac{1}{a^4}+\frac{1}{b^4}\right) \geq 8$

Thus, $\left(a^4+\frac{1}{a^4}+b^4+\frac{1}{b^4}-4\right) \geq \frac{1}{2}+8-4$

$\Rightarrow \quad \left(a^2-\frac{1}{a^2}\right)^2+\left(b^2-\frac{1}{b^2}\right)^2 \geq \frac{9}{2}$

Hence, the result

177. We have

$(m - a)(m - b)(m - c)(m - d)$

$= (b + c + d)(a + c + d)(a + b + d)(a + b + c)$

Applying AM ≥ GM, we get

$$\left(\frac{b+c+\text{d}}{3}\right) \geq \sqrt[3]{bcd}$$

$$\Rightarrow \quad \left(\frac{a+c+d}{3}\right) \geq \sqrt[3]{acd}$$

$$\Rightarrow \quad \left(\frac{a+b+d}{3}\right) \geq \sqrt[3]{abd}$$

Similarly, $\left(\frac{a+b+c}{3}\right) \geq \sqrt[3]{abc}$

On multiplication, we get

$(b + c + d)(a + c + d)(a + b + d)(a + b + c)$

$> 81abcd$

Hence, the result.

178. We know that,

$$\left(a+\frac{1}{a}\right)^2 + \left(b+\frac{1}{b}\right)^2 = (a^2+b^2) + \left(\frac{1}{a^2}+\frac{1}{b^2}\right) + 4$$

Now, $\left(\frac{a^2+b^2}{2}\right) \geq \left(\frac{a+b}{2}\right)^2$

$$\Rightarrow \quad a^2+b^2 \geq \frac{1}{2}$$

Also, $\left(\frac{(a)^{-2}+(b)^{-2}}{2}\right) \geq \left(\frac{a+b}{2}\right)^{-2}$

$$\Rightarrow \quad \left(\frac{1}{a^2}+\frac{1}{b^2}\right) \geq 8 \qquad (\because \text{ given } a+b=1)$$

Thus,

$$(a^2+b^2) + \left(\frac{1}{a^2}+\frac{1}{b^2}\right) + 4 \geq \frac{1}{2} + 8 + 4 = \frac{25}{2}$$

Hence, $\left(a+\frac{1}{a}\right)^2 + \left(b+\frac{1}{b}\right)^2 \geq \frac{25}{2}$

179. We have $(ax + by)^2 - (a^2 + b^2)(x^2 + y^2)$

$= a^2x^2 + b^2y^2 + 2abxy - a^2x^2 - a^2y^2 - b^2x^2 - b^2y^2$

$= 2abxy - a^2y^2 - b^2x^2$

$= -(a^2y^2 + b^2x^2 - 2abxy)$

$= -(ay - bx)^2 < 0$

Thus, $(ax + by)^2 < (a^2 + b^2)(x^2 + y^2) = 16$

$\Rightarrow \quad (ax + by) < 4$

Hence, the greatest value of $(ax + by)$ is 4.

180. We know that

$$\left(a+\frac{1}{a}\right)^2 + \left(b+\frac{1}{b}\right)^2 + \left(c+\frac{1}{c}\right)^2$$

$$(a^2+b^2+c^2) + \left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) + 6$$

and $\left(\frac{a^2+b^2+c^2}{3}\right) \geq \left(\frac{a+b+c}{3}\right)^2$

$$\Rightarrow \quad \left(\frac{a^2+b^2+c^2}{3}\right) \geq \frac{1}{9} \qquad \text{(given)}$$

$$\Rightarrow \quad (a^2+b^2+c^2) \geq \frac{1}{3}$$

Also, $\left(\frac{(a)^{-2}+(b)^{-2}+(c)^{-2}}{3}\right) \geq \left(\frac{a+b+c}{3}\right)^{-2}$

$$\Rightarrow \quad \left(\frac{(a)^{-2}+(b)^{-2}+(c)^{-2}}{3}\right) \geq 9$$

$$\Rightarrow \quad \left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) \geq 27$$

Thus,

$$(a^2+b^2+c^2) + \left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) + 6 \geq \frac{1}{3} + 27 + 6$$

$$\left(a+\frac{1}{a}\right)^2 + \left(b+\frac{1}{b}\right)^2 + \left(c+\frac{1}{c}\right)^2 \geq \frac{100}{3}$$

Hence, the least value is $\frac{100}{3}$.

181. It is given that

$a + 2b + 3c = 12$

Applying, AM ≥ GM, we have

$$\frac{a+b+b+c+c+c}{6} \geq \sqrt[6]{ab^2c^3}$$

$$\Rightarrow \quad \frac{a+2b+3c}{6} \geq \sqrt[6]{ab^2\text{c}^3}$$

$$\Rightarrow \quad \frac{12}{6} \geq \sqrt[6]{ab^2c^3}$$

$\Rightarrow \quad (ab^2c^3) \leq 2^6 = 64$

Hence, the maximum value is 64.

182. It is given that $4a + 3b + 2c = 45$

Applying AM ≥ GM, we have

$$\left(\frac{2a+2a+b+b+b+\frac{c}{2}+\frac{c}{2}+\frac{c}{2}+\frac{c}{2}}{9}\right) \geq \sqrt[9]{\frac{a^2b^3c^4}{4}}$$

$$\Rightarrow \quad \left(\frac{4a+3b+2c}{9}\right) \geq \sqrt[9]{\frac{a^2b^3c^4}{4}}$$

$\Rightarrow \quad \left(\frac{45}{9}\right) \geq \sqrt[9]{\frac{a^2b^3c^4}{4}}$

$\Rightarrow \quad (a^2b^3c^4) \leq 4(5)^9$

Hence, the greatest value of $(a^2b^3c^4)$ is $4(5)^9$.

183. Given $2x + 3y = 10$

Now $2x + 3y = x + x + y + y + y$

Applying AM ≥ GM, we have

$$\left(\frac{x+x+y+y+y}{5}\right) \geq \sqrt[5]{x \cdot x \cdot y \cdot y \cdot y}$$

$\Rightarrow \quad \left(\frac{2x+3y}{5}\right) \geq \sqrt[5]{x^2y^3}$

$\Rightarrow \quad \left(\frac{10}{5}\right) \geq \sqrt[5]{x^2y^3}$

$\Rightarrow \quad (2)^5 \geq x^2y^3$

$\Rightarrow \quad x^2y^3 \leq 2^5 = 32$

Hence, the maximum value of x^2y^3 is 32.

184. It is given that $x + 5y = 18$.

Applying AM ≥ GM, we have

$\Rightarrow \quad \left(\frac{x+y+y+y+y+y}{6}\right) \geq \sqrt[6]{xy^5}$

$\Rightarrow \quad \left(\frac{x+5y}{6}\right) \geq \sqrt[6]{xy^5}$

$\Rightarrow \quad \left(\frac{18}{6}\right) \geq \sqrt[6]{xy^5}$

$\Rightarrow \quad (xy^5) \leq 3^6 = 729$

Hence, the maximum value of (xy^5) is 729.

185. It is given that a, b and c are in AP.

$\Rightarrow \quad 2b = a + c$

Applying, AM ≥ GM

$\Rightarrow \quad \left(\frac{a+b+c}{3}\right) \geq \sqrt[3]{abc}$

$\Rightarrow \quad \left(\frac{3b}{3}\right) \geq \sqrt[3]{abc}$

$\Rightarrow \quad b \geq \sqrt[3]{64} = 4$

$\Rightarrow \quad b \geq 4$

Hence, the minimum value of b is 4.

186. We have

$$f(x) = (x^2+2) + \frac{1}{(x^2+1)}$$

$$= (x^2+1) + \frac{1}{(x^2+1)} + 1$$

$$\geq 2 + 1 = 3$$

Hence, the minimum value of $f(x) = x^2 + 2 + \frac{1}{x^2+1}$ is 3.

187. Clearly, the minimum value of $f(x) = x^{10} + \frac{10}{x}, x > 0$ is 11.

188. Clearly, the min. value of $f(x) = x^{2012} + \frac{2012}{x}, x > 0$ is 2013.

189. We have

$$\left(a^9 + \frac{1}{a^9}\right) + \left(a^7 + \frac{1}{a^7}\right) + \left(a^5 + \frac{1}{a^5}\right) + \left(a^3 + \frac{1}{a^3}\right) + 1$$

$$\geq 2 + 2 + 2 + 2 + 1 = 9$$

Hence, the minimum value is 9.

190. We have

$$2^{x^2+2x+2} + 2^{2-2x-x^2} = 4.2^{x^2+2x} + \frac{4}{2^{x^2+2x}}$$

$$= 4 \cdot \left(2^{x^2+2x} + \frac{1}{2^{x^2+2x}}\right)$$

$$\geq 4 \cdot 2 = 8$$

Hence, the minimum value is 8.

191. We have

$$(2^x + 2^{-x}) + (3^x + 3^{-x}) + (4^x + 4^{-x}) + (5^x + 5^{-x}) + 10$$

$$\geq 2 + 2 + 2 + 2 + 10 = 18$$

Hence, the minimum value is 18.

192. It is given that,

$$a + 2b + 3c + 4d = 10$$

We know that

$$\frac{(a+2b)+(3c+4d)}{2} \geq \sqrt{(a+2b)(3c+4d)}$$

$\Rightarrow \quad \frac{10}{2} \geq \sqrt{(a+2b)(3c+4d)}$

$$(a + 2b)(3c + 4d) \leq 25$$

Thus, $M = 25$

Also, $\frac{(a+c+2d)+2(b+c+d)}{2}$

$$\geq \sqrt{2(a+c+2d)(b+c+2d)}$$

$$\frac{10}{2} \geq \sqrt{2(a+c+2d)(b+c+2d)}$$

$$(a+c+2d)(b+c+2d) \leq \frac{25}{2}$$

Thus, $N = \frac{25}{2}$

Therefore, $M + 2N + 10$

$= 25 + 25 + 10 = 60$

193. We know that

$$\left(\frac{a^{-2}+b^{-2}+c^{-2}}{3}\right) \geq \left(\frac{a+b+c}{3}\right)^{-2}$$

$$\Rightarrow \left(\frac{a^{-2}+b^{-2}+c^{-2}}{3}\right) \geq 9 \quad \text{(given)}$$

$$\Rightarrow \left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) \geq 27$$

Hence, the minimum value is 27.

194. We know that

$$\left(\frac{x^3+y^3+z^3}{3}\right) \geq \left(\frac{x+y+z}{3}\right)^3$$

$$\Rightarrow \left(\frac{x^3+y^3+z^3}{3}\right) \geq \frac{1}{27}$$

$$\Rightarrow (x^3+y^3+z^3) \geq \frac{1}{9}$$

Also, $\left(\frac{x^6+y^6+z^6}{3}\right) \geq \left(\frac{x+y+z}{3}\right)^6$

$$\Rightarrow \left(\frac{x^6+y^6+z^6}{3}\right) \geq \frac{1}{729}$$

$$\Rightarrow (x^6+y^6+z^6) \geq \frac{1}{243}$$

Thus, the least value of $(x^3 + y^3 + z^3)(x^6 + y^6 + z^6)$ is $\frac{1}{2187}$.

195. Applying AM ≥ GM, we have

$$\left(\frac{1+a+a^2}{3}\right) \geq \sqrt[3]{a^3} = a$$

$\Rightarrow (1 + a + a^2) \geq 3a$

Similarly, $(1 + b + b^2) \geq 3b$

and $(1 + c + c^2) \geq 3c$

Thus, $(1 + a + a^2)(1 + b + b^2)(1 + c + c^2) \geq 27abc$

$\Rightarrow (1 + a + a^2)(1 + b + b^2)(1 + c + c^2) \geq 27 \times 8$

$\Rightarrow (1 + a + a^2)(1 + b + b^2)(1 + c + c^2) \geq 216$

Hence, the least value is 216.

196. We know that,

$$\left(\frac{1+x^3}{2}\right) \geq \sqrt{x^3}$$

$$\Rightarrow (1+x^3) \geq 2\sqrt{x^3}$$

Similarly, $(1+y^3) \geq 2\sqrt{y^3}$

$$(1+z^3) \geq 2\sqrt{z^3}$$

Thus, $(1+x^3)(1+y^3)(1+z^3) \geq 2\sqrt{(xyz)^3}$

$$\Rightarrow (1+x^3)(1+y^3)(1+z^3) \geq 2\sqrt{8} = 4\sqrt{2}$$

Hence, the minimum value is $4\sqrt{2}$.

197. We know that,

$$\left(1+\frac{a}{2}+\frac{a}{2}\right) \geq 3 \times \sqrt[3]{1 \cdot \frac{a}{2} \cdot \frac{a}{2}}$$

$$\Rightarrow (1+a)^3 \geq 3^3\left(\frac{a^2}{4}\right)$$

Similarly, $(1+b)^3 \geq 3^3\left(\frac{b^2}{4}\right)$

On multiplication, we get

$$\{(1+a)(1+b)\}^3 \geq \frac{3^6}{16}(a^2b^2)$$

198. We know that,

$(1 + a)(1 + b)(1 + c)$

$= 1 + (a + b + c) + (ab + bc + ca) + abc$

Now, applying AM ≥ GM

$$\Rightarrow \frac{(a+b+c)+(ab+bc+ca)+abc}{7} \geq (a^4b^4c^4)^{1/7}$$

$$\Rightarrow (a + b + c) + (ab + bc + ca) + abc \geq 7(a^4b^4c^4)^{1/7}$$

Thus, $1 + (a + b + c) + (ab + bc + ca) + abc$

$> (a + b + c) + (ab + bc + ca) + abc$

$\geq 7(a^4b^4c^4)^{1/7}$

Hence, the result.

199. We know that,

$$\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca} = \frac{a+b+c}{abc} = \frac{1}{abc}$$

$$\frac{a+b+c}{3} \geq \sqrt[3]{abc}$$

$$\Rightarrow \frac{1}{3} \geq \sqrt[3]{abc}$$

$$\Rightarrow \frac{1}{27} \geq abc$$

$$\Rightarrow abc \leq \frac{1}{27}$$

$$\Rightarrow \frac{1}{abc} \geq 27$$

Hence, $\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca} \geq 27$.

200. As we know, by Tchebycheff's Inequality

$$\left(\frac{a+b+c+d}{4}\right)^2 \leq \frac{a^2+b^2+c^2+d^2}{4}$$

$$\Rightarrow \left(\frac{8-e}{4}\right)^2 \leq \frac{16-e^2}{4}$$

$(\because a^2 + b^2 + c^2 + d^2 + e^2 = 16)$

$\Rightarrow \quad 64 + e^2 - 16e \le 4(16 - e^2)$

$\Rightarrow \quad 5e^2 - 16e \le 0$

$\Rightarrow \quad e(5e - 16) \le 0$

$\therefore \quad 0 \le e \le \frac{5}{16}$

Thus, the range of e is $\left[0, \frac{5}{16}\right]$.

201. As we know that,

$$\text{AM} \ge \text{GM}$$

$$\Rightarrow \quad \frac{ab + bc + ca}{3} \ge \sqrt[3]{(ab)(bc)(ca)}$$

$$\Rightarrow \quad \frac{8}{3} \ge \sqrt[3]{(ab)(bc)(ca)}$$

$$\Rightarrow \quad \left(\frac{8}{3}\right)^3 \ge (abc)^2$$

$$\Rightarrow \quad (abc)^2 \le \left(\frac{8}{3}\right)^3$$

$$\Rightarrow \quad (abc) \le \left(\frac{8}{3}\right)^{3/2}$$

Hence, the maximum value of abc is $\left(\frac{8}{3}\right)^{3/2}$.

202. Given equation is

$$ax^4 + bx^3 + cx^2 + dx + 6 = 0$$

It is given that α, β, γ and δ be its roots.

Thus, $\alpha + \beta + \gamma + \delta = -b$

$$\sum \alpha\beta = c$$

$$\sum \alpha\beta\gamma = -d$$

and $\quad \sum \alpha\beta\gamma\delta = 6$

Applying AM ≥ GM, we have,

$$\left(\frac{\alpha + \beta + \gamma + \delta}{4}\right)\left(\frac{\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta}{4}\right)$$

$$\ge \sqrt[4]{\alpha\beta\gamma\delta}\,\sqrt[4]{\alpha^3\beta^3\gamma^3\delta^3} = \alpha\beta\gamma\delta = 6$$

$$\Rightarrow \quad \left(-\frac{b}{4}\right)\left(-\frac{d}{4}\right) \ge 6$$

$$\Rightarrow \quad bd \ge 96$$

Hence, the minimum value of bd is 96.

203. As we know that AM ≥ GM,

where $a + 2b + c = a + b + b + c$

Thus, $\left(\frac{(a+b) + (b+c)}{2}\right) \ge \sqrt{(a+b)(b+c)}$

$$\Rightarrow \quad (a+b)(b+c) \le \left(\frac{4}{2}\right)^2 = 4$$

$$\Rightarrow \quad (ab + bc + ca + b^2) \le 4$$

$$\Rightarrow \quad (ab + bc + ca) \le 4 - b^2$$

Hence the maximum value of $(ab + bc + ca)$ is 4.

204. As we know that AM ≥ GM

Thus, $\left(\frac{a+b+c+d}{2}\right) \ge \sqrt{(a+b)(c+d)}$

Hence, $(a+b)(c+d) \le \left(\frac{2}{2}\right) = 1$

Thus the maximum value of $(a + b)(c + d)$ is 1.

205. $\left(\frac{1}{a} - 1\right) = \left(\frac{1-a}{a}\right)$

$$= \left(\frac{a+b+c-a}{a}\right)$$

$$= \left(\frac{b+c}{a}\right)$$

Similarly $\frac{1}{b} - 1 = \frac{c+a}{b}$ and $\frac{1}{c} - 1, \frac{a+b}{c}$

multiplying, we get

$$\left(\frac{1}{a} - 1\right)\left(\frac{1}{b} - 1\right)\left(\frac{1}{c} - 1\right) =$$

$$\left(\frac{b+c}{a}\right)\left(\frac{c+a}{b}\right)\left(\frac{a+b}{c}\right) \ge \left(\frac{8abc}{abc}\right) = 8$$

Hence the minimum value is 8.

206. As we know that AM ≥ HM

Thus, $\left(\frac{(b+c) + (c+a) + (a+b)}{3}\right)$

$$\ge \left(\frac{3}{\frac{1}{(b+c)} + \frac{1}{(c+a)} + \frac{1}{(a+b)}}\right)$$

$$\Rightarrow \quad \frac{2}{3}(a+b+c)\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right) \ge 3$$

$$\Rightarrow \quad (a+b+c)\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right) \ge \frac{9}{2}$$

$$\Rightarrow \quad \left(\frac{a+b+c}{b+c} + \frac{a+b+c}{c+a} + \frac{a+b+c}{a+b}\right) \ge \frac{9}{2}$$

$$\Rightarrow \quad \left(\frac{a}{b+c} + 1 + \frac{\mathrm{b}}{c+a} + 1 + \frac{\mathrm{c}}{a+\mathrm{b}} + 1\right) \ge \frac{9}{2}$$

$$\Rightarrow \quad \left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b}\right) \ge \frac{9}{2} - 3 = \frac{3}{2}$$

Hence, the result.

207. $(1 + x)(1 + y) - 2(xy + 1)$

$= 1 + x + y + xy - 2xy - 2$

$= -1 + x + y - xy$

$= -(x - 1)(y - 1)$

$=$ –ve, since $x > 1$, $y > 1$

Thus, $(1 + x)(1 + y) < 2(xy + 1)$
Similarly, $(1 + z)(1 + w) < 2(zw + 1)$
Therefore,

$$(1 + x)(1 + y)(1 + z)(1 + w) < 4(xy + 1)(zw + 1)$$

208. It can be easily proved that

$$(1 + x)(1 + y)(1 + z)(1 + w) < 4(xy + 1)(zw + 1)$$

Obviously, $(1 + xy)(1 + zw) < 2(xyzw + 1)$
Hence, $(1 + x)(1 + y)(1 + z)(1 + w) < 8(xyzw + 1)$

209. As we know that AM ≥ GM

or $\frac{a^4+b^4}{2} \geq \sqrt{a^4b^4}$

$\Rightarrow \quad (a^4 + b^4) \geq 2a^2b^2$

$\Rightarrow \quad (a^4 + b^4 + c^2) \geq 2a^2b^2 + c^2$

$\Rightarrow \quad (a^4+b^4+c^2) \geq 2a^2b^2+c^2 \geq 2\sqrt{2a^2b^2c^2}$

$\Rightarrow \quad (a^4+b^4+c^2) \geq 2\sqrt{2}\, abc$

Hence, the result.

210. Since,

$$\frac{1}{2}\left(\frac{ab}{c}+\frac{bc}{a}\right) \geq \sqrt{\frac{ab}{c}\times\frac{bc}{a}} = b \quad \ldots\text{(i)}$$

$$\frac{1}{2}\left(\frac{bc}{a}+\frac{ca}{b}\right) \geq \sqrt{\frac{bc}{a}\times\frac{ca}{b}} = c \quad \ldots\text{(ii)}$$

$$\frac{1}{2}\left(\frac{ca}{b}+\frac{ab}{c}\right) \geq \sqrt{\frac{ca}{b}\times\frac{ab}{c}} = a \quad \ldots\text{(iii)}$$

Adding relations (i), (ii) and (iii), we get

$$\frac{ab}{c}+\frac{bc}{a}+\frac{ca}{b} \geq a+b+c$$

Hence, the result.

211. By the *m*th power theorem,

$$\left(\frac{b^2+c^2}{2}\right) \geq \left(\frac{b+c}{2}\right)^2$$

$$\Rightarrow \quad \left(\frac{b^2+c^2}{b+c}\right) \geq \left(\frac{b+c}{2}\right) \quad \ldots\text{(i)}$$

similarly, $\left(\frac{c^2+a^2}{c+a}\right) \geq \left(\frac{c+a}{2}\right) \quad \ldots\text{(ii)}$

and $\left(\frac{a^2+b^2}{a+b}\right) \geq \left(\frac{a+b}{2}\right) \quad \ldots\text{(iii)}$

Adding relations (i), (ii) and (iii), we get

$$\left(\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}+\frac{a^2+b^2}{a+b}\right) \geq (a+b+c)$$

Hence, the result.

212. By the *m*th power theorem, we get

$$\left(\frac{x^{-2}+y^{-2}+z^{-2}}{3}\right) \geq \left(\frac{x^2+y^2+z^2}{3}\right)^{-1}$$

$$\Rightarrow \quad \left(\frac{x^{-2}+y^{-2}+z^{-2}}{3}\right) \geq \left(\frac{3}{x^2+y^2+z^2}\right)$$

$$\Rightarrow \quad x^{-2}+y^{-2}+z^{-2} \geq \left(\frac{9}{1}\right) = 9$$

$$\Rightarrow \quad \frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2} \geq 9$$

213. $a^{-5} + a^{-4} + 3a^{-3} + 1 + a^8 + a^{10}$

$= a^{-5} + a^{-4} + a^{-3} + a^{-3} + a^{-3} + 1 + a^8 + a^{10}$

$= \left(\frac{a^{-5}+a^{-4}+a^{-3}+a^{-3}+a^{-3}+1+a^8+a^{10}}{9}\right)$

$\geq \sqrt[9]{a^{-5}\cdot a^{-4}\cdot a^{-3}\cdot a^{-3}\cdot a^{-3}\cdot a^8\cdot a^{10}} = 1$

$$\Rightarrow \quad \left(\frac{a^{-5}+a^{-4}+a^{-3}+a^{-3}+a^{-3}+1+a^8+a^{10}}{9}\right) > 1$$

$\Rightarrow \quad (a^{-5} + a^{-4} + a^{-3} + a^{-3} + a^{-3} + 1 + a^8 + a^{10}) > 1$

Hence, the minimum value is 1.

Level III

1. Let $t_r = \left(r+\frac{1}{\omega}\right)\left(r+\frac{1}{\omega^2}\right)$

$= \left(r^2+\left(\frac{1}{\omega}+\frac{1}{\omega^2}\right)r+\frac{1}{\omega^3}\right)$

$= \left(r^2+(\omega^2+\omega)r+\frac{1}{\omega^3}\right)$

$= (r^2 - r + 1)$

Now, $S_n = \sum_{r=1}^{n} t_r$

$= \sum_{r=1}^{n}(r^2-r+1)$

$= \sum_{r=1}^{n} r^2 - \sum_{r=1}^{n} \mathrm{r} + \sum_{r=1}^{n} 1$

$= \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + n$

$= n\left(\frac{(n+1)(2n+1)}{6} - \frac{(n+1)}{2} + 1\right)$

$= \frac{1}{6}(n(2n^2+3n+1-3n-3+6))$

$= \left(\frac{n(n^2+2)}{3}\right)$

Hence, the result.

2. Let $t_r = (r - 1)(r - \omega)(r - \omega^2)$

$= r^3 - (1 + \omega + \omega^2)r^2 + (1\cdot\omega + \omega\cdot\omega^2 + \omega^2\cdot 1)r - (1\cdot\omega\cdot\omega^2)$

$= r^3 - 0\cdot r^2 + 0\cdot r - 1$

$= r^3 - 1$

Now, $S_n = \sum_{r=1}^{n} t_r$

$$= \sum_{r=1}^{n} (r^3 - 1)$$

$$= \sum_{r=1}^{n} r^3 - \sum_{r=1}^{n} 1$$

$$= \left(\frac{n(n+1)}{2}\right)^2 - n$$

Hence, the result.

3. Let $t_r = (n - r + 1).r$
$= nr - r^2 + r$
$= (n + 1)r - r^2$

Now, $S_n = \sum_{r=1}^{n} t_r$

$$= \sum_{r=1}^{n} ((n+1)r - r^2)$$

$$= (n+1)\sum_{r=1}^{n} r - \sum_{r=1}^{n} r^2$$

$$= (n+1)\left(\frac{n(n+1)}{2}\right) - \left(\frac{n(n+1)(2n+1)}{6}\right)$$

$$= \frac{n(n+1)}{2}\left((n+1) - \frac{(2n+1)}{3}\right)$$

$$= \frac{n(n+1)}{6}[3(n+1) - (2n+1)]$$

$$= \frac{n(n+1)}{6}(3n + 3 - 2n - 1)$$

$$= \frac{n(n+1)(n+2)}{6}$$

Hence, the result.

4. Let $b = ar$, $c = ar^2$, where r is the common ratio. It is given that
$a + b + c = xb$
$\Rightarrow$ $(a + ar + ar^2) = ar.x$
$\Rightarrow$ $(1 + r + r^2) = r.x$
$\Rightarrow$ $r^2 + (1 - x)r + 1 = 0$
Since r is real, so $D \geq 0$
$\Rightarrow$ $(1 - x)^2 - 4 \geq 0$
$\Rightarrow$ $(x - 1)^2 - 4 \geq 0$
$\Rightarrow$ $(x - 1)^2 - 2^2 \geq 0$
$\Rightarrow$ $(x - 1 + 2)(x - 2) \geq 0$
$\Rightarrow$ $(x + 1)(x - 3) \geq 0$
$\Rightarrow$ $x \leq -1$ or $x \geq 3$
Hence, the result.

5. Given a, b, c are in GP.
$\therefore$ $b^2 = ac$
Also, $ax^2 + 2bx + c = 0$
$\Rightarrow$ $ax^2 + 2\sqrt{ac}\,x + c = 0$
$\Rightarrow$ $(\sqrt{a}x + \sqrt{c})^2 = 0$
$\Rightarrow$ $(\sqrt{a}x + \sqrt{c}) = 0$
$\Rightarrow$ $x = -\frac{\sqrt{c}}{\sqrt{a}}$

Since both the given equations have a common root, so,

$$d\left(\frac{c}{a}\right) - 2e\left(\sqrt{\frac{c}{a}}\right) + f = 0$$

$$\Rightarrow \quad d\left(\frac{c}{a}\right) - 2e\left(\sqrt{\frac{c^2}{ac}}\right) + f = 0$$

$$\Rightarrow \quad d\left(\frac{c}{a}\right) - 2e\frac{c}{b} + f = 0 \qquad \text{(given)}$$

$$\Rightarrow \quad \frac{d}{a} - 2\frac{e}{b} + \frac{f}{c} = 0$$

$$\Rightarrow \quad \frac{d}{a} + \frac{f}{c} = 2\frac{e}{b}$$

$\therefore$ $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in AP

Hence, the result.

6. Given $a_1, a_2, a_3, a_4, \ldots, a_n$ are in HP.

$\therefore$ $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \ldots, \frac{1}{a_n}$ are in AP.

$$\Rightarrow \quad \frac{1}{a_2} - \frac{1}{a_1} = \frac{1}{a_3} - \frac{1}{a_2} = \ldots = \frac{1}{a_{n-1}} - \frac{1}{a_n} = d$$

Now, $\frac{1}{a_2} - \frac{1}{a_1} = d \Rightarrow \frac{a_1 - a_2}{a_1 a_2} = d$

$$\Rightarrow \quad a_1 a_2 = \frac{a_1 - a_2}{d}$$

Similarly, $a_2 a_3 = \frac{a_3 - a_2}{d}$

$$a_3 a_4 = \frac{a_4 - a_3}{d}$$

$$\vdots$$

$$a_{n-1} a_n = \frac{a_n - a_{n-1}}{d}$$

Thus, $a_1 a_2 + a_2 a_3 + a_3 a_4 + \ldots + a_{n-1}\, a_n$

$$= \left(\frac{a_1 - a_2}{d} + \frac{a_2 - a_3}{d} + \ldots + \frac{a_{n-1} - a_n}{d}\right)$$

$$= \frac{1}{\mathrm{d}}(a_1 - a_2 + a_2 - a_3 + \ldots + a_{n-1} - a_n)$$

$$= \frac{1}{d}(a_1 - a_n)$$

$$= (n - 1)a_1 a_n$$

Alternate method

Since $a_1, a_2, a_3, \ldots, a_n$ are in HP, so $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \cdots, \frac{1}{a_n}$ will be in AP.

It is clear that

$$\frac{1}{a_n} = \frac{1}{a_1} + (n-1)d$$

$$\Rightarrow \quad \frac{1}{a_n} - \frac{1}{a_1} = (n-1)d$$

$$\Rightarrow \quad \frac{a_1 - a_n}{a_1 a_n} = (n-1)d \Rightarrow \frac{a_1 - a_n}{d} = (n-1)a_1 a_n$$

Hence, the result.

7. We have,

$\sin^2 x + \sin^4 x + \sin^6 x + \ldots$

The series is in GP and of the form

$\therefore\ a + a^2 + a^3 + a^4 + \ldots + \infty$ or $a(1 + a + a^2 + a^3 + \ldots \infty)$.

$$S_n = \frac{1}{1-r}$$

Here $a = 1$ and $r = \sin^2 x$

$$\therefore \quad S_n = \sin^2\left(\frac{1}{1-\sin^2 x}\right)$$

$$= \frac{\sin^2 x}{1-\sin^2 x}$$

$$= \tan^2 x$$

Now, $e^{\tan^2 x \log_e 2} = 2^{\tan^2 x}$

Now, $x^2 - 9x + 8 = 0$

$\Rightarrow \quad (x-1)(x-8) = 0$

$\Rightarrow \quad x = 1$ or 8

When $\dfrac{x = 1 = 2^0}{2^{\tan^2 x} = 2^{\tan^2 1} = 1 = z^0}$

$\Rightarrow \quad \tan^2 x = 0$

$\Rightarrow \quad \tan x = 0$

When $\dfrac{x = 8 = 2^3}{2^{\tan^2 x} = 8 = 2^3}$

$\Rightarrow \quad \tan^2 x = 3$

$\Rightarrow \quad \tan x = \sqrt{3}$

Thus, $\dfrac{\cos x}{\cos x + \sin x}$

$$= \frac{1}{1+\tan x} \qquad \text{(dividing by } \cos x\text{)}$$

$$= \frac{1}{1+\sqrt{3}}$$

$$= \left(\frac{\sqrt{3}-1}{2}\right)$$

Hence, the result.

8. From the property of a triangles we have,

$a < b + c$

$\Rightarrow \quad a^2 < a(b + c)$

Similarly, $b^2 < b(c + a)$

and $\quad c^2 < c(a + b)$

Adding, we get $a^2 + b^2 + c^2 < 2(ab + bc + ca)$

$$\Rightarrow \quad \left(\frac{(ab+bc+ca)}{a^2+b^2+c^2}\right) < \frac{1}{2} \qquad \text{(i)}$$

Also, $\left(\dfrac{a^2+b^2}{2}\right) > \sqrt{a^2 \cdot b^2} = ab$

$\Rightarrow \quad (a^2 + b^2) > 2ab$

similarly, $(b^2 + c^2) > 2bc$

and $\quad (c^2 + a^2) > 2ca$

Therefore, $2(a^2 + b^2 + c^2) > 2(ab + bc + ca)$

$$\Rightarrow \quad \left(\frac{(ab+bc+ca)}{(a^2+b^2+c^2)}\right) < 1 \qquad \text{(ii)}$$

From Eqs(i) and (ii), we have

$$\frac{1}{2} < \left(\frac{(ab+bc+ca)}{(a^2+b^2+c^2)}\right) < 1$$

Hence, the result.

9. By the mth power theorem, we get

$$\left(\frac{b^2+c^2}{2}\right) \geq \left(\frac{b+c}{2}\right)^2$$

$$\Rightarrow \quad \left(\frac{b^2+c^2}{b+c}\right) \geq \left(\frac{b+c}{2}\right)$$

$$\Rightarrow \quad \frac{b+c}{b^2+c^2} \leq \frac{2}{b+c} \qquad \ldots\text{(i)}$$

Similarly, $\dfrac{c+a}{c^2+a^2} \leq \dfrac{2}{c+a} \qquad \ldots$(ii)

and $\quad \dfrac{a+b}{a^2+b^2} \leq \dfrac{2}{a+b} \qquad \ldots$(iii)

Adding Eqs (i), (ii) and (iii), we get

$$\frac{b+c}{b^2+c^2} + \frac{c+a}{c^2+a^2} + \frac{a+b}{a^2+b^2}$$

$$\leq \frac{2}{b+c} + \frac{2}{c+a} + \frac{2}{a+b} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

Hence, the result.

10. From the given polynomial, $m_1 \cdot m_2 \cdot m_3 \cdot m_4 = \dfrac{5}{4} \qquad$ (i)

Consider the terms, $\dfrac{m_1}{2}, \dfrac{m_2}{3}, \dfrac{m_3}{5}, \dfrac{m_4}{8}$

$$\text{AM} = \frac{1}{4}\left(\frac{m_1}{2} + \frac{m_2}{3} + \frac{m_3}{5} + \frac{m_4}{8}\right) = \frac{1}{4} \times 1 = \frac{1}{4}$$

$$\text{GM} = \left(\frac{m_1}{2} \cdot \frac{m_2}{3} \cdot \frac{m_3}{5} \cdot \frac{m_4}{8}\right)^{1/4}$$

$$= \left(\frac{5}{4(2.4.5.8)}\right)^{1/4} = \left(\frac{1}{2^8}\right)^{1/4} = \frac{1}{4}$$

$\therefore$ AM = GM

Therefore, all numbers are equal.

$$\Rightarrow \frac{m_1}{2} = \frac{m_2}{4} = \frac{m_3}{5} = \frac{m_4}{8} = k$$

$$\Rightarrow \prod_{i=1}^{4} m_i = (2.4.5.8)k^4$$

$$\Rightarrow \frac{5}{4} = 5 \cdot 2^6 \cdot k^4 \quad \text{(from (i))}$$

$$\Rightarrow k^4 = \left(\frac{1}{2^8}\right)$$

$$\Rightarrow k = \left(\frac{1}{2^2}\right) = \frac{1}{4}$$

Thus, $m_1 = \frac{1}{2}, m_2 = 1, m_3 = \frac{5}{4}, m_4 = 2$

Again, Sum of the roots = $\frac{a}{4}$

$$\Rightarrow \frac{m_1 + m_2 + m_3 + m_4}{4} = \frac{a}{4}$$

$$\Rightarrow \frac{a}{4} = \frac{1}{2} + 1 + \frac{5}{4} + 2 = \frac{19}{4}$$

$\Rightarrow$ $a = 19$

$\therefore$ $a + 1995 = 19 + 1995$
$= 2014$

11. We have,

$$\left(p + \frac{1}{p}\right)^2 + \left(q + \frac{1}{q}\right)^2$$

$$= \left(p^2 + \frac{1}{p^2} + 2\right) + \left(q^2 + \frac{1}{q^2} + 2\right)$$

$$= \left(p^2 + q^2 + \frac{1}{p^2} + \frac{1}{q^2} + 4\right) \quad \text{(i)}$$

By mth power theorem, we have,

$$\left(\frac{p^2 + q^2}{2}\right) \geq \left(\frac{p + q}{2}\right)^2 = \left(\frac{2}{2}\right)^2 = 1 \quad \text{(given)}$$

$$\Rightarrow (p^2 + q^2) \geq 2 \quad \text{(ii)}$$

Also $\left(\frac{p^{-2} + q^{-2}}{2}\right) \geq \left(\frac{p + q}{2}\right)^{-2} = \left(\frac{2}{2}\right)^{-2} = 1$

$$\Rightarrow (p^{-2} + q^{-2}) \geq 2$$

$$\Rightarrow \left(\frac{1}{p^2} + \frac{1}{q^2}\right) \geq 2 \quad \text{(iii)}$$

Now putting the radius of relations (ii) and (iii) in Eq. (1), we get

$$\left(p + \frac{1}{p}\right)^2 + \left(q + \frac{1}{q}\right)^2 = \left(p^2 + q^2 + \frac{1}{p^2} + \frac{1}{q^2} + 4\right) \geq 8$$

$\geq 2 + 2 + 4 = 8$

12. We have,

$$(1 + x + 2x^2 + 3x^3 + \ldots + nx^n)^2$$
$$= (1 + x + 2x^2 + 3x^3 + \ldots + nx^n$$
$$\times (1 + x + 2x^2 + 3x^3 + \ldots + nx^n$$

Therefore, the co-efficient of x^n

$$= 1 \cdot n + 2 \cdot (n - 1) + 3 \cdot (n - 2) + \ldots + n \cdot 1$$
$$= \frac{n(n+1)(n+2)}{6}$$

13. We have,

$$\sum_{r=1}^{n} r^2 - \sum_{m=1}^{n}\sum_{r=1}^{m} r = \sum_{r=1}^{n} r^2 - \sum_{m=1}^{n}\left(\frac{m^2 + m}{2}\right)$$

$$= \sum_{r=1}^{n} r^2 - \frac{1}{2}\left(\sum_{m=1}^{n} m^2 + \sum_{m=1}^{n} m\right)$$

$$= \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)(2n+1)}{12} - \frac{n(n+1)}{4}$$

$$= \frac{n(n+1)(2n+1)}{12} - \frac{n(n+1)}{4}$$

$$= \frac{n(n+1)}{4}\left(\frac{(2n+1)}{3} - 1\right)$$

$$= \frac{n(n+1)(n-2)}{6}$$

14. Let the first term be a and the common difference be d. We have,

$$\alpha = \sum_{r=1}^{2014} a_{2r}$$
$$= a_2 + a_4 + a_6 + \ldots + a_{2014}$$
$$= \frac{1007}{2}(a_2 + a_{2014})$$
$$= \frac{1007}{2}(a + d + a + 2013d)$$
$$= \frac{1007}{2}(2a + 2014d)$$
$$= 1007 \times (a + 1007d)$$

Also, $\beta = \sum_{r=1}^{2014} a_{2r-1}$

$$= a_1 + a_3 + a_5 + a_7 + \ldots + a_{2013}$$
$$= \frac{1007}{2}(a_1 + a_{2013})$$
$$= \frac{1007}{2}(a + a + 2012d)$$
$$= \frac{1007}{2}(2a + 2012d)$$
$$= 1007 \times (a + 1006d)$$

Now, $\alpha - \beta = 1007(a + 1007d - a - 1006d)$
$= 1007d$

$\Rightarrow \quad 1007d = \alpha - \beta$

$\Rightarrow \quad d = \left(\dfrac{\alpha - \beta}{1007}\right)$

Hence, the common difference is $\left(\dfrac{\alpha - \beta}{1007}\right)$

15. We have

$25(9x^2 + y^2) + 9z^2 - 15(5xy + yz + 3zx) = 0$

$\Rightarrow \quad (225x^2 + 25y^2 + 9z^2) - (75xy + 15yz + 45zx) = 0$

$\Rightarrow \quad [(15x)^2 + (5y)^2 + (3z)^2] - (15x \cdot 5y + 5y \cdot 3z + 3z \cdot 15x) = 0$

$\Rightarrow \quad \dfrac{1}{2}[(15x - 5y)^2 + (5y - 3z)^2 + (3z - 15x)^2] = 0$

$\Rightarrow \quad (15x - 5y) = 0 = (5y - 3z) = (3z - 15x)$

$\Rightarrow \quad 15x = 5y = 3z$

$\Rightarrow \quad x = \dfrac{y}{5} = \dfrac{z}{3}$

$\Rightarrow \quad x, y, z$ are in HP.

16. We have,

$(1 + 2 + 3 + \ldots + n)^2$

$= (1^2 + 2^2 + 3^2 + \ldots + n^2) + 2(1.2 + 1.3 + \ldots + 2.3 + 2.4 + \ldots)$

$\Rightarrow \quad 2(1.2 + 1.3 + \ldots + 2.3 + 2.4 + \ldots$

$= (1 + 2 + 3 + \ldots + n)^2 - (1^2 + 2^2 + 3^2 + \ldots + n^2)$

$= \left(\dfrac{n(n+1)}{2}\right)^2 - \left(\dfrac{n(n+1)(2n+1)}{6}\right)$

$= \left(\dfrac{n(n+1)}{2}\right)\left(\dfrac{n(n+1)}{2} - \dfrac{2n+1}{3}\right)$

$= \left(\dfrac{n(n+1)}{2}\right)\left(\dfrac{3n^2 + 3n - 4n - 2}{6}\right)$

$= \left(\dfrac{n(n+1)}{2}\right)\left(\dfrac{3n^2 - n - 2}{6}\right)$

$= \left(\dfrac{n(n+1)(n-1)(3n+2)}{12}\right)$

$\Rightarrow \quad 2(1.2 + 1.3 + \ldots + 2.3 + 2.4 + \ldots) = \left(\dfrac{n(n+1)(n-1)(3n+2)}{12}\right)$

$\Rightarrow \quad (1.2 + 1.3 + \ldots + 2.3 + 2.4 + \ldots) = \left(\dfrac{n(n+1)(n-1)(3n+2)}{24}\right)$

Hence, the required sum $= \left(\dfrac{n(n+1)(n-1)(3n+2)}{12}\right)$

17. We have,

$$\sum_{k=1}^{n}\left(\sum_{m=1}^{n} m^2\right) = \sum_{k=1}^{n}\left(\frac{m(m+1)(2m+1)}{6}\right)$$

$$= \sum_{k=1}^{n}\left(\frac{2m^3 + 3m^2 + m}{6}\right)$$

$$= \frac{1}{3}\sum_{k=1}^{n} m^3 + \frac{1}{2}\sum_{k=1}^{n} m^2 + \frac{1}{6}\sum_{k=1}^{n} m$$

$$= \frac{1}{3}\left(\frac{n(n+1)}{2}\right)^2 + \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6}\right) + \frac{1}{6}\left(\frac{n(n+1)}{2}\right)$$

$$= \left(\frac{n(n+1)}{12}\right)\left(\frac{n(n+1)}{2} + (2n+1) + 1\right)$$

$$= \left(\frac{n(n+1)}{12}\right)\left(\frac{n^2 + 5n + 4}{2}\right)$$

$$= \frac{1}{24}(n^4 + 6n^3 + 9n^2 + 4n)$$

Comparing the corresponding co-efficients of n^4, n^3, n^2, n and the constant term, we get

$$a = \frac{1}{24}, b = \frac{1}{4}, c = \frac{3}{8}, d = \frac{1}{6}, e = 0$$

18. Given $a_1, a_2, a_3, \ldots, a_{10}$ are in AP.

$\therefore a_{10} = a_1 + 9d$ where d is the common difference.

$\Rightarrow \quad 3 = 2 + 9d$

$\Rightarrow \quad d = \dfrac{1}{9}$

Thus, $a_4 = a_1 + 3d = 2 + \dfrac{3}{9} = 2 + \dfrac{1}{3} = \dfrac{7}{3}$

Also, $h_1, h_2, h_3, \ldots, h_{10}$ are in HP.

$\Rightarrow \quad \dfrac{1}{h_1}, \dfrac{1}{h_2}, \dfrac{1}{h_3}, \ldots, \dfrac{1}{h_{10}}$ are in AP.

$\therefore \quad \dfrac{1}{h_{10}} = \dfrac{1}{h_1} + 9D$ where D is the common difference.

$\Rightarrow \quad 9D = \dfrac{1}{3} - \dfrac{1}{2} = \dfrac{1}{6}$

$\Rightarrow \quad D = -\dfrac{1}{54}$

Thus, $\dfrac{1}{h_7} = \dfrac{1}{h_1} + 6D$

$\Rightarrow \quad \dfrac{1}{h_7} = \dfrac{1}{2} - \dfrac{1}{9} = \dfrac{7}{18} \qquad (\because a_1 = b_1 = 2)$

$\Rightarrow \quad h_7 = \dfrac{18}{7}$

Therefore,

$$a_4 h_7 + 2007 = \frac{7}{3} \times \frac{18}{7} + 2007 = 2013$$

19. Applying AM ≥ GM, we get

$$\left(\frac{x^3 + y^3 + z^3}{3}\right) \geq \left(\frac{x + y + z}{3}\right)^3 = \frac{1}{27}$$

$$\Rightarrow \quad \left(\frac{x^6 + y^6 + z^6}{3}\right) \geq \left(\frac{x + y + z}{3}\right)^6 = \frac{1}{729}$$

$$\Rightarrow \quad \left(\frac{x^3 + y^3 + z^3}{3}\right) \times \left(\frac{x^6 + y^6 + z^6}{3}\right) > \frac{1}{27} \times \frac{1}{729}$$

$\Rightarrow\quad (x^3+y^3+z^3)(x^6+y^6+z^6) > \dfrac{1}{3\times 729} = \dfrac{1}{2187}$

Hence, the least value is $\dfrac{1}{2187}$.

20. Applying AM ≥ GM, we have

$\Rightarrow\quad \dfrac{1+a+a^2}{3} \geq \sqrt[3]{1\cdot a\cdot a^2} = a$ …(i)

Similarly, $\dfrac{1+b+b^2}{3} \geq b$ …(ii)

and $\dfrac{1+c+c^2}{3} \geq c$ …(iii)

Multiplying Relations (i), (ii) and (iii), we get

$(1+a+a^2)(1+b+b^2)(1+c+c^2) \geq 27abc$
$= 27$

Hence, the minimum value is 27.

21. Applying AM ≥ GM, we get

$\left(\dfrac{1+x^3}{2}\right) \geq \sqrt{x^3} = (x)^{3/2}$ …(i)

Similarly, $\left(\dfrac{1+y^3}{2}\right) \geq (y)^{3/2}$ …(ii)

and $\left(\dfrac{1+z^3}{2}\right) \geq (z)^{3/2}$ …(iii)

Multiplying Relations (i), (ii) and (iii), we get

$\left(\dfrac{1+x^3}{2}\right)\left(\dfrac{1+y^3}{2}\right)\left(\dfrac{1+z^3}{2}\right) \geq (xyz)^{3/2}$

$\Rightarrow\quad (1+x^3)(1+y^3)(1+z^3) \geq 8(xyz)^{3/2}$

$\Rightarrow\quad (1+x^3)(1+y^3)(1+z^3) \geq 8\times(2)^{3/2} = 16\sqrt{2}$

Hence, the minimum value is $16\sqrt{2}$.

22. We have,

$(1+a)(1+b) = 1+a+b+ab$

Applying AM ≥ GM, we get

$\left(\dfrac{a+b+ab}{3}\right) \geq \sqrt[3]{a^2\cdot b^2}$

$\Rightarrow\quad (a+b+ab) \geq 3\cdot\sqrt[3]{a^2\cdot b^2}$

$\Rightarrow\quad 1+(a+b+ab) > (a+b+ab) \geq 3\cdot\sqrt[3]{a^2\cdot b^2}$

$\Rightarrow\quad (1+a)(1+b) \geq 3\cdot\sqrt[3]{a^2\cdot b^2}$

$\Rightarrow\quad ((1+a)(1+b))^3 \geq 3^3\cdot a^2b^2$

Hence, the result.

23. We have,

$\dfrac{1}{ab}+\dfrac{1}{bc}+\dfrac{1}{ca} = \dfrac{a+b+c}{abc} = \dfrac{1}{abc}$

We know that, $\dfrac{a+b+c}{3} \geq \sqrt[3]{abc}$

$\Rightarrow\quad \dfrac{1}{3} \geq \sqrt[3]{abc}$

$\Rightarrow\quad \dfrac{1}{27} \geq abc$

$\Rightarrow\quad abc \leq \dfrac{1}{27}$

$\Rightarrow\quad \dfrac{1}{abc} \geq 27$

Hence, the minimum value is 27.

24. Given $t_1 = 1$

Now, $t_2 - t_1 = 2 \Rightarrow t_2 = 3$
$t_3 - t_2 = 2^2 \Rightarrow t_3 = 7$
$t_4 - t_3 = 2^3 \Rightarrow t_4 = 15$

and so on

Now, $\sum_{r=1}^{n} t_r = t_1 + t_1 + t_3 + \ldots + t_n$
$= 1+3+7+15+\ldots$

Let $S_n = 1+3+7+15+\ldots+T_{n-1}+T_n$

$\Rightarrow\quad S_n = 1+3+7+15+\ldots+T_{n-1}+T_n$

Subtracting, we get

$0 = 1+2+4+8+\ldots+(T_n - T_{n-1}) - T_n$

$\Rightarrow\quad T_n = 1+2+4+8+\ldots+(T_n - T_{n-1})$

$\Rightarrow\quad T_n = 1+2+4+8+\ldots+2^{n-1}$

$\Rightarrow\quad T_n = \left(\dfrac{2^n-1}{2-1}\right)$

$\Rightarrow\quad T_n = (2^n - 1)$

Now, $S_n = \sum_{n=1}^{n} T_n = \sum_{n=1}^{n}(2^n - 1)$

$= \sum_{n=1}^{n} 2^n - \sum_{n-1}^{n} 1$

$= 2\left(\dfrac{2^n-1}{2-1}\right) - n$

$= (2^{n+1} - n - 2)$

25. We have,

$\dfrac{1}{1^2}+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\ldots\infty = \dfrac{\pi^2}{6}$

$\Rightarrow\quad \dfrac{1}{1^2}+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\dfrac{1}{4^2}+\dfrac{1}{5^2}+\dfrac{1}{6^2}+\ldots = \dfrac{\pi^2}{6}$

$\Rightarrow\quad \left(\dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\ldots\right)+\left(\dfrac{1}{2^2}+\dfrac{1}{4^2}+\dfrac{1}{6^2}+\ldots\right) = \dfrac{\pi^2}{6}$

$\Rightarrow\quad \left(\dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\ldots\right)+\dfrac{1}{2^2}\left(\dfrac{1}{1^2}+\dfrac{1}{2^2}+\dfrac{1}{3^2}+\ldots\right)$
$= \dfrac{\pi^2}{6}$

$\Rightarrow\quad \left(\dfrac{1}{1^2}+\dfrac{1}{3^2}+\dfrac{1}{5^2}+\ldots\right)+\dfrac{1}{2^2}\times\dfrac{\pi^2}{6} = \dfrac{\pi^2}{6}$

$$\Rightarrow \left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right)=\frac{\pi^2}{6}-\frac{1}{2^2}\times\frac{\pi^2}{6}=\frac{\pi^2}{8}$$

$$\Rightarrow \left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\right)=\frac{\pi^2}{8}$$

Hence, the result.

26. Let two quantities be a and b, respectively.

Thus, $a, A_1, A_2, b \in$ AP

Now, $b = a + 3d$ where d is the common difference.

$$\Rightarrow d = \frac{b-a}{3}$$

Now, $A_1 = a + d = a + \frac{b-a}{3} = \frac{2a+b}{3}$

and $A_2 = a + 2d = a + \frac{2(b-a)}{3} = \frac{a+2b}{3}$

So, $A_1 + A_2 = \frac{2a+b}{3} + \frac{a+2b}{3} = a + b$

Also, $a, G_1, G_2, b \in$ GP

Now, $b = ar^3 \Rightarrow r = \left(\frac{b}{a}\right)^{1/3}$ where r is the common ratio

Now, $G_1 = ar = a\left(\frac{b}{a}\right)^{1/3} = a^{2/3}b^{1/3}$

and $G_2 = ar^2 = a\left(\frac{b}{a}\right)^{2/3} = a^{1/3}b^{2/3}$

So, $G_1G_2 = ab$

Again, $a, H_1, H_2, b \in$ HP

$$\Rightarrow \frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{b} \in \text{AP}$$

Now, $\frac{1}{b} = \frac{1}{a} + 3d \Rightarrow d = \frac{1}{3}\left(\frac{1}{b}-\frac{1}{a}\right) = \frac{(a-b)}{3ab}$

$$\frac{1}{H_1} = \frac{1}{a} + d = \frac{1}{a} + \frac{(a-b)}{3ab} = \frac{3b+a-b}{3ab} = \frac{a+2b}{3ab}$$

$$\Rightarrow H_1 = \frac{3ab}{a+2b}$$

$$\frac{1}{H_2} = \frac{1}{H_1} + d = \frac{2b+a}{3ab} + \frac{a-\text{b}}{3ab} = \frac{2a+b}{3ab}$$

$$\Rightarrow H_2 = \frac{3ab}{2a+b}$$

Now,

$$H_1 + H_2 = \frac{3ab}{a+2b} + \frac{3ab}{2a+b} = \frac{9ab(a+b)}{(a+2b)(2a+b)}$$

Also, $H_1H_2 = \frac{3ab}{2a+b}\cdot\frac{3ab}{a+2b} = \frac{9a^2b^2}{(2a+b)(a+2b)}$

Now, $\frac{G_1G_2}{H_1H_2} = \dfrac{ab}{\dfrac{9a^2b^2}{(2a+b)(a+2b)}} = \frac{(2a+b)(a+2b)}{9ab}$

And, $\frac{A_1+A_2}{H_1+H_2} = \dfrac{(a+b)}{\dfrac{9ab(a+b)}{(2a+b)(a+2b)}} = \frac{(2a+b)(a+2b)}{9ab}$.

Hence, the result.

27. Let $a^x = b^y = c^z = d^u = k$

$\Rightarrow a = k^{1/x}, b = k^{1/y}, c = k^{1/z}, d = k^{1/u}$

Given that a, b, c, d are in GP.

So, $ad = bc$

$$\Rightarrow k^{\frac{1}{x}+\frac{1}{u}} = k^{\frac{1}{y}+\frac{1}{z}}$$

$$\Rightarrow \frac{1}{x}+\frac{1}{u} = \frac{1}{y}+\frac{1}{\text{z}}$$

$$\Rightarrow \frac{1}{x}-\frac{1}{y} = \frac{1}{z}-\frac{1}{u}$$

$$\Rightarrow \frac{1}{x}, \frac{1}{y}, \frac{1}{z}, \frac{1}{u} \in \text{AP}$$

$\Rightarrow a, y, z, u \in$ HP

29. We have,

$$\frac{S_n}{S'_n} = \frac{7n+1}{4n+27}$$

$$\Rightarrow \frac{S_n}{S'_n} = \frac{7n+1}{4n+27} = \frac{7n^2+n}{4n^2+27n}$$

Now, $\frac{t_n}{t'_n} = \frac{S_n - S_{n-1}}{S'_n - S'_{n-1}}$

$$= \frac{(7n^2+n)-[7(n-1)^2+(n-1)]}{(4n^2+27n)-(4(n-1)^2+27(n-1))}$$

$$= \frac{13n-6}{23-19n}$$

30. We have,

$$y = \log_{10}x + \log_{10}x^{1/2} + \log_{10}x^{1/4} + \ldots$$

$$\Rightarrow y = \left(1+\frac{1}{2}+\frac{1}{4}+\ldots\right)\log_{10}x$$

$$\Rightarrow y = \frac{1}{1-\frac{1}{2}}\times\log_{10}x = 2\log_{10}x \quad \text{(i)}$$

Now, $\frac{1+3+5+\ldots+(2y-1)}{4+7+10+\ldots+(3y+1)} = \frac{20}{7\log_{10}x}$

$$\Rightarrow \frac{y^2}{\frac{y}{2}(4+3y+1)} = \frac{20}{7\log_{10}x}$$

$\Rightarrow \quad \frac{y}{(3y+5)} = \frac{10}{7\log_{10}x}$

$\Rightarrow \quad \frac{y}{(3y+5)} = \frac{10}{\frac{7y}{2}} = \frac{20}{7y}$ from (i)

$\Rightarrow \quad 7y^2 = 60y + 100$

$\Rightarrow \quad 7y^2 - 60y - 100 = 0$

$\Rightarrow \quad 7y^2 - 70y + 10y - 100 = 0$

$\Rightarrow \quad 7y(y - 70) + 10(y - 70) = 0$

$\Rightarrow \quad (y - 70)(7y + 10) = 0$

$\Rightarrow \quad y = 70, -\frac{10}{7}.$

When $y = 70$,

$2 \log_{10} x = 70$

$\Rightarrow \quad \log_{10} x = 35$

$\Rightarrow \quad x = 10^{35}$

When $y = -\frac{10}{7}$

$2 \log_{10} x = -\frac{10}{7}$

$\Rightarrow \quad \log_{10} x = -\frac{5}{7}$

$\Rightarrow \quad x = 10^{\left(-\frac{5}{7}\right)}$

31. Let a be the first term and d be the common difference of the given AP, whereas b and r are the first term and the common ratio of the given GP.

It is given that,

$\frac{10}{2}(2a + 9d) = 155$

$(2a + 9d) = 31$ …(i)

and $b + br = 9$

$b(1 + r) = 9$ …(ii)

Also, it is given that,

$b = d$ and $a = r$

Putting the value of b and r in Eq. (ii), we get,

$d + 9d = 9$ and $2a + ad = 31$

Solving, we get

$a = 2$ or $\frac{25}{2}$ and $d = 3$ or $\frac{3}{2}$

Hence, the AP is 2, 3, 8, 11, … or $\frac{25}{2}, \frac{79}{6}, \frac{83}{6}, \ldots$

and the GP is 3, 6, 12, 24, … or $\frac{2}{3}, \frac{25}{3}, \frac{625}{6}, \ldots$

32. Let a be the first term and r be the common ratio of the given GP. By the hypothesis, we have

$\frac{a}{1-r} = 2$ …(i)

and $\frac{a^3}{1-r^3} = 2$ …(ii)

Cubing Eq. (i) and dividing by Eq. (ii), we get

$$\frac{\frac{a^3}{(1-r)^3}}{\frac{a^3}{1-r^3}} = \frac{8}{24}$$

$\Rightarrow \quad \frac{1-r^3}{(1-r)^3} = \frac{1}{3}$

$\Rightarrow \quad \frac{1+r+r^2}{1-2r+r^2} = \frac{1}{3}$

$\Rightarrow \quad 2r^2 + 5r + 2 = 0$

$\Rightarrow \quad (2r + 1)(r + 2) = 0$

$\Rightarrow \quad r = -\frac{1}{2}, -2$

When $r = -\frac{1}{2}$,

$a = 2\left(1 + \frac{1}{2}\right) = 3$

When $r = -2$,

$a = 2(1 + 2) = 6$

Hence, the required GP is

$$\begin{cases} 3, -\frac{3}{2}, \frac{3}{4}, \ldots \\ \quad \text{or} \\ 6, -12, 24, \ldots \end{cases}.$$

Note *No questions asked in 1990, 1991.*

33. Let a be the first term and d be the common difference of the given AP.

Then $S_n = \frac{n}{2}(2a + (n-1)d)$

Also, sum of cubes of first n terms,

$S_3 = a^3 + (a + d)^3 + (a + 2d)^3 + \ldots + (a + (n - 1)d)^3$

$= na^3 + 3a^2(1 + 2 + 3 + \ldots + (n - 1))d$

$+ 3a(1^2 + 2^2 + 3^2 + \ldots + (n - 1)^2)d^2$

$+ (1^3 + 2^3 + 3^3 + \ldots + (n - 1)^3)d^3$

$= na^3 + 3a^2\left(\frac{n(n-1)}{2}\right)d$

$+ 3a\left(\frac{n(n-1)(2n-1)}{6}\right)d^2 + \left(\frac{n(n-1)}{2}\right)^3 d^3$

$= \frac{n}{2}[2a^3 + 3a^2d(n-1) + (n-1)(2n-1)$

$+ d^3\left(\frac{n(n-1)^2}{2}\right)$

which is a multiple of S_n.

34. Let a be the first term and d be the of the given AP, whereas b and r are the first term and the common ratio of the given GP.

It is given that, $\frac{10}{2}(2a + 9d) = 155$

$(2a + 9d) = 31$ …(i)

and $b + br = 9$

$b(1 + r) = 9$ …(ii)

Also, it is given that,

$b = d$ and $a = r$

Putting the value of b and r in Eq. (ii), we get

$d + 9d = 9$ and $2a + ad = 31$

Solving, we get

$$a = 2 \text{ or } \frac{25}{2} \text{ and so } d = 3 \text{ or } \frac{3}{2}$$

Hence, the AP is 2, 3, 8, 11, … or $\frac{25}{2}, \frac{79}{6}, \frac{83}{6}, \ldots$

and the GP is 3, 6, 12, 24, … or $\frac{2}{3}, \frac{25}{3}, \frac{625}{6}, \ldots$

35. Let a be the first term and d be the common difference of the AP

So, $a + md$, $a + nd$, $a + rd$ are in GP

$\Rightarrow \quad (a + nd)^2 = (a + md)(a + rd)$

$\Rightarrow \quad a^2 + 2a \cdot nd + n^2d^2 = a^2 + a(m + r)d + mr \cdot d^2$

$a(2n - m - r) = d(mr - n^2)$

$$\frac{a}{d} = \frac{(mr - n^2)}{(2n - m - r)} \quad \ldots(i)$$

Also, m, n, r are in HP. So,

$$n = \frac{2mr}{m + r} \quad \ldots(ii)$$

From Eqs (i) and (ii), we get

$$\frac{a}{d} = \frac{\frac{n}{2}(m + r) - n^2}{2n - (m + r)}$$

$$= \frac{n(m + r) - 2n^2}{2[2n - (m + r)]}$$

$$= \frac{n((m + \mathrm{r}) - 2n)}{2[2n - (m + r)]}$$

$$= -\frac{n}{2}$$

36. Given $x = 1 + 3a + 6a^2 + 10a^3 + \ldots$ …(i)

$y = 1 + 4b + 10b^2 + 20b^3 + \ldots$ …(ii)

and $S = 1 + 3(ab) + 5(ab)^2 + \ldots$ …(iii)

Multiplying Eq. (i) by a, we get

$ax = a + 3a^2 + 6a^3 + \ldots$ …(iv)

Subtracting Eq. (iv) from Eq. (i), we get

$x - ax = 1 + 2a + 3a^2 + 4a^3 + \ldots$

$\Rightarrow \quad (1 - a)x = 1 + 2a + 3a^2 + 4a^3 + \ldots$

$\Rightarrow \quad (1 - a)^2 x = (1 + 2a + 3a^2 + 4a^3 + \ldots)(1 - a)$

$\Rightarrow \quad (1 - a)^2 x = (1 + 2a + 3a^2 + 4a^3 + \ldots)$

$- (a + 2a^2 + 3a^3 + 4a^4 + \ldots)$

$= 1 + a + a^2 + a^3 + a^4 + \ldots$

$$= \frac{1}{1 - a}$$

$$\Rightarrow \quad x = \frac{1}{(1 - a)^3}$$

$$\Rightarrow \quad a = \left(1 - \frac{1}{x^{1/3}}\right) \quad \ldots(v)$$

Similarly, $b = \left(1 - \frac{1}{y^{1/4}}\right)$ …(vi)

Now, multiplying Eq. (iii) by ab, we get

$abS = ab + 3(ab)^2 + 5(ab)^3 + \ldots$ …(vii)

Subtracting Eq. (vii) from Eq. (iii), we get

$(1 - ab)S = 1 + 2ab + 2(ab)^2 + \ldots$

$$\Rightarrow \quad (1 - ab)S = 1 + \frac{2ab}{1 - ab}$$

$$\Rightarrow \quad (1 - ab)S = \frac{1 + ab}{1 - ab}$$

$$\Rightarrow \quad S = \frac{1 + ab}{(1 - ab)^2} \quad (viii)$$

From Eqs (v), (vi) and (viii), we get

$$\Rightarrow \quad S = \frac{1 + (1 - x^{-1/3})(1 - y^{-1/4})}{[1 - (1 - x^{-1/3})(1 - y^{-1/4})]^2}$$

37. Let t_n be the first term of the cube n^3 and S_n be the sum of first terms of each of n^3.

$S_n = 1 + 3 + 7 + 13 + \ldots + t_{n-1} + t_n$

$S_n = \quad 1 + 3 + 7 + 13 + \ldots + t_{n-1} + t_n$

Subtracting, we get

$0 = (1 + 2 + 4 + 6 + \ldots \text{ to } n \text{ terms}) - t_n$

$t_n = (1 + 2 + 4 + 6 + \ldots \text{ to } n \text{ terms})$

$$= 1 + 2\left(\frac{n-1}{2}\right)[2.1 + (n - 1 - 1) \cdot 1]$$

$= 1 + (n - 1)n$

$= n^2 - n + 1$

Again, let T_n be the last term of n^3.

Thus, $S_n = 1 + 5 + 11 + 19 + \ldots + T_{n-1} + T_n$

$S_n = \quad 1 + 5 + 11 + 19 + \ldots + T_{n-1} + T_n$

Subtracting, we get

$0 = (1 + 4 + 6 + 8 + \ldots \text{ to } n \text{ terms}) - T_n$

$T_n = (1 + 4 + 6 + 8 + \ldots \text{ to } n \text{ terms})$

$$= 1 + 2\left(\frac{n-1}{2}\right)[2.1 + (n - 1 - 1) \cdot 1]$$

$= 1 + (n - 1)n$

$= n^2 - n + 1$

and rest of the part you try it from mathematical induction.

38. Let r be the common ratio of the given GP.

$$\text{Given } 12 = \frac{2ab}{a + b} = \frac{2a \cdot ar}{a + ar} = \frac{2ar}{a + r}$$

$\Rightarrow \quad ar = 6(1 + r)$ …(i)

$$\text{Also, } 36 = \frac{2bc}{b + c} = \frac{2ar \cdot ar^2}{ar + ar^2} = \frac{2ar^2}{1 + \mathrm{r}}$$

$\Rightarrow \quad ar^2 = 18(1 + r)$ …(ii)

Dividing Eq. (ii) by Eq. (i), we get,

$r = 3$

Thus, $a = \frac{6(1+3)}{3} = 8$

Hence, the required GP is 8, 24, 72, 216, 648.

39. Let the series be $a + ar + ar^2 + ar^3 + \ldots \infty$

Given $\frac{a}{1-r} = 162$ …(i)

and $\frac{a(1-r^n)}{1-r} = 160$ …(ii)

$\Rightarrow \quad (1-r^n) = \frac{160}{162} = \frac{80}{81}$ from (i)

$\Rightarrow \quad (1-r^n) = \frac{80}{81}$

$\Rightarrow \quad r^n = 1 - \frac{80}{81} = \frac{1}{81}$

$\Rightarrow \quad \left(\frac{1}{r}\right)^n = 81$

Since $\left(\frac{1}{r}\right)$ is an integer and n is also an integer, the possible values of $\left(\frac{1}{r}\right)$ are 3, 9, 81.

If $r = \frac{1}{3}$, $n = 4$,

$a = 162\left(1 - \frac{1}{3}\right) = 108$

If $r = \frac{1}{9}$, $n = 2$,

$a = 162\left(1 - \frac{1}{9}\right) = 144$

If $r = \frac{1}{81}$, $n = 1$,

$a = 162\left(1 - \frac{1}{81}\right) = 160$

40. Let the three numbers of the GP be $\frac{a}{r}, a, ar$.

Given $\frac{a}{r} + a + ar = 42$ …(i)

and $\left(\frac{a}{r} + 2\right), (a+2), (ar-4)$ are in AP.

So, $2(a+2) = \left(\frac{a}{r} + 2\right) + (ar - 4)$

$\Rightarrow \quad 2(a+2) + 2 = \left(\frac{a}{r} + ar\right)$

$\Rightarrow \quad \left(\frac{a}{r} + ar\right) = 2a + 6$ …(ii)

From Eqs (i) and (ii), we get

$2a + 6 + a = 42$

$\Rightarrow \quad 3a = 42 - 6 = 36$

$\Rightarrow \quad a = 12$

Put the value of a in Eq. (i), we get

$\Rightarrow \quad \frac{12}{r} + 12 + 12r = 42$

$\Rightarrow \quad \frac{2}{r} + 2 + 2r = 7$

$\Rightarrow \quad \frac{2}{r} + 2r = 5$

$\Rightarrow \quad 2r^2 - 5r + 2 = 0$

$\Rightarrow \quad (r-2)(2r-1) = 0$

$\Rightarrow \quad r = 2$ or $1/2$

Hence, the sequence of GP are 6, 12, 24 or 24, 12, 6.

41. $\sqrt{1 + \frac{1}{1^2} + \frac{1}{2^2}} = \sqrt{1 + 1 + \frac{1}{4}} = \frac{3}{2}$

$\sqrt{1 + \frac{1}{2^2} + \frac{1}{3^2}} = \sqrt{1 + \frac{1}{4} + \frac{1}{9}} = \sqrt{\frac{49}{36}} = \frac{7}{6}$

$\sqrt{1 + \frac{1}{3^2} + \frac{1}{4^2}} = \sqrt{1 + \frac{1}{9} + \frac{1}{16}} = \sqrt{\frac{169}{144}} = \frac{13}{12}$

$\vdots$

$\sqrt{1 + \frac{1}{1999^2} + \frac{1}{2000^2}} = \frac{3998001}{3998000}$

and now given expression,

$$= \frac{3}{2} + \frac{7}{6} + \frac{13}{12} + \ldots + \frac{3998001}{3998000}$$

$$= \left(1 + \frac{1}{2}\right) + \left(1 + \frac{1}{6}\right) + \left(1 + \frac{1}{12}\right) + \ldots\ldots + \left(1 + \frac{1}{3998000}\right)$$

$$= 1999 + \left(\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \ldots + \frac{1}{3998000}\right)$$

$$= 1999 + \left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \ldots + \left(\frac{1}{1999} - \frac{1}{2000}\right)\right]$$

$$= 1999 + \left(1 - \frac{1}{2000}\right)$$

Hence, the value of n is 2000.

42. We have,

$$\sum_{k=0}^{359} k \cdot \cos(k°)$$

$= 1 \cdot \cos(1°) + 2\cos(2°) + 3\cos(3°) + \ldots + 179\cos(179°)$
$+ 180\cos(180°) + 181\cos(181°) + 182\cos(182°)$
$+ \ldots + 359\cos(359°)$

$= 1 \cdot \cos(1°) + 2\cos(2°) + 3\cos(3°) + \ldots + 179\cos(179°)$
$-180 - 181\cos(1°) - 182\cos(2°) - \ldots -359\cos(179°)$
$= -180 - 180\cos(1°) - 180\cos(2°) - \ldots -180\cos(179°)$
$= -180 - 180[\cos(1°) + \cos(2°) + \ldots + \cos(179°)]$

$$= -180 - 180\left(\frac{\sin\left(\frac{179°}{2}\right)}{\sin\left(\left(\frac{1}{2}\right)^{\circ}\right)}\right)\left(\cos\left(1° + \frac{(179-1)°}{2}\right)\right)$$

$$= -180 - 180\left(\frac{\sin\left(\frac{179°}{2}\right)}{\sin\left(\left(\frac{1}{2}\right)^{\circ}\right)}\right)[\cos(90°)]$$

$= -180$

43. We have

$$S = 1 + 2\left(1+\frac{1}{n}\right) + 3\left(1+\frac{1}{n}\right)^2 + 4\left(1+\frac{1}{n}\right)^3 + \ldots$$

$$S = 1 + 2x + 3x^2 + 4x^3 + \ldots + nx^{n-1} \quad \ldots(i)$$

where $x = \left(1+\frac{1}{n}\right)$

$$xS = x + 2x^2 + 3x^3 + \ldots + (n-1)x^{n-1} + nx^n \quad \ldots(ii)$$

Subtracting Eq. (ii) from Eq. (i), we get

$$(1-x)S = 1 + x + x^2 + x^3 + \ldots + x^{n-1} - nx^n$$

$$(1-x)S = \left(\frac{1-x^n}{1-x}\right) - nx^n$$

$$S = \left(\frac{1-x^n}{(1-x)^2}\right) - \frac{nx^n}{(1-x)}, \text{ where } x = \left(1+\frac{1}{n}\right)$$

44. We have

$$f(x) = \sum_{n=1}^{\infty} \sin\left(\frac{2x}{3^n}\right)\sin\left(\frac{x}{3^n}\right)$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\left(2\sin\left(\frac{2x}{3^n}\right)\sin\left(\frac{x}{3^n}\right)\right)$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\left(\cos\left(\frac{x}{3^n}\right) - \cos\left(\frac{x}{3^{n-1}}\right)\right)$$

$$= \frac{1}{2}\sum_{n=1}^{\infty}\cos\left(\frac{x}{3^n}\right) - \frac{1}{2}\sum_{n=1}^{\infty}\cos\left(\frac{x}{3^{n-1}}\right)$$

$$= \frac{1}{2}\left(\cos\left(\frac{x}{3}\right) + \cos\left(\frac{x}{3^2}\right) + \cos\left(\frac{x}{3^3}\right) + \ldots\right)$$

$$- \frac{1}{2}\left(\cos(x) + \cos\left(\frac{x}{3}\right) + \cos\left(\frac{x}{3^2}\right) + \ldots\right)$$

$$= \frac{1}{2}(1 - \cos x)$$

45. Given equation is

$$10x^3 - cx^2 - 54x - 27 = 0$$

Replace is x by $1/x$, we get

$$\frac{10}{x^3} - \frac{c}{x^2} - \frac{54}{x} - 27 = 0$$

$\Rightarrow \quad 10 - cx - 54x^2 - 27x^3 = 0$

$\Rightarrow \quad 27x^3 + 54x^2 + cx - 10 = 0 = 0 \quad \ldots(A)$

Thus, the roots of the above equation are in AP.

Let α, β, γ be the roots.

$$\alpha + \beta + \gamma = -2 \quad \ldots(i)$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{27} \quad \ldots(ii)$$

$$\alpha\beta\gamma = \frac{10}{27} \quad \ldots(iii)$$

Let $\alpha = a - d, \beta = a, \gamma = a + d$

We have,

$\alpha + \beta + \gamma = -2$

$\Rightarrow \quad 3a = -2$

$\Rightarrow \quad a = -\frac{2}{3}$

Put $a = -\frac{2}{3}$ in Eq. (A), we get

$\Rightarrow \quad 27\left(-\frac{2}{3}\right)^3 + 54\left(-\frac{2}{3}\right)^2 + c\left(-\frac{2}{3}\right) - 10 = 0$

$\Rightarrow \quad -27 \times \frac{8}{27} + 54 \times \frac{4}{9} - \frac{2}{3}c - 10 = 0$

$\Rightarrow \quad -8 + 24 - \frac{2}{3}c - 10 = 0$

$\Rightarrow \quad \frac{2}{3}c = 6$

$\Rightarrow \quad c = 9$

Further, $\alpha\beta\gamma = \frac{10}{27}$

$\Rightarrow \quad a(a^2 - d^2) = \frac{10}{27}$

$\Rightarrow \quad -\frac{2}{3}\left(\frac{4}{9} - d^2\right) = \frac{10}{27}$

$\Rightarrow \quad \left(d^2 - \frac{4}{9}\right) = \frac{5}{9}$

$\Rightarrow \quad d^2 = \frac{5}{9} + \frac{4}{9} = 1$

$\Rightarrow \quad d = \pm 1$

$\therefore \quad \alpha = a - d = -\frac{2}{3} - 1 = -\frac{5}{3}$

Thus, $\gamma = a + d = -\frac{2}{3} + 1 = \frac{1}{3}$

Hence, the roots are $\left(-\frac{3}{5}, -\frac{3}{2}, 3\right)$.

Level IV

1. Given cot A, cot B, cot C are in A.P

$$\frac{\cos A}{\sin A}, \frac{\cos B}{\sin B}, \frac{\cos C}{\sin C} \in \text{A.P}$$

$$\frac{(b^2+c^2-a^2)}{2abc\,k}, \frac{(a^2+c^2-b^2)}{2abc\,k}, \frac{(a^2+b^2-c^2)}{2abc\,k} \in \text{A.P}$$

$(b^2 + c^2 - a^2)$, $(a^2 + c^2 - b^2)$, $(a^2 + b^2 - c^2) \in$ A.P

$(a^2 + b^2 + c^2 - 2a^2)$, $(a^2 + b^2 + c^2 - 2b^2)$,

$(a^2 + b^2 + c^2 - 2c^2) \in$ A.P

$-a^2, -b^2 -c^2 \in$ A.P

$a^2, b^2, c^2 \in$ A.P

Hence, the result.

2. Let three terms be a, ar, ar^2

Given $a + ar + ar^2 = \alpha S$ …(i)

and $a^2 + a^2r^2 + a^2r^4 = S^2$ …(ii)

Dividing (i) and (ii), we get,

$$\frac{(1+r+r^2)^2}{(1+r^2+r^4)} = \frac{\alpha^2 S^2}{S^2} = \alpha^2$$

$$\Rightarrow \quad \frac{(1+r+r^2)}{(1-r+r^2)} = \alpha^2$$

$\Rightarrow$ $\alpha^2(1 - r + r^2) = (1 + r + r^2)$

$\Rightarrow$ $(\alpha^2 - 1)r^2 - (\alpha^2 + 1)r + (\alpha^2 - 1) = 0$

As r is real, so $(\alpha^2 + 1)^2 - 4(\alpha^2 - 1)^2 \geq 0$

$\Rightarrow$ $(\alpha^2 + 1)^2 - (2\alpha^2 - 2)^2 \geq 0$

$\Rightarrow$ $(\alpha^2 + 1 + 2\alpha^2 - 2)(\alpha^2 + 1 - 2\alpha^2 + 2) \geq 0$

$\Rightarrow$ $(3\alpha^2 - 1)(-\alpha^2 + 3) \geq 0$

$\Rightarrow$ $(3\alpha^2 - 1)(\alpha^2 - 3) \leq 0$

$$\Rightarrow \quad \frac{1}{3} \leq \alpha^2 \leq 3$$

But $\alpha^2 = 1 \Rightarrow r = 0$.

It is not possible.

Thus, $\frac{1}{3} \leq \alpha^2 \leq 3 - \{1\}$

$$\Rightarrow \quad \alpha^2 \in \left(\frac{1}{3}, 1\right) \cup (1, 3)$$

Hence, the result.

3. We have

$$\frac{s_n}{s_{n'}} = \frac{7n+1}{4n+27}$$

$$\frac{s_n}{s_{n'}} = \frac{7n^2+n}{4n^2+27n}$$

$$\frac{t_n}{t_{n'}} = \frac{s_n - s_{n-1}}{s_{n'} - s_{n-1'}}$$

$$= \frac{(7n^2+n)-(7(n-1)^2+(n-1))}{(4n^2+27n)-(4(n-1)^2+27(n-1))}$$

$$= \frac{14n-6}{8n+23}$$

4. We have $y = \log_{10}x + \log_{10}x^{1/2} + \log_{10}x^{1/4} + \ldots$

$$\Rightarrow \quad y = \left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots\right)\log_{10}x$$

$$\Rightarrow \quad y = \left(\frac{1}{1-\frac{1}{2}}\right)\log_{10}x = 2\log_{10}x$$

Again $\frac{1+3+5+\ldots+(2y-1)}{4+7+10+\ldots+(3y+1)} = \frac{20}{7\log_{10}x}$

$$\Rightarrow \quad \frac{y^2}{\frac{y}{2}(4+3y+1)} = \frac{20}{7\log_{10}x}$$

$$\Rightarrow \quad \frac{2y}{(3y+5)} = \frac{20}{7\log_{10}x}$$

$$\Rightarrow \quad \frac{y}{(3y+5)} = \frac{10}{7\log_{10}x}$$

$$\Rightarrow \quad \frac{y}{(3y+5)} = \frac{10}{7y/2} = \frac{20}{7y}$$

$\Rightarrow$ $7y^2 - 60y - 100 = 0$

$\Rightarrow$ $7y^2 - 70y + 10y - 100 = 0$

$\Rightarrow$ $7y(y - 10) + 10(y - 10) = 0$

$\Rightarrow$ $(y - 10)(7y + 10) = 0$

$$\Rightarrow \quad y = 10, -\frac{10}{7}$$

Thus, $2\log_{10}x = 10, -\frac{10}{7}$

$$\Rightarrow \quad \log_{10}x = 5, -\frac{5}{7}$$

$$\Rightarrow \quad x = 10^5, 10^{-\frac{5}{7}}$$

Hence, the solutions are

$$\Rightarrow \quad x = 10^5, 10^{-\frac{5}{7}};\ y = 5, -\frac{5}{7}$$

5. We have $\left(\frac{1+a_1}{2}\right) \geq \sqrt{1 \cdot a_1} = \sqrt{a_1}$

$$(1+a_1) \geq 2\sqrt{a_1}$$

Similarly $(1+a_2) \geq 2\sqrt{a_2}$

$(1+a_3) \geq 2\sqrt{a_3}$

....

....

$(1+a_n) \geq 2\sqrt{a_n}$

Multiplying all, we get,

$$(1+a_1)(1+a_2)\ldots(1+a_n) \geq 2^n\sqrt{a_1a_2\ldots a_n}$$

$(1 + a_1)(1 + a_2) \ldots (1 + a_n) \geq 2^n$

Hence, the result.

6. We have $\frac{1}{x-1}-\frac{1}{x+1}=\frac{2}{x^2-1}$

$$\frac{2}{x^2-1}-\frac{2}{x^2+1}=\frac{4}{x^4-1}$$

$$\frac{4}{x^4-1}-\frac{4}{x^4+1}=\frac{8}{x^8-1}$$

$$\vdots$$

$$\frac{2^n}{x^{2n}-1}-\frac{2^n}{x^{2n}+1}=\frac{2^{n+1}}{x^{2^{n+1}}-1}$$

Thus, $\frac{1}{x-1}-\left(\frac{1}{x+1}+\frac{2}{x^2+1}+\ldots+\frac{2^n}{x^{2n}+1}\right)$

$$=\left(\frac{1}{x-1}-\frac{2^{n+1}}{x^{2^{n+1}}-1}\right)$$

7. We have $S_1=\frac{1}{1-(1/2)}=2$

$$S_2=\frac{2}{1-(1/3)}=3$$

$$S_3=\frac{3}{1-(1/4)}=4$$

$$\vdots$$

$$S_{2n-1}=\frac{(2n-1)}{1-(1/2n)}=2n$$

Now,

$$S_1^2+S_2^2+S_3^2+\ldots+S_{2n-1}^2$$

$$= 2^2 + 3^2 + 4^2 + \ldots + (2n)^2$$

$$= (1^2 + 2^2 + 3^2 + \ldots + (n)^2 - 1)$$

$$+ (n + 1)^2 + (n + 2)^2 + \ldots + (2n)^2$$

$$=\left(\frac{n(n+1)(2n+1)}{6}-1\right)$$

$$+S_n^2+S_{n+1}^2+S_{n+2}^2+\ldots+S_{2n-1}^2$$

8. We have,

$$\sin^2 x + \sin^4 x + \sin^6 x + \ldots$$

$$=\frac{\sin^2 x}{1-\sin^2 x}$$

$$= \tan^2 x$$

Now, $e^{\tan^2 x \log_e 2}=2^{\tan^2 x}$

Now, $x^2 - 9x + 8 = 0$

$\Rightarrow \quad (x - 1)(x - 8) = 0$

$\Rightarrow \quad x = 1, 8$

when $2^{\tan^2 x}=1=2^0$

$\Rightarrow \quad \tan^2 x = 0$

$\Rightarrow \quad \tan x = 0$

when $2^{\tan^2 x}=8=2^3$

$\Rightarrow \quad \tan^2 x = 3$

$\Rightarrow \quad \tan x=\sqrt{3}$

Thus, $\frac{\cos x}{\cos x+\sin x}=\frac{1}{1+\tan x}$

$$=\frac{1}{1+\sqrt{3}}$$

$$=\left(\frac{\sqrt{3}-1}{2}\right)$$

9. The given series is in the form

$$1 + a + a^2 + a^3 + \ldots \text{ to } \infty$$

where $a=\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)$

$$S_n=\frac{1}{1-a}$$

$$=\frac{1}{1-\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)}$$

$$=\frac{2\sqrt{2}}{2\sqrt{2}-\sqrt{2}+1}$$

$$=\frac{2\sqrt{2}}{\sqrt{2}+1}$$

$$=2\sqrt{2}(\sqrt{2}-1)$$

10. Given $f(1) = 2$

$$f(2) = f(1)f(1) = 2^2$$

$$f(3) = f(2)f(1) = 2^3$$

$$\vdots$$

$$f(n) = f(n - 1)f(1) = 2^n$$

Now,

$$\sum_{k=1}^{n} f(a+k)=16(2^n-1)$$

$$= f(a + 1) + f(a + 2) + \ldots + f(a + n)$$

$$= f(a)(f(1) + f(2) + \ldots + f(n))$$

$\Rightarrow \quad 2^a(2 + 2^2 + \ldots + 2^n) = 16(2^n - 1)$

$\Rightarrow \quad 2^a \cdot 2(2^n - 1) = 16(2^n - 1)$

$\Rightarrow \quad 2^{a+1} = 16 = 2^4$

$\Rightarrow \quad a + 1 = 4$

$\Rightarrow \quad a = 3$

Hence, the value of a is 3.

11. From the given polynomial, $x_1 + x_2 + x_3 = 1$ …(i)

$x_1x_2 + x_2x_3 + x_3x_1 = \beta$ …(ii)

$x_1x_2x_3 = -\gamma$ …(iii)

From Eq. (i), we get

$$2x_2+x_2=1 \Rightarrow x_2=\frac{1}{3}$$

From Eq. (ii), we get

$$\beta = x_2(x_1 + x_3) + x_1x_3$$
$$= 2x_2^2 + x_1x_3$$
$$= \frac{2}{9} + \left(\frac{1}{3} - d\right)\left(\frac{1}{3} + d\right)$$
$$= \frac{2}{9} + \frac{1}{9} - d^2$$
$$= \frac{1}{3} - d^2$$

$\Rightarrow \quad \beta = \frac{1}{3} - d^2$

$\Rightarrow \quad \left(\beta - \frac{1}{3}\right) = -d^2 \le 0$

$\Rightarrow \quad \beta \le \frac{1}{3}$

$\Rightarrow \quad \beta \in \left(-\infty, \frac{1}{3}\right]$

From Eq. (iii), we get,

$$x_1x_2x_3 = -\gamma$$

$\Rightarrow \quad \left(\frac{1}{3} - d\right)\frac{1}{3} \cdot \left(\frac{1}{3} + d\right) = -\gamma$

$\Rightarrow \quad \gamma = \frac{1}{3}\left(d^2 - \frac{1}{9}\right)$

$\Rightarrow \quad \gamma + \frac{1}{27} = \frac{d^2}{3} \ge 0$

$\Rightarrow \quad \gamma \ge -\frac{1}{27}$

$\therefore \quad \gamma \in \left[-\frac{1}{27}, \infty\right)$

12. Let $x = 1 + 3a + 6a^2 + 10a^3 + \ldots, |a| < 1,$
$y = 1 + 4b + 10b^2 \; + 20b^3 + \ldots, |b| < 1.$
find $S = 1 + 3(ab) + 5(ab)^2 + \ldots$
in terms of x and y.

13. Let $(1 + x^2)^2(1 + x)^n = \sum_{k=0}^{n+4} a_k x^k$

If a_1, a_2 and a_3 are in A.P, find n.

14. We have

$$\cos x = \frac{2\cos(x + y)\cos(x - y)}{\cos(x + y) + \cos(x - y)}$$
$$= \frac{2(\cos^2 x - \sin^2 y)}{2\cos x \cos y}$$

$\Rightarrow \quad \cos^2 x \cos y = (\cos^2 x - \sin^2 y)$

$\Rightarrow \quad \sin^2 y = \cos^2 x\,(1 - \cos y)$

$\Rightarrow \quad \left(2\sin\left(\frac{y}{2}\right)\cos\left(\frac{y}{2}\right)\right) = \cos^2 x \times 2\sin^2\left(\frac{y}{2}\right)$

$\Rightarrow \quad 4\sin^2\left(\frac{y}{2}\right)\cos^2\left(\frac{y}{2}\right) = \cos^2 x \times 2\sin^2\left(\frac{y}{2}\right)$

$\Rightarrow \quad 4\cos^2\left(\frac{y}{2}\right) = 2\cos^2 x$

$\Rightarrow \quad \cos^2 x \sec^2\left(\frac{y}{2}\right) = 2$

$\Rightarrow \quad \left|\cos x \sec\left(\frac{y}{2}\right)\right| = \sqrt{2}$

Applying, AM ≥ HM, we get

$$\left(\frac{a + b + c}{3}\right) \ge \frac{3}{\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)}$$

$\Rightarrow \quad (a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \ge 9$

Clearly, $n = 9$

Hence, the value of $(n + m^2 - 4)$ is 7.

15. We have,

$$1 + x + x^2 + x^3 + \ldots + x^{200}$$
$$= x^{100}\left(\frac{1}{x^{100}} + \frac{1}{x^{99}} + \frac{1}{x^{98}} + \ldots + x^{98} + x^{99} + x^{100}\right)$$
$$= x^{100}\left(\left(x^{100} + \frac{1}{x^{100}}\right) + \left(x^{99} + \frac{1}{x^{99}}\right) + \ldots + \left(x + \frac{1}{x}\right) + 1\right)$$
$$\ge x^{100}\left(\frac{2 + 2 + \ldots + 2}{100 \text{ times}} + 1\right)$$
$$= x^{100}(200 + 1)$$
$$= (200 + 1)x^{100}$$

Thus, $\frac{1}{1 + x + x^2 + \ldots + x^{200}} \le \frac{1}{201x^{100}}$

$\Rightarrow \quad \frac{x^{100}}{1 + x + x^2 + \ldots + x^{200}} \le \frac{1}{201}$

Hence, the greatest value is $\frac{1}{201}$.

16. Given $\frac{2ab}{a + b} = 12$

$\Rightarrow \quad \frac{ab}{a + b} = 6$

$\Rightarrow \quad \frac{a \cdot ar}{a + ar} = 6$

$\Rightarrow \quad \frac{ar}{1 + r} = 6 \quad \ldots(i)$

Also, $\frac{2bc}{b + c} = 36$

$\Rightarrow \quad \frac{bc}{b + c} = 18$

$\Rightarrow \quad \dfrac{ar \cdot ar^2}{ar + ar^2} = 18$

$\Rightarrow \quad \dfrac{ar^2}{1+r} = 18$ …(ii)

Dividing Eq. (ii) by Eq. (i), we get

$\dfrac{ar^2}{1+r} \div \dfrac{ar}{1+r} = \dfrac{18}{6}$

$\Rightarrow \quad r = 3$

Putting $r = 3$ in Eq. (i), we get

$\Rightarrow \quad \dfrac{3a}{1+3} = 6$

$\Rightarrow \quad a = 8$

$\Rightarrow \quad b = ar = 24; c = ar^2 = 72$

Hence, the series is 8, 24, 72, 216, 648.

17. Given $\alpha + \beta = \dfrac{4}{A}, \alpha\beta = \dfrac{1}{A}$

and $\gamma + \delta = \dfrac{6}{B}, \gamma\delta = \dfrac{1}{B}$

It is also given that α, β, γ and δ are in HP.

$\Rightarrow \quad \dfrac{1}{a}, \dfrac{1}{\beta}, \dfrac{1}{\gamma}, \dfrac{1}{\delta} \in \text{AP}$

$\Rightarrow \quad \dfrac{1}{\alpha} + \dfrac{1}{\delta} = \dfrac{1}{\beta} + \dfrac{1}{\gamma}$

Also, $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\delta} + \dfrac{1}{\gamma} = 6 + 4 = 10$

$\Rightarrow \quad \left(\dfrac{1}{\alpha} + \dfrac{1}{\delta}\right) + \left(\dfrac{1}{\beta} + \dfrac{1}{\gamma}\right) = 10$

$\Rightarrow \quad \left(\dfrac{1}{\alpha} + \dfrac{1}{\delta}\right) = 5 = \left(\dfrac{1}{\beta} + \dfrac{1}{\gamma}\right)$

$\Rightarrow \quad \left(\dfrac{1}{\alpha} + \dfrac{1}{\delta}\right) = 5$

$\Rightarrow \quad \left(\dfrac{1}{\alpha} + \dfrac{1}{\alpha} + 3D\right) = 5$

$\Rightarrow \quad \left(\dfrac{2}{\alpha} + 3D\right) = 5$ …(i)

Again, $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = 4$

$\Rightarrow \quad \dfrac{1}{\alpha} + \dfrac{1}{\alpha} + D = 4$

$\Rightarrow \quad \dfrac{2}{\alpha} + D = 4$ …(ii)

Solving, we get,

$D = \dfrac{1}{2}, \alpha = \dfrac{4}{7}$

$\dfrac{1}{\beta} = \dfrac{1}{\alpha} + D = \dfrac{7}{4} + \dfrac{1}{2} = \dfrac{9}{4} \Rightarrow \beta = \dfrac{4}{9}$

$\dfrac{1}{\gamma} = \dfrac{1}{\alpha} + 2D = \dfrac{7}{4} + 1 = \dfrac{11}{4} \Rightarrow \gamma = \dfrac{4}{11}$

$\dfrac{1}{\delta} = \dfrac{1}{\alpha} + 3D = \dfrac{7}{4} + \dfrac{3}{2} = \dfrac{13}{4} \Rightarrow \delta = \dfrac{4}{13}$

Thus, $A = \dfrac{1}{\alpha\beta} = \dfrac{1}{\alpha} \cdot \dfrac{1}{\beta} = \dfrac{63}{16}$

and $B = \dfrac{1}{\gamma\delta} = \dfrac{1}{\gamma} \cdot \dfrac{1}{\delta} = \dfrac{143}{16}$

18. Let three numbers be a, b and c

Given $a + b + c = 42$ …(i)

Also, $a + 2, b + 2, c - 4 \in$ AP

$\Rightarrow \quad a + 2 + c - 4 = 2(b + 2)$

$\Rightarrow \quad a + c - 2 = 2b + 4$

$\Rightarrow \quad a + c = 2b + 6$ …(ii)

From Eq. (i) and Eq. (ii), we get

$2b + 6 + b = 42$

$\Rightarrow \quad 3b = 42 - 6 = 36$

$b = 12$

$\Rightarrow \quad ar = 12$

From Eq. (i), we get

$a + b + c = 42$

$\Rightarrow \quad a + ar + ar^2 = 42$

$\Rightarrow \quad a + 12 + 12r = 42$

$\Rightarrow \quad \dfrac{12}{r} + 12 + 12r = 42$

$\Rightarrow \quad \dfrac{1}{r} + 1 + r = \dfrac{42}{12} = \dfrac{7}{2}$

$\Rightarrow \quad 2(r^2 + r + 1) = 7r$

$\Rightarrow \quad 2r^2 - 5r + 2 = 0$

$\Rightarrow \quad 2r^2 - 4r - r + 2 = 0$

$\Rightarrow \quad 2r(r - 2) - (r - 2) = 0$

$\Rightarrow \quad (r - 2)(2r - 1) = 0$

$\Rightarrow \quad r = 2, \dfrac{1}{2}$

When $r = 2$,

$a = 6$

Thus, the sequence is {6, 12, 24}.

When $r = 1/2$,

$a = 24$

Thus, the sequence is {24, 12, 6}.

20. We have $\left(p + \dfrac{1}{p}\right)^2 + \left(q + \dfrac{1}{q}\right)^2$

$= (p^2 + q^2) + \left(\dfrac{1}{p^2} + \dfrac{1}{q^2}\right) + 4$

Now, $\left(\frac{p^2+q^2}{2}\right) \geq \left(\frac{p+q}{2}\right)^2 = \frac{1}{4}$

$\Rightarrow \quad (p^2+q^2) \geq \frac{1}{2}$ ...(i)

Also, $\left(\frac{p^{-2}+q^{-2}}{2}\right) \geq \left(\frac{p+q}{2}\right)^{-2}$

$\Rightarrow \quad \left(\frac{p^{-2}+q^{-2}}{2}\right) \geq 4$

$\Rightarrow \quad \left(\frac{1}{p^2}+\frac{1}{q^2}\right) \geq 8$ ...(ii)

Adding Eqs (i) and (ii), we get

$\Rightarrow \quad (p^2+q^2)+\left(\frac{1}{p^2}+\frac{1}{q^2}\right) \geq \frac{1}{2}+8$

$(p^2+q^2)+\left(\frac{1}{p^2}+\frac{1}{q^2}\right)+4 \geq \frac{1}{2}+8+4 = \frac{25}{2}$

$\Rightarrow \quad (p^2+q^2)+\left(\frac{1}{p^2}+\frac{1}{q^2}\right)+4 \geq \frac{25}{2}$

$\Rightarrow \quad \left(p+\frac{1}{p}\right)^2+\left(q+\frac{1}{q}\right)^2 \geq \frac{25}{2}$

Hence, the result.

21. We have,

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right)$$
$$=1+\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right)+\frac{1}{abc}$$
$$=1+\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\left(\frac{a+b+c}{abc}\right)+\frac{1}{abc}$$
$$=1+\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\frac{2}{abc}$$

As we know that AM ≥ HM,

$$\left(\frac{a+b+c}{3}\right) \geq \frac{3}{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}$$

$\Rightarrow \quad \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq \frac{9}{2}$

Also, $\left(\frac{a+b+c}{3}\right) \geq \sqrt[3]{abc}$

$\Rightarrow \quad \frac{2}{3} \geq \sqrt[3]{abc}$

$\Rightarrow \quad (abc) \leq \frac{8}{27}$

$\Rightarrow \quad \frac{1}{(abc)} \geq \frac{27}{8}$

$\Rightarrow \quad \frac{2}{(abc)} \geq \frac{54}{8}$

Thus, $1+\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+\frac{2}{abc} \geq 1+\frac{9}{2}+\frac{54}{8} = \frac{49}{4}$

22. We have

$$(x-1)(x-2)(x-3) \ldots (x-100)$$
$$= x^{100} - (1+2+3+\ldots+100)x^{99} + \ldots$$

Co-efficient of $x^{99} = -(1+2+3+\ldots+100)$

$$= -\frac{100}{2}(1+100)$$
$$= -50 \times 101$$
$$= -5050$$

23. We have

$$(2x-1)(2x-3)(2x-5) \ldots (2x-199)$$
$$= (2x)^{100} - (1+3+5+\ldots+199)(2x)^{99} + \ldots$$

Thus, the co-efficient of x^{99}

$$= -(1+3+5+\ldots+199)(2)^{99}$$
$$= -(100)^2(2)^{99}$$
$$= -(2)^{99} \times 10000$$

24. Given $S_n = cn^2$

Now, $t_n = S_n - S_{n-1}$

$$= cn^2 - c(n-1)^2$$
$$= (2n-1)c$$

Hence, the required sum

$$= \sum_{n=1}^{n} c^2(2n-1)^2$$
$$= c^2\sum_{n=1}^{n}(4n^2-4n+1)$$
$$= c^2\left(\frac{4n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n\right)$$
$$= nc^2\left(\frac{2(n+1)(2n+1)}{3} - 2(n+1)+1\right)$$
$$= \frac{nc^2}{3}(4n^2+6n+2-6n-6+3)$$
$$= \frac{nc^2 \times (4n^2-1)}{3}]$$

25. We have,

$$(1+a)(1+b)(1+c)$$
$$= 1 + (a+b+c) + (ab+bc+ca) + abc$$

As we know that AM ≥ GM

$$\frac{(a+b+c+ab+bc+ca+abc)}{7} \geq \sqrt[7]{a^4b^4c^4}$$

$\Rightarrow \quad (a+b+c+ab+bc+ca+abc) \geq 7\sqrt[7]{a^4b^4c^4}$

$\Rightarrow \quad (1+a+b+c+ab+bc+ca+abc)$

$$> (a+b+c+ab+bc+ca+abc) \geq 7\sqrt[7]{a^4b^4c^4}$$

$\Rightarrow \quad (1+a)(1+b)(1+c) \geq 7\sqrt[7]{a^4b^4c^4}$

$\Rightarrow \quad (1+a)^7(1+b)^7(1+c)^7 \geq 7^7(a^4b^4c^4)$

Thus $k = 7$ and $m = 7$

Hence, the value of $k + m$ is 14.

26. We have,

$$t_r = \left(\frac{r+3}{r(r+1)(r+2)}\right)$$

$$= \left(\frac{(r+2)+1}{r(r+1)(r+2)}\right)$$

$$= \left(\frac{1}{r(r+1)} + \frac{1}{r(r+1)(r+2)}\right)$$

$$= \frac{1}{r(r+1)} + \frac{1}{r(r+1)} - \frac{1}{(r+1)(r+2)}$$

$$= \frac{2}{r(r+1)} - \frac{1}{(r+1)(r+2)}$$

$$= 2\left(\frac{1}{r} - \frac{1}{(r+1)}\right) - \left(\frac{1}{(r+1)} - \frac{1}{(r+2)}\right)$$

Now, $S_n = t_1 + t_2 + t_3 + \ldots + t_n$

$$= 2\left(1 - \frac{1}{n+1}\right) - \left(\frac{1}{2} - \frac{1}{n+2}\right)$$

When $n \to \infty$,

$$S_\infty = \left(2 - \frac{1}{2}\right) = \frac{3}{2}$$

27. Let $t_k = k \cdot (2^{n-k+1})$

Now,

$S_n = t_1 + t_2 + t_3 + \ldots + t_n$

$= 2^n + 2.2^{n-1} + 3.2^{n-2} + \ldots + (n-1)\cdot 2^2 + n \cdot 2 \quad \ldots$(i)

$\frac{S_n}{2} = 2^{n-1} + 2.2^{n-2} + \ldots + (n-1)\cdot 2 + n \quad \ldots$(ii)

Subtracting we get,

$$\frac{S_n}{2} = 2^n + 2^{n-1} + 2^{n-2} + \ldots + 2 - n$$

$\Rightarrow \quad \frac{S_n}{2} = 2(2^n - 1) - n = (2^{n+1} - n - 2)$

$\Rightarrow \quad S_n = 2(2^{n+1} - n - 2)$

It is given that the sum is $\frac{1}{4}(n+1)(2^{n+1} - n - 2)$

Thus, $2(2^{n+1} - n - 2) = \frac{1}{4}(n+1)(2^{n+1} - n - 2)$

$\Rightarrow \quad 2 = \frac{1}{4}(n+1)$

$\Rightarrow \quad n + 1 = 8$

$\Rightarrow \quad n = 7$

Hence, the value of n is 7.

28. Given $t_r - t_{r-1} = 2^{r-1}$

$t_2 - t_1 = 2$

$t_3 - t_2 = 2^2$

$t_4 - t_3 = 2^3$

$\vdots$

$t_n - t_{n-1} = 2^{n-1}$

Adding, we get

$t_n - t_1 = 2 + 2^2 + 2^3 + \ldots + 2n$

$t_n - t_1 = 2(2^n - 1)$

$t_n = 2^{n+1} - 1$

Now, $\sum_{r=1}^{n} t_r$.

$= t_1 + t_2 + t_3 + \ldots + t_n$

$= (2-1) + (2^2-1) + (2^3-1) + \ldots + (2^{n+1}-1)$

$= 2(1 + 2 + 2^2 + 2^3 + \ldots + 2^n) - (n+1)$

$= 2(2^{n+1} - 1) - (n+1)$

$= (2^{n+2} - n - 3)$

29. We have $\sum_{q=1}^{n}\sum_{p=1}^{q}\sum_{k=1}^{p} 1$

$$= \sum_{q=1}^{n}\sum_{p=1}^{q} p$$

$$= \sum_{q=1}^{n}\left(\frac{q(q+1)}{2}\right)$$

$$= \frac{1}{2}\sum_{q=1}^{n}(q^2 + q)$$

$$= \frac{1}{2}\left(\sum_{q=1}^{n} q^2 + \sum_{q=1}^{n} q\right)$$

$$= \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\right)$$

$$= \frac{n(n+1)}{4}\left(\frac{(2n+1)}{3} + 1\right)$$

$$= \frac{n(n+1)}{4}\left(\frac{(2n+4)}{3}\right)$$

$$= \frac{n(n+1)(n+2)}{6}$$

30. We have,

$2000x^6 + 100x^5 + 10x^3 + x - 2 = 0$

$\Rightarrow \quad 2000x^6 + x(1 + 10x^2 + 100x^4) - 2 = 0$

$\Rightarrow \quad x(1 + 10x^2 + 100x^4) = 2 - 2000x^6$

$\Rightarrow \quad x\left(\frac{1000x^6 - 1}{10x^2 - 1}\right) = 2(1 - 1000x^6)$

$\Rightarrow \quad \left(\frac{x}{10x^2 - 1}\right) = -2$

$\Rightarrow \quad -x = 20x^2 - 2$

$\Rightarrow \quad 20x^2 + x - 2 = 0$

$$\Rightarrow \quad x = \frac{-1 \pm \sqrt{1+160}}{40}$$

$$\Rightarrow \quad x = \frac{-1+\sqrt{161}}{40} \text{ or } \frac{-1-\sqrt{161}}{40}$$

$$\Rightarrow \quad x = \frac{-1+\sqrt{161}}{40}$$

Thus, $m = -1$, $n = 161$, $r = 40$

Hence, the value of $m + n + r$ is 200.

31. Let $S = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots$ to ∞

We have

$$\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} \ldots \text{ to } \infty$$

$$= \left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\right) + \left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} \ldots \text{ to } \infty\right)$$

$$= S + \left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} \ldots \text{ to } \infty\right)$$

$$= S + \frac{1}{2^4}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots \text{ to } \infty\right)$$

$$= S + \frac{1}{2^4} \times \frac{\pi^4}{90}$$

$$S = \frac{\pi^4}{90} - \frac{1}{2^4} \times \frac{\pi^4}{90} = \frac{\pi^4}{90}\left(1 - \frac{1}{2^4}\right)$$

$$S = \frac{\pi^4}{90}\left(1 - \frac{1}{16}\right) = \frac{\pi^4}{96}$$

32. We have,

$$\sum_{m=1}^{n}\left(\sum_{k=1}^{m}(2^k + 3k)\right)$$

$$= \sum_{m=1}^{n}\left(2(2^m - 1) + 3\frac{m(m+1)}{2}\right)$$

$$= 2\sum_{m=1}^{n}(2^m - 1) + \frac{3}{2}\sum_{m=1}^{n}(m^2 + m)$$

$$= 2\sum_{m=1}^{n}(2^m - 1) + \frac{3}{2}\left(\sum_{m=1}^{n} m^2 + \sum_{m=1}^{n} m\right)$$

$$= 2(2(2^n - 1) - n) + \frac{3}{2}\left(\left(\frac{n(n+1)(2n+1)}{6}\right) + \left(\frac{n(n+1)}{2}\right)\right)$$

$$= 2(2^{n+1} - n - 2) + \frac{3n(n+1)}{4}\left(\frac{2n+1}{3} + 1\right)$$

$$= 2(2^{n+1} - n - 2) + \frac{3n(n+1)(n+2)}{6}$$

33. We have,

$$\tan^{-1}\left(\frac{1}{2r^2}\right)$$

$$= \tan^{-1}\left(\frac{2}{4r^2}\right)$$

$$= \tan^{-1}\left(\frac{2}{1+(4r^2-1)}\right)$$

$$= \tan^{-1}\left(\frac{2}{1+(2r+1)(2r-1)}\right)$$

$$= \tan^{-1}\left(\frac{(2r+1)-(2r-1)}{1+(2r+1)(2r-1)}\right)$$

$$= \tan^{-1}(2r+1) - \tan^{-1}(2r-1)$$

Now, $\sum_{r=1}^{n}(\tan^{-1}(2r+1) - \tan^{-1}(2r-1))$

$$= (\tan^{-1} 3 - \tan^{-1} 1) + (\tan^{-1} 5 - \tan^{-1} 3) + (\tan^{-1} 7 - \tan^{-1} 5) + \ldots + (\tan^{-1}(2n+1) - \tan^{-1}(2n-1))$$

$$= \tan^{-1}(2n+1) - \tan^{-1}(1)$$

$$= \tan^{-1}\left(\frac{2n+1-1}{1+(2n+1)}\right)$$

$$= \tan^{-1}\left(\frac{2n}{2n+2}\right)$$

$$= \tan^{-1}\left(\frac{1}{1+\frac{1}{n}}\right)$$

When $n \to \infty$,

$$= \tan^{-1}(1)$$

$$= \frac{\pi}{4}$$

34. Let $t_n = \dfrac{nx^{n-1}}{(x+1)(x+2)(x+3)\ldots(x+n)}$

$$= \frac{(x+n)x^{n-1} - x^n}{(x+1)(x+2)(x+3)\ldots(x+n)}$$

$$= \frac{x^{n-1}}{(x+1)(x+2)(x+3)\ldots(x+n-1)} - \frac{x^n}{(x+1)(x+2)(x+3)\ldots(x+n)}$$

Now,

$$S_n = t_1 + t_2 + t_3 + \ldots + t_n$$

$$= \left(1 - \frac{1}{x+1}\right) + \left(\frac{x}{x+1} - \frac{x^2}{(x+1)(x+2)}\right) + \ldots + \left(\frac{x^{n-1}}{(x+1)(x+2)\ldots(x+n-1)} - \frac{x^n}{(x+1)(x+2)\ldots(x+n)}\right)$$

$$= \left(1 - \frac{x^n}{(x+1)(x+2)\ldots(x+n)}\right)$$

35. We have $U_n = \sum_{k=0}^{n} \left(\frac{1}{2^k}\right)$

$$= \left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^n}\right)$$

$$= \left(\frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}}\right) = 2\left(1 - \frac{1}{2^{n+1}}\right)$$

Now, $\sum_{n=1}^{n} U_n = \sum_{n=1}^{n} 2\left(1 - \frac{1}{2^{n+1}}\right)$

$$= \sum_{n=1}^{n} 2 - \sum_{n=1}^{n} \frac{1}{2^n}$$

$$= 2n - \left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots + \frac{1}{2^n}\right)$$

$$= 2n - \frac{1}{2}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}}\right)$$

$$= 2n - \frac{1}{2}\left(\frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}}\right) = 2n - \left(1 - \left(\frac{1}{2}\right)^n\right)$$

$$= (2^{-n} + 2n - 1)$$

36. We have,

$$\left(a^2 - \frac{1}{a^2}\right)^2 + \left(b^2 - \frac{1}{b^2}\right)^2$$

$$= \left(a^4 + \frac{1}{a^4} - 2\right) + \left(b^4 + \frac{1}{b^4} - 2\right)$$

$$= \left(a^4 + b^4 + \frac{1}{a^4} + \frac{1}{b^4} - 4\right)$$

We know that, $\left(\frac{a^4 + b^4}{2}\right) \geq \left(\frac{a+b}{2}\right)^4 = 1$

$\Rightarrow \quad (a^4 + b^4) \geq 2$

Also,

$$\left(\frac{a^{-4} + b^{-4}}{2}\right) \geq \left(\frac{a+b}{2}\right)^{-4} = 1$$

$\Rightarrow \quad \left(\frac{a^{-4} + b^{-4}}{2}\right) \geq 1$

$\Rightarrow \quad \left(\frac{1}{a^4} + \frac{1}{b^4}\right) \geq 2$

Hence, the minimum value of

$$\left(a^4 + b^4 + \frac{1}{a^4} + \frac{1}{b^4} - 4\right) \geq 2 + 2 - 4 = 0$$

$\Rightarrow \quad \left(a^2 - \frac{1}{a^2}\right)^2 + \left(b^2 - \frac{1}{b^2}\right)^2 \geq 0$

Hence, the minimum value is 0.

37. We have,

$$\sum_{x=1}^{100} f(x)$$

$$= f(1) + f(2) + f(3) + f(4) + \ldots + f(100)$$

$$= \left[\frac{1}{2} + \frac{1}{100}\right] + \left[\frac{1}{2} + \frac{2}{100}\right] + \left[\frac{1}{2} + \frac{3}{100}\right] + \ldots + \left[\frac{1}{2} + \frac{100}{100}\right]$$

$$= \left[\frac{1}{2} + \frac{1}{100}\right] + \left[\frac{1}{2} + \frac{2}{100}\right] + \ldots + \left[\frac{1}{2} + \frac{49}{100}\right] + \left[\frac{1}{2} + \frac{50}{100}\right] + \left[\frac{1}{2} + \frac{51}{100}\right] + \ldots + \left[\frac{1}{2} + \frac{100}{100}\right]$$

$$= 49 \times 0 + 51 \times 1$$

$$= 51$$

38. We have,

$$\frac{1}{\log_2 4} + \frac{1}{\log_4 4} + \frac{1}{\log_8 4} + \ldots + \frac{1}{\log_{2^n} 4}$$

$$= \frac{1}{\log_2 4} + \frac{1}{\log_{2^2} 4} + \frac{1}{\log_{2^3} 4} + \ldots + \frac{1}{\log_{2^n} 4}$$

$$= \log_4 2 + \log_4 4 + \log_4 8 + \ldots + \log_4 2^n$$

$$= \log_4 (2 \cdot 4 \cdot 8 \cdot 16 \ldots 2^n$$

$$= \log_4 (2 \cdot 2^2 \cdot 2^3 \cdot 2^4 \ldots 2^n)$$

$$= \log_4 (2^{1+2+3+\ldots+n})$$

$$= (1 + 2 + 3 + \ldots + n)\log_4 (2)$$

$$= \frac{n(n+1)}{2} \times \log_{2^2}(2)$$

$$= \frac{n(n+1)}{2} \times \frac{1}{2}\log_2(2)$$

$$= \frac{n(n+1)}{4}$$

39. We have

$$S_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}}$$

$$= \left(\frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}}\right) = 2\left(1 - \frac{1}{2^n}\right)$$

It is given that

$$2 - S_n < \frac{1}{100}$$

$\Rightarrow \quad 2 - 2\left(1 - \frac{1}{2^n}\right) < \frac{1}{100}$

$\Rightarrow \quad \frac{1}{2^{n-1}} < \frac{1}{100}$

Clearly $n - 1 = 7$, where $\frac{1}{128} < \frac{1}{100}$

$\Rightarrow \quad n = 8$

Hence, the least value of n is 8.

40. As we know that AM ≥ GM

$$\frac{s+(s-a)+(s-b)+(s-c)}{4} \geq \sqrt[4]{s(s-a)(s-b)(s-c)}$$

$$\Rightarrow \frac{4s-(a+b+c)}{4} \geq \sqrt[4]{\Delta^2}$$

$$\Rightarrow \frac{4s-2s}{4} \geq \sqrt[4]{\Delta^2}$$

$$\Rightarrow \frac{s}{2} \geq \sqrt[4]{\Delta^2} = (\Delta)^{2/4} = (\Delta)^{1/2}$$

$$\Rightarrow \left(\frac{s}{2}\right)^2 \geq \Delta$$

$$\Rightarrow \Delta \leq \frac{s^2}{4}$$

Hence, the result.

Integer Type Questions

1. We have

$$(1-x) = (x+y+z-x) = y+z$$

Now, $\frac{y+z}{2} \geq \sqrt{yz}$

$$\Rightarrow (y+z) \geq 2\sqrt{yz}$$

$$\Rightarrow (1-x) = (y+z) \geq 2\sqrt{yz} \quad \ldots(i)$$

Similarly, $(1-y) = (x+z) \geq 2\sqrt{xz} \quad \ldots(ii)$

$$(1-z) = (x+y) \geq 2\sqrt{xy} \quad \ldots(iii)$$

Multiplying Eqs (i), (ii) and (iii), we get

$$(1-x)(1-y)(1-z) \geq 8xyz$$

$$\Rightarrow \frac{1}{(1-x)(1-y)(1-z)} \leq \frac{1}{8xyz}$$

$$\Rightarrow \frac{16xyz}{(1-x)(1-y)(1-z)} \leq \frac{16xyz}{8xyz}$$

$$\Rightarrow \frac{16xyz}{(1-x)(1-y)(1-z)} \leq 2$$

Hence, the maximum value is 2.

2. We know that, $\left(\frac{\sin x + \cos x + \operatorname{cosec} 2x}{3}\right) \geq \sqrt[3]{\sin x \cdot \cos x \cdot \operatorname{cosec} 2x}$

$$\Rightarrow \left(\frac{\sin x + \cos x + \operatorname{cosec} 2x}{3}\right)^3 \geq \frac{1}{2}$$

$$\Rightarrow (\sin x + \cos x + \operatorname{cosec} 2x)^3 \geq \frac{27}{2}$$

Thus $m^n = 27 = 3^3$

$\Rightarrow$ $m = 3 = n$

Hence, the value of $(m+n+2)$ is 8.

3. We have,

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right)$$

$$= 1 + \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) + \left(\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right) + \frac{1}{abc}$$

$$= 1 + \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) + \left(\frac{a+b+c}{abc}\right) + \frac{1}{abc}$$

$$= 1 + \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) + \frac{2}{abc}$$

As we know that AM ≥ HM

$$\left(\frac{a+b+c}{3}\right) \geq \frac{3}{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)}$$

$$\Rightarrow \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$$

Also, $\left(\frac{a+b+c}{3}\right) \geq \sqrt[3]{abc}$

$$\Rightarrow \frac{1}{3} \geq \sqrt[3]{abc}$$

$$\Rightarrow (abc) \leq \frac{1}{27}$$

$$\Rightarrow \frac{1}{(abc)} \geq 27$$

$$\Rightarrow \frac{2}{(abc)} \geq 54$$

Thus, $1 + \left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) + \frac{2}{abc} \geq 1 + 9 + 54 = 64$

$$\Rightarrow \left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)\left(1+\frac{1}{c}\right) \geq 64 = 4^3 = p^2$$

$\Rightarrow$ $p = 4$ and $q = 3$

Hence, the value of $p + q$ is 7.

4. It is given that, $a + b + c = 1$ so,

$$1 - a = a + b + c - a = b + c$$
$$1 - b = a + b + c - b = c + a$$
$$1 - c = a + b + c - c = a + b$$

Applying AM ≥ GM, we have

$$\frac{b+c}{2} \geq \sqrt{bc}$$

$$\Rightarrow (b+c) \geq 2\sqrt{bc}$$

Similarly $(c+a) \geq 2\sqrt{ca}$

$$(a+b) \geq 2\sqrt{ab}$$

After multiplying, we get

$$(b+c)(c+a)(a+b) \geq 8(abc)$$

$\Rightarrow$ $(1-a)(1-b)(1-c) \geq 8(abc)$

$\Rightarrow$ $k = 8$

Hence, the value of $(\sqrt[3]{k}+1)$ is 3.

5. We know that
$$\left(\frac{1+a}{2}\right) \geq \sqrt{a}$$
$\Rightarrow \quad (1+a) \geq 2\sqrt{a}$

Similarly, $(1+b) \geq 2\sqrt{b}$
$$(1+c) \geq 2\sqrt{c}$$
$$(1+d) \geq 2\sqrt{d}$$
Multiplying, we get
$$(1+a)(1+b)(1+c)(1+d) \geq 16\sqrt{abcd}$$
Thus, $\lambda = 16$

Hence, the value of $(\sqrt[4]{\lambda}+1)$ is 3.

6. Let the common ratio be r.

Given $a_1 + a_n = 66$

$\Rightarrow \quad a + ar^{n-1} = 66$ …(i)

Also, $a_2 \cdot a_{n-1} = 128$

$\Rightarrow \quad ar \cdot ar^{n-2} = 128$

$\Rightarrow \quad a^2 r^{n-1} = 128$ …(ii)

From Eqs (i) and (ii), we get
$$a\left(1+\frac{128}{a^2}\right) = 66$$
$\Rightarrow \quad a + \frac{128}{a} = 66$

$\Rightarrow \quad a^2 - 66a + 128 = 0$

$\Rightarrow \quad (a-2)(a-64) = 0$

$\Rightarrow \quad a = 2$ or 64

From Eq. (i), we get,
$$r^{n-1} = 32$$
Again, $\sum_{i=1}^{n} a_i = 126$

$\Rightarrow \quad (a_1 + a_2 + a_3 + \ldots + a_n) = 126$

$\Rightarrow \quad (a + ar + ar^2 + \ldots + ar^{n-1}) = 126$

$\Rightarrow \quad a(1 + r + r^2 + \ldots + r^{n-1}) = 126$

$\Rightarrow \quad a\left(\frac{1-r^n}{1-\mathrm{r}}\right) = 126$

$\Rightarrow \quad 2\left(\frac{1-r^n}{1-r}\right) = 126$

$\Rightarrow \quad \left(\frac{1-r^n}{1-r}\right) = 63$

$\Rightarrow \quad \left(\frac{1-32r}{1-r}\right) = 63$

$\Rightarrow \quad 1 - 32r = 63 - 63r$

$\Rightarrow \quad 31r = 62$

$\Rightarrow \quad r = 2$

Now, from Eq. (iii), we get,
$$2^{n-1} = 32 = 2^5$$
$\Rightarrow \quad n - 1 = 5$

$\Rightarrow \quad n = 6$

7. Given $\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} = \frac{10}{3}$

$\Rightarrow \quad \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} = \frac{10}{3}$

$\Rightarrow \quad \frac{3}{b+c} + \frac{3}{c+a} + \frac{3}{a+b} = 10$

$\Rightarrow \quad \frac{a+b+c}{b+c} + \frac{a+b+c}{c+a} + \frac{a+b+c}{a+b} = 10$

$\Rightarrow \quad \frac{a+(b+c)}{b+c} + \frac{b+(a+c)}{c+a} + \frac{c+(a+b)}{a+b} = 10$

$\Rightarrow \quad \frac{a}{b+c} + 1 + \frac{b}{c+a} + 1 + \frac{c}{a+b} + 1 = 10$

$\Rightarrow \quad \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} = 10 - 3 = 7$

Hence, the value of
$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \text{ is } 7.$$

8. Let $s = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots + \frac{1}{4.2^{n-1}}$
$$= \frac{1}{4}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots + \frac{1}{2^{n-1}}\right)$$
$$= \frac{1}{4}\left(\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}\right) = \frac{1}{2}\left(1-\left(\frac{1}{2}\right)^n\right)$$
Now, $b_n = \left(\frac{1}{5}\right)^{\log_{\sqrt{5}} \frac{1}{2}\left(1-\left(\frac{1}{2}\right)^n\right)}$
$$= \left(\frac{1}{5}\right)^{2\log_5 \frac{1}{2}\left(1-\left(\frac{1}{2}\right)^n\right)}$$
$$= (5)^{-2\log_5 \frac{1}{2}\left(1-\left(\frac{1}{2}\right)^n\right)}$$
$$= (5)^{-2\log_5 \frac{1}{2}}, \text{ when } n \to \infty$$
$$= (5)^{\log_5 (4)} = 4$$
Thus, $\lim_{n\to\infty} (b_n) + 3 = 7$

9. Given a, x, y, z, b are in AP.

Clearly x, y, z are the AM between a and b

Thus, $x + y + z = 3\left(\frac{a+b}{2}\right)$

$\Rightarrow \quad 3\left(\frac{a+b}{2}\right) = 15$

$\Rightarrow \quad (a+b) = 10$

Also, a, x, y, z, b are in HP.

$\Rightarrow \quad \frac{1}{a}, \frac{1}{x}, \frac{1}{y}, \frac{1}{z}, \frac{1}{b}$ are in HP.

$$\Rightarrow \quad \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{3}{2}\left(\frac{1}{a}+\frac{1}{b}\right)$$

$$=\frac{3}{2}\left(\frac{a+b}{ab}\right)=\frac{3}{2}\left(\frac{10}{ab}\right)$$

$$\Rightarrow \quad \frac{3}{2}\left(\frac{10}{ab}\right)=\frac{5}{3}$$

$\Rightarrow \quad ab = 9$

Clearly a, b are the roots of

$t^2 - (a+b)t + ab = 0$

$\Rightarrow \quad t^2 - 10t + 9 = 0$

$\Rightarrow \quad (t-9)(t-1) = 0$

$\Rightarrow \quad t = 9$ or 1

$\Rightarrow \quad a = 9, b = 1$

Hence, the value of $(a - b - 2)$ is 6

10. We have

$$\frac{1-p^6}{1-p}=(1+3x+(3x)^2+(3x)^3+\ldots+(3x^5))$$

$$\Rightarrow \quad \frac{1-p^6}{1-p}=\frac{1-(3x)^6}{1-(3x)}$$

Comparing, we get

$p = 3x$

$$\Rightarrow \quad \left|\frac{p}{x}+2\right|=3+2=5$$

11. We have

$$\sum_{x=5}^{n+5}4(x-3) = 4\sum_{x=5}^{n+5}(x-3)$$

$$= 4[2+3+4+\ldots+(n+2)]$$

$$= 4\cdot\frac{n}{2}(2+n+2)$$

$$= 2n(n+4)$$

$$= 2n^2 + 8n$$

Comparing with $An^2 + B_n + C$ we get

$A = 2, B = 8, C = 0$

Hence, the value of $(A + B - C - 4)$ is 6.

12. We have $\sum_{r=1}^{n} r(r+1)(2r+3)$

$$=\sum_{r=1}^{n}(2r^3+5r^2+3r)$$

$$=2\sum_{r=1}^{n}r^3+5\sum_{r=1}^{n}r^2+3\sum_{r=1}^{n}r$$

$$=2\left(\frac{n(n+1)}{2}\right)^2+5\left(\frac{n(n+1)(2n+1)}{6}\right)+3\left(\frac{n(n+1)}{2}\right)$$

$$=\left(\frac{n(n+1)}{2}\right)\left[2\left(\frac{n(n+1)}{2}\right)+5\left(\frac{(2n+1)}{3}\right)+3\right]$$

$$=\frac{1}{6}n(n+1)(3n^2+13n+14)$$

$$=\frac{1}{6}(n+1)(3n^3+13n^2+14n)$$

$$=\frac{1}{6}(3n^4+16n^3+27n^2+14n)$$

$$\text{Now, } (a+c+1)=\left(\frac{3+27}{6}+1\right)$$

$$=5+1=6$$

13. As we know that

$$\left(\frac{x^2+y^2}{2}\right)\geq\left(\frac{x+y}{2}\right)^2$$

$$\Rightarrow \quad \left(\frac{8}{2}\right)\geq\left(\frac{x+y}{2}\right)^2$$

$$\Rightarrow \quad 4\geq\left(\frac{x+y}{2}\right)^2$$

$\Rightarrow \quad (x+y)^2 \leq 16$

$\Rightarrow \quad (x+y) \leq 4$

Hence, the maximum value of $(x + y)$ is 4.

14. We have

$$\sum_{k=1}^{n}\left(\sum_{m=1}^{k}m^2\right)$$

$$=\sum_{k=1}^{n}\left(\frac{k(k+1)(2k+1)}{6}\right)$$

$$=\frac{1}{6}\sum_{k=1}^{n}(k(2k^2+3k+1))$$

$$=\frac{1}{6}\sum_{k=1}^{n}(2k^3+3k^2+k)$$

$$=\frac{1}{6}\left[2\left(\frac{n(n+1)}{2}\right)^2+3\left(\frac{n(n+1)(2n+1)}{6}\right)+\left(\frac{n(n+1)}{2}\right)\right]$$

$$=\frac{1}{6}\left(\frac{n(n+1)}{2}\right)\left[2\left(\frac{n(n+1)}{2}\right)+2n+1+1\right]$$

$$=\frac{1}{6}\left(\frac{n(n+1)}{2}\right)(n^2+3n+2)$$

$$=\frac{1}{12}(n^2+n)(n^2+3n+2)$$

$$=\frac{1}{24}(n^4+4n^3+5n^2+2n)$$

Comparing with $an^4 + bn^3 + cn^2 + d_n + e$,

We have

$$a=\frac{1}{12} \text{ and } d=\frac{1}{6}$$

Hence, the value of $12(a + d)$

$$= 12\left(\frac{1}{12} + \frac{1}{6}\right)$$

$$= 12 \times \frac{1+2}{12}$$

$$= 3.$$

15. Given equation $ab + bc + ca = 12$ will provide us the greatest value if

$$ab = bc = ca = \frac{12}{3} = 4$$

Hence, the greatest value of

$$(abc)^2 = 4^3 = 8^2$$

$\Rightarrow \quad abc = 8$

Previous Years' JEE-Advanced Examinations

1. Given $x + y + z = 15$

Now, $x + y + z = \frac{5}{2}(a + b)$

$\Rightarrow \quad \frac{5}{2}(a + b) = 15$

$\Rightarrow \quad (a + b) = 6$ …(i)

Also, $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{2}{\frac{1}{a} + \frac{1}{b}}$

$\Rightarrow \quad \frac{2ab}{a+b} = \frac{5}{3}$

$\Rightarrow \quad \frac{2ab}{6} = \frac{5}{3}$

$\Rightarrow \quad ab = 5$

Solving Eqs (i) and (ii), we get

$a = 1, b = 5.$

2. Given

$\log(x + z) + \log(x + z - y)$
$= 2\log(x - z)$

$\Rightarrow \quad \log\{(x + z)(x + z - 2y)\} = \log\{(x - z)^2\}$

$\Rightarrow \quad (x + z)(x + z - 2y) = (x - z)^2$

$\Rightarrow \quad (x + z)^2 - 2y(x + z) = (x - z)^2$

$\Rightarrow \quad 4xz = 2y(x + z)$

$\Rightarrow \quad y = \frac{2xz}{(y + z)}$

Thus, x, y, z are in HP.

3. Let a and d be the first term and the common difference, respectively of the given AP whereas b and r be the first term and common ratio, respectively of the given GP.

Thus, $x = a + (m - 1)d$, $x = br^{m-1}$
$y = a + (n - 1)d$, $y = br^{n-1}$
$z = a + (p - 1)d$, $z = br^{p-1}$

Now, $x - y = (m - n)d$
$y - z = (n - p)d$
$z - x = (p - m)d$

We have $x^{y-z} \cdot y^{z-x} \cdot x^{x-y}$

$= (br^{m-1})^{(n-p)d} \cdot (br^{n-1})^{(p-m)d} \cdot (br^{p-1})^{(m-n)d}$

$= b^{(n-p+p-m+m-n)d} \cdot r^{(m-1)(n-p)+(n-1)(p-m)+(p-1)(m-n)}$

$= b^0 \cdot r^0$

$= 1$

4. Let the number of sides be n.

Here, $a = 120°$, $d = 5°$

$n[2a + \{(n - 1)d\}] = (2n - 4) \times 180°$

$\Rightarrow \quad n[2 \cdot 120° + \{(n - 1)5°\}] = (2n - 4) \times 180°$

$\Rightarrow \quad 240n + 5(n^2 - n) = 360n - 720$

$\Rightarrow \quad 5n^2 - 125n + 720 = 0$

$\Rightarrow \quad n^2 - 25n + 144 = 0$

$\Rightarrow \quad (n - 9)(n - 16) = 0$

$\Rightarrow \quad n = 9$ and 16

$\Rightarrow \quad n = 9$ is the required solution.

5. Given A, B, C are in AP.

$\Rightarrow \quad 2B = A + C$

$\Rightarrow \quad 3B = A + B + C = 180°$

$\Rightarrow \quad B = 60°$

Given $b : c = \sqrt{3} : \sqrt{2}$

Now, by sine rule,

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$\Rightarrow \quad \frac{b}{c} = \frac{\sin B}{\sin C}$

$\Rightarrow \quad \frac{\sin(60°)}{\sin(120° - A)} = \frac{\sqrt{3}}{\sqrt{2}}$

$\Rightarrow \quad \frac{\frac{\sqrt{3}}{2}}{\sin(120° - A)} = \frac{\sqrt{3}}{\sqrt{2}}$

$\Rightarrow \quad \frac{1}{2\sin(120° - A)} = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \sin(120° - A) = \frac{1}{\sqrt{2}}$

$\Rightarrow \quad \sin(120° - A) = \sin(45°)$

$\Rightarrow \quad A = 85°$

6. Given $a_2 - a_1 = a_3 - a_2 = \ldots = a_n - a_{n-1} = d$

$a_n = a_1 + (n - 1)d$

$(n - 1)d = a_n - a_1$

We have

$$\frac{1}{\sqrt{a_1} + \sqrt{a_2}} + \frac{1}{\sqrt{a_2} + \sqrt{a_3}} + \ldots + \frac{1}{\sqrt{a_{n-1}} + \sqrt{a_n}}$$

$$= \frac{1}{d}\left(\frac{d}{\sqrt{a_1} + \sqrt{a_2}} + \frac{d}{\sqrt{a_2} + \sqrt{a_3}} + \ldots + \frac{d}{\sqrt{a_{n-1}} + \sqrt{a_n}}\right)$$

$$= \frac{1}{d}\left(\frac{a_2 - a_1}{\sqrt{a_1} + \sqrt{a_2}} + \frac{a_3 - a_2}{\sqrt{a_2} + \sqrt{a_3}} + \ldots + \frac{a_n - a_{n-1}}{\sqrt{a_{n-1}} + \sqrt{a_n}}\right)$$

$$= \frac{1}{d}[(\sqrt{a_2} - \sqrt{a_1}) + (\sqrt{a_3} - \sqrt{a_2}) + \ldots + (\sqrt{a_n} - \sqrt{a_{n-1}})]$$

$$= \frac{1}{d}(\sqrt{a_n} - \sqrt{a_1})$$

$$= \frac{1}{d}\left(\frac{(a_n - a_1)}{\sqrt{a_n} + \sqrt{a_1}}\right)$$

$$= \frac{1}{d}\left(\frac{(n-1)d}{\sqrt{a_n} + \sqrt{a_1}}\right)$$

$$= \left(\frac{(n-1)}{\sqrt{a_n} + \sqrt{a_1}}\right)$$

7. Let $t_n = 27$, $t_m = 8$, $t_p = 12$

Now, $\frac{t_n}{t_m} = \frac{27}{8} \qquad \frac{t_m}{t_p} = \frac{8}{12}$

$\Rightarrow \quad \frac{ar^{n-1}}{ar^{m-1}} = \frac{27}{8} \qquad \frac{ar^{m-1}}{ar^{p-1}} = \frac{8}{12}$

$\Rightarrow \quad r^{n-m} = \frac{27}{8} \qquad r^{m-p} = \frac{8}{12}$

$\Rightarrow \quad r^{n-m} = \left(\frac{3}{2}\right)^3 \qquad r^{m-p} = \left(\frac{2}{3}\right)$

$\Rightarrow \quad r^{n-m} = \left(\frac{3}{2}\right)^3 \qquad r^{p-m} = \left(\frac{3}{2}\right)$

$\Rightarrow \quad r^{\frac{n-m}{3}} = \left(\frac{3}{2}\right) \qquad r^{p-m} = \left(\frac{3}{2}\right)$

$\Rightarrow \quad r^{\frac{n-m}{3}} = r^{p-m}$

$\Rightarrow \quad \frac{n-m}{3} = p - m$

$\Rightarrow \quad n - m = 3p - 3m$

$\Rightarrow \quad 3p - 2m = n$

$\Rightarrow \quad 3p - 2m - n = 0$

Thus, there exist infinite number of triplets (n, m, p) satisfying $3p - 2m - n = 0$, where $m, n, p \in N$.

8. Let the first term = a and the common ratio = r.

Given $ar^2 = 4$

Let $P = a \cdot ar \cdot ar^2 \cdot ar^3 \cdot ar^4$

$= a^5 r^{1+2+3+4}$

$= a^5 r^{10}$

$= (ar^2)^5$

$= 4^5$

9. Given $2a = 2 + b \Rightarrow a = 1 + \frac{b}{2}$

$18b = c^2 \Rightarrow b = \frac{c^2}{18}$

$\therefore \quad a = 1 + \frac{b}{2} = 1 + \frac{c^2}{36}$

Given $a + b + c = 25$

$2a = 2 + b \Rightarrow a = 1 + \frac{b}{2} = 1 + \frac{c^2}{36}$,

$18b = c^2 \Rightarrow b = \frac{c^2}{18}$

Now, $a + b + c = 25$

$\Rightarrow \quad 1 + \frac{c^2}{36} + \frac{c^2}{18} + c = 25$

$\Rightarrow \quad \frac{3c^2}{36} + c = 24$

$\Rightarrow \quad \frac{c^2}{12} + c = 24$

$\Rightarrow \quad c^2 + 12c - 288 = 0$

$\Rightarrow \quad (c + 24)(c - 12) = 0$

$\Rightarrow \quad c = 12$ or -24

$\Rightarrow \quad c = 12$, since $2 \le c \le 18$

Thus, $a = 1 + 4 = 5$, $b = 8$, $c = 12$.

13. It is a false statement.

Given $\cot A$, $\cot B$, $\cot C$ are in AP.

$\Rightarrow \quad \frac{\cos A}{\sin A}, \frac{\cos B}{\sin B}, \frac{\cos C}{\sin C} \in \text{AP}$

$\Rightarrow \quad \frac{b^2 + c^2 - a^2}{2abck}, \frac{c^2 + a^2 - b^2}{2abck}, \frac{a^2 + b^2 - c^2}{2abck} \in \text{AP}$

$\Rightarrow \quad (b^2 + c^2 - a^2), (c^2 + a^2 - b^2), (a^2 + b^2 - c^2) \in \text{AP}$

$\Rightarrow \quad (a^2 + b^2 + c^2 - 2a^2), (b^2 + c^2 + a^2 - 2b^2),$

$(a^2 + b^2 + c^2 - 2c^2) \in \text{AP}$

$\Rightarrow \quad (-2a^2), (-2b^2), (-2c^2) \in \text{AP}$

$\Rightarrow \quad a^2, b^2, c^2 \in \text{AP}$

14. It is a false statement.

Let z_1, z_2, z_3 be three complex numbers, which are in AP.

Thus, $2z_2 = z_1 + z_3$

$\Rightarrow \quad z_2 = \frac{z_1 + z_3}{2}$

Therefore, z_1, z_2, z_3 lie on a straight line.

15. Given a, b, c are in GP.

$\Rightarrow \quad b^2 = ac$

$\Rightarrow \quad b = \sqrt{ac}$

Given equation is $ax^2 + 2bx + c = 0$

$\Rightarrow \quad ax^2 + 2\sqrt{ac}\,x + c = 0$

$\Rightarrow \quad (\sqrt{a}x + \sqrt{c})^2 = 0$

$\Rightarrow \quad (\sqrt{a}x + \sqrt{c}) = 0$

$\Rightarrow \quad x = -\frac{\sqrt{c}}{\sqrt{a}}$

Since the equations have a common root, so

$$d\left(\frac{c}{a}\right)-2e\left(\frac{\sqrt{c}}{\sqrt{a}}\right)+f=0$$

$$\Rightarrow \quad d\left(\frac{c}{a}\right)-2e\left(\frac{\sqrt{c^2}}{\sqrt{ac}}\right)+f=0$$

$$\Rightarrow \quad d\left(\frac{c}{a}\right)-2e\left(\frac{c}{b}\right)+f=0$$

$$\Rightarrow \quad \left(\frac{d}{a}\right)-2\left(\frac{e}{b}\right)+\frac{f}{c}=0$$

$$\Rightarrow \quad \left(\frac{d}{a}\right)+\frac{f}{c}=2\left(\frac{e}{b}\right)$$

$\Rightarrow \quad \frac{d}{a},\frac{e}{b},\frac{f}{c}$ are in AP.

16 Let three terms be a, ar, ar^2.
Given $a + ar + ar^2 = \alpha S$ …(i)
and $a^2 + a^2r^2 + a^2r^4 = S^2$ …(ii)
Dividing Eqs (i) and (ii), we get,

$$\frac{(1+r+r^2)^2}{(1+r^2+r^4)}=\frac{\alpha^2 S^2}{S^2}=\alpha^2$$

$$\Rightarrow \quad \frac{(1+r+r^2)}{(1-r+r^2)}=\alpha^2$$

$\Rightarrow \quad \alpha^2(1-r+r^2)=(1+r+r^2)$
$\Rightarrow \quad (\alpha^2-1)r^2-(\alpha^2+1)r+(\alpha^2-1)=0$

As r is real, so
$(\alpha^2+1)^2-4(\alpha^2-1)^2 \geq 0$
$\Rightarrow \quad (\alpha^2+1)^2-(2\alpha^2-2)^2 \geq 0$
$\Rightarrow \quad (\alpha^2+1+2\alpha^2-2)(\alpha^2+1-2\alpha^2+2) \geq 0$
$\Rightarrow \quad (3\alpha^2-1)(-\alpha^2+3) \geq 0$
$\Rightarrow \quad (3\alpha^2-1)(\alpha^2-3) \leq 0$

$$\Rightarrow \quad \frac{1}{3}\leq \alpha^2 \leq 3$$

But $\alpha^2 = 1 \Rightarrow r = 0$. Which it is not possible.

Thus, $\frac{1}{3}\leq \alpha^2 \leq 3-\{1\}$

$$\Rightarrow \quad \alpha^2 \in \left(\frac{1}{3},1\right)\cup(1,3)$$

Hence, the result.

17. We have $(a^2+b^2+c^2)p^2$
$-2(ab+bc+cd)p+(b^2+c^2+d^2) \leq 0$
$\Rightarrow \quad (ap-b)^2+(bp-c)^2+(cp-a)^2 \leq 0$
$\Rightarrow \quad (ap-b)^2+(bp-c)^2+(cp-a)^2 = 0$
$\Rightarrow \quad (ap-b)^2=0, (bp-c)^2=0, (cp-a)^2=0$
$\Rightarrow \quad (ap-b)=0, (bp-c)=0, (cp-a)=0$

$$\Rightarrow \quad \frac{a}{b}=\frac{b}{c}=\frac{c}{a}=p$$

Thus, a, b, c, d are in GP.

18. Given $\alpha+\beta+\gamma=2\pi$
Now,

$$A=\frac{1}{3}\left[\cos\left(\alpha+\frac{\pi}{2}\right)\cdot\cos\left(\beta+\frac{\pi}{2}\right)\cdot\cos\left(\gamma+\frac{\pi}{2}\right)\right]$$

$$=-\frac{1}{3}(\sin\alpha+\sin\beta+\sin\gamma)$$

$$=-\frac{1}{3}(\sin\alpha+\sin\beta+\sin(2\pi-(\alpha+\beta)))$$

$$=-\frac{1}{3}(\sin\alpha+\sin\beta-\sin(\alpha+\beta))$$

$$=-\frac{2}{3}\sin\left(\frac{\alpha+\beta}{2}\right)\left(\cos\left(\frac{\alpha-\beta}{2}\right)-\cos\left(\frac{\alpha+\beta}{2}\right)\right)$$

$$=-\frac{4}{3}\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)$$

$$=-\frac{4}{3}\sin\left(\pi-\frac{\gamma}{2}\right)\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)$$

$$=-\frac{4}{3}\sin\left(\frac{\gamma}{2}\right)\sin\left(\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}\right)$$

A will be least only when, $\alpha=\frac{2\pi}{3}=\beta=\gamma$

Thus, the least value of A is

$$=-\frac{1}{3}\left(3\times\sin\left(\frac{2\pi}{3}\right)\right)=-\frac{\sqrt{3}}{2}$$

19. Given

$$\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+\ldots$$

$$=\left(1-\frac{1}{2}\right)+\left(1-\frac{1}{4}\right)+\left(1-\frac{1}{8}\right)+\left(1-\frac{1}{16}\right)+\ldots$$

$$=n-\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots+\frac{1}{2^n}\right)$$

$$=n-\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}\right)$$

$$=n-\frac{1}{2}\left(\frac{1-\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}\right)$$

$$=n-\left(1-\left(\frac{1}{2}\right)^n\right)$$

$$=n-1+2^{-n}.$$

20. Let $n = 2$m.
Then $1^2+2.2^2+3^2+2.4^2+5^2+2.6^2+\ldots$
$\ldots+(2m-1)^2+2\cdot(2m)^2$

$$=\frac{2m(2m+1)^2}{2}$$

$\Rightarrow \quad 1^2 + 2.2^2 + 3^2 + 2.4^2 + 5^2 + 2.6^2 + \ldots$

$\ldots + (2m-1)^2$

$$= \frac{2m(2m+1)^2}{2} - 2\cdot(2m)^2$$

$= m((2m+1)^2 - 4m^2)$

$= m(2m-1)^2$

Put $2m - 1 = n$, we get

$$1^2 + 2.2^2 + 3^2 + 2.4^2 + 5^2 + 2.6^2 + \ldots + n^2 = \frac{n^2(n+1)}{2}.$$

21. Let $a_1, a_2, \ldots, a_n \in$ AP with the common difference d and $a_1 = A$ and $a_{2n-1} = B$.
Then $a_{2n-1} - a_1 = B - A$

$(2n - 1 - 1)d = B - A$

$2(n - 1)d = B - A$

$$d = \frac{B-A}{2(n-1)}$$

Thus, $a = a_n = a_1 + (n-1)d$

$$= A + \frac{(B-A)}{2} = \left(\frac{A+B}{2}\right)$$

Next, if $b_1, b_2, \ldots, b_{2n-1} \in$ GP with the common ratio r and $b_1 = A$, $b_{2n-1} = B$

Thus, $\dfrac{b_{2n-1}}{b_1} = \dfrac{B}{A}$

$$r^{2n-2} = \frac{B}{A}$$

$$r^{n-1} = \sqrt{\frac{B}{A}}$$

Now, $b = b_n = b_1 r^{n-1} = \sqrt{AB}$

Similarly, $c = \dfrac{2ab}{a+b}$

It is easy to show that $ac - b^2 = 0$.

Now, $a - b = \dfrac{1}{2}(A+B) - \sqrt{AB}$

$$= \frac{1}{2}(\sqrt{A} - \sqrt{B})^2 \geq 0$$

Similarly, $\dfrac{1}{c} - \dfrac{1}{b} \geq 0 \Rightarrow b \geq c$

Thus, $a \geq b \geq c$.

23. Given

$$\log_3 2, \log_3(2^x - 5), \log_3\left(2^x - \frac{7}{2}\right) \text{ are in AP.}$$

$$\Rightarrow \quad 2\log_3(2^x - 5) = \log_3 2 + \log_3\left(2^x - \frac{7}{2}\right)$$

$$\Rightarrow \quad \log_3(2^x - 5)^2 = \log_3\left\{2\cdot\left(2^x - \frac{7}{2}\right)\right\}$$

$$\Rightarrow \quad (2^x - 5)^2 = \left\{2\cdot\left(2^x - \frac{7}{2}\right)\right\}$$

$$\Rightarrow \quad (a-5)^2 = \left\{2\cdot\left(a - \frac{7}{2}\right)\right\}, a = 2^x$$

$\Rightarrow \quad a^2 - 10a + 25 = 2a - 7$

$\Rightarrow \quad a^2 - 12a + 32 = 0$

$\Rightarrow \quad (a-8)(a-4) = 0$

$\Rightarrow \quad a = 4$ or 8

When $a = 4$

$\Rightarrow \quad 2^x = 4 = 2^2 \Rightarrow x = 2$

When $a = 8$

$\Rightarrow \quad 2^x = 2^3$

$\Rightarrow \quad x = 3$

But $x = 2$ does not satisfy the given equation.

Thus, the solution is $x = 3$.

24. Let a and b be two numbers respectively.

Then $p = a + \dfrac{b-a}{n+1} = \dfrac{na+b}{n+1}$

and $\dfrac{1}{q} = \dfrac{1}{n+1}\left(\dfrac{n}{a} + \dfrac{1}{b}\right)$

Thus, $(n+1)p - na = b$ and $\left(\dfrac{n+1}{q} - \dfrac{n}{a}\right) = \dfrac{1}{b}$

Eliminating b, we get

$$((n+1)p - na)\left(\frac{n+1}{q} - \frac{n}{a}\right) = 1$$

$\Rightarrow \quad ((n+1)p - na)(a(n+1) - nq) = qa$

$\Rightarrow \quad (n+1)^2 ap - npq(n+1) - na^2(n+1) + (n^2-1)aq = 0$

$\Rightarrow \quad (n+1)ap - npq - na^2 + (n-1)aq = 0$

$\Rightarrow \quad na^2 - ((n-1)q + (n+1)p)a + npq = 0$

As a is real, so

$((n-1)q + (n+1)p)^2 - 4n^2pq \geq 0$

$\Rightarrow \quad (n-1)^2q^2 - 2(n^2+1)pq + (n+1)^2p^2 \geq 0$

Thus q cannot lie between the roots of

$(n-1)^2x^2 - 2(n^2+1)xp + (n+1)^2p^2 = 0$

One of the roots of this equation is p and the other is $\left(\dfrac{n+1}{n-1}\right)^2 p$.

Also product of the roots $= \left(\dfrac{n+1}{n-1}\right)^2 p^2$

Thus q cannot lie between the roots of p and $\left(\dfrac{n+1}{n-1}\right)^2 p$.

25. We have $S_1 = \dfrac{1}{1-\dfrac{1}{2}} = 2$

$$S_2 = \frac{2}{1-\frac{1}{3}} = 3$$

$$S_3 = \frac{3}{1-\frac{1}{4}} = 4$$

$$\vdots$$

$$S_n = \frac{n}{1-\frac{1}{n+1}} = n+1$$

Thus, $S_1^2 + S_2^2 + \ldots + S_{2n-1}^2$

$= 2^2 + 3^3 + 4^2 + \ldots + (2n)^2$

$= [1^2 + 2^2 + 3^3 + 4^2 + \ldots + (2n)^2] - 1$

$= \dfrac{2n(2n+1)(4n+1)}{6} - 1$

26. Let two positive numbers be a and b respectively.

Given $\dfrac{\frac{2ab}{a+b}}{\sqrt{ab}} = \dfrac{4}{5}$

$\Rightarrow \quad \dfrac{\sqrt{ab}}{a+b} = \dfrac{2}{5}$

$\Rightarrow \quad \dfrac{a+b}{\sqrt{ab}} = \dfrac{5}{2}$

$\Rightarrow \quad (a+b)^2 = \dfrac{25ab}{4}$ …(i)

Now, $(a-b)^2 = (a+b)^2$

$= \dfrac{25ab}{4} - 4ab = \dfrac{9ab}{4}$ …(ii)

Dividing Eq. (i) and Eq. (ii), we get

$\dfrac{(a+b)^2}{(a-b)^2} = \dfrac{25}{9}$

$\Rightarrow \quad \dfrac{(a+b)}{(a-b)} = \dfrac{5}{3}$

$\Rightarrow \quad \dfrac{(a+b)+(a-b)}{(a+b)-(a-b)} = \dfrac{5+3}{5-3}$

$\Rightarrow \quad \dfrac{a}{b} = \dfrac{8}{2} = \dfrac{4}{1}$

27. Given $x = \sum_{n=0}^{\infty} \cos^{2n}\varphi$

$\Rightarrow \quad x = 1 + \cos^2\varphi + \cos^4\varphi + \ldots$

$\Rightarrow \quad x = \dfrac{1}{1-\cos^2\varphi} = \dfrac{1}{\sin^2\varphi} = \operatorname{cosec}^2\varphi$

Similarly,

$y = \sec^2\varphi$

and $\quad z = \dfrac{1}{1-\cos^2\varphi \sin^2\varphi}$

Thus, $z = \dfrac{1}{1-\frac{1}{y}\cdot\frac{1}{x}} = \dfrac{xy}{xy-1}$

$\Rightarrow \quad xyz - z = xy$

$\Rightarrow \quad xyz = xy + z$

Again, $\dfrac{1}{x} + \dfrac{1}{y} = \sin^2\varphi + \cos^2\varphi = 1$

$\Rightarrow \quad x + y = xy = xyz - z$

$\Rightarrow \quad x + y + z = xyz$

28. Given

$\ln(a+c), \ln(a-c), \ln(a-2b+c) \in$ AP

$\Rightarrow \quad (a+c), (a-c), (a-2b+c) \in$ GP

$\Rightarrow \quad (a-c)^2 = (a+c)(a+c-2b)$

$\Rightarrow \quad a^2 + c^2 - 2ac = a^2 + ac - 2ab + ac + c^2 - 2bc$

$\Rightarrow \quad 2ab + 2bc = 4ac$

$\Rightarrow \quad ab + bc = 2ac$

$\Rightarrow \quad b(a+c) = 2ac$

$\Rightarrow \quad b = \dfrac{2ac}{a+c}$

Thus, a, b, c are in HP

31. Now, $x_1 + x_2 + x_3 = 1$ …(i)

$x_1x_2 + x_2x_3 + x_3x_1 = \beta$ …(ii)

$x_1x_2x_3 = -\gamma$ …(iii)

From Eq. (i), we get,

$2x_2 + x_2 = 1 \Rightarrow x_2 = \dfrac{1}{3}$

From Eq. (ii), we get

$\beta = x_2(x_1 + x_3) + x_1x_3$

$= 2x_2^2 + x_1x_3$

$= \dfrac{2}{9} + \left(\dfrac{1}{3} - d\right)\left(\dfrac{1}{3} + d\right)$

$= \dfrac{2}{9} + \dfrac{1}{9} - d^2$

$= \dfrac{1}{3} - d^2$

$\Rightarrow \quad \left(\beta - \dfrac{1}{3}\right) = -d^2 \le 0$

$\Rightarrow \quad \beta \le \dfrac{1}{3}$

$\Rightarrow \quad \beta \in \left(-\infty, \dfrac{1}{3}\right]$

From Eq. (iii), we get

$x_1x_2x_3 = -\gamma$

$\Rightarrow \quad \left(\dfrac{1}{3} - d\right)\dfrac{1}{3}\cdot\left(\dfrac{1}{3} + d\right) = -\gamma$

$\Rightarrow \quad \gamma = \frac{1}{3}\left(d^2 - \frac{1}{9}\right)$

$\Rightarrow \quad \gamma + \frac{1}{27} = \frac{d^2}{3} \geq 0$

$\Rightarrow \quad \gamma \geq -\frac{1}{27}$

$\Rightarrow \quad \gamma \in \left[-\frac{1}{27}, \infty\right)$

32. Given $p + q = 2$, $pq = A$
and $r + s = 18$, $rs = B$
Let $p = a - 3d$, $q = a - d$, $r = a + d$, $s = a + 3d$
Thus, $p + q + r + s = 20$
$\Rightarrow \quad 4a = 20$
$\Rightarrow \quad a = 5$
Also, $(r + s) - (p + q) = 18 - 2 = 16$
$\Rightarrow \quad 8d = 16$
$\Rightarrow \quad d = 2$
Thus, $p = 5 - 6 = -1$, $q = 5 - 2 = 3$
and $r = 5 + 2 = 7$, $s = 5 + 6 = 11$
Therefore, $A = pq = -3$ and $B = rs = 77$

33. Let a and b be two positive numbers.
Given $x = \frac{a+b}{2}$
Also, $a, y, z, b \in$ GP
Let r be the common ratio.
Then $t_4 = b \Rightarrow ar^3 = b$

$\Rightarrow \quad r = \left(\frac{b}{a}\right)^{1/3}$

Now, $y = ar = a \cdot \left(\frac{b}{a}\right)^{1/3} = a^{2/3}b^{1/3}$

$\Rightarrow \quad y^3 = a^2b$

and $z = ar^2 = a\left(\frac{b}{a}\right)^{2/3} = a^{1/3}b^{2/3}$

$\Rightarrow \quad z^3 = ab^2$
Now, $y^3 + z^3 = ab(a + b)$
and $yz = ab$

Therefore, $\frac{y^3 + z^3}{xyz} = \frac{ab(a+b)}{\frac{ab(a+b)}{2}} = 2$

34. Let the first term = a, and the common difference = d.

Given $T_m = \frac{1}{n} \Rightarrow a + (m-1)d = \frac{1}{n}$ …(i)

$T_n = \frac{1}{m} \Rightarrow a + (n-1)d = \frac{1}{m}$ …(ii)

From Eqs (i) and (ii), we get

$(m-n)d = \frac{1}{n} - \frac{1}{m} = \frac{(m-n)}{mn}$

$\Rightarrow \quad d = \frac{1}{mn}$

Put $d = \frac{1}{mn}$ in Eq. (i), we get,

$a = \frac{1}{mn}$

Now, $T_{mn} = a + (mn - 1)d$

$= \frac{1}{mn} + (mn-1)\frac{1}{mn}$

$= \frac{1 + (mn-1)}{mn}$

$= 1$

35. Given x, y, z are in GP.
$\Rightarrow \quad y^2 = xz$
$\Rightarrow \quad \log(y^2) = \log(xz)$
$\Rightarrow \quad 2 \log y = \log x + \log z$
$\Rightarrow \quad \log x, \log y, \log z \in$ AP
$\Rightarrow \quad 1 + \log x, 1 + \log y, 1 + \log z \in$ AP

$\Rightarrow \quad \frac{1}{1+\log x}, \frac{1}{1+\log y}, \frac{1}{1+\log z} \in$ HP

36. Given $a_1 = 2$. So, $a_{10} = a_1 + 9d$
$\Rightarrow \quad 9d = a_{10} - a_1 = 3 - 2 = 1$

$\Rightarrow \quad d = \frac{1}{9}$

Now, $a_4 = a_1 + 3d = 2 + \frac{1}{3} = \frac{7}{3}$

Also, $h_{10} = 3$

$\Rightarrow \quad \frac{1}{h_{10}} = \frac{1}{3}$

$\Rightarrow \quad \frac{1}{h_1} + 9D = \frac{1}{3}$

$\Rightarrow \quad 9D = \frac{1}{3} - \frac{1}{h_1}$

$\Rightarrow \quad 9D = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}$

$\Rightarrow \quad D = -\frac{1}{54}$

Now, $\frac{1}{h_7} = \frac{1}{h_1} + 6D$

$\Rightarrow \quad \frac{1}{h_7} = \frac{1}{2} - \frac{1}{9} = \frac{9-2}{18} = \frac{7}{18}$

$\Rightarrow \quad h_7 = \frac{18}{7}$

Thus, $a_4 h_7 = \frac{7}{3} \times \frac{18}{7} = 6$

37. Let α and β be the roots of the given equation
Thus, HM of the roots of α and β

$$= \frac{2\alpha\beta}{\alpha + \beta}$$

$$= \frac{2\left(\frac{8 + 2\sqrt{5}}{5 + \sqrt{2}}\right)}{\left(\frac{4 + \sqrt{5}}{5 + \sqrt{2}}\right)}$$

$$= \frac{2(8 + 2\sqrt{5})}{(4 + \sqrt{5})} = \frac{4(4 + \sqrt{5})}{(4 + \sqrt{5})} = 4$$

38. Given $\frac{a}{1 - r} = 4, ar = \frac{3}{4}$

$\Rightarrow \quad \frac{\frac{3}{4r}}{1 - \text{r}} = 4$

$\Rightarrow \quad \frac{3}{4r} = 4(1 - r)$

$\Rightarrow \quad 16r(1 - r) = 3$

$\Rightarrow \quad 16r^2 - 16r + 3 = 0$

$\Rightarrow \quad 16r^2 - 12r - 4r + 3 = 0$

$\Rightarrow \quad 4r(4r - 3) - 1(4r - 3) = 0$

$\Rightarrow \quad (4r - 3)(4r - 1) = 0$

$\Rightarrow \quad r = \frac{3}{4} \text{ or } \frac{1}{4}$

When $r = \frac{1}{4}, a = 3.$

When $r = \frac{3}{4}, a = 1.$

40. Given $T_{n+1} - T_n = 21$

$\Rightarrow \quad {}^{n+1}C_3 - {}^nC_3 = 21$

$\Rightarrow \quad \frac{(n+1)!}{3!(n+1-3)!} - \frac{(n)!}{3!(n-3)!} = 21$

$\Rightarrow \quad \frac{(n+1)!}{(n-2)!} - \frac{(n)!}{(n-3)!} = 126$

$\Rightarrow \quad (n + 1)n(n - 1) - n(n - 1)(n - 2) = 126$

$\Rightarrow \quad n(n^2 - 1) - n(n^2 - 3n + 2) = 126$

$\Rightarrow \quad (n^3 - n - n^3 + 3n^2 - 2n) = 126$

$\Rightarrow \quad 3n^2 - 3n = 126$

$\Rightarrow \quad n^2 - n - 42 = 0$

$\Rightarrow \quad (n - 7)(n + 6) = 0$

$\Rightarrow \quad n = 7 \text{ or } -6$

$\Rightarrow$ Since n should be a natural number, so $n = 7$

41. Given $a, b, c, d \in$ AP

$\Rightarrow \quad \frac{1}{a}, \frac{1}{\text{b}}, \frac{1}{\text{c}}, \frac{1}{\text{d}} \in \text{HP}$

$\Rightarrow \quad \frac{abcd}{a}, \frac{abcd}{b}, \frac{abcd}{c}, \frac{abcd}{d} \in \text{HP}$

$\Rightarrow \quad bcd, acd, abd, abc \in$ HP

42.

43.

44. Given $c = a_1 \cdot a_2 \cdot a_3 \ldots a_n$
As we know that, AM ≥ GM

$$\frac{a_1 + a_2 + \ldots + 2a_{n-1} + a_n}{n} \geq \sqrt[n]{2(a_1 \cdot a_2 \ldots a_n)}$$

$\Rightarrow \quad \frac{a_1 + a_2 + \ldots + 2a_{n-1} + a_n}{n} \geq \sqrt[n]{2c}$

$\Rightarrow \quad (a_1 + a_2 + \ldots + 2a_{n-1} + a_n) \geq n(2c)^{1/n}$

Hence, the minimum value of
$a_1 + a_2 + \ldots + a_{n-1} + 2a_n$ is $n(2c)^{1/n}$.

45. Given a, b, c are in AP

$\Rightarrow \quad 2b = a + c$ …(i)

Also, a^2, b^2, c^2 are in GP.

$\Rightarrow \quad b^4 = a^2c^2$ …(ii)

$\Rightarrow \quad a + b + c = \frac{3}{2}$

$\Rightarrow \quad 3b = \frac{3}{2}$

$\Rightarrow \quad b = \frac{1}{2}$

From Eq. (ii), we get,

$$a^2c^2 = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$$

$\Rightarrow \quad ac = \pm\frac{1}{4}$

Also $a + c = 2b = 1$

Thus, a and c are the roots of

$$x^2 - x \pm \frac{1}{4} = 0$$

$\Rightarrow \quad x^2 - x + \frac{1}{4} = 0, x^2 - x - \frac{1}{4} = 0$

$\Rightarrow \quad \left(x - \frac{1}{2}\right)^2 = 0, \left(x - \frac{1}{2}\right)^2 = \frac{1}{2}$

$\Rightarrow \quad x = \frac{1}{2}, x = \frac{1}{2} \pm \frac{1}{\sqrt{2}}$

$\Rightarrow \quad x = \frac{1}{2} \pm \frac{1}{\sqrt{2}}$, since $a < b < c$

Thus, $a = \frac{1}{2} - \frac{1}{\sqrt{2}}$, as $a < b < c$

47. Given $a, b, c \in$ AP

$\Rightarrow \quad 2b = a + c$

Also, $a^2, b^2, c^2 \in$ HP

$\Rightarrow \quad \frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2} \in \text{AP}$

$\Rightarrow \quad \frac{2}{b^2} = \frac{1}{a^2} + \frac{1}{c^2}$

$\Rightarrow \quad b^2 = \dfrac{2a^2 \cdot c^2}{a^2 + c^2}$

$\Rightarrow \quad \left(\dfrac{a+c}{2}\right)^2 = \dfrac{2a^2 \cdot c^2}{a^2 + c^2}$

$\Rightarrow \quad (a+c)^2 = \dfrac{8a^2 \cdot c^2}{a^2 + c^2}$

$\Rightarrow \quad (a^2 + c^2 + 2ac)(a^2 + c^2) = 8a^2 \cdot c^2$

$\Rightarrow \quad (a^2 + c^2)^2 + 2ac(a^2 + c^2) - 8a^2 \cdot c^2 = 0$

$\Rightarrow \quad (a^2 + c^2 + ac)^2 - 9a^2c^2 = 0$

$\Rightarrow \quad (a^2 + c^2 + ac)^2 - (3ac)^2 = 0$

$\Rightarrow \quad (a^2 + c^2 + ac + 3ac)(a^2 + c^2 + ac - 3\,ac) = 0$

$\Rightarrow \quad (a^2 + c^2 + 4ac)(a^2 + c^2 - 2ac) = 0$

$\Rightarrow \quad (a - c)^2((a + c)^2 + 2ac) = 0$

$\Rightarrow \quad (a - c)^2 = 0,\ (a + c)^2 + 2ac = 0$

When $(a - c)^2 = 0$

$\Rightarrow \quad a = c$

$\Rightarrow \quad a = b = c$

When $((a + c)^2 + 2ac) = 0$

$\Rightarrow \quad 4b^2 + 2ac = 0$

$\Rightarrow \quad 2b^2 + 4ac = 0$

$\Rightarrow \quad b^2 = a\left(-\dfrac{c}{2}\right)$

$\Rightarrow \quad a, b, \left(-\dfrac{c}{2}\right) \in \text{GP}$

48. Let r be the common ratio, where $-1 < r < 1$.

We have $\dfrac{x}{1-r} = 5$

$\Rightarrow \quad \dfrac{1}{1-r} = \dfrac{5}{x}$

$\Rightarrow \quad (1-r) = \dfrac{x}{5}$

Now, $-1 < r < 1$

$\Rightarrow \quad -1 < r < 1$

$\Rightarrow \quad 0 < 1 - r < 2$

$\Rightarrow \quad 0 < \left(\dfrac{x}{5}\right) < 2$

$\Rightarrow \quad 0 < x < 10$

49. We have $(1 + a)(1 + b)(1 + c)$

$= 1 + (a + b + c) + (ab + bc + ca) + abc$

As we know that, AM ≥ GM

$\Rightarrow \quad \dfrac{(a+b+c+ab+bc+ca+abc)}{7} \geq \sqrt[7]{a^4b^4c^4}$

$\Rightarrow \quad (a+b+c+ab+bc+ca+abc) \geq 7\sqrt[7]{a^4b^4c^4}$

$\Rightarrow \quad (1 + a + b + c + ab + bc + ca + abc)$

$> (a+b+c+ab+bc+ca+abc) \geq 7\sqrt[7]{a^4b^4c^4}$

$\Rightarrow \quad (1+a)(1+b)(1+\text{c}) \geq 7\sqrt[7]{a^4b^4c^4}$

$\Rightarrow \quad (1 + a)^7(1 + b)^7(1 + c)^7 \geq 7^7\,(a^4b^4c^4)$

50. Given $\alpha + \beta$, $\alpha^2 + \beta^2$, $\alpha^3 + \beta^3$ are in G.P

$\Rightarrow \quad (\alpha^2 + \beta^2)^2 = (\alpha + \beta)(\alpha^3 + \beta^3)$

$\Rightarrow \quad \alpha^4 + \beta^4 + 2\alpha^2\beta^2 = \alpha^4 + \beta^4 + \alpha\beta(\alpha^2 + \beta^2)$

$\Rightarrow \quad \alpha\beta(\alpha^2 + \beta^2) - 2\alpha^2\beta^2 = 0$

$\Rightarrow \quad \alpha\beta(\alpha - \beta)^2 = 0$

$\Rightarrow \quad \alpha\beta\{(\alpha + \beta)^2 - 4\alpha\beta\} = 0$

$\Rightarrow \quad \dfrac{c}{a}\left\{\left(-\dfrac{b}{a}\right)^2 - 4\dfrac{c}{a}\right\} = 0$

$\Rightarrow \quad \dfrac{c}{a}\left\{\dfrac{b^2}{a^2} - \dfrac{4c}{a}\right\} = 0$

$\Rightarrow \quad c(b^2 - 4ac) = 0$

$\Rightarrow \quad cD = 0$

51. Let S denotes the total number of runs scored by the cricketer in the n matches, then

$S = 1.2^n + 2.2^{n-1} + 3.2^{n-2} + \ldots + n \cdot 2^1$ …(i)

$\Rightarrow \quad \dfrac{1}{2}S = 1.2^{n-1} + 2.2^{n-2} + 3.2^{n-3} + \ldots + n.$ …(ii)

Subtracting Eq. (i) and Eq. (ii), we get

$\Rightarrow \quad \left(1 - \dfrac{1}{2}\right)S = 2^n + 2^{n-1} + 2^{n-2} + \ldots + 2 - n$

$\Rightarrow \quad \dfrac{1}{2}S = 2\left(\dfrac{2^n - 1}{2 - 1}\right) - n$

$\Rightarrow \quad S = 2^2(2^n - 1) - 2n = 2(2^{n+1} - 1 - 2n)$

Therefore,

$2(2^{n+1} - 1 - 2n) = \left(\dfrac{n+1}{4}\right)(2^{n+1} - n - 2)$

$\Rightarrow \quad \dfrac{(n+1)}{4} = 2$

$\Rightarrow \quad n = 7$

52. Given

$$a_n = \frac{3}{4} - \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 - \ldots + (-1)^{n-1}\left(\frac{3}{4}\right)^n$$

$$= \frac{3}{4}\left(\frac{\left(1 - \left(-\frac{3}{4}\right)^n\right)}{\left(1 - \left(-\frac{3}{4}\right)\right)}\right) = \frac{3}{7}\left(1 - \left(-\frac{3}{4}\right)^n\right)$$

Now, $b_n > a_n$

$\Rightarrow \quad 1 - a_n > a_n$

$\Rightarrow \quad 2a_n < 1$

$\Rightarrow \quad a_n < \dfrac{1}{2}$

$\Rightarrow \quad \dfrac{3}{7}\left(1 - \left(-\dfrac{3}{4}\right)^n\right) < \dfrac{1}{2}$

$$\Rightarrow \quad \left(1-\left(-\frac{3}{4}\right)^n\right) < \frac{7}{6}$$

$$\Rightarrow \quad \left(-\frac{3}{4}\right)^n > -\frac{1}{6}$$

If n is even, it is true for all even n.
If n is odd, it is true for all $n \geq 7$.
Thus, the least natural number n_0 such that $b_n > a_n$ for every $n \geq n_0$ is 7.

53. Given

$$\theta = \sum_{k=1}^{\infty} \tan^{-1}\left(\frac{1}{2k^2}\right)$$

$$= \sum_{k=1}^{\infty} \tan^{-1}\left(\frac{2}{4k^2}\right)$$

$$= \sum_{k=1}^{\infty} \tan^{-1}\left(\frac{2}{1+(4k^2-1)}\right)$$

$$= \sum_{k=1}^{\infty} \tan^{-1}\left(\frac{2}{1+(2k-1)(2k+1)}\right)$$

$$= \sum_{k=1}^{\infty} \tan^{-1}\left(\frac{(2k+1)-(2k-1)}{1+(2k-1)(2k+1)}\right)$$

$$= \sum_{k=1}^{\infty} (\tan^{-1}(2k+1) - \tan^{-1}(2k-1))$$

$= [\tan^{-1}(3) - \tan^{-1}(1)] + [\tan^{-1}(5) - \tan^{-1}(3)] + \ldots + [\tan^{-1}(2n+1) - \tan^{-1}(2n-1)]$

$= \tan^{-1}(2n+1) - \tan^{-1}(1)$, as $n \to \infty$

$$= \tan^{-1}\left(\frac{2n+1-1}{1+(2n+1)\cdot 1}\right), \text{ as } n \to \infty$$

$$= \tan^{-1}\left(\frac{2n}{2n+2}\right), \text{ as } n \to \infty$$

$$= \tan^{-1}\left(\frac{n}{n+1}\right), \text{ as } n \to \infty$$

$$= \tan^{-1}\left(\frac{1}{1+\left(\frac{1}{n}\right)}\right), \text{ as } n \to \infty$$

$$= \tan^{-1}(1)$$

$$= \frac{\pi}{4}.$$

Hence, $\tan\left(\frac{\pi}{4}\right) = 1$.

54. Let a be the first term and r be the common ratio of the GP.

As GP is of distinct positive terms, $a > 0$ and $r \neq 1$.

We have $b_1 = a$, $b_2 = a + ar = a(1 + r)$

$b_3 = b_2 + a_3 = a(1 + r + r^2)$

$b_4 = b_3 + a_4 = a(1 + r + r^2 + r^3)$

Now, $b_2 - b_1 = a(1 + r) - a = ar$

$b_3 - b_2 = a(1 + r + r^2) - a(1 + r) = ar^2$

Thus, b_1, b_2, b_3, b_4 are not in AP

$$\frac{b_2}{b_1} = \frac{a(1+r)}{a} = (1+r)$$

Also, $\dfrac{b_3}{b_2} = \dfrac{a(1+r+r^2)}{a(1+r)} = \dfrac{(1+r+r^2)}{(1+r)} \neq \dfrac{b_2}{b_1}$

Therefore, b_1, b_2, b_3, b_4 are not in GP.

Thus, the statement I is true.

Now, $\dfrac{1}{b_2} - \dfrac{1}{b_1} = \dfrac{1}{a(1+r)} - \dfrac{1}{a} = -\dfrac{r}{a(1+r)}$

and $\dfrac{1}{b_3} - \dfrac{1}{b_2} = \dfrac{1}{a(1+r+r^2)} - \dfrac{1}{a(1+r)}$

$$= -\frac{r^2}{a(1+r)(1+r+r^2)} \neq \frac{1}{b_2} - \frac{1}{b_1}$$

Thus $\dfrac{1}{b_1}, \dfrac{1}{b_2}, \dfrac{1}{b_3}, \dfrac{1}{b_4}$ are not in AP

So, b_1, b_2, b_3, b_4 are not in HP

Therefore, the statement II is false.

55. We have

$t_n = c[n^2 - (n-1)^2] = (2n-1)c$

$t_n^2 = (2n-1)^2 c^2$

$t_n^2 = (4n^2 - 4n + 1)c^2$

Now, $\displaystyle\sum_{n=1}^{n} t_n^2 = \sum_{n=1}^{n} (4n^2 - 4n + 1)c^2$

$$= c^2\left\{\frac{4n(n+1)(2n+1)}{6} - \frac{4n(n+1)}{2} + n\right\}$$

$$= \frac{c^2 n}{6}\{4(n+1)(2n+1) - 12(n+1) + 6\}$$

$$= \frac{c^2 n}{3}\{2(n+1)(2n+1) - 6(n+1) + 3\}$$

$$= \frac{c^2 n}{3}\{2(2n^2+3n+1) - 6(n+1) + 3\}$$

$$= \frac{c^2 n}{3}\{(4n^2+6n+2) - 6(n+1) + 3\}$$

$$= \frac{n(4n^2-1)c^2}{3}$$

56. Given $a_k = 2a_{k-1} - a_{k-2}$

$\Rightarrow \quad a_k + a_{k-2} = 2a_{k-1}$

$\Rightarrow \quad a_1, a_2, a_3, \ldots, a_{11}$ are in AP

Given $\dfrac{a_1^2 + a_2^2 + \ldots + a_{11}^2}{11} = 90$

$$\Rightarrow \quad \frac{a^2 + (a+d)^2 + (a+2d)^2 + \ldots + (a+10d)^2}{11} = 90$$

$\Rightarrow \quad 11a^2 + 2(1 + 2 + 3 + \ldots + 10)ad + d^2(1^2 + 2^2 + \ldots + 10^2) = 990$

$\Rightarrow \quad 11a^2 + 2.55ad + 55.7 \cdot d^2 = 990$

$\Rightarrow \quad a^2 + 10ad + 35d^2 = 90$

$\Rightarrow \quad 225 + 150d + 35d^2 = 90$

$\Rightarrow \quad 35d^2 + 150d + 135 = 0$

$\Rightarrow \quad 7d^2 + 30d + 27 = 0$

$\Rightarrow \quad 7d^2 + 21d + 9d + 27 = 0$

$\Rightarrow \quad 7d(d + 3) + 9(d + 3) = 0$

$\Rightarrow \quad (d + 3)(7d + 9) = 0$

$\Rightarrow \quad d = -3 \text{ or } -9/7$

Since $a_2 < \frac{27}{2}$, so $d = -3$.

Now, $\frac{a_1 + a_2 + \ldots + a_{11}}{11}$

$$= \frac{a + (a + d) + (a + 2d) + \ldots + (a + 10d)}{11}$$

$$= \frac{11a + d(1 + 2 + 3 + \ldots + 10)}{11}$$

$$= \frac{11a + 55d}{11}$$

$$= a + 5d$$

$$= 15 - 15$$

$$= 0.$$

57. Given $a_1, a_2, \ldots, a_{100}$ are in AP.

Here, $a_1 = 3$ and $S_p = \sum_{i=1}^{p}(a_i), 1 \le p \le 100$

Now, $\frac{S_m}{S_n} = \frac{S_{5n}}{S_n} = \frac{\frac{5n}{2}(6 + (5n - 1)d)}{\frac{n}{2}(6 + (n - 1)d)}$

$$= 5 \times \frac{(6 - d) + 5nd}{(6 - d) + nd}$$

It is independent of n only when $6 - d = 0$

Thus, $d = 6$

Now, $a_2 = a_1 + d = 3 + 6 = 9$

58. Given $a_1, a_2, \ldots, a_n \in$ HP

$\Rightarrow \quad \frac{1}{a_1}, \frac{1}{a_2}, \ldots, \frac{1}{a_n} \in$ AP

Given $\frac{1}{a_{20}} = \frac{1}{a_1} + 19d$

$\Rightarrow \quad 19d = \frac{1}{a_{20}} - \frac{1}{a_1}$

$\Rightarrow \quad 19d = \frac{1}{25} - \frac{1}{5} = -\frac{4}{25}$

$\Rightarrow \quad d = -\frac{4}{25 \times 19}$

Also, $\frac{1}{a_n} < 0$, so

$$\frac{1}{a_1} + (n - 1)d < 0$$

$\Rightarrow \quad \frac{1}{5} + (n - 1)\left(-\frac{4}{25 \times 19}\right) < 0$

$\Rightarrow \quad \frac{4(n - 1)}{25 \times 19} > \frac{1}{5}$

$\Rightarrow \quad (n - 1) > \frac{19 \times 5}{4}$

$\Rightarrow \quad n > 1 + \frac{19 \times 5}{4} \ge 25$

59. We have $S_n = \sum_{k=1}^{4n} (-1)^{\frac{k(k+1)}{2}} \cdot k^2$

$= -1^2 - 2^2 + 3^2 + 4^2 - 5^2 - 6^2 + 7^2 + 8^2 - 9^2 - 10^2 + \ldots - (4n - 1)^2 + (4n)^2$

$= (3^2 - 1^2) + (4^2 - 2^2) + (7^2 - 5^2) + (8^2 - 6^2) + \ldots + ((4n)^2 - (4n - 2)^2)$

$= 2(3 + 1) + 2(4 + 2) + 2(7 + 5) + \ldots + 2(4n + 4n - 2)$

$= 2(3 + 1) + 2(4 + 2) + 2(7 + 5) + \ldots + 2(4n - 1 + 4n - 3) + 2(4n + 4n - 2)$

$= 2(1 + 2 + 3 + 4 + \ldots + (4n - 1) + 4n)$

$= \frac{2.4n(4n + 1)}{2}$

$= 4n(4n + 1)$

When $4n(4n + 1) = 1056$

$\Rightarrow \quad 4n^2 + n - 264 = 0$

$\Rightarrow \quad n = 8$

When $4n(4n + 1) = 1132$

$\Rightarrow \quad n(4n + 1) = 283$

$\Rightarrow \quad 4n^2 + n - 283 = 0$

$\Rightarrow \quad n = 9$

60. Given a, b, c are in GP.

$\Rightarrow \quad b^2 = ac$

$\Rightarrow \quad c = \frac{b^2}{a}$

Now, $\frac{a + b + c}{3} = b + 2$

$\Rightarrow \quad a + b + c = 3b + 6$

$\Rightarrow \quad a - 2b + c = 6$

$\Rightarrow \quad a - 2b + \frac{b^2}{a} = 6$

$\Rightarrow \quad 1 - 2\frac{b}{a} + \frac{b^2}{a^2} = \frac{6}{a}$

$\Rightarrow \quad \left(\frac{b}{a} - 1\right)^2 = \frac{6}{a}$

$\Rightarrow \quad$ Since $\frac{b}{a}$ is an integer, so $a = 6$

Now, $\left(\frac{a^2 + a - 14}{a + 1}\right) = \frac{36 + 6 - 14}{7} = \frac{28}{7} = 4$

CHAPTER

2

Quadratic Equations and Expressions

QUADRATIC EQUATIONS

CONCEPT BOOSTER

1.1 ALGEBRAIC EXPRESSION

It is a collection of symbols + and –. Such as $x + 2y + 3z - 4u$, $p + q - r$, $2a + 3b - 4c$ are algebraic expressions.

1.2 POLYNOMIAL

It is an algebraic expression in which the exponent of the variable contain whole number.

For example, $2x + 4$, $3x^2 + 4x + 5$, $4x^3 + 3x^2 + 4x + 50$, etc., are polynomials.

1.3 EQUATION

It is an equality, which satisfies for some values of the unknown quantity. For example, $3x + 6 = 0$, $x^2 - 3x + 2 = 0$, $x^3 - 6x^2 + 11x - 6 = 0$, etc., are equations.

1.4 IDENTITY

The full form of identity is identical equation. It is true for every values of the unknown quantities.

For example, $a^2 - b^2 = (a + b)(a - b)$ is an algebraic identity. Similarly $\sin^2\theta + \cos^2\theta = 1$ is a trigonometrical identity.

Those are true for every values of the variable.

1.5 TYPES OF EQUATIONS

There are following four types of equations:

(i) Linear Equation
$ax + b = 0$, $a \neq 0$

(ii) Quadratic Equation
$ax^2 + bx + c = 0$, $a \neq 0$

(iii) Cubic Equation
$ax^3 + bx^2 + cx + d = 0$, $a \neq 0$

(iv) Quartic Equation
$ax^4 + bx^3 + cx^2 + dx + e = 0$, $a \neq 0$

1.6 DIFFERENCE BETWEEN AN EQUATION AND AN IDENTITY

An identity in x is satisfied lay all permissible values of x, whereas an equation is satisfied by some particular values of the variable x.

For example, $(x + 1)^2 = x^2 + 2x + 1$ is an identity in x and is satisfied for all values of x. Whereas a quadratic equation satisfies by only two values of x.

Notes

1. If a linear equation is satisfied by more than one root, it is an identity. For example, $ax + b = 0$ is an identity in x only if $a = 0$, $b = 0$.
2. If a quadratic equation is satisfied by more than two roots, it is also an identity.
   For example, $ax^2 + bx + c = 0$ is an identity in x only when $a = 0$, $b = 0$ and $c = 0$.

1.7 QUADRATIC FORMULA

The solution of a quadratic equation $ax^2 + bx + c = 0$ is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

which is known as **Shiddarth Achariya Formula** and also **D-formula**.

Proof: The quadratic equation is

$$ax^2 + bx + c = 0, a \neq 0$$

$$\Rightarrow \quad x^2 + \left(\frac{b}{a}\right)x + \frac{c}{a} = 0.$$

$$\Rightarrow \quad x^2 + 2\left(\frac{b}{2a}\right)x + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0$$

$$\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\Rightarrow \quad \left(x + \frac{b}{2a}\right) = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

$$\Rightarrow \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Notes

1. The expression $b^2 - 4ac$ is called the discriminant and is denoted as D defined as $D = b^2 - 4ac$. The value of D determines whether the quadratic equation has two real solutions, one real solution and no real solution according as D is positive (+ve), zero and negative (–ve).

1.8 Nature of the Roots

Let $ax^2 + bx + c = 0$, $a \neq 0$, whereas $D = b^2 - 4ac$.
The nature of the roots depends on D.

(i) If $D > 0$, the roots are real and distinct.
(ii) If $D = 0$, the roots are real and equal.
(iii) If $D < 0$, the roots are imaginary and distinct.
(iv) If $D > 0$ and a perfect square, the roots are rational.
(v) If $D > 0$ and not a perfect square, the roots are irrational.
(vi) If $a, b, c \in R$ and one of the roots is imaginary, say $\alpha + i\beta$, its other root will be its conjugate, i.e. $\alpha - i\beta$.
(vii) If $a, b, c \in Q$ and one of the roots is irrational, say $p + \sqrt{q}$, its other root will be its conjugate, i.e. $p - \sqrt{q}$.
(viii) If $a = 1$, $b, c \in Q$ and $D = b^2 - 4ac$ is a perfect square, both the roots are integers.
(ix) If $a + b + c = 0$ (i.e. the sum of the co-efficients is zero), 1 is one root and the other root will be $\frac{c}{a}$.
(x) If $a - b + c = 0$, –1 is one root and the other root is $-\frac{c}{a}$.
(xi) If the equation $ax^2 + bx + c = 0$ has real roots α and β, we write

$$ax^2 + bx + c = a(x - \alpha)(x - \beta).$$

1.9 Sum and Product of the Roots

(i) If α and β be the roots of $ax^2 + bx + c = 0$, then
(a) $\alpha + \beta = \text{Sum of the roots} = -\frac{b}{a}$
(b) $\alpha\beta = \text{product of the roots} = \frac{c}{a}$.

(ii) If α, β, γ are the roots of $ax^3 + bx^2 + cx + d = 0$, then
(a) $\alpha + \beta + \gamma = -\frac{b}{a}$
(b) $\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a}$
(c) $\alpha\beta\gamma = -\frac{d}{a}$

(iii) If $\alpha, \beta, \gamma, \delta$ are the roots of $ax^4 + bx^3 + cx^2 + dx + c = 0$, then
(a) $\alpha + \beta + \gamma + \delta = -\frac{b}{a}$
(b) $\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \frac{c}{a}$
(c) $\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = -\frac{d}{a}$
(d) $\alpha\beta\gamma\delta = -\frac{e}{a}$

1.10 Symmetic Functions of the Roots

In order to find the value of the symmetric function of the roots α and β, we should express the given function in terms of $\alpha + \beta$ and $\alpha\beta$.

(i) $(\alpha - \beta) = \sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$
(ii) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
(iii) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$
(iv) $\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2(\alpha\beta)^2$
$= \{(\alpha + \beta)^2 - 2\alpha\beta\}^2 - 2(\alpha\beta)^2$
(v) $\alpha^5 + \beta^5 = (\alpha^3 + \beta^3)(\alpha^2 + \beta^2) - \alpha^2\beta^2(\alpha + \beta)$
(vi) $\alpha^7 + \beta^7 = (\alpha^4 + \beta^4)(\alpha^3 + \beta^3) - \alpha^3\beta^3(\alpha + \beta)$

1.11 Formation of an Equation

(i) If α and β be the roots of $ax^2 + bx + c = 0$, then

$$x^2 - \left(-\frac{b}{a}\right)x + \frac{c}{a} = 0$$

$$\Rightarrow \quad x^2 - (\alpha + \beta)x + \alpha\beta = 0.$$

(ii) If α, β and γ are the roots of $ax^3 + bx^2 + cx + d = 0$, then

$$x^3 - \left(-\frac{b}{a}\right)x^2 + \frac{c}{a}x - \frac{d}{a} = 0$$

$$\Rightarrow \quad x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma = 0.$$

(iii) If $\alpha, \beta, \gamma, \delta$ are the roots of $ax^4 + bx^3 + cx^2 + dx + e = 0$, then

$$x^4 - \left(-\frac{b}{a}\right)x^3 + \left(\frac{c}{a}\right)x^2 - \left(-\frac{d}{a}\right)x + \frac{e}{a} = 0.$$

$$\Rightarrow \quad x^4 - (\alpha + \beta + \gamma + \delta)x^3 + (\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)x^2 - (\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta)x + \alpha\beta\gamma\delta = 0$$

1.12 Common Roots of Quadratic Equations

Let $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ be two quadratic equations.

(i) **When one root is common**
Let α be a common root between the two equations. Then

$$a_1\alpha^2 + b_1\alpha + c_1 = 0$$

and

$$a_2\alpha^2 + b_2\alpha + c_2 = 0$$

Solving by cross-multiplication method, we get

$$\frac{\alpha^2}{b_1c_2 - b_2c_1} = \frac{\alpha}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$

We know that,

$$\alpha^2 = (\alpha)^2$$

$$\Rightarrow \quad \left(\frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)^2 = \left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}\right)$$

$$\Rightarrow \quad (c_1a_2 - c_2a_1)^2 = (b_1c_2 - b_2c_1)(a_1b_2 - a_2b_1)$$

which is the required condition.

(ii) **When two roots are common**

In this case two equations are identical. So the co-efficients of x^2, x and constant terms are proportional, i.e.

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

(iii) When a_is, b_is, c_is are real and one imaginary root is common, both the roots are common, i.e.

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

(iv) When a_is, b_is, c_is are rational and one irrational root is common, both roots are common, i.e.

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$$

Notes

1. To find the common root between the equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$, make the co-efficient of x^2 same by multiplying the equations by a_2 and a_1 respectively and subtract the resulting equations.
2. If $f(x) = 0$ and $g(x) = 0$ be two polynomials having some common roots, those common root(s) is/are also the roots of $h(x) = af(x) + bg(x) = 0$ but not all the roots of $h(x)$ are necessarily common roots.

1.13 Graph of a Quadratic Polynomial

Let $\quad y = ax^2 + bx + c, a \neq 0$

$$= a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a}$$

$$= a\left(x + \frac{b}{2a}\right)^2 - \frac{D}{4a}$$

$$\Rightarrow \quad y + \frac{D}{4a} = a\left(x + \frac{b}{2a}\right)^2$$

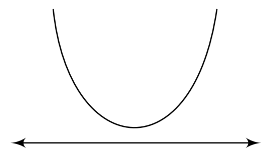
$$\Rightarrow \quad Y = aX^2, \quad X = \left(x + \frac{b}{2a}\right), Y = y + \frac{D}{4a}$$

which represents a parabola with vertex

$$\left(-\frac{b}{2a}, -\frac{D}{4a}\right).$$

Case I: When $a > 0$

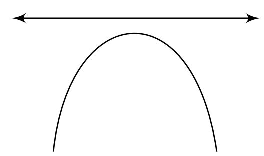
In this case, we get an upward parabola.



This quadratic polynomial will provide us a minimum value which is $-\frac{D}{4a}$, where the point of minima is $x = -\frac{b}{2a}$.

Case II: When $a < 0$

In this case, we get a downward parabola.

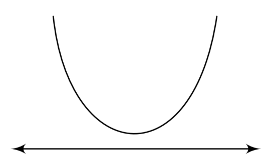


This quadratic polynomial will provide us a maximum value which is also $-\frac{D}{4a}$, where the point of maxima is $x = -\frac{b}{2a}$.

Note

1. Let $f(x) = ax^2 + bx + c$.
   If $a > 0$, the range of the quadratic polynomial is [min value, ∞)
   If $a < 0$, the range of the quadratic polynomial is ($-\infty$, max value].

Case III: If $a > 0, D < 0 \Leftrightarrow f(x) > 0$

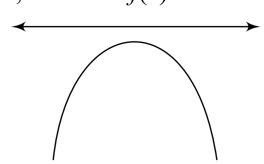


Example 1: Let $f(x) = x^2 + 2x + 4$.

Solution: Here $a = 1 > 0$ and $D = b^2 - 4ac = 4 - 4.1.4 = 4 - 16 = -12 < 0$

Thus $f(x) > 0$ for every x in R.

Case IV: If $a < 0, D > 0 \Leftrightarrow f(x) < 0$



Example 2: Let $f(x) = -x^2 + 3x - 8$.

Solution: Here, $a = -1 < 0$, $D = 9 - 32 = -23 < 0$

Thus, $f(x) < 0$ for every x in R

Case V: Range in restricted domain.

Consider $f(x) = ax^2 + bx + c \rightarrow x \in [d, e]$.

First we find the values of

$$f(d),\ f(e),\ f\left(-\frac{b}{2a}\right).$$

Maximum value $= M = \max\left\{f(d),\ f(e),\ f\left(-\frac{b}{2a}\right)\right\}$

Minimum value $= m = \min\left\{f(d),\ f(e),\ f\left(-\frac{b}{2a}\right)\right\}$

Case VI: Maximum and Minimum values of a rational function

Let $y = \dfrac{ax^2 + bx + c}{px^2 + qx + r}$

To find the maximum and minimum values of y, we should remember the following points:

(i) First we make it a quadratic equation of x.
(ii) Since x, y in R , then $D \geq 0$
(iii) Then solve the quadratic equation of y.
(iv) If $y \in [A, B]$, the maximum value $= A$ and Minimum Value $= B$
(v) If $y \in (-\infty, A] \cup [B, \infty)$, the maximum and minimum values are not defined.
(vi) If $y \in (-\infty, \infty)$, the maximum and minimum values are also not defined.

1.14 Resolution of a Second Degree Expression in *x* and *y*

Let $f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$.
Now, $f(x, y) = 0$ gives

$$ax^2 + 2(hy + g)x + (by^2 + 2fy + c) = 0$$

$$\Rightarrow \quad x = \frac{-2(hy + g) \pm 2\sqrt{(hy + g)^2 - a(by^2 + 2fy + \text{c})}}{2a}$$

$$= \frac{-(hy + g) \pm \sqrt{(hy + g)^2 - a(by^2 + 2fy + c)}}{a}$$

$$\Rightarrow \quad ax + hy + g = \pm\sqrt{(h^2 - ab)y^2 + 2(hg - af)y + (g^2 - ac)}$$

Now in order that $f(x, y)$ may be the product of two linear factors of the form $px + qy + r$, the quantity under the radical sign must be a perfect square, hence

$$(hg - af)^2 = (h^2 - ab)(g^2 - bc)$$

$$\Rightarrow \quad (h^2g^2 + a^2f^2 - 2afgh) = (g^2h^2 - g^2ab - h^2bc + a^2bc)$$

$$\Rightarrow \quad abc + 2fgh - af^2 - bg^2 - ch^2 = 0$$

$$\Rightarrow \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

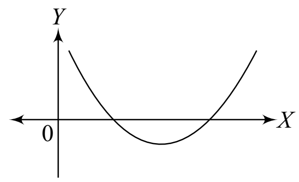
1.15 Location of the Roots

Let $f(x) = ax^2 + bx + c$, where $a \neq 0$ and α and β be the roots of $f(x) = 0$.

For simplicity, we can assume that $\alpha \leq \beta$.

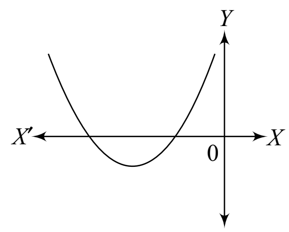
where $\alpha = \dfrac{-b - \sqrt{D}}{2a}$ and $\beta = \dfrac{-b + \sqrt{D}}{2a}$, $D = b^2 - 4ac$

1. When both roots are +ve



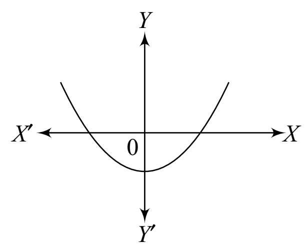
(i) Sum of the roots > 0
(ii) Product of the roots > 0
(iii) For real roots, $D \geq 0$.

2. When both roots are negative



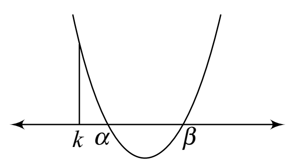
(i) Sum of the roots < 0.
(ii) Product of the roots > 0.
(iii) For real roots, $D \geq 0$

3. When roots are of opposite signs

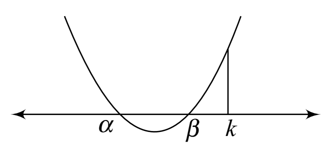


(i) Product of the roots < 0

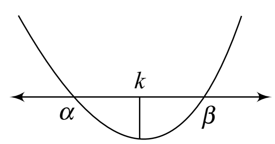
4. When both roots are greater than k.
(i) $D \geq 0$
(ii) $af(k) > 0$
(iii) $\alpha + \beta > 2k$



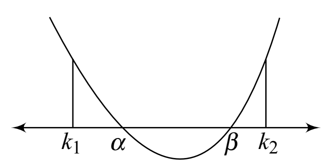
5. When both roots are less than k
(i) $D \geq 0$
(ii) $af(k) > 0$
(iii) $\alpha + \beta < 2k$



6. When k lies between the roots
(i) $D > 0$
(ii) $af(k) < 0$



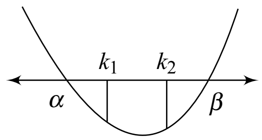
7. When both roots are confined by k_1 and k_2 such that $k_1 < k_2$
(i) $D \geq 0$
(ii) $af(k_1) > 0$



(iii) $af(k_2) > 0$

(iv) $k_1 < \frac{\alpha+\beta}{2} < k_2$

8. When (k_1, k_2) lies in between the roots.

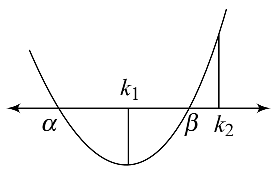


(i) $D > 0$

(ii) $af(k_1) < 0$

(iii) $af(k_2) < 0$

9. When exactly one root lies in between (k_1, k_2).



(i) $D > 0$

(ii) $f(k_1) f(k_2) < 0$

1.16 Some Special Types of Quadratic Equations

I. An equation is of the form

$$(x-a)(x-b)(x-c)(x-d) = k,$$

where $a < b < c < d$, $b - a = d - c$.

Rule: Put

$$y = \frac{(x-a)+(x-b)+(x-c)+(x-d)}{4}$$

$$= x - \frac{(a+b+c+d)}{4}$$

II. An equation is of the form

$$(x-a)(x-b)(x-c)(x-d) = kx^2,$$

where $ab = cd$.

Rule: Put $y = x + \frac{ab}{x}$.

III. An equation is of the form $(x-a)^4 + (x-b)^4 = k$

Rule: Put $y = \frac{(x-a)+(x-\mathrm{b})}{2}$.

IV. An equation is of the form

$$ax^4 + bx^3 + cx^2 + bx + a = 0,$$

where $a, b, c \in R - \{0\}$

Rule:

1. Divide by x^2 of both the sides.
2. Put $x + \frac{1}{x} = y$.

V. An equation is of the form

$$ax^5 + bx^4 + cx^3 - cx^2 - bx - a = 0$$

Rule: $x - 1$ is factor of the given equation.

VI. An equation is of the form

$$ax^{2n} + bx^n + c = 0, a \in 0, n \in N.$$

Rule: Put $x^n = y$.

1.17 Transformation of Polynomial Equation

Rule: Let $\alpha, \beta, \gamma, \ldots$ be the roots of $f(x) = 0$, then

1. the equation whose roots are $-\alpha, -\beta, -\gamma$ is $f(-x) = 0$.
2. the equation whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ is $f\left(\frac{1}{x}\right) = 0$
3. the equation whose roots are $k\alpha, k\beta, k\gamma$ is $f(xk) = 0$.
4. the equation whose roots area $\alpha - h, \beta - h, \gamma - h$ is $f(x+h) = 0$
5. the equation whose roots are $\sqrt{\alpha}, \sqrt{\beta}, \sqrt{\gamma}$ is $f(x^2) = 0$.
6. the equation whose roots are $\alpha^2, \beta^2, \gamma^2$ is $f(\sqrt{x}) = 0$.
7. the equation whose roots are $\alpha^3, \beta^3, \gamma^3$ is $f(\sqrt[3]{x}) = 0$.
8. the equation whose roots are $\frac{1}{\alpha\beta}, \frac{1}{\beta\gamma}, \frac{1}{\gamma\alpha}$ is $f(\alpha\beta\gamma \times x) = 0$.
9. the equation whose roots are $\frac{\alpha+1}{\alpha-1}, \frac{\beta+1}{\beta-1}, \frac{\gamma+1}{\gamma-1}$ is $f\left(\frac{x+1}{x-1}\right) = 0$.

1.18 Intermediate Value Theorem

If $f(a)$ and $f(b)$ are of opposite signs, then there is a root between a and b.

If $f(a)$ and $f(b)$ are of opposite signs, then there is a root between a and b.

If $f(a)$ and $f(b)$ are of the same sign, then either no root or even number of roots lie in between a and b.

Notes Let $f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n$

1. Every equation of an odd degree has at least one real root whose sign is opposite to that of its last term, provided the leading co-efficients is positive.

   In this function, in place of x we shall substitute $-\infty$, 0, $+\infty$, respectively.

   Thus, $f(-\infty) = -\infty$, $f(0) = a_n$, $f(\infty) = \infty$.

   If a_n is positive, then $f(x) = 0$ has a root lying in between $-\infty$ and 0 and if a_n is negative then $f(x)$ has a root lying in between 0 and ∞.
2. Every equation, which is of an even degree and has its last term negative, has at least two real roots, one positive and one negative, providing the leading co-efficents is positive.

   In this function, in place of x we shall substitute $-\infty$, 0, $+\infty$, respectively.

   Thus, $f(-\infty) = \infty$, $f(0) = a_n$, $f(\infty) = \infty$.

   Since a_n is negative, $f(x) = 0$ has a root lying in between $-\infty$ and 0 and a root lying in between 0 and ∞.

3. To determine the nature of the roots we should remember the following rules:
   (a) If the co-efficients are all positive, the equation has no positive root. Thus, the equation $x^5 + 4x^3 + 2x + 1 = 0$ has no positive root.
   (b) If the co-efficients of the even powers of x are all of the same sign and the co-efficients of the odd powers of x are all of opposite sign, the equation has no negative root.
   For example, the equation $x^7 + x^5 - 3x^4 + x^3 - 3x^2 + 2x - 5 = 0$ has no negative root.
   (c) If the equation contains only even powers of x and the co-efficients are all of the same sign, the equation has no real root. For example, the equation $x^5 + 3x^4 + 2x^2 + 1 = 0$ cannot have a real root.
   (d) If the equation contains only odd powers of x, and the co-efficients are all of the same sign, the equation has no real roots other than $x = 0$. For example, the equation $x^9 + 4x^5 + 5x^3 + 3x = 0$ has no real root other than $x = 0$.

1.19 Descartes Rule of Signs

Consider a polynomial

$$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \dots + a_n$$

with real co-efficients and $a_n \neq 0$, $n \in W$.

We have

$$f(-x) = a_0(-x)^n + a_1(-x)^{n-1} + a_2(-x)^{n-2} + \dots\dots + a_n$$

By observing the sign variation, in the co-efficients of $f(x)$ and $f(-x)$, we can predict the following things about the nature of the roots.

(i) The maximum number of positive roots of $f(x)$ is equal to the number of sign changes in the co-efficients of $f(x)$. Let this number be p.

(ii) The maximum number of negative roots of $f(x)$ is equal to the number of sign changes in the co-efficients of $f(-x)$. Let this number be q.

(iii) The maximum number of imaginary roots of
$f(x) = n - (p + q)$.

In-equations

2.1 Concept of solving Algebraic In-equation

Solving an algebraic in-equation $f(x) > 0$, where $f(x)$ is a product (or quotient or both) of distinct linear factors. We proceed methodically

(i) Make sure that the coefficients of x in all the linear factors are positive.

(ii) Equate all factors to zero to get the critical points (the points where these factors change their signs) $c_1, c_2, c_3, \dots$ where $c_1 < c_2 < c_3 < \dots$

(iii) Write + and – sign alternately on the number line starting from the extreme right.

(iv) If $f(x) > 0$, we shall use the open interval with the critical points.

(v) If $f(x) \geq 0$ or ≤ 0, we shall use the closed interval with the critical points.

(vi) If $f(x)$ contains an even power factors, say x^2, $(x - 2)^2$, $(x - 3)^{10}$, $(x - 4)^{100}$, we shall not represent them on the number line, we neglect them.

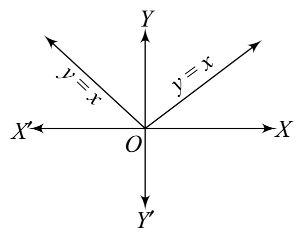
(vii) If $f(x)$ contains an irreducible quadratic factor $(ax^2 + bx + c)$, where $a > 0$ and $D < 0$, we discard it.

2.2 Equation Containing Absolute Values

Definition

Let for every x in R, the magnitude of x is called its modulus or absolute value. It is denoted by $|x|$.

Thus , $f(x) = |x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$.



Properties of Modulus

(i) Geometrically, the modulus of x means, it is the distance between x and the origin, O.

(ii) $||x|| = |-x| = |x|$

(iii) $|x| = \max(-x, x)$

(iv) $-|x| = \min(-x, x)$

(v) $\sqrt{x^2} = |x|$

(vi) $|x + y| = |x| + |y|$, when x and y are of the opposite sign. $\Rightarrow xy \leq 0$

(vii) $|x + y| = |x| + |y|$, when x and y are of the same sign. $\Rightarrow xy \geq 0$

(viii) $|xy| = |x|\,|y|$.

(ix) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, $y \neq 0$.

(x) $|x| = 1 \Rightarrow x = \pm 1$

(xi) $|x| = -2 \Rightarrow x = \varphi$

(xii) $|x| < 1 \Rightarrow -1 < x < 1$

(xiii) $|x| > 1 \Rightarrow x > 1$ and $x < -1$

Inequalities with the Absolute Value

Basic rules for the absolute value of the inequality.

1. $|x| = 2 \Rightarrow x = \pm 2$
2. $|x| = -3 \Rightarrow x = \phi$
3. $|x| < 4 \Rightarrow -4 < x < 4$
4. $|x| \le 5 \Rightarrow -5 \le x \le 5$
5. $|x| > 2 \Rightarrow x > 2, x < -2 \Rightarrow x \in (-\infty, -2) \cup (2, \infty)$
6. $|x| \ge 3 \Rightarrow x \ge 3, x \le -3 \Rightarrow x \in (-\infty, 3] \cup [3, \infty)$

2.3 Irrational Equations

Introduction

In an equation, if the unknown quantities are under radical sign, it is known as an irrational equation.

For example, $\sqrt{x} + \sqrt{x-2} = 1, \sqrt{x-3} = -2, \sqrt{x} - \sqrt{x+1} = 2$, etc., are irrational equations.

The concept of irrational equation comes from irrational expression. This expression contains at least one fractional power of the unknown quantity.

For example,

1. $f(x) = \sqrt{x}$ is an irrational expression.
2. $f(x) = \sqrt[3]{x}$ is an irrational expression.
3. $f(x) = \sqrt[2013]{x}$ is an irrational expression.

Thus, we get the even root as well as odd root of the unknown quantity of an irrational expression.

If roots of an irrational expression are even, i.e. $\sqrt{x}, \sqrt[4]{x}, \sqrt[6]{x}, \sqrt[10]{x}, \sqrt[100]{x}, \ldots$, it is defined for non-negative real values of the radicand.

If the radicand is negative $(\sqrt{-x}, \sqrt[4]{-x}, \ldots)$, the roots are imaginary.

If all the roots are odd, that is, $\sqrt[3]{x}, \sqrt[5]{x}, \sqrt[9]{x}, \sqrt[2013]{x} \ldots$, it is defined for all real values of the radicand.

If the odd radicand is negative, its roots are also negative.

2.4 Irrational In-equations

Type I: An in-equation is of the form

$$\sqrt[2n+1]{f(x)} < \sqrt[2n+1]{g(x)}, n \in N$$

$\Rightarrow \quad f(x) < g(x)$

Type II: An in-equation is of the form

$$\sqrt[2n]{f(x)} < \sqrt[2n]{g(x)}, n \in N$$

$$\Rightarrow \begin{cases} f(x) \ge 0 \\ g(x) > 0 \\ g(x) > f(x) \end{cases}$$

Type III: An in-equation is of the form

$$\sqrt[2n+1]{f(x)} < g(x), n \in N.$$

$\Rightarrow \quad f(x) < g^{2n+1}(x)$

Type IV: An in-equation is of the form

$$\sqrt[2n]{f(x)} < g(x), n \in N$$

Type V: An in-equation is of the form

$$\sqrt[2n+1]{f(x)} > g(x), n \in N.$$

$\Rightarrow \quad f(x) > g^{2n+1}(x)$

$$\Rightarrow \begin{cases} f(x) \ge 0 \\ g(x) > 0 \\ f(x) < g^{2n}(x) \end{cases}$$

Type VI: An in-equation is of the form $\sqrt[2n]{f(x)} > g(x), n \in N.$

$$\Rightarrow \begin{cases} g(x) \ge 0 \\ f(x) > g^{2n}(x) \end{cases} \text{ and } \begin{cases} g(x) < 0 \\ f(x) \ge 0 \end{cases}$$

2.5 Exponential Equations

Type I: An equation is of the form

$a^{f(x)} = 1, a > 0, a \ne 1$

$\Rightarrow \quad f(x) = 0$

Type II: An equation is of the form $f(ax) = 0$

$\Rightarrow \quad f(t) = 0$, where $t = a^x$.

If $t_1, t_2, t_3, \ldots t_n$ are the roots of $f(t) = 0$, then $ax = t_1, \ldots, a_x = t_n$.

Type III: An equation is of the form

$$\alpha \cdot a^{f(x)} + \beta \cdot b^{f(x)} + \gamma \cdot c^{f(x)} = 0,$$

where $\alpha, \beta, \gamma \in R$ and $\alpha, \beta, \gamma \ne 0$ and the bases satisfy the condition $b^2 = ac$.

$\Rightarrow \quad \alpha \cdot t^2 + \beta \cdot t + \gamma = 0$, where $t = \left(\frac{a}{b}\right)^{f(x)}$.

Type IV: An equation is of the form

$$\alpha \cdot a^{f(x)} + \beta \cdot b^{f(x)} + c = 0,$$

where $\quad \alpha, \beta, c \in R, a, b, c \ne 0$

and $ab = 1$ (where a and b are inverse positive numbers).

$\Rightarrow \quad \alpha \cdot t^2 + c \cdot t + \beta = 0$, where $t = a^{f(x)}$

Type V: An equation is of the form

$a^{f(x)} + b^{f(x)} = c,$

where $\quad a, b, c \in R$

and $a^2 + b^2 = c$.

The solution of the given equation is $f(x) = 2$.

Type VI: An equation is of the form

$a^{f(x)} + b^{f(x)} + c^{f(x)} = d,$

where $a, b, c, d \in R$ and $a^3 + b^3 + c^3 = d$. The required solution is $f(x) = 3$.

Type VII: An equation is of the form

$a^{f(x)} + a^{g(x)} = c,$

where $\quad f(x) + g(x) = 1$, and

$a, c\ (\ne 0)$

We shall put $a^{f(x)} = t$

Type VIII: An equation is of the form

$a^{f(x)} + a^{f(x)-1} + a^{f(x)-2} = b^{f(x)} + b^{f(x)-1} + b^{f(x)-2}$

Then the required solution is

$$f(x) - 2 = \log_{a/b}\left(\frac{b^2+b+1}{a^2+a+1}\right)$$

Type IX: An equation is of the form

$$(a+\sqrt{b})^{L(x),Q(x)} + (a-\sqrt{b})^{L(x),Q(x)} = c,$$

where $a^2 - b = 1$.

We shall put

$$(a+\sqrt{b})^{L(x),Q(x)} = t$$

Type X: Different types of equation on LHS = RHS

We shall use the graphical method or somewhere we can also use the concept of AM ≥ GM.

2.6 Exponential In-equations

Rule to solve the exponential inequations.

Type I: An exponential in-equation is of the form

$$a^{f(x)} > b$$

$$\Rightarrow \begin{cases} a > 1 \\ f(x) > \log_a b \end{cases}$$

$$\Rightarrow \begin{cases} 0 < a < 1 \\ f(x) < \log_a b \end{cases}$$

Type II: An exponential in-equation is of the form

$$a^{f(x)} < b$$

$$\Rightarrow \begin{cases} a > 1 \\ f(x) < \log_a b \end{cases}$$

$$\Rightarrow \begin{cases} 0 < a < 1 \\ f(x) > \log_a b \end{cases}$$

Type III: An exponential in-equation is of the form

$$f(x)^{g(x)} > f(x)^{h(x)}$$

$$\Rightarrow \begin{cases} f(x) > 1 \\ g(x) > h(x) \end{cases}$$

$$\Rightarrow \begin{cases} 0 < f(x) < 1 \\ g(x) < h(x) \end{cases}.$$

Type IV: An exponential in-equation is of the form

$$f(x)^{g(x)} < f(x)^{h(x)}$$

$$\Rightarrow \begin{cases} f(x) > 1 \\ g(x) < h(x) \end{cases}$$

$$\Rightarrow \begin{cases} 0 < f(x) < 1 \\ g(x) > h(x) \end{cases}.$$

Exercises

Level I

(Questions Based on Fundamentals)

ABC OF QUADRATIC EQUATION

1. Solve for x
   (i) $x^2 - 4x + 3 = 0$
   (ii) $4x^2 - 3x - 1 = 0$
   (iii) $5x^2 - 15x + 11 = 0$
   (iv) $x^2 - 3x + 5 = 0$
   (v) $\sqrt{\dfrac{x}{1-x}} + \sqrt{\dfrac{1-x}{x}} = \dfrac{13}{6}$
   (vi) $4x - 5.2^x + 4 = 0$
   (vii) $x^{2/3} + x^{1/3} - 2 = 0$
   (viii) $7^{1+x} + x^{1/3} - 2 = 0$
   (ix) $2x = 42^{x-1}$
   (x) $\sqrt{x} + \sqrt{x - \sqrt{1-x}} = 1.$
2. If $x = \sqrt{2+\sqrt{2+\sqrt{2+\ldots \text{to } \infty}}}$, then find x.
3. If $x = 2 + 2^{2/3} + 2^{1/3}$, then find the value of $x^3 - 6x^2 + 6x$.
4. Solve for x:
   $$x^4 - x^3 + 2x^2 - x + 1 = 0$$
5. Solve for x:
   $$(x+1)(x+2)(x+3)(x+4) = 120$$
6. If a, b, c are three real numbers , then find the roots of the equation
   $$\frac{(x-b)(x-c)}{(a-b)(a-c)}a^2 + \frac{(x-a)(x-c)}{(b-a)(b-c)}b^2 + \frac{(x-b)(x-a)}{(c-b)(c-a)}c^2 - x^2 = 0$$
7. Find the number of real solutions of
   $$x - \frac{1}{x^2-4} = 2 - \frac{1}{x^2-4}$$
8. Find the number of real roots of the equation
   $$(x-1)^2 + (x-2)^2 + (x-3)^2 = 0.$$
9. Find the roots of x , if
   $$\frac{x-ab}{a+b} + \frac{x-ac}{a+c} + \frac{x-bc}{b+c} = a+b+c$$
10. Find the roots of x , if $\dfrac{x-a}{b+c} + \dfrac{x-b}{c+a} + \dfrac{x-c}{a+b} = 3$.
11. Solve for x:
    $$\sqrt{5x^2-6x+8} - \sqrt{5x^2-6x-7} = 1$$

12. Find the number of roots of x, if $\frac{x-b}{a-b}+\frac{x-a}{b-a}=1$.
13. Find the values of a, if
$$(a^2 - 3a + 2)x^2 + (|a| - 1)x + (a^2 - 5a + 4) = 0$$
gives more than two roots.

NATURE OF ROOTS

14. Prove that the roots of the equation $x^2 - 2(a + b)x + 2(a^2 + b^2) = 0$ are imaginary and distinct.
15. If the roots of the equation $(c^2 - ab)x^2 - 2(a^2 - bc)x + (b^2 - ac) = 0$ be equal, then prove that either $a = 0$ or $a^3 + b^3 + c^3 = 3abc$.
16. For what values of m , the roots of the equation $x^2 - 2(1 + 3m)x + 7(3 + m) = 0$ will be equal?
17. The product of the roots of the equation $x^2 - 3\lambda\lambda + 2.e^{2\log\lambda} - 1 = 0$ is 7. If the roots be real ,then prove that λ =2.
18. If the roots of the equation $(b - c)x^2 + (c - a)x + (a - b) = 0$ be equal, then prove that a, b, c are in A.P.
19. If the roots of the equation $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$ has equal roots , prove that a,b, c are in H.P.
20. If a, b, c are positive and are in A.P , prove that the roots of the quadratic equation $ax^2 + bx + c = 0$ are real for $\left|\frac{c}{a}-7\right| \ge 4\sqrt{3}$.
21. Discuss the nature of the roots of the equation $4ax^2 + 3bx + 2c = 0$, where $a, b, c \in R$ and are connected by the relation $a + b + c = 0$.
22. If the roots of the equation $x^2 - 2cx + ab = 0$ be real and unequal , then prove that the roots of $x^2 - 3(a + b)x + (a^2 + b^2 + 2c^2) = 0$.
23. If the roots of the equation $(a - 1)(x^2 + x + 1)2 = (a + 1)(x^4 + x^2 + 1)$ are real and distinct, then prove that $a^2 - 4 > 0$.

SUM AND PRODUCT OF THE ROOTS

24. If α and β are the roots of the equation $x^2 - 5x + 6 = 0$, then find the value of $\beta^2 + \beta^2$, $\beta^3 + \beta^3$, $\alpha - \beta$, $\alpha^5 + \beta^5$, $\alpha^7 + \beta^7$.
25. If the product of the roots of the equation $mx^2 - 2x + (2m - 1) = 0$ is 3, then find the value of m.
26. If the equation $(k - 2)x^2 - (k - 4)x - 2 = 0$ has difference of roots as 3, then find the value of k.
27. If α, β and the roots of the equation $ax^2 + bx + c = 0$, then find the value of
$$\frac{1}{(a\alpha+b)^2}+\frac{1}{(a\beta+b)^2}$$
28. If r be the ratio of the roots of the equation $ax^2 + bx + c = 0$, then prove that $\frac{(r+1)^2}{r}=\frac{b^2}{ac}$
29. If the roots of the equation $\frac{1}{x+\text{p}}+\frac{1}{x+q}=\frac{1}{r}$ are equal in magnitude but in opposite sign , then prove that $p + q = 2r$, and the product of the roots is $-\frac{1}{2}(p^2+q^2)$.
30. For the equation $3x^2 + px + 3 = 0$, $p > 0$, if one of the roots is square of the other, then find p.
31. If one root of the equation $ax^2 + bx + c = 0$ be the square of the other , then prove that $b^3 + ac^2 + a^2c = 3abc$.
32. If the roots of the equation $3x^2 + 2(k^2 + 1)x + (k^2 - 3k + 2) = 0$ be of opposite signs, then find k.
33. In a triangle PQR, $\angle R = \pi/2$. If $\tan\left(\frac{P}{2}\right)$ and $\tan\left(\frac{Q}{2}\right)$ are the roots of the equation $ax^2 + bx + c = 0$, $a \neq 0$, then prove that $a + b + c$.
33. Let p and q be the roots of the equation $x^2 - 2x + A = 0$ and let r and s be the roots of the equation $x^2 - 18x + B = 0$. If $p < q < r < s$ are in A.P., then Find A and B.
34. If the roots of the equation $ax^2 + cx + c = 0$ are in the ratio $p : q$, then prove that
$$\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{c}{a}}=0$$
35. If α, β are the roots of the equation $x^2 + px + q = 0$ and γ, δ are the roots of $x^2 + px - r = 0$, then find the value of $(\alpha - \gamma)(\alpha - \delta)$
36. If α, β are the roots of the equation $x^2 + px + 1 = 0$ and γ, δ are the roots of the equation $x^2 + qx + 1 = 0$, then find the value of $(\alpha - \gamma)(\beta - \gamma)(\alpha + \delta)(\beta + \delta)$.
37. If α, β are the roots of $x^2 + px - q = 0$ and γ, δ are the roots of $x^2 + px + r = 0$, prove that $(\alpha - \gamma)(\alpha - \delta) = (\beta - \gamma)(\beta - \delta) = q + r$.
38. If α, β are the roots of $x^2 - x + p = 0$ and γ, δ are the roots of $x^2 - 4x + q = 0$.
If α, β, γ, δ are in G.P , then find the integral values of p and q.
39. If one root of the equation $x^2 - px + q = 0$ be twice the other, show that $2p^2 = 9q$.
40. If the difference of the roots of $x^2 - px + q = 0$ is unity, then prove that
(a) $p^2 - 4q = 1$
(b) $p^2 + 4q^2 = (1 + 2q)^2$
41. If α, β are the roots of the equation $x^2 - x - 1 = 0$, then find the value of $\sum\left(\frac{1-\alpha}{1+\alpha}\right)$.
42. If α, β, γ are the roots of $x^3 + px^2 + qx + r = 0$, then prove that
$$\left(\alpha-\frac{1}{\beta\gamma}\right)\left(\beta-\frac{1}{\gamma\alpha}\right)\left(\gamma-\frac{1}{\alpha\beta}\right)=-\frac{(1+r)^3}{r^2}$$

43. If α, β are the roots of $6x^2 - 6x + 1 = 0$, then prove that

$$\frac{1}{2}(p + q\alpha + r\alpha^2 + s\alpha^3) + \frac{1}{2}\left(p + q\beta + r\beta^2 + s\beta^3\right) \text{ is } \frac{p}{1} + \frac{q}{2} + \frac{r}{3} + \frac{s}{4}.$$

44. If α, β are the roots of $ax^2 + 2bx + c = 0$ and $\delta + \delta, \beta + \delta$ are the roots of $Ax^2 + 2Bx + C = 0$, then prove that $\frac{b^2 - ac}{B^2 - AC} = \left(\frac{a}{A}\right)^2$.

45. Let a, b, c, d be real numbers in G.P. If u, v, w satisfy the system of equations $u + 2v + 3w = 6$, $4u + 5v + 6w = 12$, $6u + 9v = 4$, then prove that the roots of the equation $\left(\frac{1}{u} + \frac{1}{v} + \frac{1}{w}\right)x^2 + [(b - c)^2 + (c - a)^2 + (a - b)^2]x + u + v + w = 0$ and $20x^2 + 10(a - d)2x - 9 = 0$ are reciprocals to each other.

FORMATION OF QUADRATIC EQUATION WITH GIVEN ROOTS

46. Find the quadratic equation whose one root is $2 - i\sqrt{3}$.
47. If α, β are the roots of a quadratic equation $x^2 - 3x + 5 = 0$, then find the equation whose roots are $\alpha^2 - 3\alpha + 7$ and $\beta^2 - 3\beta + 7$
48. If α, β are roots of the equation $x^2 - 5x + 6 = 0$ then find the equation whose roots are $\alpha + 3$ and $\beta + 3$
49. If α, β, γ are the roots of the equation $9x^3 - 7x + 6 = 0$, then find the equation whose roots are $3\alpha + 2$, $3\beta + 2$, $3\gamma + 2$.
50. If α, β are the roots of $2x^2 - 3x - 6 = 0$ then find the equation whose roots are $\alpha^2 + 2$ and $\beta^2 + 2$
51. If $\alpha \neq \beta$ and $\alpha^2 = 5\alpha - 3$, $\beta^2 = 5\beta - 3$ form the quadratic equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.
52. If the roots of the equation $(x - a)(x - b) = 0$ be c and d, then prove that the roots of the equation $(x - c)(x - d) + k = 0$ are a and b.
53 If α, β are the roots of the equation $(x - a)(x - b) + c = 0$, then find the roots of the equation $(x - \beta)(x - \beta) = c$.
54 α, β are the roots of the equation $\gamma(x^2 - x) + 5 = 0$. If λ_1 and λ_2 be the two values of λ, determined from the equation $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{4}{5}$, then prove that the value of $\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} = 254$.
55. If α, β are the roots of $x^2 - (x + 1)p - c = 0$ then find the value of

$$\frac{\alpha^2 + 2\alpha + 1}{\alpha^2 + 2\alpha + c} + \frac{\beta^2 + 2\beta + 1}{\beta^2 + 2\beta + c}$$

COMMON ROOTS OF TWO QUADRATIC EQUATIONS

56. If the equation $x^2 + 2x + 3\lambda = 0$, $2x^2 + 3x + 5\lambda = 0$ have a non zero common root, then find λ
57. If the equations $ax^2 + bx + c = 0$ and $cx^2 + bx + a = 0$, $a \neq$ c have a negative common root, then find the values of $a - b + c$.
58. If the equation $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ have a common root, then find the value of $a + b$.
59. If the equations $ax^2 + 2cx + b = 0$ and $ax^2 + 2bx + c = 0$ (b ≠ c) have a common root, then prove that $a + 4b + 4c = 0$.
60. If $x^2 + mx + 1 = 0$ and $(a - b)x^2 + (b - c)x + (c - a) = 0$ have both roots common, then prove that
   (i) $m = -2$
   (ii) b, a, c are in A.P.
   (iii) $2a - b - c = 0$.
61. If the equations $ax^2 + bx + c = 0$ and $x^2 + 2x + 3 = 0$ have common root, then prove that $a : b : c = 1 : 2 : 3$.
62. If the equations $ax^2 + bx + c = 0$ and $x^3 + 3x^2 + 3x + 2 = 0$ have two common roots then prove that $a = b = c$.
63. Find the value of a for which the equation $x^3 + ax + 1 = 0$ and $ax^4 + ax^2 + 1 = 0$ have a common root.

SIGN OF A QUADRATIC EXPRESSION

64. Sove for x:
   (i) $x^2 - 2x - 3 < 0$
   (ii) $x^2 - 3x + 2 > 0$
65. For what values of k is the inequality $x^2 - (k - 3)x - k + 6 > 0$ valid for all real x?
66. For what values of k, the inequality $(k - 2)x^2 + 8x + k + 4 < 0$ satisfies for all real values of x. ?
67. For what values of m, the equation $mx^2 - (m + 1)x + 2m - 1 = 0$ does not possesses any real roots.
68. For what values of p the curves $y = 2px + 1$ and $y = (p - 6)x^2 - 2$ do not intersect?
69. For what values of k , the curve $y = x^2 + kx + 4$ touches the x-axis.?
70. Find the integral values of k for which the equation $(k - 12)x^2 + 2(k - 12)x + 2 = 0$ possess no real roots.
71 Find the value of 'b' $x^2 + bx + 1 > 0$.
72 Find a, if $x^2 + 2ax + 10 - 3a > 0$, ® $x \in$ R.
73 Find λ, if $x^2 - 2(4\lambda - 1)x + (15\lambda^2 - 2\lambda - 7) > 0$, ® $x \in R$.

RANGE OF A QUADRATIC POLYNOMIAL

74. Find the maximum and minimum values of
   (i) $f(x) = x^2 + 2x + 4$
   (ii) $f(x) = x^2 + 4x + 4$
   (iii) $f(x) = x^2 - 5x + 4$
   (iv) $f(x) = -x^2 + x - 4$
   (v) $f(x) = -x^2 + 6x - 9$
   (vi) $f(x) = -x^2 + 6x - 8$
75. Find the ranges of
   (i) $f(x) = x^2 + x + 1$
   (ii) $f(x) = x^2 + 3x + 2$

(iii) $f(x) = -x^2 + 3^x - 10$

76. Let $P(x) = ax^2 + bx + 8$ is a quadratic polynomial. If the minimum value of $P(x)$ is 6 at $x = 2$ then find the value of a and b.
77. Find the range of $f(x) = 2x^2 - 3x + 2$ in [0, 2]
78. Find the range of $f(x) = -x^2 + 6x - 1$ in [0, 4]

MAXIMUM AND MINIMUM VALUES OF A RATIONAL EXPRESSION

79. If $x \in R$, find the least and greatest value of the expression $\dfrac{x^2 - 6x + 1}{x^2 + 6x + 1}$.
80. If $y = \dfrac{x^2 - 2x + 4}{x^2 + 2x + 4}$, ® $x \in$ R, then find the max and min value of y.
81. If $y = \dfrac{x^2 - x + 1}{x^2 + x + 1}$, ® $x \in$ R , then find y.
82. Find the greatest and least value of $y = \dfrac{x^2 + x + 1}{x^2 - x + 1}$.
83. If x is real, find the maximum value of $\dfrac{3x^2 + 9x + 17}{3x^2 + 9x + 7}$.
84. If $x \in R$, then prove that, $\dfrac{x^2 + 2x + a}{x^2 + 4x + 3a}$ can take all real values if $a \in [0, 1]$.
85. If $y = \tan x \cot 3x$, ® $x \in R$, then find y.
86. Find the range of the expression
$$f(\theta) = \frac{\tan^2\theta - 2\tan\theta - 8}{\tan^2\theta - 4\tan\theta - 5}$$
87. Find the range of the expression
$$f(\theta) = \frac{(\cot^2\theta + 5)(\cot^2\theta + 10)}{(\cot^2\theta + 1)}$$
88. Find the range of values of a, such that $f(x) = \dfrac{ax^2 + 2(a+1)x + 9a - 4}{x^2 - 8x + 32}$ is always negative.
89. Show that the function $z = 2x^2 + 2xy + y^2 - 2x + 2y + 2$ is not smaller that –3.
90. Find the minimum value of
$$f(x) = \frac{\left(x + \frac{1}{x}\right)^6 - \left(x^6 + \frac{1}{x^6}\right) - 2}{\left(x + \frac{1}{x}\right)^3 + \left(x^3 + \frac{1}{x^3}\right)} \text{ for } x > 0$$
91. Let x be a positive real . Find the maximum possible value of the expression
$$y = \frac{x^2 + 2 - \sqrt{x^4 + 4}}{x}$$

RESOLUTION INTO TWO LINEAR FACTORS

92. Find the values of m for which the expression $2x^2 + mxy + 3y^2 - 5y - 2$ may be resolve into two linear factors.
93. Find the value of k for which the expression $x^2 + 2xy + ky^2 + 2x + k = 0$ may resolve into two linear factors.
94. If the expression $ax^2 + by^2 + cz^2 + 2ayz + 2bzx + 2cxy$ be resolved into two rational factors, then prove that, $a^3 + b^3 + c^3 = 3abc$.
95. If x, y and z are real and distinct , then prove that $f(x, y) = x^2 + 4y^2 + 9z^2 - 6yz - 3zx - 2xy$ is always non negative.
96. If x is real and $4y^2 + 4xy + x + 6 = 0$, then find the complete set of values of x for which y is real.
97. Find the greatest and least values of x and y satisfying the relation $x^2 + y^2 = 6x - 8y$.

LOCATION OF ROOTS

98. Find the values of m for which both roots of equation $x^2 - mx + 1 = 0$ are less than unity.
99. For what real values of m both roots of the equation $x^2 - 6mx + 9m^2 - 2m + 2 = 0$ exceed 3?
100. Find all values of p so that 6 lies between roots of the equation $x^2 + 2(p - 3)x + 9 = 0$
101. Find the values of m for which exactly one root of the equation $x^2 - 2mx + m^2 - 1 = 0$ lies in the interval (–2, 4).
102. Find all values of a for which the equation $4x^2 - 2x + a = 0$ has two roots lie in the interval (–1, 1)
103. Find the values of a for which one root of the equation $(a - 5)x^2 - 2ax + a - 4 = 0$ is smaller than 1 and the other greater than 2.
104. If the roots of the equation $x^2 - 2ax + a^2 + a - 3 = 0$ are real and less than 3 then find a.
105. If both the roots of the equation $x^2 - 12kx + k^2 + k - 5 = 0$ are less than 5, then prove that
106. If both the roots of the equation $x^2 - 6ax + 2 - 2a + 9a^2 = 0$ exceed 3, find a
107. If the equation $ax^2 + bx + c = 0$ $(a > 0)$ has two roots α and β such that $\alpha < -2$ and $\beta > 2$ then
(a) $b^2 - 4ac > 0$
(b) $4a + 2|b| + c < 0$
(c) $a + |b| + c = 0$
(d) $c < 0$
108. Find the value of 'λ' for which $2x^2 - 2(2\lambda + 1)x + \lambda(\lambda + 1) = 10$ may have one root less than λ and other root greater than λ.
109. Find the value of 'a' for which the equation $2x^2 - 2(2a + 1)x + a(a - 1) = 0$ has roots α and β such that $\alpha < a < \beta$.

LOCATION OF ROOTS IN INEQUALITIES

110. Find all values of m for which the inequality $mx^2 - 4x + 3m + 1 > 0$ is satisfied for all +ve values of x.

111. Find all values of b for which $x^2 - x + b - 3 < 0$ for atleast one –ve x.
112. Find all values of k for which the inequality $k.4^x + (k - 1).2^{x+2} + (k - 1) > 1$ is satisfied for all x in R.
113. Find all possible parameters 'a' for which $f(x) = (a^2 + a - 2)x^2 - (a + 5)x - 2$ is non +ve for every $x \in [0, 1]$.

SOME SPECIAL TYPE EQUATIONS

114. $x^4 - 5x^2 + 4 = 0$
115. $2x^4 + 2x^2 + 3 = 0$.
116. $(x - 1)^4 + (x - 5)^4 = 82$
117. $2x^4 - x^3 - 11x^2 - x + 2 = 0$
118. $(x - 1)(x - 2)(x - 3)(x - 4) = 8$.
119. $x^2 + \dfrac{x^2}{(x+1)^2} = 3$.

TRANSFORMATION EQUATIONS

120. Solve : $3x^3 - 22x^2 + 48x - 72 = 0$ when the roots are in H.P.
121. Solve : $6x^3 - 11x^2 + 6x - 1 = 0$, when the roots are in H.P.
122. If α, β, γ be the roots of $x^3 + px^2 + qx + r = 0$, then find a cubic equation whose roots are $\alpha(\beta + \gamma), \beta(\gamma + \alpha), \gamma(\alpha + \beta)$.
123. If α, β, γ be the roots of $x^3 - px^2 + r = 0$, then find a cubic equation whose roots are $\dfrac{\beta+\gamma}{\alpha}, \dfrac{\gamma+\alpha}{\beta}, \dfrac{\alpha+\beta}{\gamma}$.
124. If α, β, γ be the roots of $x^3 + x^2 - 4x + 7 = 0$, then find a cubic equation whose roots are $\beta + \gamma, \gamma + \alpha, \alpha + \beta$.
125. If α, β, γ be the roots of $x^3 + x + 2 = 0$, then find a cubic equation whose roots are $(\alpha - \beta)^2, (\beta - \gamma)^2, (\gamma - \alpha)^2$.

DESCARTES RULES OF SIGN

126. Find the number of +ve and –ve roots of $x^5 - x^4 + x^3 + 8x^2 + 2x - 2 = 0$.
127. Find the number and position of the real roots of $x^4 - 41x^2 + 40x + 126 = 0$.
128. Find the number and position of the real roots of $x^4 - 14x^2 + 16x + 9 = 0$.
129. Find the nature of the roots of $3x^4 + 12x^2 + 5x - 4 = 0$.
130. Find the least possible number of imaginary roots of $x^9 - x^5 + x^4 + x^2 + 1 = 0$.
131. Show that $x^7 - 3x^4 + 2x^3 - 1 = 0$ has atleast four imaginary roots..

IN-EQUATION

Type I: ABC of In-equation

Solve for $\boldsymbol{x}$:

132. $x^2 - 4x + 3 > 0$.
133. $x^2 + x - 2 \geq 0$.
134. $x^2 + 2x - 8 < 0$.
135. $x^2 - 7x + 10 \leq 0$.
136. $x^2 - 3x + 2 > 0$.
137. $x^2 - 5x + 4 \leq 0$.
138. $x^2 - 7x + 12 \leq 0$.
139. $x^2 - 8x + 12 < 0$.
140. $x^3 - 6x^2 + 11x - 6 < 0$.
141. $x^3 - 4x \leq 0$.
142. $x^3 - 6x^2 + 12x - 8 \geq 0$.

Type II: Co-efficient of the highest power of x is positive.

Q. Solve for $\boldsymbol{x}$:

143. $-x^2 + 5x - 4 > 0$.
144. $-x^2 + 7x - 6 \leq 0$.
145. $-x^2 + 3x - 2 > 0$.
146. $-x^2 + 5x - 6 < 0$.
147. $-x^2 + 10x - 21 \leq 0$.
148. $-x^3 + x \leq 0$.
149. $-x^3 + 4x \geq 0$.
150. $(x - 2)(1 - x^2) \geq 0$.
151. $-x^3 + x^2 - 2x \geq 0$.

Type III: Transpose is allowed but cross-multiplication is not allowed.

Q. Solve for x:

152. $\dfrac{1}{x} > 2$
153. $\dfrac{x-2}{4-x} \geq 0$
154. $\dfrac{(x-1)(x-3)}{(x+1)(x-5)} \geq 0$
155. $\dfrac{1}{x-2} \geq \dfrac{1}{x+1}$
156. $\dfrac{1}{x} > 1$
157. $\dfrac{2}{x} \geq 1$
158. $\dfrac{1}{x} > \dfrac{1}{x-1}$
159. $\dfrac{1}{x+2} \leq \dfrac{1}{x-3}$
160. $\dfrac{1}{x} - 1 \geq 0$
161. $\dfrac{(x-1)(x-2)}{(x-3)(x-4)} \geq 0$
162. $\dfrac{(x-1)(x-2)}{(x-2)(x-4)} \leq 0$
163. $\dfrac{1}{x+1} - 1 \geq 2 - \dfrac{1}{x-2}$

Type IV: If $f(x)$ contains an even power factors, we shall not represent on the number line, we neglect them.

Q. Solve for $\boldsymbol{x}$:

164. $x^2(x - 2)^4(x - 3) \geq 0$
165. $(x - 1)^4(x - 3)^6(x - 2)^5(x - 4)^{10} > 0$

166. $\dfrac{(x-1)^{2013}(x-2)^{2014}}{(x-3)^{2016}(x-5)^{2010}} > 0$

167. $x^2(x-1) > 0$

168. $x^2(x-1)^{10}(x+2)^{11} > 0$

169. $\dfrac{(x-1)^4(x-2)^{12}}{(x-3)^{17}(x-5)^{2012}} > 0$

170. $\dfrac{(x+1)^{201}(x-4)^{2012}}{(x+3)^{2012}(x-2)^{2013}} > 0$

171. $\dfrac{1}{x^2} - 1 \geq 0$

172. $x^3 - 3x + 2 \geq 0$

173. $x^3 + 7x^2 - 36 \geq 0$

Type V: If $f(x) = ax^2 + bx + c$, where $a > 0$ and $D < 0$, then we discard it

Solve for x:

174. $x^2 + x + 2 > 0$

175. $x^2 - x + 3 < 0$

176. $(x + 2_(x^4 + x^2 + 1) > 0$

177. $x^4 - 4 \leq 0$

178. $(x^2 + 4x + 1)(x^2 + 1) \geq 0$

179. $(x - 2)(x^2 + x + 2) > 0$

180. $(x - 3)(-x^2 + x + 1) > 0$

181. $x^3 + 4x \geq 0$

182. $x^4 - 9 \geq 0$

183. $x^3 - 5x + 4 \geq 0$

184. $x^3 - 3x^2 + 3x - 9 \geq 0$

185. $x^3 - 6x^2 + 12x - 9 \geq 0$

186. $2x^2 + 5x > 12$

187. $4x^2 + 4x + 1 \leq 0$

Type VI: Common values of x in

$f(x) \geq 0$ and $g(x) \geq 0$

or

$f(x) \geq 0$ and $g(x) \leq 0$

Q. Solve for x:

188. $x^2 - 3x + 2 > 0,\ x^2 + 2x - 8 < 0$

189. $x^2 - 9 \leq 0,\ x^2 - 1 \geq 0$

190. $x^3 - 9x \geq 0$ and $x^3 + 4x \leq 0$

191. $x^2 - 4x + 3 \geq 0$ and $x^2 - 4 \leq 0$

192. $x^2 - 9 \geq 0$ and $x^2 - 4 \leq 0$

193. $x^2 - 6x + 8 \geq 0$ and $x^2 - 3x + 2 \geq 0$

194. $x^2 - 5x + 6 \geq 0$ and $x^2 - 10x + 24 \leq 0$

195. $x^2 - x \geq 0$ and $x^2 - 12x + 27 \leq 0$

196. $x^3 + x \geq 0$ and $x^2 - 9 \leq 0$

197. $x^3 - 9x \geq 0$ and $x^3 + 4x \leq 0$

EQUATION CONTAINING ABSOLUTE VALUES

198. Solve for x: $|x|^2 - 3|x| + 2 = 0$

199. Find the sum of the roots of $x^2 - 4|x| + 3 = 0$

Solve for $\boldsymbol{x}$:

200. $|3x - 1| = |x + 5|$

201. $|2x - 5| = x - 3$

202. $|x^2 - x - 6| = x + 2$

203. $2|x - 2| + 3|x - 4| = 3$

204. $|x| + |x - 2| = 4$

205. $|x - 1| + |x - 3| = 2$

206. $|x^2 - 1| + |x^2 - 4| = 3$

207. $\sqrt{x + 2\sqrt{x-1}} + \sqrt{x - 2\sqrt{x-1}} = 2$

208. $\left|\dfrac{x}{x-1}\right| + |x| = \dfrac{x^2}{|x-1|}$

209. $2^{|x+1|} - 2^x = |2^x - 1| + 1$

210. $|x^2 + x - 20| = -(x^2 + x - 20)$

211. $\left|\left(\dfrac{x^2-6x+8}{x^2-4x+3}\right)\right| = -\left(\dfrac{x^2-6x+8}{x^2-4x+3}\right)$

INEQUALITIES WITH THE ABSOLUTE VALUE

Solve for x:

212. $|x - 1| < 3$

213. $|x - 4| \leq 3$

214. $|x + 2| > 5$

215. $|x - 2| \geq 3$

216. $\left|\dfrac{4x-2}{3}\right| \leq 2$

217. $x^2 - |3x - 2| > 0$

218. $|x| + |x - 2| > 3$

219. $|x + 2| + |x| + |x - 2| > 6.$

220. $\left|\dfrac{2}{x+3}\right| \leq 1$

221. $|3x + 2| > |2x - 1|$

IRRATIONAL EQUATIONS

Solve for $\boldsymbol{x}$:

222. $\sqrt{(2x+7)} + \sqrt{(x+4)} = 0$

223. $\sqrt{(x-4)} = -5$

224. $\sqrt{(x-6)} - \sqrt{(8-x)} = 2$

225. $\sqrt{(-2-x)} = \sqrt[5]{(x-7)}$

226. $\sqrt{x} + \sqrt{(x+16)} = 3$

227. $7\sqrt{x} + 8\sqrt{-x} + \dfrac{15}{x^3} = 98.$

228. $\sqrt{x-2} + \sqrt{4-x} = \sqrt{6-x}$

229. $\sqrt{2x-4} - \sqrt{x+5} = 1$

230. $\sqrt{3x+4} + \sqrt{x-4} = 2\sqrt{x}$

231. $\sqrt{x-1} + \sqrt{2x+6} = 6$

IRRATIONAL INEQUATIONS

Solve for x:

232. $\sqrt[5]{\left[\frac{3}{x+1}+\frac{7}{x+2}\right]}<\sqrt[5]{\frac{6}{x-1}}$

233. $\sqrt[3]{\frac{x-2}{x-1}}<\sqrt[3]{\frac{1}{x-1}}$

234. $\sqrt{3x-2}<\sqrt{x+4}$

235. $\sqrt{4x-3}<\sqrt{2x+5}$

236. $\sqrt[3]{(3x-5)}<(x-1)$

237. $\sqrt[3]{3x-2}<x$

238. $\sqrt{(x+14)}<(x+2)$

239. $\sqrt{2x-2}<x-1$

240. $\sqrt[3]{5x-4}>x$

241. $\sqrt[3]{x^3-7x^2}>(-36)^{1/3}$

242. $\sqrt{-x^2+4x-3}>(6-2x)$

243. $\sqrt{x-2}+\sqrt{x-1}>2$

244. $\sqrt{x-6}-\sqrt{10-x}\geq 1$

EXPONENTIAL EQUATIONS

Solve for $\boldsymbol{x}$:

245. $5^{x^2+3x+2}=1$

246. $3^{x^2+5|x|+6}=1$

247. $5^{2x}-24.5^x-25=0$

248. $64.9^x-84.12^x+27.16^x=0$

249. $15.2^{x+1}+15.2^{2-x}=135$

250. $3^{x-4}+5^{x-4}=34.$

251. $1+3^{x/2}=2^x$

252. $3^x+4^x+5^x=6^x$

253. $1^x+6^x+8^x=9^x$

254. $16^{\sin^2 x}+16^{\cos^2 x}=10$

255. $2^x+2^{x-1}+2^{x-2}=5^x+5^{x-1}+5^{x-2}$

256. $(5+2\sqrt{6})^{x^2-3}+(5-2\sqrt{6})^{x^2-3}=10.$

257. $(15+4\sqrt{14})^x+(15-4\sqrt{14})^x=30$

258. $(5^x+5^{-x})=\log_{10}25, x\in R$

259. $\sin x=x^2+x+1.$

260. Find x, for every real x, $p^x+p^{-x}=\log_q q^2-1$where $p, q\in R$.

261. Find the number of solutions of x , $n^{-|x|}|m-|x||=1$, where $m, n>1$ and $n>m$.

262. Find the number of solution of x for which $2x+x^2=1$.

EXPONENTIAL IN-EQUATIONS

Solve for x:

263. $2^x>1$

264. $\left(\frac{1}{3}\right)^x>1$

265. $5^{x^2 3x+3}>5$

266. $\left(\frac{1}{2}\right)^{x^2-5x+8}>\left(\frac{1}{4}\right)$

267. $4^x+2^{x+1}-8\leq 0$

268. $|x|^{x^2-x-2}\geq 1, -1<x<1$

269. $\left(\frac{1}{3}\right)^{\sqrt{x+4}}>\left(\frac{1}{3}\right)^{\sqrt{x^2+3x+4}}$

270. $\frac{2^{1-x}-2^x+1}{2^x-1}\leq 0$

271. $2^{x+1}-2^{x+2}-2^{x+3}>5^{x+1}-5^{x+2}$.

272. $3^{x^2-4}<2^{x^2-x}$

LEVEL II

(Mixed Problems)

1. The roots of the quadratic equation $(a+b-2c)x^2-(2a-b-c)x+(a-2b+c)=0$ are
   (a) $a+b+c, a-b+c$ (b) $\frac{1}{2}, a-2b+c$
   (c) $a-2b+c, \frac{1}{a+b-c}$ (d) none of these

2. If the equation $x^2+ax+b=0$ and $x^2+bx+a=0$ have a common root, the numerical value of $a+b$ is
   (a) 1 (b) 0
   (c) –1 (d) none of these

3. If $(1-p)$ is a root of quadratic equation $x^2+px+(1-p)=0$, its roots are:
   (a) 0, 1 (b) –1, 1
   (c) 0, –1 (d) –1, 2

4. The set of values of p for which the roots of the equation $3x^2 + 2x + p(p - 1) = 0$ are of opposite sign is:
   (a) $(-\infty, 0)$ (b) $(0, 1)$
   (c) $1, \infty$ (d) $(0, \infty)$
5. The value of p, for which both the roots of the equation $4x^2 - 20px + (25p^2 + 15p - 66) = 0$ are less than 2, lies in
   (a) $\left(\frac{4}{5}, 2\right)$ (b) $(2, \infty)$
   (c) $\left(-1, \frac{-4}{5}\right)$ (d) $(-\infty, -1)$
6. The number of values of k for which the equation $x^2 - 3x + k = 0$ has two distinct roots lying in the interval (0, 1) is:
   (a) three
   (b) two
   (c) infinitely many
   (d) no value of k satisfies the requirement
7. If α, β be the roots of $(x - a)(x - b) + c = 0$, $c \neq 0$, the roots of $(\alpha\beta - c)x^2 + (\alpha + \beta)x + 1 = 0$ are
   (a) $\frac{1}{a}, \frac{1}{b}$ (b) $-\frac{1}{a}, \frac{1}{b}$ (c) $\frac{1}{a}, -\frac{1}{b}$ (d) $-\frac{1}{a}, -\frac{1}{b}$
8. The quadratic equation with real co-efficients one of whose complex roots has the real part 12 and modulus 13 is
   (a) $x^2 - 12x + 13 = 0$ (b) $x^2 - 24x + 13 = 0$
   (c) $x^2 - 24x + 169 = 0$ (d) $x^2 - 24x - 169 = 0$
9. The root of the equation
   $x = \sqrt{6 + \sqrt{6 + \sqrt{6 + \ldots\ldots \infty}}}$ is
   (a) 3 (b) 2 (c) –3 (d) 2
10. If $x^2 + mx + 1 = 0$ and $(a - b)x^2 + (b - c)x + (c - a) = 0$ have both roots common, which of the following is true?
    (a) $m = -1$ (b) $m = 2$
    (c) $m = -2$ (d) a, b, c are in AP
11. If the quadratic equation
    $p(q - r)x^2 + q(r - p)x + r(p - q) = 0$
    has equal roots, $\frac{2}{q}$ is equal to
    (a) $\frac{1}{p} + \frac{1}{r}$ (b) $p + r$
    (c) $\frac{1}{p} + r$ (d) $p + \frac{1}{r}$
12. If x, y, z are real and unequal, the expression $x^2 + 4y^2 + 9z^2 - 6yz - 3zx - 2xy$ is always
    (a) non-negative (b) non-positive
    (c) zero (d) none of these
13. Both the roots of the equation:
    $(x - 1)(x - 2) + (x - 2)(x - 3) + (x - 3)(x - 1) = 0$ are
    (a) complex numbers (b) always real
    (c) integers (d) rational numbers
14. The number of real roots of the equation
    $(x - 3)^2 + (x - 5)^2 + (x - 7)^2 = 0$ is
    (a) 2 (b) 0
    (c) 4 (d) infinitely many
15. The equation $x^2 + 5|x| + 4 = 0$ has
    (a) 4 real roots
    (b) 2 real roots
    (c) no real root
    (d) 6 real roots
16. The roots of $x^2 - 8|x| + 12 = 0$
    (a) do not form a progression
    (b) form an AP with 0 sum
    (c) form an AP with non-zero sum
    (d) form a GP
17. If $\alpha, \beta, \gamma, \delta$ be the roots of $16x^4 + 4x^2 + 1 = 0$, the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$ is
    (a) –1/8 (b) 0
    (c) 1/8 (d) 1
18. If $\alpha, \beta, \gamma, \delta$ be the roots of $x^4 + x^2 + 1 = 0$, the value of $(\alpha + \beta)(\alpha + \gamma)(\alpha + \delta)(\beta + \gamma)(\beta + \delta)(\gamma + \delta)$
    (a) –1 (b) 0
    (c) 1 (d) none of these
19. The value of a for which the equation
    $(a^2 + 4a + 3)x^2 + (a^2 - a - 2)x + (a + 1) = 0$ in x has more than two roots
    (a) 1 (b) –1
    (c) 2 (d) no value of a
20. The number of values of the triplet (a, b, c) for which $a \cos 2x + b \sin^2 x + c = 0$ is satisfied by all real x is
    (a) 2 (b) 4
    (c) 6 (d) infinite
21. If the roots of $ax^2 + bx + c = 0$ be the Reciprocals of those of $lx^2 + mx + n = 0$, where all the co-efficients are non-zero real numbers, then
    (a) $\frac{a}{l} = \frac{b}{m} = \frac{c}{n}$ (b) $\frac{a}{n} = \frac{b}{m} = \frac{c}{l}$
    (c) $al = bm = cn$ (d) none of these
22. If the equations $x^2 + ax + 12 = 0$; $x^2 + bx + 15 = 0$; $x^2 + (a + b)x + 36 = 0$ have a common positive root, the values of a and b, respectively, are
    (a) –7, –8 (b) –8, –7
    (c) –5, –3 (d) no value of a exist
23. If x is any real number, the value of the expression
    $$f(x) = \frac{x^2 - 5x + 4}{x^2 + 5x + 4}$$
    (a) lies between –9 and –1/9
    (b) do not lies between –9 and –1/9
    (c) may be any real number
    (d) is always positive

24. The greatest and least values of $\frac{x^2 - x + 1}{x^2 + x + 1}$ for real values of x are
(a) 4 and 1/4 (b) 2 and 1/2
(c) 3 and 1/3 (d) –1/3 and –3

25. Given that for all real x, the expression $\frac{x^2 - 2x + 4}{x^2 + 2x + 4}$ lies between 1/3 and 3. The values between which the expression $\frac{9.3^{2x} + 6.3^x + 4}{9.3^{2x} - 6.3^x + 4}$ lies are
(a) 1/3 and 3 (b) –2 and 0
(c) –1 and 1 (d) 0 and 2

26. If α, β, γ are the roots of the equation $x^3 + 4x + 1 = 0$, the value of $(\alpha + \beta)^{-1} + (\beta + \gamma)^{-1} + (\gamma + \alpha)^{-1}$ is
(a) 2 (b) 3 (c) 4 (d) 5

27. If a, b, c are sides of a triangle ABC such that
$x^2 - 2(a + b + c)x + 3\lambda(ab + bc + ca) = 0$
has real roots, then:
(a) $\lambda < \frac{4}{3}$ (b) $\lambda > \frac{5}{3}$
(c) $\lambda \in \left(\frac{4}{3}, \frac{5}{3}\right)$ (d) $\lambda \in \left(\frac{1}{3}, \frac{5}{3}\right)$

28. If $a^2 + b^2 = 2$ and a, b be real, $a + b$ lies in the interval
(a) [–1, 1] (b) [–2, 1]
(c) [1, 2] (d) [–2, 2]

29. If a, b, c be real numbers and $a^2 + b^2 + c^2 = 3$, the value of $ab + bc + ca$ lies in the interval
(a) $\left[-\frac{3}{2}, 1\right]$ (b) $\left[-\frac{3}{2}, 3\right]$
(c) $\left[-\frac{1}{2}, 1\right]$ (d) [–1, 2]

30. The number of real solutions of the equation $(5 + 2\sqrt{6})^{x^2-3} + (5 - 2\sqrt{6})^{x^2-3} = 10$ is
(a) 2 (b) 4
(c) 6 (d) none of these

31. The number of real solutions of the equation
$$(2 + \sqrt{3})^{x^2-2x+1} + (2 - \sqrt{3})^{x^2-2x+1} = \frac{2}{2 - \sqrt{3}}$$ is
(a) 0 (b) 2 (c) 4 (d) 6

32. The number of real solutions of the equation
$$\left(\frac{9}{10}\right)^x = (-3 + x - x^2)$$ is
(a) 0 (b) 1
(c) 2 (d) more than 2

33. The co-efficient of x^9 in the polynomial given by $(x + 1)(x + 2) \ldots (x + 10) + (x + 2)(x + 3) \ldots (x + 11) + \ldots + (x + 11)(x + 12) \ldots (x + 20)$ is
(a) 1155 (b) 1515 (c) 5151 (d) 5511

34. How many roots does the following equation $3^{|x|}\,|2 - |x|| = 1$ possess?
(a) 1 (b) 2 (c) 3 (d) 4

35. The co-efficient of x^6 in $(x - 1)(x - 2)(x - 3) \ldots (x - 8)$ is
(a) 236 (b) 216 (c) 546 (d) 36

36. The number of real roots of the equation $x^8 - x^5 + x^2 - x + 1 = 0$ is equal to
(a) 0 (b) 2 (c) 4 (d) 6

37. The number of real roots of the equation $2x^3 + 5x^2 + 3 = 0$ is
(a) 4 (b) 1 (c) 0 (d) 3

38. The number of values of x satisfying
$$(\sqrt{2})^x + (\sqrt{3})^x = (\sqrt{13})^x$$ is
(a) 1 (b) 2 (c) 4 (d) infinite

39. The number of values of x satisfying $3x + 4x = 5x$ is
(a) 0 (b) 1 (c) 2 (d) 3

40. The largest interval in which x lies satisfying $x^{12} - x^9 + x^4 - x + 1 > 0$ is
(a) $[0, \infty)$ (b) $(-\infty, 0]$
(c) $(-\infty, \infty)$ (d) None

41. The number of real solutions of $2x = x^2 + 1$ is
(a) 1 (b) 2 (c) 0 (d) 4

42. The number of real solutions $\sqrt{x - 2} + \sqrt{6 - x} = 2$ is
(a) 1 (b) 0 (c) 2 (d) none

43. The number of real solutions of $|x| + |x - 2| = 2$ is
(a) 1 (b) 2 (c) 0 (d) infinite

44. The number of real solutions $x - \frac{1}{x - 2} = 2 - \frac{1}{x - 2}$ is
(a) 1 (b) 2 (c) 3 (d) 0

45. The number of real solutions $3x + 4x + 5x = 6x$ is
(a) 1 (b) 2
(c) 4 (d) more than 4

46. If $a, b, c > 0$, $a^2 = b$ and $a + b + c = abc$, the least value of $a^4 + a^2 + 7$ is
(a) 18 (b) 19 (c) 20 (d) 21

47. Let α, β be the roots of $x^2 - 8x + A = 0$ and γ, δ the roots of $x^2 - 72x + B = 0$. If $\alpha < \beta < \gamma < \delta$ are in GP, the value of $A + B$ is
(a) 980 (b) 982 (c) 984 (d) 986

48. Given that for all x in R, the expression $\frac{x^2 - 2x + 4}{x^2 + 2x + 4}$ lies between 1/3 and 3, the value between which the expression $\frac{9.3^{2x} + 6.3^x + 4}{9.3^{2x} - 6.3^x + 4}$ lies in
(a) 1/3 and 3 (b) 1/2 and 2
(c) –1 and 1 (d) 0 and 2

49. The total number of solutions of $\sin(\pi x) = \ln||x||$ is
(a) 2 (b) 4 (c) 6 (d) 8

50. The minimum value of $y = \dfrac{x^3 + x + 2}{x}, x > 0$ is
(a) 3 (b) 4 (c) 5 (d) 2

51. If $P(x) = ax^2 + bx + c$ and $Q(x) = -ax^2 + dx + c$, where $ac \neq 0$, $P(x) \cdot Q(x) = 0$ has
(a) exactly one root (b) at least two real root
(c) exactly three real roots (d) all four are real roots.

52. If $a > 0$, $b > 0$ and $c > 0$, both the roots of the equation $ax^2 + bx + c = 0$
(a) are real and negative (b) have negative real parts
(c) are rational numbers (d) none

53. The real values of a for which the quadratic equation $2x^2 - (a^3 + 8a - 1)x - a^2 - 4a = 0$ possesses roots of opposite signs is given by
(a) $a > 5$ (b) $0 < a < 4$
(c) $a > 0$ (d) $a > 7$

54. Suppose a, b and c are positive numbers such that $a + b + c = 1$. The maximum value of $ab + bc + ca$ is
(a) 1/3 (b) 1/4 (c) 1/2 (d) 2/3

55. The roots of $(x - 1)(x - 3) + k(x - 2)(x - 4) = 0, k > 0$ are
(a) real (b) real and equal
(c) imaginary (d) one real and other imaginary

56. The largest integral value of m for which the quadratic expression $y = x^2 - (2m + 6)x + 4m + 12$ is always positive, for every x in R, is
(a) –1 (b) –2 (c) 0 (d) 2

57. Let r_1, r_2 and r_3 be the solutions of the equation $x^3 - 2x^2 + 4x + 5074 = 0$, the value of $(r_1 + 2)(r_2 + 2)(r_3 + 2)$ is
(a) 5050 (b) 5066 (c) –5050 (d) –5066

58. The equation whose roots are $\sec^2 \alpha$ and $\text{cosec}^2 \alpha$ can be
(a) $2x^2 - x - 1 = 0$ (b) $x^2 - 3x + 3 = 0$
(c) $x^2 - 9x + 9 = 0$ (d) None

59. Let a, b, c be the three roots of the equation $x^3 + x^2 - 333x - 1102 = 0$, the value of $a^3 + b^3 + c^3$ is
(a) 2006 (b) 2005 (c) 2003 (d) 2002

60. The absolute term in the quadratic expression $\sum_{k=1}^{n} \left(x - \dfrac{1}{k+1}\right)\left(x - \dfrac{1}{k}\right)$ as $n \to \infty$, is
(a) 1 (b) –1 (c) 0 (d) 1/2

61. The number of values of the parameter $\alpha \in [0, 2\pi]$, for which the quadratic function

$$\sin\alpha \cdot x^2 + 2x \cdot \cos\alpha + \frac{1}{2}(\cos\alpha + \sin\alpha)$$

is the square of a linear function, is
(a) 2 (b) 3 (c) 4 (d) 1

62. The set of values of a for which the inequality $(x - 3a)(x - a - 3) < 0$ is satisfied for all $x \in [1, 3]$ is
(a) (1/3, 3) (b) (0, 1/3)
(c) (–2, 0) (d) (–2, 3)

63. If α, β and γ are the roots of the equation, $x^3 - x - 1 = 0$, the value of $\dfrac{1+\alpha}{1-\alpha} + \dfrac{1+\beta}{1-\beta} + \dfrac{1+\gamma}{1-\gamma}$ is
(a) 0 (b) –1 (c) –7 (d) 1

64. For every x in R, the polynomial $x^8 - x^5 + x^2 - x + 1$ is
(a) positive (b) never positive
(c) positive and negative (d) negative

65. If the roots of the cubic equation, $x^3 + ax^2 + bx + c = 0$ are three consecutive positive integers, the value of $\dfrac{a^2}{b+1}$ is
(a) 3 (b) 2 (c) 1 (d) 1/3

66. If both roots of $(3a + 1)x^2 - (2a + 3b)x + 3 = 0$ are infinite , then
(a) $a = \infty, b = 0$ (b) $a = 0, b = \infty$
(c) $a = 1/3, b = 2/9$ (d) $a = \infty, b = \infty$

67. If $\tan a$, $\tan b$ and $\tan g$ are the roots of the equation
$$x^3 - (a + 1)x^2 + (b - a)x - b = 0, (b - a \neq 1),$$
where $\alpha + \beta + \gamma$ lies between 0 and π, the value of $(\alpha + \beta + \gamma)$ is
(a) $\pi/4$ (b) $\pi/2$ (c) $3\pi/4$ (d) None

68. Three roots of the equation $x^4 - px^3 + qx^2 - rx + s = 0$ are $\tan A$, $\tan B$ and $\tan C$, where A, B, C are the angles of a triangle. The fourth root of the biquadratic is
(a) $\dfrac{q-r}{1-q+s}$ (b) $\dfrac{p-r}{1+q-s}$
(c) $\dfrac{p+r}{1-q+s}$ (d) $\dfrac{p+r}{1+q-s}$

69. The number of real roots of $(x + 3)^4 + (x + 5)^4 = 16$ is
(a) 0 (b) 2 (c) 4 (d) None

70. If α, β be the roots of the equation $x^2 - ax + b = 0$ and $A_n = \alpha^n + \beta^n$, $A_{n+1} - aA_n + bA_{n-1}$ is equal to
(a) $-a$ (b) b (c) 0 (d) $a - b$

71. If α, β, γ are such that $\alpha + \beta + \gamma = 2$, $\alpha^2 + \beta^2 + \gamma^2 = 6$, $\alpha^3 + \beta^3 + \gamma^3 = 8$, the value of $\alpha^4 + \beta^4 + \gamma^4$ is
(a) 5 (b) 18 (c) 12 (d) 36

72. The number of irrational solutions of the equation $\sqrt{x^2 + \sqrt{x^2 + 11}} + \sqrt{x^2 - \sqrt{x^2 + 11}} = 4$ is
(a) 0 (b) 2 (c) 4 (d) 11

73. The number of solutions of $10^{2x} + 25^{1/x} = \left(\dfrac{65}{8}\right) \times 50^{1/x}$ is
(a) 0 (b) 2 (c) 4 (d) infinite

74. The equation $(2.4)^x = (2.6)^x - 1$ has
(a) no solution (b) exactly one sol
(c) atleast 2 solution (d) infinite solution

75. If $n \in N$, the number of real roots of $1 + x + \dfrac{x^2}{2!} + ... + \dfrac{x^{2n}}{(2n)!} = 0$ is
(a) n (b) 2 (c) 0 (d) none

76. The number of real roots of the equation $2^x + 2^{x-1} + 2^{x-2} = 7^x + 7^{x-1} + 7^{x-2}$ is
(a) 4 (b) 2 (c) 1 (d) 0

77. The number of real solutions of $x - \frac{1}{x^2-4} = 2 - \frac{1}{x^2-4}$ is
(a) 0 (b) 1 (c) 2 (d) infinite

78. The number of real solutions of the equation $2^{x/2} + (\sqrt{2}+1)^x = (5+2\sqrt{2})^{x/2}$ is
(a) infinite (b) 6 (c) 4 (d) 1

79. The number of values of a for which
$$(a^2 - 3a + 2)x^2 + (a^2 - 5a + 6)x + a^2 - 4 = 0$$
is an identity in x is
(a) 0 (b) 1 (c) 2 (d) 3

80. If $y = 2[x] + 3 = 3[x - 2] + 5$, the value of $[x + y]$ is, where [] = GIF
(a) 10 (b) 12 (c) 15 (d) none

81. If $0 < x < 1000$ and $\left[\frac{x}{2}\right] + \left[\frac{x}{3}\right] + \left[\frac{x}{5}\right] = \frac{31}{30}x$, where [] = GIF, the possible values of x is
(a) 34 (b) 33 (c) 32 (d) none

82. If $|x| + |x - 2| = 2$, the value of x lies in
(a) $(-\infty, 0]$ (b) $(1, 2]$ (c) $[0, 2]$ (d) $[2, \infty)$

83. If $|x| + |x + 2| + |x - 2| = k$, the equation has only one solution if k is
(a) 2 (b) 4 (c) 6 (d) 8

84. Let $y = \frac{x}{x^2 + x + 4}$. The minimum value of y is
(a) 1/3 (b) 1/5 (c) 1/4 (d) 1/8

85. The number of real solutions of $|x| + |x^2 - 1| = \frac{7}{6}$ is
(a) 6 (b) 4 (c) 5 (d) 8.

86. The real values of x of
$\sqrt{x + 3 - 4\sqrt{x-1}} + \sqrt{x + 8 - 6\sqrt{x-1}} = 1$ lies in
(a) (4, 5) (b) [5, 10] (c) [4, 10] (d) [4, 5]

87. The real values of x of $|x^2 - 1| + |x^2 - 4| = 3$ lies in
(a) $[-1, 1] \cup [-2, 2]$
(b) $[-2, -1] \cup [1, 2]$
(c) $[-4, -1] \cup [1, 4]$
(d) none

88. If $a = \log_{24}12$, $b = \log_{36}24$ and $c = \log_{48}36$, the value of $abc + 1$ is
(a) $2ac$ (b) $2bc$ (c) $2ca$ (d) None

89. If $x = 1 + \log_a bc$, $y = 1 + \log_b ac$ and $z = 1 + \log_c ab$, the value of $\frac{xyz}{xy + yz + xz}$ is
(a) 0 (b) –1 (c) 2 (d) 1

90. If $[x]^2 - 5[x] + 6 = 0$, where [] = GIF, the value of x lies in
(a) [3, 4) (b) [2, 3) (c) [4, 5) (d) None

Level III

(Problems For JEE-Advanced)

1. Let α and β be the roots of $(x - 2)(x - 3) + (x - 3)(x - 1) + (x + 1)(x - 2) = 0$, find the value of
$$\frac{1}{(\alpha+1)(\beta+1)} + \frac{1}{(\alpha-2)(\beta-2)} + \frac{1}{(\alpha-3)(\beta-3)}$$
2. If α and β are the roots of $x^2 - 3x + 1 = 0$, find the value of $(\alpha^5 + \beta^5)(\alpha^4 + \beta^4)$.
3. If α is a root of $4x^2 + 2x - 1 = 0$, prove that $4\alpha^3 - 3\alpha$ is the other root.
4. If α, β, γ are the roots of $x^3 - 3x + 2 = 0$, find an equation whose roots are $\alpha^2 + 2, \beta^2 + 2, \gamma^2 + 2$.
5. If the equations $3x^2 + px + 1 = 0$ and $2x^2 + qx + 1 = 0$ have a common root, prove that $2p^2 + 3q^2 - 5pq + 1 = 0$.
6. If roots of $ax^2 + bx + c = 0$ are of opposite sign lying in the interval (–2, 2), prove that $1 + \frac{c}{4a} - \left|\frac{b}{2a}\right| > 0$.

Solve for x:

7. $\sqrt{x + \sqrt{x+11}} + \sqrt{x - \sqrt{x+11}} = 4$
8. $\sqrt{2x^2 + 5x - 2} - \sqrt{2x^2 + 5x - 9} = 1$
9. $\sqrt{x} + \sqrt{x - \sqrt{1-x}} = 1$
10. $\sqrt[3]{(2x-1)} + \sqrt[3]{(x-1)} = 1$
11. $\sqrt{x^2 + 2x + 1} + \sqrt{x^2 - 4x + 4} = 3$
12. $|x^2 - 1| + |x^2 - 2| = 1$
13. $|x^2 - x| + 2|x - 1| = |x^2 + x - 2|$.
14. $\sqrt{x + 3 - 4\sqrt{x-1}} + \sqrt{x + 8 - 6\sqrt{x-1}} = 1$
15. Find out the range in which the value of the function $\frac{x^2 + 34x - 71}{x^2 + 2x - 7}$ lies for all real values of x. Justify your answer. **[Roorkee, 1983]**

Solve for x:

16. $\sqrt{5x^2 - 6x + 8} - \sqrt{5x^2 - 6x - 7} = 1$ **[Roorkee, 1985]**
17. $\frac{2\sqrt{x+1}}{3 - \sqrt{x}} = \frac{11 - 3\sqrt{x}}{5\sqrt{x-9}}$ **[Roorkee, 1985]**
18. $(x^2 + 2)^2 + 8x^2 = 6x(x^2 + 2)$ **[Roorkee, 1986]**
19. What is wrong with the following calculation?
$$1 = \sqrt{1} = \sqrt{(-1)\times(-1)}$$
$$= \sqrt{(-1)} \times \sqrt{(-1)} = i \times i = -1$$ **[Roorkee, 1987]**

20. Solve for x:
$$3x^3 = (x^2 + x\sqrt{18} + \sqrt{32})(x^2 - x\sqrt{18} - \sqrt{32}) - 4x^2$$
[Roorkee, 1988]
21. Solve for x: $2^{|x+1|} - 2^x = |2^x - 1| + 1$ **[Roorkee, 1989]**
22. Let there be a quotient of two natural numbers in which the denominator is one less than the square of the numerator. If we add 2 to both the numerator and the denominator, the quotient will exceed 1/3, and if subtract 3 from numerator, the quotient will lie between 0 and 1/10. Determine the quotient. **[Roorkee, 1990]**
23. Solve for x: $(15 + 4\sqrt{14})^t + (15 - 4\sqrt{14})^t = 30$ where $t = x^2 - 2|x|$. **[Roorkee, 1991]**
24. Find the positive solutions of the system of equations $x^{x+y} = y^n$, $y^{x+y} = x^{2n}y^n$, where $n > 0$. **[Roorkee, 1992]**
25. Obtain real solutions of the simultaneous equations
$$xy + 3y^2 - x + 4y - 7 = 0;$$
$$2xy + y^2 - 2x - 2y + 1 = 0.$$ **[Roorkee, 1993]**
26. If α and β are the roots of the equation $x^2 - px + q = 0$, find the quadratic equation the roots of which are $(\alpha^2 - \beta^2)(\alpha^3 - \beta^3)$ and $\alpha^3\beta^2 + \alpha^2\beta^3$. **[Roorkee, 1994]**
27. If the roots of $10x^3 - cx^2 - 54x - 27 = 0$ are in HP, find the value of c and all the roots. **[Roorkee, 1995]**

No questions asked in 1996, 1997 and 1998.

28. Let $\alpha + i\beta$, $\alpha, \beta \in R$ be a root of the equation $x^3 + px + r = 0$; $q, r \in R$. Find a real cubic equation, independent of α and β, whose one roots is 2α. **[Roorkee, 1999]**
29. If α, β be the roots of the equation $(x - a)(x - b) + c = 0$, find the roots of the equation $(x - \alpha)(x - \beta) - c = 0$. **[Roorkee, 2000]**
30. Given that α, γ be the roots of $Ax^2 - 4x + 1 = 0$ and β, δ the roots of $Bx^2 - 6x + 1 = 0$, find the values of A and B such that $\alpha, \beta, \gamma, \delta$ are in HP. **[Roorkee, 2000]**
31. The sum of the roots of the equation is equal to the sum of squares of their reciprocals. Find whether bc^2, ca^2 and ab^2 are in AP, GP or HP? **[Roorkee, 2001]**
32. If α is a root of $4x^2 + 2x - 1 = 0$, prove that $(4\alpha^3 - 3\alpha)$ is the other root.
33. Let $P(x) = ax^2 + bx + c$, where b and c are integers. If $(x^4 + 6x^2 + 25)$ and $3x^4 + 4x^2 + 28x + 5$ both are divisible by $P(x)$, find the value $P(1)$.
34. If α, β; β, γ; γ, α are the roots of $a_ix^2 + b_ix + c_i = 0$, $i = 1, 2, 3$, find the value of $(1 + \alpha)(1 + \beta)(1 + \gamma)$.
35. If x be real number such that $x^3 + 4x = 8$, find the value of $(x^7 + 6x^2 + 2)$.
36. Suppose a, b and c are the roots of $x^3 - x^2 - 672 = 0$. Find the value of $(a^3 + b^3 + c^3)$.
37. If a, b and c are the roots of $x^3 - 10x + 11 = 0$ such that $m = \tan^{-1}(a) + \tan^{-1}(b) + \tan^{-1}(c)$, find the value of $\tan\left(\frac{m}{2}\right)$.
38. For every x in R, if $a \le \frac{x}{x^2 + x + 4} \le b$, find the value of $(5a + 10b + 2)$.
39. If $x^2 - x - 1 = 0$, find the value of $\left(x^8 + \frac{1}{x^8} + 3\right)$.
40. If $x^2 - 2x - 1 = 0$, find the value of $\left[\sqrt{2}\left(x^5 + \frac{1}{x^5}\right) + 42\right]$.
41. If x be a real number satisfying $x^3 + \frac{1}{x^3} = 18$, find the value of $\left(x^7 + \frac{1}{x^7} + 4\right)$.
42. If $a + b + c = 0$ and $a^2 + b^2 + c^2 = 1$, find the value of $(a^4 + b^4 + c^4)$.
43. If a, b and c be the roots of $x^3 + px^2 + qx + r = 0$, find the value of $(b + c - a)(c + a - b)(a + b - c)$.
44. Find the common roots of
$$\begin{cases} x^5 - x^3 + x^2 - 1 = 0 \\ x^4 - 1 = 0 \end{cases}.$$
45. Find the greatest value of
$$f(x) = \frac{\left(x + \frac{1}{x}\right)^4 - \left(x^4 + \frac{1}{x^4}\right) - 1}{\left(x + \frac{1}{x}\right)^2 + \left(x^2 + \frac{1}{x^2}\right)}, x \in R - \{0\}.$$
46. If α, β be the roots of $x^2 - 2x - a + 1 = 0$ and γ, δ the roots of $x^2 - 2(a + 1)x + a(a - 1) = 0$ such that α and β lie in (γ, δ), find the value of a.
47. If α, β, γ be the roots of $x^3 - x^2 - x - 1 = 0$, find $\alpha^3 + \beta^3 + \gamma^3$.
48. If α, β, γ be the roots of $x^3 + 3x + 9 = 0$, find the value of $(\alpha^9 + \beta^9 + \gamma^9)$.
49. If all the roots of a biquadratic equation $x^4 + px^3 + qx^2 + rx + s = 0$ such that $p^3 = 2^m r$ and $p^4 = 2^n s$, where $m, n \in N$, find the value of $(m + n - 4)$.
50. If the product of two roots of $x^4 + x^3 - 16x^2 - 4x + 48 = 0$ is 6, find its roots.

Level IV

(Tougher Problems for JEE-Advanced)

1. Find the range of the expression
$$y = \frac{\tan^2\theta - 2\tan\theta - 8}{\tan^2\theta - 4\tan\theta - 5}.$$
2. Find the range of the expression
$$y = \frac{(\cot^2\theta + 5)(\cot^2\theta + 10)}{(\cot^2\theta + 1)}.$$
3. If x is real, find the maximum or minimum values of
$$y = \frac{2x^2 - 3x + 2}{2x^2 + 3x + 2}.$$
4. Find the value of k for which the expression $12x^2 - 10xy + 2y^2 + 11x - 5y + k$ is the product of two linear factors.

5. Find the greatest and the least values of x and y satisfying the relation $x^2 + y^2 = 6x - 8y$
6. If a, b, c, d be in GP, prove that $ax^3 + bx^2 + cx + d$ is divisible by $(ax^2 + c)$.
7. Prove that $ax^3 + bx + c$ is divisible by $x^2 + px + 1$ if $a^2 - c^2 = ab$
8. Prove that $x^3 + px^2 + qx + r$ will be a perfect cube if $p^2 = 27r$ and $3pr = q^2$.
9. Find the integral roots of
$$x^4 - x^3 - 19x^2 + 49x - 30 = 0.$$
10. Solve for x: $(6 - x)^4 + (8 - x)^4 = 16$.
11. Let α and β be the roots of $x^3 + ax^2 + bx + c = 0$ satisfying the relation $\alpha\beta + 1 = 0$. Prove that $c^2 + ac + b + 1 = 0$.
12. If α, β, γ be the roots of $x^3 + x + 2 = 0$, find an equation whose roots are
$$(\alpha - \beta)^2, (\beta - \gamma)^2, (\gamma - \alpha)^2.$$
13. If α, β, γ be the roots of $x^3 + qx + r = 0$, prove that
$$\left(\frac{\alpha^5 + \beta^5 + \gamma^5}{5}\right) = \left(\frac{\alpha^2 + \beta^2 + \gamma^2}{2}\right) \times \left(\frac{\alpha^3 + \beta^3 + \gamma^3}{3}\right).$$
14. If a line $y = mx + 1$ is a tangent to the curve $y^2 = 4x$, find the value of $(m^2 + m + 1)$.
15. Find the value of x which satisfies the equation
$$x = 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots$$
16. Let a and b be the roots of the equation $x^2 - 10cx - 11d = 0$ and those of $x^2 - 10ax - 11b = 0$ are c, d, find the value of $a + b + c + d$ when $a \neq b \neq c \neq d$.
17. If p and q be real numbers such that $p \neq 0, p^3 = -q$. If α and β are non-zero complex numbers satisfying $\alpha + \beta = -p$ and $\alpha^3 + \beta^3 = q$, find a quadratic equation having roots $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.
18. Let $\beta(a)$ and $\beta(a)$ be the roots of
$$(\sqrt[3]{a+1} - 1)x^2 + (\sqrt{1+a} - 1)x + (\sqrt[6]{a+1} - 1) = 0,$$
where $a > -1$, such that $L = \lim_{\alpha \to a^+} \alpha(a)$ and $M = \lim_{\alpha \to a^+} \beta(a)$. Find the value of $L + 2M + 3$.
19. Find the smallest integral value of k for which both the roots of $x^2 - 8kx + 16(k^2 - k + 1) = 0$ are real and distinct and have value at-least 6.
20. If α, β, γ be the roots of $x^3 + qx + r = 0$, prove that the equation whose roots are $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}, \frac{\beta}{\gamma} + \frac{\gamma}{\beta}, \frac{\gamma}{\alpha} + \frac{\alpha}{\gamma}$ is $r^2(x + 1)^3 + q^3(x + 1) + q^3 = 0$.
21. If $(x^2 + x + 1) + (x^2 + 2x + 3) + (x^2 + 3x + 5) + \ldots + (x^2 + 20x + 39) = 4500$, find the value of x.
22. Let α, β, γ are the roots of $x^3 + 2x^2 - x - 3 = 0$.
If the absolute value of $\left(\frac{\alpha+3}{\alpha-2} + \frac{\beta+3}{\beta-2} + \frac{\gamma+3}{\gamma-2}\right)$ is expressed as $\frac{p}{q}$, where p and q are co-prime, find the value of $(p + q - 2)$.
23. Find the number of integral values of a for which the graph of $y = 16x^2 + 8(a + 5)x - (7a + 5)$ is always above the x axis.
24. If the biquadratic $x^4 + 4x^3 + 6px^2 + 4qx + r$ is divisible by $x^3 + 3x^2 + 9x + 3$, find the value of $2(p + q)r$.
25. If α, β, γ be the roots of $2016x^3 + 2x^2 + 1 = 0$, find the value of
$$\left(\frac{1}{12}\left(\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\beta^2}\right)\left(\frac{1}{\alpha^3} + \frac{1}{\beta^3} + \frac{1}{\beta^3}\right) + 4\right).$$
26. Find the minimum value of
$$f(x) = \frac{\left(x + \frac{1}{x}\right)^6 - \left(x^6 + \frac{1}{x^6}\right) - 2}{\left(x + \frac{1}{x}\right)^3 + \left(x^3 + \frac{1}{x^3}\right)} \text{ for } x > 0.$$
27. Let x be a positive real number. Find the maximum possible value of the expression
$$y = \frac{x^2 + 2 - \sqrt{x^4 + 4}}{x}.$$
28. Find the range of values of a, such that
$$f(x) = \frac{ax^2 + 2(a+1)x + 9a - 4}{x^2 - 8x + 32} \text{ is always negative.}$$
29. If α, β, γ be such that $\alpha + \beta + \gamma = 2$, $\alpha^2 + \beta^2 + \gamma^2 = 6$, $\alpha^3 + \beta^3 + \gamma^3 = 8$, find value of $\alpha^4 + \beta^4 + \gamma^4$.
30. If $\tan\alpha$, $\tan\beta$ and $\tan\gamma$ be the roots of the equation $x^3 - (a + 1)x^2 + (b - a)x - b = 0$ $(b - a \neq 1)$, where $\alpha + \beta + \gamma$ lies between 0 and π, find the value of $(\alpha + \beta + \gamma)$.

Integer Type Questions

1. If $\alpha, \beta, \gamma, \delta$ be the roots of $16x^4 + 4x^2 + 1 = 0$, find the value of $8(\alpha^4 + \beta^4 + \gamma^4 + \delta^4) + 4$.
2 If $\alpha, \beta, \gamma, \delta$ be the roots of $x^4 + x^2 + 1 = 0$, find $(\alpha + \beta)(\alpha + \gamma)(\alpha + \delta)(\beta + \gamma)(\beta + \delta)(\gamma + \delta)$.
3. If $x^2 - 10ax - 11b = 0$ have roots c and d, $x^2 - 10cx - 11d = 0$ have roots a and b, find the value of $(a + b + c + d - 1208)$.
4. If α, β be the roots of $x^2 + px - q = 0$ and γ, δ be the roots of $x^2 + px + r = 0$, find the value of $\frac{(\alpha - \gamma)(\alpha - \delta)}{(\beta - \gamma)(\beta - \delta)}$.
5. If both the roots of $x^2 - 2ax + x^2 - 1 = 0$ lie between -3 and 4, prove that $[a]$ cannot be 4, where [,] = GIF.
6. If $\log_3 2, \log_3(2^x - 5), \log_3\left(2^x - \frac{7}{2}\right)$ are in AP, find the value of x.

7. If p be the number of solutions of $n^{-|x|}\left|m-|x|\right|=1$, where $m, n > 1$ and $n > m$. and q is the number of integral values of a for which the equation $x^2 + ax + a + 1 = 0$ has integral roots, find the value of $p + q + 2$.
8. If $x^2 + 3x + 5 = 0$ and $ax^2 + ax + a + 1$ have a common root and $a, b, c \in N$, find the minimum value of $(a + b + c)$.
9. If m is the number of real solutions of $\sin x = \frac{x}{10}$ and n is the number of real solutions of $\sin(\pi x) = |\ln|x||$, find the value of $(m - n + 1)$
10. If α and α be the roots of $x^2 - (x + 1)p - q = 0$, find the value of $\frac{\alpha^2+2\alpha+1}{\alpha^2+2\alpha+q}+\frac{\beta^2+2\beta+1}{\beta^2+2\beta+q}$.
11. Find the number of real roots of
$$6x^6 - 25x^5 + 31x^4 - 31x^2 + 25x - 6 = 0.$$
12. The set of real values of a for which the equation $x^4 - 2ax^2 + x + a^2 - a = 0$ has all real solutions, is given by $\left[\frac{p}{q}, \infty\right)$, where p and q are relatively prime positive integers, find the value of $(p + q + 1)$.
13. If α, β, γ be the roots of $x^3 - 3x - 1 = 0$, find the value of $\left(\frac{\alpha+1}{\alpha-1}+\frac{\beta+1}{\beta-1}+\frac{\gamma+1}{\gamma-1}\right)$.
14. If m is the number of real solutions of
$$\sin\left(\frac{\pi x}{2\sqrt{3}}\right) = x^2 - 2\sqrt{3}x + 4$$
and n is the number of values of a for which the equation $(a^2 - 3a + 2)x + (a^2 - 5a + 4)x + (|a| - 1) = 0$ has an identity in x, find the value of $(m + n + 2)$.
15. If α, β be the roots of $x^2 - 3x + A = 0$ and γ, δ the roots of $x^2 - 12x + B = 0$ such that $\alpha, \beta, \gamma, \delta$ are in GP find the value of $\left(\frac{B}{A^2+A+2}+1\right)$.

Comprehensive Link Passage (For JEE-Advanced Examination Only)

Passage I

Let the quadratic equation be $ax^2 + bx + c = 0$, where a, b, c are all real and α, β be its roots.

Also if , $a_1, a_2, a_3, \ldots$ be in AP, then $a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \ldots$ and if $b_1, b_2, b_3, \ldots$ are in GP, then $\frac{b_2}{b_1}=\frac{b_3}{b_2}=\frac{b_4}{b_3}=\ldots$ and also, if, $c_1, c_2, c_3, \ldots$ are in HP, then $\frac{1}{c_2}-\frac{1}{c_1}=\frac{1}{c_3}-\frac{1}{c_2}=\frac{1}{c_4}-\frac{1}{c_3}=\ldots$, where $a_i \neq 0, b_i \neq 0, c_i \neq 0$, $i = 1, 2, 3, \ldots$

On the basis of the above information, answer the following questions.

1. Let p and q be the roots of $x^2 - 2x + A = 0$ and r, s be the roots of $x^2 - 18x + B = 0$.
If $p < q < r < s$ are in AP, the values of A and B are
(a) –5, 67 (b) –3, 77 (c) 67, –5 (d) 77, –3
2. Let α, β be the roots of $x^2 - x + p = 0$ and γ, δ be the roots of $x^2 - 4x + q = 0$. If $\alpha, \beta, \gamma, \delta$ are in GP, the integral values of p and q are
(a) –2, –32 (b) –2, 3 (c) –6, 3 (d) –6, –32.
3. Let α, β be the roots of $Ax^2 - 4x + 1 = 0$ and γ, δ be the roots of $Bx^2 - 6x + 1 = 0$. If $\alpha, \beta, \gamma, \delta$ are in HP, the integral values of A and B are
(a) –3, 8 (b) –3, 16 (c) 3, 8 (d) 3, 16.

Passage II

Consider a quadratic equation $(1 + m)x^2 - 2(1 + 3m)x + (1 + 8m) = 0$, where $m \in R - \{-1\}$. Then

1. The number of integral values of m, such that given quadratic equation has imaginary roots, is
(a) 0 (b) 1 (c) 2 (d) 3
2. The set of values of m such that the given quadratic equation has at least one roots is negative, is
(a) $m \in (-\infty, -1)$ (b) $m \in \left(-\frac{1}{8}, \infty\right)$
(c) $m \in \left(-1, -\frac{1}{8}\right)$ (d) $m \in R$
3. The set of values of m, such that the given quadratic equation has both roots are positive, is
(a) $m \in R$
(b) $m \in (-1, 3)$
(c) $m \in [3, \infty)$
(d) $m \in (-\infty, -1) \cup [3, \infty)$.
4. The set of value of m, such that the given equation has at least one root is positive, is
(a) $m \in (-\infty, -1) \cup \left(-1, -\frac{1}{8}\right) \cup \left(-\frac{1}{3}, \infty\right)$
(b) $m \in (-\infty, -1) \cup \left(-\frac{1}{3}, -\frac{1}{8}\right) \cup [3, \infty)$
(c) $m \in \left(-\frac{1}{3}, -\frac{1}{8}\right)$
(d) $m \in (-\infty, -1) \cup \left(-1, -\frac{1}{8}\right) \cup [3, \infty)$

Passage III

Let $(a+\sqrt{b})^{Q(x)} + (a+\sqrt{b})^{Q(x)-2\lambda} = A$, where $\lambda \in N, A \in R$ and $a^2 - b = 1$

Then we write in place of $(a-\sqrt{b})$ is $(a+\sqrt{b})^{-1}$

1. If $(4+\sqrt{15})^{[x]} + (4-\sqrt{15})^{[x]} = 62$, where [] = GIF, then
(a) $x \in [-3, -2) \cup [1, 2)$ (b) $x \in [-3, -2)$
(c) $x \in [-2, -1) \cup [2, 3)$ (d) $x \in [-2, 3)$

2. The solutions of

$(2+\sqrt{3})^{x^2-2x+1}+(2-\sqrt{3})^{x^2-2x+1}=\dfrac{4}{(2-\sqrt{3})}$ are

(a) $(1\pm\sqrt{3}),1$ (b) $(1\pm\sqrt{2}),1$

(c) $(1\pm\sqrt{3}),2$ (d) $(1\pm\sqrt{2}),2$

3. The number of real solutions of the equation $(15+4\sqrt{14})^t+(15-4\sqrt{14})^t=30$ is, where $t=x^2-2|x|$

(a) 0 (b) 2 (c) 4 (d) 6

4. If α, β are the roots of the equation $1!+2!+3!+\ldots+(x-1)!+x!=k^2$ and $k \in I$, where $\alpha<\beta$ and if $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are the roots of the equation

$(a+\sqrt{b})^{x^2-[1+2\alpha+3\alpha^2+4\alpha^3+5\alpha^4]}+(a-\sqrt{b})^{x^2+[-5\beta]}=2a$,

where $a^2-b=1$ and [] = GIF, the value of $|\alpha_1+\alpha_2+\alpha_3+\alpha_4-\alpha_1\alpha_2\alpha_3\alpha_4|$ is

(a) 216 (b) 221 (c) 224 (d) 209

Passage IV

Consider the cubic equation $f(x)=ax^3+bx^2+cx+d=0$, where a, b, c, d are real numbers.

Then

1. If two roots are equal in magnitude but opposite in sign, then

(a) $bc+ad=0$ (b) $bc-ad=0$

(c) $ab=cd$ (d) $ab+cd=0$.

2. If the equation has one and only one positive real roots, then

(a) $b^2<3ac, ad>0$ (b) $b^2>3ac, ad>0$

(c) $b^2<3ac, ad<0$ (d) $b^2>3ac, ad<0$

3. If $a=1$ and one root of the given equation is unity, the value of $b+c+d$ is

(a) 36 (b) 0 (c) –1 (d) 1

Passage V

Let α, β be the roots of $x^2+px+q=0$ and γ, δ the roots of $x^2+rx+s=0$.

On the basis of the above information, answer the following questions.

1. If $\alpha, \beta, \gamma, \delta$ are in GP, then

(a) $q^2r^2=p^2s^2$ (b) $q^2r^2+p^2s^2=0$

(c) $qr^2+ps^2=0$ (d) $qr^2=ps^2$

2. If $\alpha, \beta, \gamma, \delta$ are in AP, then

(a) $p^2+r^2=4(s+q)$ (b) $p^2-r^2=4(s-q)$

(c) $p^2-r^2=2(s-q)$ (d) $p^2+r^2=2(s+q)$

3. The value of $(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)$ is

(a) $p^2s^2-pr(q+s)+s(p^2-2q)+qr^2$

(b) $q^2-s^2-pr(q+s)+s(p^2-2q)+qr^2$

(c) $q^2+s^2-pr(q+s)+s(p^2-2q)+qr^2$

(d) $q^2+s^2-pr(q-s)+s(p^2-2q+qr^2)$

Match Matrix
(For JEE-Advanced Examination Only)

1. Observe the following Columns:

Column I		Column II	
(A)	If the number of solutions of the system of equations $x+2y=6$ and $\|x-3\|=y$ is m,	(P)	m is the AM of n and p.
(B)	If x and y are integers and $(x-8)(x-10)=2^y$, and the number of solutions be n,	(Q)	n is the GM of m and p
(C)	If the number of integral solutions for the equation $x+2y=2xy$ is p,	(R)	p is the HM of m and n
(S)	$n=\dfrac{m^p+p^m}{mp}$	(T)	$m=\sqrt{n\sqrt{p\sqrt{n\sqrt{p\sqrt{n\ldots..}}}}}$

2. Observe the following Columns:

Column I		Column II	
(A)	If $a+b+2c=0$, $c\neq 0$, the equation $ax^2+bx+c=0$ has	(P)	at least one root in (–2, 0)
(B)	If $a, b, c \in R$ such that $2a+3b+6c=0$, the equation $ax^2+bx+c=0$ has	(Q)	at least one root in (–1, 0)
(C)	Let a, b, c be non-zero real numbers such that $\int_0^1(1+\cos^8 x)(ax^2+bx+c)dx=\int_0^2(1+\cos^8 x)(ax^2+bx+c)dx$, the equation $ax^2+bx+c=0$ has	(R)	at least one root in (–1, 1)
		(S)	at least one root in (0, 1)
		(T)	at least one root in (0, 2)

3. Observe the following Columns:

Column I		Column II	
(A)	If N be the number of solutions and S be the sum of all roots of the equation $\|x-\|4-x\|\|-2x=4$,	(P)	$S=0$

(B)	If N be the number of solutions and S be the sum of all roots of the equation $\|x^2 - x - 6\| = x + 2$,	(Q)	$N = 1$
(C)	If N be the number of solutions and S be the sum of all roots of the equation $x^2 - 3\|x\| - 4 = 0$,	(R)	$N = 2$
		(S)	$N = 3$
		(T)	$S = 4$

4. Observe the following Columns:

Column I		Column II	
(A)	If a, b, c, d are four non-zero real numbers such that $(d + a - b)^2 + (d + b - c)^2 = 0$ and roots of the equation $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$ are real and unequal, then	(P)	$a + b + c \neq 0$
(B)	If a, b, c are three non-zero real numbers such that the equation $(b - c)x^2 + b(c - a)x + c(a - b) = 0$ are real and equal,	(Q)	a, b, c are in AP.
(C)	If the three equations $x^2 + px + 12 = 0$, $x^2 + qx + 15 = 0$ and $x^2 + (p + q)x + 36 = 0$ have a common positive root and a, b, c be their other roots,	(R)	a, b, c are in GP
		(R)	a, b, c are in HP
		(T)	$a = b = c$

Matching List Type
(Only one Option is correct)

This section contains four questions, each having two matching list. Choices for the correct combination of elements from List I and List II are given as options (A), (B), (C) and (D), out of which only one is correct.

5. Let $\cos^2\left(\frac{\pi}{8}\right)$ is a root of the equation $x^2 + ax + b = 0$, where $a, b \in Q$.

List I	List II
(P) The value of a is	(1) 1/8
(Q) The value of b is	(2) –7/8
(R) The value of $(a - b)$ is	(3) –9/8
(S) The value of $(a + b)$ is	(4) –1

Codes:

	P	Q	R	S
(A)	1	3	4	2
(B)	4	1	3	2
(C)	4	3	1	2
(D)	1	4	2	3

6. Let three equations $x^2 + ax + 12 = 0$, $x^2 + bx + 15 = 0$ and $x^2 + (a + b)x + 36 = 0$ have a common positive root.

List I	List II
(P) The value of a is	(1) –8
(Q) The value of b is	(2) 9
(R) The value of $a + b + 20$ is	(3) –7
(S) The value of $(10 - a + b)$ is	(4) 5

Codes:

	P	Q	R	S
(A)	3	4	1	2
(B)	3	4	1	2
(C)	3	1	4	2
(D)	1	4	2	3

7. Let m is the minimum value of $y = \frac{x^3 + x + 2}{x}$, $x > 0$, n is the maximum value of $y = \frac{x}{x^2 + x + 4}$, $x > 0$, p is the number of solutions of $3^x + 4^x + 5^x = 6^x$ and q is the number of real roots of $(x + 3)4 + (x + 5)^4 = 16$.

List I	List II
(P) The value of m is	(1) 5
(Q) The value of p is	(2) 3
(R) The value of $(p + 5n + 3)$ is	(3) 4
(S) The value of $(p + q)$ is	(4) 1

Codes:

	P	Q	R	S
(A)	1	4	3	2
(B)	3	4	2	1
(C)	3	1	4	2
(D)	3	4	1	2

8. Let α and γ are the roots of $Ax^2 - 4x + 1 = 0$; and β and δ are the roots of $Bx^2 - 6x + 1 = 0$. If $\alpha, \beta, \gamma, \delta$ are in HP.

List I	List II
(P) The value of A is	(1) $\frac{3}{2}$
(Q) The value of B is	(2) 3
(R) The value $(\alpha + \beta)$ is	(3) $\frac{7}{12}$
(S) The value of $(\gamma + \delta)$ is	(4) 8

Codes:

	P	Q	R	S
(A)	2	4	1	3
(B)	4	2	3	1
(C)	4	3	1	2
(D)	3	4	1	2

Assertion (A) and Reason (R)

(A) Both A and R are true and R is the correct explanation of A.
(B) Both A and R are true and R is not the correct explanation of A.
(C) A is true and R is false.
(D) A is false and R is true.

1. *Assertion (A):* If the equations $ax^2 + bx + c = 0$, where a, b, c are real numbers, and $x^2 + 2x + 3 = 0$ have a common root, then $a : b : c = 1 : 2 : 3$
Reason (R): Roots of $x^2 + 2x + 3 = 0$ are imaginary.
2. *Assertion (A):* The number of real solutions of the equation $\sin(2^x)\cos(2^x) = \frac{1}{4}(2^x + 2^{-x})$ is 2.
Reason (R): AM ≥ GM
3. *Assertion (A):* If a, b, c are rationals and $2^{1/3}$ satisfies $a + bx + cx^2 = 0$, then $a = 0 = b = c$.
Reason (R): A polynomial equation with rational co-efficients have irrational roots.
4. *Assertion (A):* If x, y, z are real and $2x^2 + y^2 + z^2 = 2x - 4y + 2xz - 5$, the maximum possible value of $x - y + z$ is 4.
Reason (R): The above equation re-arranges as such of three squareds equated to zero.
5. *Assertion (A):* If $(a - b)x^2 + (b - c)x + (c - a) = 0$, then $x = 1$ is a root.
Reason (R): If sum of the co-efficients of $ax^2 + bx + c = 0$ is zero, then 1 is a root.
6. *Assertion (A):* If $a + b + c > 0$ and $a < 0 < b < c$, both the roots of $a(x - b)(x - c) + b(x - a)(x - b) + c(x - a)(x - b) = 0$ are negative.
Reason (R): If both the roots are negative, the sum of the roots is negative as well as product of the roots.
7. *Assertion (A):* The co-efficient of x^{49} in $(x - 1)(x - 2) \ldots (x - 50)$ is $-(1 + 2 + 3 + \ldots + 50)$.
Reason (R): The number of real solutions of $\sin x = x^2 + x + x + 1$ is zero.
8. *Assertion (A):* If $(a^2 + b^2 + c^2)p^2 - 2(ab + bc + cd)p + (b^2 + c^2 + d^2) \le 0$, then a, b, c, d are in GP.
Reason (R): If a, b, c, d are in GP, then $\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$.
9. *Assertion (A):* If $\left|\frac{x}{x-1}\right| + |x| = \frac{x^2}{|x-1|}$, then $x \in (1, \infty)$.
Reason (R): If $f(x) + g(x) = f(x)\, g(x)$, then $f(x)\, g(x) \ge 0$
10. *Assertion (A):* The number of real solutions of $2^x + 2^{x-1} + 2^{x-2} = 7^x + 7^{x-1} + 7^{x-2}$ is 1
Reason (R): If $a^x + a^{x-1} + a^{x-2} = b^x + b^{x-1} + b^{x-2}$, then $x = 2 + \log_{(a/b)}\left(\frac{b^2 + b + 1}{a^2 + a + 1}\right)$.

Questions asked in Previous Years' JEE-Advanced Examinations

1. Show that the square of $\frac{\sqrt{26 - 15\sqrt{3}}}{5\sqrt{2} - \sqrt{38 + 5\sqrt{3}}}$ is a rational number. **[IIT-JEE, 1978]**
2. If α, β be the roots of the equation $x^2 + px + 1 = 0$ and γ, δ the roots of $x^2 + qx + 1 = 0$, find the value of $(\alpha - \gamma)(\beta - \gamma)(\alpha + \delta)(\beta + \delta)$. **[IIT-JEE, 1978]**
3. If α, β are the roots of $x^2 + px + q = 0$ and γ, δ are the roots of $x^2 + rx + s = 0$, find the value of $(\alpha - \gamma)(\beta - \gamma)(\alpha - \delta)(\beta - \delta)$ in terms of p, q, r and s. **[IIT-JEE, 1979]**
4. Show that for any triangle with sides a, b, c; $3(ab + bc + ca) \le (a + b + c)^2 \le 4(ab + bc + ca)$ **[IIT-JEE, 1979]**
5. Find the integral solutions of the following systems of inequalities
(i) $5x - 1 < (x + 1)^2 < 7x - 3$
(ii) $\frac{x}{2x+1} \ge \frac{1}{4}; \frac{6x}{4x-1} < \frac{1}{2}$ **[IIT-JEE, 1979]**
6. If x, y, z are all real and distinct,
$U = x^2 + 4y^2 + 9z^2 - 6yz - 3zx - 2xy$ is always
(a) non-negative (b) non-positive
(c) zero (d) none. **[IIT-JEE, 1979]**
7. Let $a > 0, b > 0$ and $c > 0$. Both the roots of the equation $ax^2 + bx + c = 0$
(a) are real and negative
(b) have negative real parts
(c) have positive real parts
(d) none **[IIT-JEE, 1979]**
8. For what values of m does the system of equations $3x + my = m$ and $2x - 5y = 20$ has solution satisfying the condition $x > 0, y > 0$. **[IIT-JEE, 1980]**
9. Show that the equation $e^{\sin x} - e^{-\sin x} - 4 = 0$ has no real solution. **[IIT-JEE, 1980]**
10. Both roots of the equation
$(x - b)(x - c) + (x - c)(x - a) + (x - a)(x - b) = 0$
are always
(a) positive (b) negative (c) real (d) none **[IIT-JEE, 1980]**
11. *No question asked in 1981.*
12. The number of real solutions of the equation $|x|^2 - 3|x| + 2 = 0$ is
(a) 4 (b) 1 (c) 3 (d) 2 **[IIT-JEE, 1982]**
13. mn squares of equal size are arranged to form a rectangle of dimension m by n, where m and n are natural numbers. Two squares will be called neighbours if they have exactly one common side. A natural number is written in each square such that the number written in any square is the arithmetic mean of the numbers written in its neighbouring squares. Show that this is possible only if all the numbers used are equal. **[IIT-JEE, 1982]**
14. If $x_1, x_2, \ldots, x_n$ are any real numbers and n is any positive integer, then
(a) $n\sum_{i=1}^{n} x_i^2 < \left(\sum_{i=1}^{n} x_i\right)^2$ (b) $\sum_{i=1}^{n} x_i^2 < \left(\sum_{i=1}^{n} x_i\right)^2$
(c) $\sum_{i=1}^{n} x_i^2 < n\left(\sum_{i=1}^{n} x_i\right)^2$ (d) None **[IIT-JEE, 1982]**

15. The largest interval for which $x^{12} - x^9 + x^4 - x + 1 > 0$ is
(a) $-4 < x < \leq 0$ (b) $0 < x < 1$
(c) $-100 < x < 100$ (d) R **[IIT-JEE, 1982]**

16. Two towns A and B are 60 km apart. A school is to be built to serve 150 students in town A and 50 students in town B. If the total distance to be travelled by all 200 students is to be as small as possible, the school should be built at
(a) Town B (b) 45 km from town A
(c) Town A (d) 45 km from town B
[IIT-JEE, 1982]

17. If p, q, r are any real numbers, then
(a) max $(p, q) <$ max (p, q, r)
(b) $\max(p, q) = \frac{1}{2}(p + q - |p - q|)$
(c) max $(p, q) <$ min (p, q, r)
(d) None **[IIT-JEE, 1982]**

18. If $a + b + c = 0$, the quadratic equation $3ax^2 + 2bx + c = 0$ has
(a) at least one root in (0, 1)
(b) one root in (2, 3) and the other in (–2, –1)
(c) imaginary roots
(d) none **[IIT-JEE, 1983].**

19. If one root of the equation $ax^2 + bx + c = 0$ is equal to the nth power of the other, show that
$$(ac^n)^{\frac{1}{n+1}} + (a^n c)^{\frac{1}{n+1}} + b = 0.$$ **[IIT-JEE, 1983]**

20. Find all real values of x which satisfy $x^2 - 3x + 2 \geq 0$ and $x^2 - 3x - 4 \leq 0$. **[IIT-JEE, 1983]**

21. If $(2 + i\sqrt{3})$ is a root of the equation $x^2 + px + q = 0$ where p and q are real, then $(p, q) = \ldots$
[IIT-JEE, 1982]

22. The equation $2x^2 + 3x + 1 = 0$ has an irrational root. Is it true/false? **[IIT-JEE, 1983]**

23. If $a < b < c < d$, the roots of the equation $(x - a)(x - c) + 2(x - b)(x - d) = 0$ are real and distinct. Is it true or false? **[IIT-JEE, 1984]**

24. The equation $x - \frac{2}{x-1} = 1 - \frac{2}{x-1}$ has
(a) no root (b) one root
(c) two equal root (d) infinitely many roots.
[IIT-JEE, 1984]

25. If $a^2 + b^2 + c^2 = 1$, $ab + bc + ca$ lies in the interval
(a) $\left[\frac{1}{2}, 2\right]$ (b) $[-1, 2]$ (c) $\left[-\frac{1}{2}, 1\right]$ (d) $\left[-1, \frac{1}{2}\right]$
[IIT-JEE, 1984]

26. For real x, the function $\frac{(x-a)(x-b)}{(x-c)}$ will assume all real values provided
(a) $a > b > c$ (b) $a < b < c$
(c) $a > c > b$ (d) $a < c < b$
[IIT-JEE, 1984]

27. If the product of the roots of the equation $x^2 - 2kx + 2e^{2\ln k} - 1 = 0$ is 7, the roots are real for $k = \ldots$
[IIT-JEE, 1984]

28. Solve for x; $(5 + 2\sqrt{6})^{x^2-3} + (5 - 2\sqrt{6})^{x^2-3} = 10$
[IIT-JEE, 1984]

29. If $P(x) = ax^2 + bx + c$ and $Q(x) = -ax^2 + dx + c$ where $ac \neq 0$, then $P(x)Q(x)$ has at least two real roots.
[IIT-JEE, 1985]

30. The solution of the equation $\log_7(\log_5(\sqrt{x+5} + \sqrt{x})) = 0$ is …
[IIT-JEE, 1986]

31. If the quadratic equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ $(a \neq b)$ have a common root, the numerical value of $a + b$ is … **[IIT-JEE, 1986]**

32. If a, b and c are distinct positive numbers, the expression $(b + c - a)(c + a - b)(a + b - c) - abc$ is
(a) positive (b) negative
(c) non-positive (d) none
[IIT-JEE, 1986]

33. If S be the set of all real x, such that $\left(\frac{2x-1}{2x^3 + 3x^2 + x}\right)$ is positive, S contains
(a) $\left(-\infty, -\frac{3}{2}\right)$ (b) $\left(-3, -\frac{1}{4}\right)$
(c) $\left(-\frac{1}{4}, \frac{1}{2}\right)$ (d) $\left(\frac{1}{2}, 3\right)$
[IIT-JEE, 1986]

34. For $a \leq 0$, determine all real roots of the equation $x^2 - 2a|x - a| - 3a^2 = 0$. **[IIT-JEE, 1986]**

35. Find the set of all x for which
$$\frac{2x}{2x^2 + 5x + 2} > \frac{1}{x+1}$$ **[IIT-JEE, 1987]**

36. Let $\alpha_1, \alpha_2, \beta_1, \beta_2$ be the roots of $ax^2 + bx + c = 0$ and $px^2 + qx + r = 0$ respectively. If the system of equations $\alpha_1 y + \alpha_2 z = 0$ and $\beta_1 y + \beta_2 z = 0$ has non-trivial solution, prove that $\frac{b^2}{q^2} = \frac{ac}{pr}$. **[IIT-JEE, 1987]**

37. Solve for x:
$\log_{(2x+3)}(6x^2 + 23x + 21) = 4 - \log(3x + 7)(4x^2 + 12x + 9)$
[IIT-JEE, 1987]

38. Solve for x : $|x^2 + 4x + 3| + 2x + 5 = 0$
[IIT-JEE, 1988]

39. The equation $x^{3/4}\log_2 x\left(2 + \log_2 x - \frac{5}{4}\right) = \sqrt{2}$ has
(a) at least one real solution
(b) exactly three real solutions
(c) exactly one irrational solution
(d) complex roots **[IIT-JEE, 1988]**

40. If x and y are positive real numbers, and m and n are any positive integers, $\frac{x^n y^m}{(1 + x^{2n})(1 + y^{2m})} < \frac{1}{4}$. Is it true or false?
[IIT-JEE, 1989]

41. There are exactly two distinct linear function … and … which maps (–1, 1) into (0, 2). **[IIT-JEE, 1989]**

42. If α and β be the roots of $x^2 + px + q = 0$ and α^4 and β^4 are the roots of $x^2 - rx + s = 0$, the equation $x^2 - 4qx + 2q^2 - r = 0$ has always
(a) two real roots (b) two positive roots
(c) two negative roots (d) one positive and one negative root
[IIT-JEE, 1989]

43. Let a, b, c be real numbers, $a \neq 0$. If α is a root of $a^2x^2 + bx + c = 0$, β is the root of $a^2x^2 - bx - 2c = 0$ and $0 < \alpha < \beta$, the equation $a^2x^2 + 2bx + 2c = 0$ has a root γ that always satisfies **[IIT-JEE, 1989]**
(a) $\gamma = \frac{\alpha + \beta}{2}$ (b) $\gamma = \alpha + \frac{\beta}{2}$
(c) $\gamma = \alpha$ (d) $\alpha < \gamma < \beta$

44. If $x < 0$, $y < 0$, $x + y + \frac{x}{y} = \frac{1}{2}$ and $(x + y) \cdot \frac{x}{y} = -\frac{1}{2}$, then $x = \ldots$ and $y = \ldots$ **[IIT-JEE, 1990]**

45. *No question asked in 1991.*

46. Let α, β be the roots of the equation $(x - a)(x - b) - c = 0$, $c \neq 0$. The roots of the equation $(x - \alpha)(x - \beta) + c = 0$ are
(a) a, c (b) b, c
(c) a, b (d) $a + c, b + c$.
[IIT-JEE, 1992]

47. *No question asked in 1993.*

48. The number of points of intersection of two curves $y = 2 \sin x$ and $y = 5x^2 + 2x + 3$ is
(a) 0 (b) 1 (c) 2 (d) infinite
[IIT-JEE, 1994]

49. If p, q, r are positive and in AP, the roots of quadratic equation $px^2 + px + r$ are all real for
(a) $\left|\frac{r}{p} - 7\right| \geq 4\sqrt{3}$ (b) $\left|\frac{p}{r} - 7\right| \geq 4\sqrt{3}$
(c) all p and r (d) no p and r
[IIT-JEE, 1994]

50. Let $p, q \in \{1, 2, 3, 4\}$. The number of equations of the form $px^2 + qx + r = 0$ having real roots is
(a) 15 (b) 9 (c) 7 (d) 8
[IIT-JEE, 1994]

51. Let $a, b, c \in R$ and α, β be the roots of $ax^2 + bx + c = 0$ such that $\alpha < -1$ and $\beta > 1$, show that $1 + \frac{c}{a} + \left|\frac{b}{a}\right| = 0$.
[IIT-JEE, 1995]

52. *No question asked in 1996.*

53. The equation $\sqrt{x+1} - \sqrt{x-1} = \sqrt{4x-1}$ has
(a) no solution (b) one solution
(c) two solutions (d) more than two solution
[IIT-JEE, 1997]

54. Find the set of all solutions of the equation $2^{|y|} - |2^{y-1} - 1| = 2^{y-1} + 1$. **[IIT-JEE, 1997]**

55. The sum of all the real roots of the equation $|x - 2|^2 + |x - 2| - 2 = 0$ is …… **[IIT-JEE, 1997]**

56. *No question asked in 1998.*

57. Let a, b, c, d be real numbers in GP. If u, v, w satisfy the system of equations
$U + 2v + 3w = 6$, $4u + 5v + 6w = 12$,
$6u + 9v = 4$, show that the roots of the equation
$\left(\frac{1}{u} + \frac{1}{v} + \frac{1}{w}\right)x^2$
$-((b - c)^2 + (c - a)^2 + (a - b)^2)x + (u + v + w) = 0$
and $20x^2 + 10(a - d)2x - 9 = 0$ are reciprocal to each other. **[IIT-JEE, 1999]**

58. The roots of the equation
$x^2 - 2ax + a^2 + a - 3 = 0$
are real less than 3, then
(a) $a < 2$ (b) $2 \leq a \leq 3$
(c) $3 \leq a \leq 4$ (d) $a > 4$ **[IIT-JEE, 1999]**

59. If α and β are the roots of the equation $x^2 + bx + c = 0$, where $c < 0 < b$, then
(a) $0 < \alpha < \beta$ (b) $\alpha < 0 < \beta < |\alpha|$
(c) $\alpha < \beta < 0$ (d) $\alpha < 0 < |\alpha| < \beta$
[IIT-JEE, 2000]

60. If $b > a$, the equation $(x - a)(x - b) - 1 = 0$ has
(a) both roots in $[a, b]$ (b) both roots in $(-\infty, a)$
(c) both roots in (b, ∞) (d) one roots in and other in
[IIT-JEE, 2000]

61. For the equation $3x^2 + px + 3 = 0$, $p > 0$, if one of the roots is square of the other, then p is
(a) 1/3 (b) 1 (c) 3 (d) 2/3
[IIT-JEE, 2000]

62. If α, β are the roots of $ax^2 + bx + c = 0$, $(a \neq 0)$ and $\alpha + \delta, \beta + \delta$ are the roots of $Ax^2 + Bx + C = 0$, $(A \neq 0)$ for some constant δ, prove that $\frac{b^2 - 4ac}{a^2} = \frac{B^2 - 4AC}{A^2}$.
[IIT-JEE, 2000]

63. Let a, b, c be real numbers with $a \neq 0$ and let α, β be the roots of the equation $ax^2 + bx + c = 0$. Express the roots of $a^2x^2 + abc + c^2 = 0$ in terms of α, β.
[IIT-JEE, 2001]

64. Let α, β be the roots of $x^2 - x + p = 0$ and γ, δ be the roots of $x^2 - 4x + q = 0$. If $\alpha, \beta, \gamma, \delta$ are in GP, the integral values of p and q respectively, are
(a) –2, –32 (b) –2, 3 (c) –6, 3 (d) –6, –32
[IIT-JEE, 2001]

65. Let $f(x) = (1 + b^2)x^2 + 2bx + 1$ and let $m(b)$ be the minimum value of $f(x)$. As b varies, the range of $m(b)$ is
(a) (0, 1] (b) [0, 1/2] (c) [1/2, 1] (d) [0, 1]
[IIT-JEE, 2001]

66. The number of solutions of $\log_4(x - 1) = \log_2(x - 3)$ is
(a) 3 (b) 1 (c) 2 (d) 0
[IIT-JEE, 2001]

67. The number of values of k, for which the system of equations $(k + 1)x + 8y = 4k$, $kx + (k + 3)y = 3k - 1$ has infinitely many solutions is
(a) 0 (b) 1 (c) 2 (d) infinite
[IIT-JEE, 2002]

68. The set of all real numbers x for which $x^2 - |x + 2| + x > 0$ is
(a) $(-\infty, -2) \cup (2, \infty)$ (b) $(-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$
(c) $(-\infty, -1) \cup (1, \infty)$ (d) $(\sqrt{2}, \infty)$
[IIT-JEE, 2002]

69. If $x^2 + (a - b)x + (1 - a - b) = 0$, where $a, b \in R$, find the values of a for which the equation has unequal real roots for all values of b. **[IIT-JEE, 2003]**

70. For all x, $x^2 + 2ax + 10 - 3a > 0$, the interval in which a lies is
(a) $a < -3$ (b) $-5 < a < 2$
(c) $a > 5$ (d) $2 < a < 5$
[IIT-JEE, 2004]

71. If one root is square of the other root of the equation $x^2 + px + q = 0$, the relation between p and q is
(a) $p^3 - q(3p - 1) + q^2 = 0$
(b) $p^3 - q(3p + 1) + q^2 = 0$
(c) $p^3 + q(3p - 1) + q^2 = 0$
(d) $p^3 + q(3p + 1) + q^2 = 0$ **[IIT-JEE, 2004]**

72. The second degree polynomial $f(x)$ is satisfying $f(0) = 0, f(1) = 1, f'(x) > 0$ for all $x \in (0, 1)$,
(a) $ax + (1 - a)x^2, a \in R$ (b) $ax + (1 - a)x^2, a \in R^+$
(c) no polynomial (d) $ax + (1 - a)x^2, a \in (0, 2)$
[IIT-JEE, 2005]

73. Let α, β be the roots of $ax^2 + bx + c = 0$ and $\Delta = b^2 - 4ac$. If $\alpha + \beta, \alpha^2 + \beta^2 \cdot \alpha^3 + \beta^3$ are in GP, then
(a) $\Delta = 0$ (b) $\Delta \neq 0$ (c) $b\Delta = 0$ (d) $c\Delta = 0$
[IIT-JEE, 2005]

74. Let a, b, c be the sides of a triangle where $a \neq c$ and $\lambda \in R$. If the roots of the equation
$x^2 + 2(a + b + c)x + 3\lambda(ab + bc + ca) = 0$ are real, then
(a) $\lambda < \dfrac{4}{3}$ (b) $\lambda > \dfrac{5}{3}$
(c) $\lambda \in \left(\dfrac{1}{3}, \dfrac{5}{3}\right)$ (d) $\lambda \in \left(\dfrac{4}{3}, \dfrac{5}{3}\right)$
[IIT-JEE, 2006]

75. Let a and b be the roots of the equation $x^2 - 10cx - 11d = 0$ and those of $x^2 - 10ax - 11b = 0$ are c, d, find the value of $a + b + c + d$ when $a \neq b \neq c \neq d$.
[IIT-JEE, 2006]

76. If α, β be the roots of the equation $x^2 - px + r = 0$ and $\dfrac{\alpha}{2}, 2\beta$ be the roots of the equation $x^2 - qx + r = 0$, the value of r is
(a) $\dfrac{2}{9}(p - q)(2q - p)$ (b) $\dfrac{2}{9}(p - q)(2p - q)$
(c) $\dfrac{2}{9}(q - 2p)(2q - p)$ (d) $\dfrac{2}{9}(2p - q)(2q - p)$
[IIT-JEE, 2007]

77. Let $f(x) = \dfrac{x^2 - 6x + 5}{x^2 - 5x + 6}$.

Column I	Column II
(i) If $-1 < x < 1$, then $f(x)$ satisfies	(A) $0 < f(x) < 1$
(ii) If $1 < x < 2$, then $f(x)$ satisfies	(B) $f(x) < 0$
(iii) If $3 < x < 5$, then $f(x)$ satisfies	(C) $f(x) > 0$
(iv) If $x > 5$, then $f(x)$ satisfies	(D) $f(x) < 1$

[IIT-JEE, 2007]

78. Let a, b, c, p, q be real numbers. Suppose α, β are the roots of the equation $x^2 + 2px + q$ and $\alpha, \dfrac{1}{\beta}$ are the roots of the equation $ax^2 + 2bx + c = 0$, where $b^2 \in \{-1, 0, 1\}$.
Statement 1: $(p^2 - q)(b^2 - ac) \geq 0$
Statement 2: $b \neq pa, c \neq qa$ **[IIT-JEE, 2008]**

79. The smallest integral value of k for which both the roots of $x^2 - 8kx + 16(k^2 - k + 1) = 0$ are real and distinct and have value at least 4, is **[IIT-JEE, 2009]**

80. Let p and q be real numbers such that $p \neq 0, \neq^3 \neq q$, $p^3 \neq -q$ and α and β are non-zero complex numbers satisfying $\alpha + \beta = -p, \alpha^3 + \beta^3 = q$, a quadratic equation having roots $\dfrac{\alpha}{\beta}$ and $\dfrac{\beta}{\alpha}$ is
(a) $(p^3 + q)x^2 - (p^3 + 2q)x + (p^3 + q) = 0$
(b) $(p^3 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0$
(c) $(p^3 - q)x^2 - (5p^3 - 2q)x + (p^3 - q) = 0$
(d) $(p^3 - q)x^2 - (5p^3 + 2q)x + (p^3 - q) = 0$
[IIT-JEE, 2010]

81. Let α and β be the roots of $x^2 - 6x - 2 = 0$ with $\alpha > \beta$. If $a_n = \alpha^n - \beta^n$ for $n \geq 1$, the value of $\dfrac{a_{10} - 2a_8}{2a_9}$ is
(a) 1 (b) 2 (c) 3 (d) 4
[IIT-JEE, 2011]

82. A value of b for which the equations $x^2 + bx - 1 = 0$ and $x^2 + x + b = 0$ have one root in common is
(a) $-\sqrt{2}$ (b) $-i\sqrt{3}$ (c) $i\sqrt{5}$ (d) $\sqrt{2}$
[IIT-JEE, 2011]

83. The value of $(6 + \log_{3/2} x)$ where
$$x = \left(\frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\ldots\ldots}}}\right)$$
is ... **[IIT-JEE, 2012]**

84. *No question asked in 2013.*

85. Let $\alpha(a)$ and $\beta(a)$ be the roots of
$(\sqrt[3]{a+1} - 1)x^2 + (\sqrt{1+a} - 1)x + (\sqrt[6]{a+1} - 1) = 0$,
where $a > -1$. Then $\lim\limits_{\alpha \to a^+} \alpha(a)$ and $\lim\limits_{\alpha \to a^+} \beta(a)$ are
(a) $-5/2, 1$ (b) $-1/2, -1$ (c) $-7/2, 2$ (d) $-9/2, 3$
[IIT-JEE, 2012]

86. The quadratic equation $p(x) = 0$ with real co-efficients has purely imaginary roots.
The equation $p(p(x)) = 0$ has
(a) only purely imaginary roots
(b) all real roots
(c) two real and two purely imaginary roots
(d) neither real nor purely imaginary roots
[IIT-JEE, 2014]

87. The number of polynomials $f(x)$ with non-negative co-efficients of degree ≤ 2 satisfying $f(0) = 0$ and $\int_0^1 f(x)\,dx = 1$. **[IIT-JEE, 2014]**

ANSWERS

LEVEL I

2. 2
3. 2
4. $\left\{\pm i, \frac{1 \pm i\sqrt{3}}{2}\right\}$
5. $\left\{1, -6, \frac{5 \pm i\sqrt{39}}{2}\right\}$
6. infinite
7. 0
8. 0
9. $x = ab + bc + ca$
10. $x = a + b + c$
11. $\left\{4, -\frac{14}{5}\right\}$
12. infinite
13. $a = 1$
31. $p = 3$.
32. $1 < k < 2$
33. $A = -3, B = 77$
35. $-(q + r)$
36. $q^2 - p^2$.
38. $p = -2, q = -32$.
53. a, b
55. 1
56. –3
57. 0
58. –1
64. $a = -2$
65. $-3 < k < 5$
66. $-6 < k < 4$.
67. $m > 1$ and $m < -\frac{1}{7}$
68. $-3 < p < 6$.
69. $k = \pm 4$
70. $12 < k < 14$.
71. $-2 < b < 2$
72. $-5 < a < 2$
73. $2 < \lambda < 4$.
74. (i) Min value = 3
    (ii) Min Value = 0
    (iii) Min value = –9/4
    (iv) Max Value= –15/4
    (v) Max value = 0
    (vi) Max value = –1
75. (i) $\left[\frac{3}{4}, \infty\right)$
    (ii) $\left[-\frac{1}{4}, \infty\right)$
    (iii) $\left(-\infty, -\frac{31}{4}\right]$
76. $a = 1/2$ and $b = -2$.
77. $\left[\frac{7}{8}, 4\right]$
78. $[-1, 8]$
79. $-2 \le y \le -1/2$
80. $1/3 \le y \le 3$
81. $1/3 \le y \le 3$
82. $1/3 \le y \le 3$
83. 41.
85. $y \ge 3$ & $y \le 1/3$
86. R
87. $[25, \infty)$.
88. $\left(-\infty, -\frac{1}{2}\right)$
90. 20.
91. $2(\sqrt{2} - 1)$, when $x = \sqrt{2}$.
92. $m = 7, -7$
93. $k = 0, 2$.
96. $x \in (-8, -2] \cup [3, \infty)$.
97. $x \in [-2, 8]$ and $y \in [-9, 1]$.
98. $m \in (-\infty, -2]$
99. $m \in \left(\frac{11}{9}, \infty\right)$
100. $p \in \left(-\infty, -\frac{3}{4}\right)$.
101. $m \in (-3, -1) \cup (3, 5)$.
102. $a \in \left(-2, \frac{1}{4}\right)$
103. $a \in (5, 24)$.
104, $a < 2$.
105. $k \in (-2, 4)$
106. $a > \frac{11}{9}$
107. (a, b)
108. $\lambda \in R$
109. $-3 < a < 0$.
110. $m < -\frac{4}{3}$ and $m > 1$
111. $b > 13/4$
112. $k \in [1, \infty)$
113. $a \in [-3, 3]$.
126. +ve root = 1, –ve root = 2.
127. +ve root = 2, , which lie between 2 and 3 and another one between 5 and 6.

–ve root = 2, which lie between –1 and –2 and another one between –6 and –7.

128. +ve root = 2 , which lie between 2 and 3, $f(2) > 0,\ f\left(2\frac{1}{4}\right) < 0,\ f(3) > 0$.

–ve root = 2, which lie between 0 and –1 and between –4 and –5.

132. $x \in (-\infty, 1] \cup [3, \infty)$
133. $x \in (-\infty, -2] \cup [2, \infty)$
134. $x \in (-4, 2)$
135. $x \in [2, 5]$
136. $x \in (-\infty, 1] \cup [2, \infty)$
137. $x \in [1, 4]$
138. $x \in [3, 4]$
139. $x \in (2, 6)$
140. $x \in (-\infty, 1) \cup (2, 3)$
141. $x \in (-\infty, -2) \cup [0, 2]$
142. $x \in (2, \infty)$
143. $x \in (1, 4)$
144. $x \in (-\infty, -2] \cup (0, 2)$
145. $x \in (1, 2)$
146. $x \in (-\infty, 2) \cup (3, \infty)$
147. $x \in (-\infty, 3] \cup [7, \infty)$
148. $x \in [-1, 0] \cup [1, \infty)$
149. $x \in (-\infty, -1] \cup [1, 2]$
150. $x \in (-\infty, -1] \cup [1, 2]$
151. $x \in (-\infty, 0]$
152. $x \in \left(0, \frac{1}{2}\right)$
153. $x \in [2, 5)$
154. $x \in (-\infty, -1] \cup [1, 3] \cup [5, \infty)$
155. $x \in (-\infty, -1] \cup (2, \infty)$
156. $x \in (0, 1)$
157. $x \in (0, 2]$
158. $x \in (0, 1)$
159. $x \in (-\infty, -2) \cup (3, \infty)$
160. $x \in (0, 1]$
161. $x \in (-\infty, 1] \cup [2, 3) \cup (4, \infty)$
162. $x \in [1, 4) - \{2\}$
163. $x \in (-\infty, 1] \cup [2, \infty)$
164. $x \in [3, \infty)$
165. $x \in (2, \infty) - \{3, 4\}$
166. $x \in (1, \infty) - \{1, 3, 5\}$
167. $x \in [1, \infty)$
168. $x \in (-2, \infty) - \{0, 1\}$
169. $x \in (3, \infty) - \{5\}$
170. $x \in (-\infty, -1) \cup (2, \infty) - \{3, 4\}$
171. $x \in [-1, 1] - \{0\}$
172. $x \in [-2, \infty)$
173. $x \in [-6, -3] \cup [2, \infty)$
174. $x \in R$
175. $x \in \phi$
176. $x \in (-2, \infty)$
177. $x \in [-\sqrt{2}, \sqrt{2}]$
178. $x \in (-\infty, -2-\sqrt{3}] \cup [-2+\sqrt{3}, \infty)$
179. $x \in [2, \infty)$
180. $x \in \left(-\infty, \frac{1-\sqrt{5}}{2}\right) \cup \left(\frac{1+\sqrt{5}}{2}, 3\right)$
181. $x \in [0, \infty)$
182. $x \in (-\infty, -\sqrt{3}] \cup [\sqrt{3}, \infty)$
183. $x \in \left[\frac{-1-\sqrt{17}}{2}, 1\right] \cup \left[\frac{-1+\sqrt{17}}{2}, \infty\right)$
184. $x \in [3, \infty)$
185. $x \in [3, \infty)$
186. $x \in (-\infty, -4] \cup \left[\frac{3}{2}, \infty\right)$
187. $x = \varphi$
188. $x \in (-4, 1)$
189. $x \in (-3, -1) \cup [2, 3]$
190. $x \in [-3, 0]$
191. $x \in [-2, -1]$
192. $x = \varphi$
193. $x \in (-\infty, 1) \cup [4, \infty]$
194. $x \in [4, 6]$
195. $x \in [0, 3]$
196. $x \in [0, 3]$
197. $x \in [-3, 0]$
198. $x = \pm 1, \pm 2$
199. 0
200. $x = -1, 3$
201. $\left\{2, \frac{8}{3}\right\}$
202. $\{-2, 2, 4\}$
203. $x = \phi$
204. $x = -1, 3$
205. $x \in [1, 3]$
206. $x \in [-2, -1] \cup [1, 2]$
207. $x\ [1, 2]$
208. $x \in [1, \infty] \cup \{0\}$
209. $x \in [0, \infty)$
210. $x \in [-5, 4]$
211. $x \in (1, 2] \cup (3, 4]$
212. $x \in [-2, 4)$
213. $x \in [1, 4]$
214. $x \in (-\infty\ -7) \cup (3, \infty)$
215. $x \in (-\infty\ -1] \cup (5, \infty)$
216. $x \in [-1, 2]$
217. $x \in (-\infty\ -2) \cup (1, 2) \cup (2, \infty)$
218. $x \in \left(-\infty, -\frac{1}{2}\right) \cup \left(\frac{5}{2}, \infty\right)$
219. $x \in (-\infty\ -2) \cup (2, \infty)$
220. $x \in (-\infty\ -5] \cup [-1, \infty)$
221. $x \in (-\infty, -3) \cup \left(-\frac{1}{5}, \infty\right)$
222. No Solution

223. No Solution
224. No Solution
225. No Solution
226. No Solution
227. No Solution
228. $\left\{4, \frac{12}{5}\right\}$
229. $x = 20$
230. $x = 4$
231. $x = 5$
232. $x \in (-\infty, -2) \cup \left(-\frac{5}{4}, -1\right) \cup (1, 5)$
233. $x \in (1, 3)$
234. $x \in \left[\frac{2}{3}, 6\right)$
235. $x \in \left[\frac{3}{4}, 1\right)$
236. $x \in (-1, \infty) - \{2\}$
237. $x \in (-2, 1) \cup (1, \infty)$
238. $x \in (2, \infty)$
239. $x \in (3, \infty)$
240. $x \in \left(\frac{-1-\sqrt{17}}{2}, 1\right) \cup \left(\frac{-1+\sqrt{17}}{2}, \infty\right)$
241. $x \in (2, 3) \cup (6, \infty)$
242. $x \in \left(\frac{13}{5}, 3\right)$
243. $x \in \left(\frac{41}{16}, \infty\right)$
244. $x \in \left[8 + \frac{\sqrt{7}}{2}, 10\right]$
245. $x = -1, 2$
246. $x = \phi$
247. $x = 2$
248. $x = 1, 2$
249. $x = 1, 2$
250. $x = 6$
251. $x = 2$
252. $x = 3$
253. $x = 3$
254. $x = n\pi \pm \frac{\pi}{6}, n\pi \pm \frac{\pi}{3}, n \in I$
255. $x = \log_{2/5}\left(\frac{124}{175}\right)$
256. $x = \pm 2, \pm\sqrt{2}$
257. $x = \pm 1$
258. No Solution
259. No Solution
260. No Solution
261. One Solution
262. One Solution
263. $x \in (0, \infty)$
264. $x \in (-\infty, 0)$
265. $x \in (-\infty, 1) \cup (2, \infty)$
266. $x \in (-\infty, 2) \cup (3, \infty)$
267. $x \in (-\infty, 1]$
268. $x \in [-1, 2]$
269. $x \in [-4, 2] \cup [0, \infty)$
270. $x \in [-\infty, 0] \cup [1, \infty)$
271. $x \in (0, \infty)$
272. $x \in (0, 1)$

Level II

1. (d)	2. (c)	3. (c)	4. (b)	5. (d)
6. (d)	7. (d)	8. (b)	9. (a)	10. (c)
11. (a)	12. (a)	13. (b)	14. (b)	15. (c)
16. (b)	17. (a)	18. (b)	19. (b)	20. (d)
21. (b)	22. (a)	23. (b)	24. (c)	25. (a)
26. (c)	27. (a)	28. (d)	29. (b)	30. (b)
31. (b)	32. (a)	33. (a)	34. (d)	35. (c)
36. (a)	37. (c)	38. (a)	39. (b)	40. (c)
41. (b)	42. (c)	43. (d)	44. (d)	45. (a)
46. (b)	47. (c)	48. (a)	49. (c)	50. (b)
51. (b)	52. (b)	53. (b)	54. (a)	55. (a)
56. (c)	57. (c)	58. (c)	59. (a)	60. (a)
61. (a)	62. (b)	63. (c)	64. (a)	65. (a)
66. (c)	67. (a)	68. (a)	69. (b)	70. (c)
71. (b)	72. (c)	73. (b)	74. (c)	75. (c)
76. (c)	77. (a)	78. (d)	79. (b)	80. (c)
81. (b)	82. (c)	83. (b)	84. (b)	85. (a)
86. (b)	87. (b)	88. (b)	89. (d)	90. (a,b)

Level III

1. R
2. $[25, \infty)$
3. Maximum value = 7, Minimum value = 1/7
4. $k = 2$
5. $-2 \le x \le 8$ and $-9 \le y \le 1$
9. 1, 2, 3 and –5
10. 6, 8
12. $x^3 + 6x^2 + 9x + 12 = 0$
14. $6\cos\left(\frac{2\pi}{9}\right), 6\cos\left(\frac{4\pi}{9}\right), 6\cos\left(\frac{8\pi}{9}\right)$
15. $\cos\left(\frac{\pi}{7}\right), \cos\left(\frac{3\pi}{7}\right), \cos\left(\frac{5\pi}{7}\right)$
16. 1210
17. $(p^2 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0$
18. 1
19. 2

INTEGER TYPE QUESTIONS

1. 3	2. 0	3. 2	4. 1	5. 4
6. 3	7. 7	8. 9	9. 9	10. 1
11. 6	12. 8	13. 7	14. 4	15. 5

COMPREHENSIVE LINK PASSAGES

Passage I:	1. (b)	2. (c)	3. (a)	
Passage II:	1. (c)	2. (c)	3. (c)	4. (d)
Passage III:	1. (c)	2. (b)	3. (c)	4. (c)
Passage IV:	1. (b)	2. (c)	3. (c)	
Passage V:	1. (d)	2. (b)	3. (c)	

MATCH MATRIX

1. $A \to (P, T)$; $B \to (Q, S)$; $C \to (R)$
2. $A \to (R, S, T)$; $B \to (P, Q, R)$; $C \to (R, S, T)$
3. $A \to (P, Q)$; $B \to (S, T)$; C((P, R)
4. $A \to (P, Q, R, S, T)$; $B \to (P, Q)$; $C \to (R)$
5. (B)
6. (C)
7. (D)
8. (A)

ASSERTION AND REASON

1. (b)	2. (d)	3. (c)	4. (b)	5. (a)
6. (d)	7. (b)	8. (b)	9. (a)	10. (b)

HINTS AND SOLUTIONS

LEVEL I

2. Let $x = \sqrt{2 + \sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots\ldots \text{ to } \infty}}}}$

$$x = \sqrt{2 + x}$$
$$x^2 = 2 + x$$
$$x^2 - x - 2 = 0$$
$$(x - 2)(x + 1) = 0$$
$$x = 2, -1$$

since square always provide us non negative values, so $x = 2$

3. Given $x = 2 + 2^{2/3} + 2^{1/3}$

$$x - 2 = 2^{2/3} + 2^{1/3}$$
$$(x - 2)^3 = (2^{2/3} + 2^{1/3})^3$$
$$x^3 - 6x^2 + 12x - 8$$
$$= (2^{2/3})^3 + (2^{1/3})^3 + 3.2^{2/3}2^{1/3}(2^{2/3} + 2^{1/3})$$
$$= 4 + 2 + 3.2(x - 2)$$
$$= 6 + 6x - 12$$
$$= 6x - 6$$
$$x^3 - 6x^2 + 6x = 8 - 6 = 3$$

4. Divide by x^2, we get

$$\left(x^2 + \frac{1}{x^2}\right) - \left(x + \frac{1}{x}\right) + 2 = 0$$
$$\left(x + \frac{1}{x}\right)^2 - 2 - \left(x + \frac{1}{x}\right) + 2 = 0$$
$$\left(x + \frac{1}{x}\right)^2 - \left(x + \frac{1}{x}\right) = 0$$

Put $\left(x + \frac{1}{x}\right) = t$

Then $t^2 - t = 0$

$$t = 0 \text{ and } t = 1$$
$$x + \frac{1}{x} = 0 \text{ and } x + \frac{1}{x} = 1$$
$$x^2 + 1 = 0 \text{ and } x^2 - x + 1 = 0$$
$$x = \pm i \text{ and } x = \frac{1 \pm i\sqrt{3}}{2}$$

Hence, the solutions are

$$\left\{\pm i, \frac{1 \pm i\sqrt{3}}{2}\right\}$$

5. $(x + 1)(x + 2)(x + 3)(x + 4) = 120$

$\{(x + 1)(x + 4)\}\{(x + 2)(x + 3)\} = 120$

$\{(x^2 + 5x + 4)\}\{(x^2 + 5x + 6)\} = 120$

Put $x^2 + 5x = a$

$$(a + 4)(a + 6) = 120$$
$$a^2 + 10a - 96 = 0$$
$$(a + 16)(a - 6) = 0$$
$$a = 6, -16$$
$$x^2 + 5x = 6, -16$$
$$x^2 + 5x - 6 = 0, x^2 + 5x + 16 - 0$$
$$(x + 6)(x - 1) = 0, x^2 + 5x + 16 - 0$$
$$x = 1, -6, \frac{-5 \pm i\sqrt{39}}{2}$$

Hence, the solutions are

$$\left\{1, -6, \frac{-5 \pm i\sqrt{39}}{2}\right\}$$

6. It is true for all real values of x.

So it is an identity in x

Thus the number of real solution is infinite.

7. $x - \frac{1}{x^2 - 4} = 2 - \frac{1}{x^2 - 4}$

No real values of x satisfying the given equation

Thus, $x = \phi$

8. $(x - 1)^2 + (x - 1)^2 + (x - 3)^2 = 0$

$$3x^2 - 12x + 14 = 0$$

Clearly, $D < 0$

Number of real solutions is zero

9. $\frac{x - ab}{a + b} + \frac{x - bc}{b + c} + \frac{x - ca}{c + a} = a + b + c$

$$\frac{x-ab}{a+b}-c+\frac{x-bc}{b+c}-a+\frac{x-ca}{c+a}-b=0$$

$$\frac{x-ab-ac-bc}{a+b}+\frac{x-ab-ac-bc}{b+c}$$

$$+\frac{x-ab-ac-bc}{c+a}=0$$

$$(x-ab-ac-bc)\left(\frac{1}{a+\text{b}}+\frac{1}{b+c}+\frac{1}{c+a}\right)=0$$

$(x - ab - ac - bc) = 0$

$x = (ab + bc + ca)$

10. $\frac{x-a}{b+c}+\frac{x-b}{c+a}+\frac{x-c}{a+b}=3$

$$\frac{x-a}{b+c}-1+\frac{x-b}{c+a}-1+\frac{x-c}{a+b}-1=0$$

$$\frac{x-a-b-c}{b+c}+\frac{x-b-c-a}{c+a}+\frac{x-c-a-b}{a+b}=0$$

$$(x-a-b-c)\left(\frac{1}{b+\text{c}}+\frac{1}{c+a}+\frac{1}{a+b}\right)=0$$

$(x - a - b - c) = 0$

$x = (a + b + \underline{c})$

11. Put $u=\sqrt{5x^2-6x+8}, v=\sqrt{5x^2-6x-7}$

Then $u - v = 1$

$u^2 - v^2 = 15$

and then solve it

12. Given equation is $\frac{x-b}{a-b}+\frac{x-a}{b-a}=1$

Clearly , it is true for all real values of x.

So It is an identity in x

Thus, the number of real roots = infinite

13. Since the given equation is an identity in x, so

$a^2 - 3a + 2 = 0, |a| - 1 = 0, a^2 - 5a + 4 = 0$

$(a - 1)(a - 2) = 0, |a| = 1, (a - 1)(a - 4) = 0$

$a = 1, 2 ; a = \pm 1 ; a = 1, 4$

Hence, the value of a is 1. ie. $a = 1$

14. To prove $D < 0$
15. Do yourself
16. Apply $D = 0$
17. It is given that , product of the roots = 7

$2e^{2\log\lambda} - 1 = 7$

$2e^{\log\lambda^2} = 8$

$e^{\log\lambda^2} = 4$

$\lambda^2 = 4$

$\lambda = \pm 2$

$\lambda = 2$, since the logarithm of a negative number is not defined.

18. S = Sum of the co-efficients

$= b - c + c - a + a - b = 0$

So, 1 is a root

Now, product of the roots = $1 \cdot 1 = 1$

$$\frac{a-b}{b-c}=1$$

$a - b = b - c$

$a + c = 2b$

a, b and c are in A.P

19. S = Sum of the co-efficients

$= a(b - c) + b(c - a) + c(a - b)$

$= ab - ac + bc - ba + ca - ab = 0$

Thus, 1 is a root

Now, product of the roots = 1. 1 = 1

$$\frac{c(a-b)}{a(b-c)}=1$$

$ac - bc = ab - ac$

$2ac = ab + bc$

$b(a + c) = 2ac$

$$b=\frac{2ac}{(a+c)}$$

Thus, a, b and c are in H.P

20. Given a, b, c are in A.P

$a + c = 2b$

Also, it is given that, $D \geq 0$

$b^2 - 4ac \geq 0$

$$\left(\frac{a+c}{2}\right)^2-4ac\geq 0$$

$a^2 + c^2 - 14ac \geq 0$

$$\left(\frac{c}{a}\right)^2-14\left(\frac{c}{a}\right)+1\geq 0$$

$$\left(\frac{c}{a}-7\right)^2\geq(4\sqrt{3})^2$$

$$\left|\left(\frac{c}{a}-7\right)\right|\geq(4\sqrt{3})$$

21. Do yourself
22. Given roots of $x^2 - 2cx + ab = 0$ are real

So, $4c^2 - 4ab \geq 0$

$c^2 - ab \geq 0$

Now, $D = 4(a + b)^2 - 4(a^2 + b^2 + 2c^2)$

$= 4\{(4(a + b)^2 - 4(a^2 + b^2 + 2c^2)\}$

$= 4\{a^2 + b^2 + 2ab - (a^2 + b^2 + c^2)\}$

$= 4(2ab - 2c^2)$

$= 8(ab - c^2) < 0$

Thus, the roots are imaginary

23. Given equation is

$(a - 1)(x^2 + x + 1)^2 = (a + 1)(x^4 + x^2 + 1)$

$(a - 1)(x^2 + x + 1)^2 = (a + 1)(x^2 - x + 1)(x^2 + x + 1)$

$(a - 1)(x^2 + x + 1) = (a + 1)(x^2 - x + 1)$

$-2x^2 + 2ax - 2 = 0$

$x^2 - ax + 1 = 0$

Since roots are real and distinct,

so, $D > 0$

$a^2 - 4 > 0$

24. Do yourself
25. Do yourself
26. Do yourself
27. Do yourself
28. Given $\frac{\alpha}{\beta} = r$

$$\alpha + \beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}$$

$$\beta(1+r) = -\frac{b}{a}$$

$$\beta = -\frac{b}{a(1+r)}$$

Also, $\alpha\beta = \frac{c}{a}$

$$r\beta^2 = \frac{c}{a}$$

$$r\frac{b^2}{a^2(1+r)^2} = \frac{c}{a}$$

$$\frac{b^2 r}{(1+r)^2} = ac$$

$$\frac{b^2}{ac} = \frac{(1+r)^2}{r}$$

Hence, the result.

29. Given equation is

$$\frac{1}{x+p} + \frac{1}{x+q} = \frac{1}{r}$$

$$\frac{x+q+x+p}{(x+p)(x+q)} = \frac{1}{r}$$

$$(x+p)(x+q) = (2x+p+q)r$$

$$x^2 + (p+q)x + pq = (2x+p+q)r$$

$$x^2 + (p+q-2r)x + pq - r(p+q)) = 0$$

Sum of the roots = 0

$$(p+q-2r) = 0$$

$$p+q = 2r$$

Now, product of the roots

$$= pq - r(p+q)$$

$$= pq - \frac{1}{2}(p+q)^2$$

$$= -\frac{1}{2}(p^2+q^2)$$

30. Let the roots be α, α^2

Now, product of the roots =1

$$\alpha \cdot \alpha^2 = 1$$

$$\alpha^3 = 1$$

$\alpha = 1, \omega, \omega^2$, where ω is the cube root of unity.

Let $\alpha = \omega, \omega^2$

Now, sum of the roots = $-\frac{p}{3}$

$$(\omega + \omega^2) = -\frac{p}{3}$$

$$-1 = -\frac{p}{3}$$

$$p = 3$$

Hence, the value of p is 3

31. Let the roots be α, α^2

$$\alpha + \alpha^2 = -\frac{b}{a}, a \cdot \alpha^2 = \frac{c}{a}$$

$$\alpha + \alpha^2 = -\frac{b}{a}, \alpha^3 = \frac{c}{a}$$

Now, $\alpha + \alpha^2 = -\frac{b}{a}$

$$(\alpha + \alpha^2)^3 = \left(-\frac{b}{a}\right)^3$$

$$\alpha^3 + \alpha^6 + 3\alpha^3(\alpha + \alpha^2) = \left(-\frac{b}{a}\right)^3$$

$$\frac{c}{a} + \frac{c^2}{a^2} + 3 \cdot \frac{c}{a}\left(-\frac{b}{a}\right) = -\frac{b^3}{a^3}$$

$$b^3 + ac^2 + a^2c = 3abc$$

Hence, the result

32. Product of the roots < 0

$$(k^2 - 3k + 2) < 0$$

$$(k-1)(k-2) < 0$$

$$1 < k < 2$$

33. It is given that, $\angle P + \angle Q = \frac{\pi}{2}$

$$\frac{P}{2} + \frac{Q}{2} = \frac{\pi}{4}$$

$$\tan\left(\frac{P}{2} + \frac{Q}{2}\right) = \tan\left(\frac{\pi}{4}\right) = 1$$

$$\frac{\tan\left(\frac{P}{2}\right) + \tan\left(\frac{Q}{2}\right)}{1 - \tan\left(\frac{P}{2}\right) \cdot \tan\left(\frac{Q}{2}\right)} = 1$$

$$\frac{-\frac{b}{a}}{1 - \frac{c}{a}} = 1$$

$$-b = a - c$$

$$a + b = c$$

Hence, the result.

33. Let $p = a - 3d$, $q = a - d$, $r = a + d$, $s = a + 3d$

Now, $p + q = 2$, $pq = A$; $r + s = 18$, $rs = B$

Now, $p + q + r + s = 20$

$$4a = 20$$

$$a = 5$$

Also, $p = q = 2$

$$2a - 4d = 2$$

$$10 - 4d = 2$$

$4d = 8$
$d = 2$

Therefore, $p = a - 3d = 5 - 6 = 1$
$q = a - d = 5 - 2 = 3$
$r = a + d = 5 + 2 = 7$
$s = a + 3d = 5 + 6 = 11$

Thus, $A = pq = -3$
$B = rs = 77$

34. Given $\frac{\alpha}{\beta} = \sqrt{\frac{p}{q}}$

Here, $\alpha + \beta = -\frac{c}{a}, \alpha\beta = \frac{c}{a}$

Now, $\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{c}{a}}$

$= \sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{c}{a}}$

$= \frac{\alpha + \beta}{\sqrt{\alpha\beta}} + \sqrt{\frac{c}{a}}$

$= \frac{-\frac{c}{a}}{\sqrt{\frac{c}{a}}} + \sqrt{\frac{c}{a}}$

$= -\sqrt{\frac{c}{a}} + \sqrt{\frac{c}{a}}$

$= 0$

Hence, the result.

35. $x^2 + px + q = 0$ $\quad$ $x^2 + px - r = 0$
a, β $\quad$ γ, δ
$\alpha + \beta = -p, \alpha\beta = q \quad \gamma + \delta = -p, \gamma\delta = -r$

Now, $(\alpha - \gamma)(\alpha - \delta)$
$= \alpha^2 - (\gamma + \delta)\alpha + \gamma\delta$
$= \alpha^2 + p\alpha - \text{r}$
$= -q - \text{r}$
$= -(q + r)$

36. $x^2 + px + 1 = 0$ $\quad$ $x^2 + qx + 1 = 0$
α, β $\quad$ γ, δ
$\alpha + \beta = -p, \alpha\beta = 1 \quad \gamma + \delta = -q, \gamma\delta = 1$

Now, $(\alpha - \gamma)(\beta - \gamma)(\alpha + \delta)(\beta + \delta)$
$= (\gamma - \alpha)(\gamma - \beta)(\delta + \alpha)(\delta + \beta)$
$= \{\gamma^2 - (\alpha + \beta)\gamma + \alpha\beta\}\{\delta^2 + (\alpha + \beta)\delta + \alpha\beta\}$
$= \{\gamma^2 + p\gamma + 1\}\{\delta^2 - p\delta + 1\}$
$= (p\gamma - q\gamma)(-q\delta - p\delta)$
$= -(p - q)(p + q)\gamma\delta$
$= -(p - q)(p + q)$
$= -(p^2 - q^2)$
$= (q^2 - p^2)$

37. Similiar to Q.36

38. $x^2 - x + p = 0$ $\quad$ $x^2 - 4x + q = 0$
α, β $\quad$ γ, δ
$\alpha + \beta = 1, \alpha\beta = p \quad \gamma + \delta = 4, \gamma\delta = q$

It is given that , $\alpha, \beta, \gamma, \delta$ are in G.P
$\alpha, \alpha r, \alpha r^2, \alpha r^3 \in$ G.P
Now, $\alpha + \beta = 1$
$\alpha + \alpha r = 1$
$\alpha(1 + r) = 1$
$\alpha = \frac{1}{(1 + r)}$
Also, $\gamma + \delta = 4$
$\alpha r^2 + \alpha r^3 = 4$
$\alpha r^2 (1 + r) = 4$
$r^2 = 4$
$r = \pm 2$

when $r = 2, \alpha = \frac{1}{3}$ $\quad$ when $r = -2, \alpha = -1$

$\beta = \alpha r = \frac{2}{3}$ $\quad$ $\beta = \alpha r = 2$

$\gamma = \alpha r^2 = \frac{4}{3}$ $\quad$ $\gamma = \alpha r^2 = -4$

$\delta = \alpha r^3 = \frac{8}{3}$ $\quad$ $\delta = \alpha r^3 = 8$

Therefore, $p = \alpha\beta = -2$
$q = \gamma\delta = -32$

39. Let the roots be α, β
It is given that $\alpha = 2\beta$
Now, $\alpha + \beta = p, \alpha\beta = q$
$2\beta + \beta = p, \alpha\beta = q$
$3\beta = p, 2\beta^2 = q$
$\beta = \frac{p}{3}, 2\beta^2 = q$
$2\beta^2 = q$
$2\left(\frac{p}{3}\right)^2 = q$
$2p^2 = 9q$

40. It is given that , $|\alpha - \beta| = 1$
$|\alpha - \beta|^2 = 1$
$(\alpha + \beta)^2 - 4\alpha\alpha = 1$
$p^2 - 4q = 1$
Also, $p^2 = 4q + 1$
$p^2 + 4q^2 = 4q^2 + 4q + 1 = (2q + 1)^2$
Hence, the result.

41. Here, $\alpha + \beta = -1, \alpha\beta = -1$

Now, $\sum\left(\frac{1-\alpha}{1+\alpha}\right) = \left(\frac{1-\alpha}{1+\alpha}\right) + \left(\frac{1-\beta}{1+\beta}\right)$

$= \frac{2 - 2\alpha\beta}{1 + (\alpha + \beta) + \alpha\beta}$

$= \frac{2 + 2}{1 - 1 - 1} = -4$

42. Here,

$$\alpha + \beta + \gamma = -p,\ \alpha\beta + \alpha\gamma + \beta\gamma = q,\ \alpha\beta\gamma = -r$$

Now, $\left(\alpha - \frac{1}{\beta\gamma}\right)\left(\beta - \frac{1}{\gamma\alpha}\right)\left(\gamma - \frac{1}{\alpha\beta}\right)$

$$= \left(\alpha - \frac{\alpha}{\alpha\beta\gamma}\right)\left(\beta - \frac{\beta}{\alpha\beta\gamma}\right)\left(\gamma - \frac{\gamma}{\alpha\beta\gamma}\right)$$

$$= \alpha\beta\gamma\left(1 - \frac{1}{\alpha\beta\gamma}\right)^3$$

$$= -r\left(1 + \frac{1}{r}\right)^3$$

$$= -\frac{(1+r)^3}{r^2}$$

43. Here, $\alpha + \beta = 1, \alpha\beta = \frac{1}{6}$

Now,

$$\frac{1}{2}(p + q\alpha + r\alpha^2 + s\alpha^3) + \frac{1}{2}(p + q\beta + r\beta^2 + s\beta^3)$$

$$= \frac{1}{2}\{(p+p) + q(\alpha+\beta) + r(\alpha^2+\beta^2) + s(\alpha^3+\beta^3)\}$$

$$= \frac{1}{2}\left(2p + q + \frac{2}{3}r + \frac{s}{2}\right)$$

$$= \frac{p}{1} + \frac{q}{2} + \frac{r}{3} + \frac{s}{4}$$

44. Here, $\alpha + \beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}$

and $(\alpha+\delta) + (\beta+\delta) = -\frac{2B}{A}, (\alpha+\delta)(\beta+\delta) = \frac{C}{A}$

Now, $\alpha - \beta = (\alpha + \delta) - (\beta + \delta)$

$(\alpha - \beta)^2 = \{(\alpha + \delta) - (\beta + \delta)\}^2$

$(\alpha + \beta)^2 - 4\alpha\beta = \{(\alpha + \delta) + (\beta + \delta)\}^2 - 4(\alpha + \delta)(\beta + \delta)$

$$\frac{4b^2}{a^2} - \frac{4c}{a} = \frac{4B^2}{A^2} - \frac{4C}{A}$$

$$\frac{b^2 - ac}{a^2} = \frac{B^2 - AC}{A^2}$$

$$\frac{b^2 - ac}{B^2 - AC} = \left(\frac{a}{A}\right)^2$$

Hence, the result.

45. See the solution of Q.57 (IIT-JEE-1999)

46. $x^2 - Sx + P = 0$

$x^2 - 4x + 7 = 0$

47. $\alpha^2 - 3\alpha + 5 = 0 \Rightarrow \alpha^2 - 3\alpha + 7 = 2$

$\beta^2 - 3\beta + 5 = 0 \Rightarrow \beta^2 - 3\beta + 7 = 2$

Hence, the required equation is

$x^2 - Sx + P = 0$

$x^2 - 4x + 4 = 0$

48. $x^2 - 5x + 6 = 0$ gives $x = 2, x = 3$

Let $\alpha = 2, \beta = 3$

Hence, the required equation is

$x^2 - Sx + P = 0$

$x^2 - 11x + 30 = 0$

49. Let $y = 3\alpha + 2$

$$\alpha = \frac{y-2}{3}$$

Now, $9\alpha^3 - 7\alpha + 2 = 0$

$$9\left(\frac{y-2}{3}\right)^3 - 7\left(\frac{y-2}{3}\right) + 2 = 0$$

$9(y - 2)^3 - 7(y - 2) + 18 = 0$

$9y^3 - 54y^2 + 101y - 50 = 0$

Hence, the required equation is

$9x^3 - 54x^2 + 101x - 50 = 0$

50. Do yourself

51. Do yourself

52. We have

$(x - a)(x - b) - k = (x - c)(x - d)$

$(x - c)(x - d) + k = (x - a)(x - b)$

Hence, the roots are a and b.

53. We have

$(x - a)(x - b) + c = (x - \alpha)(x - \beta)$

$(x - a)(x - b) - c = (x - a)(x - b)$

Hence, the roots are a and b

54. The given equation reduces to

$\lambda x^2 + (1 - \lambda)x + 5 = 0$

Let its roots are α and β

$$\alpha + \beta = \frac{\lambda - 1}{\lambda}, \alpha\beta = \frac{5}{\lambda}$$

Now, $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{4}{5}$

$$\frac{\alpha^2 + b^2}{\alpha\beta} = \frac{4}{5}$$

$$\frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta} = \frac{4}{5}$$

$$\frac{\frac{(\lambda-1)^2}{\lambda^2} - \frac{10}{\lambda}}{\frac{5}{\lambda}} = \frac{4}{5}$$

$$\frac{(\lambda-1)^2 - 10\lambda}{5\lambda} = \frac{4}{5}$$

$\lambda^2 - 16\lambda + 1 = 0$

Let its roots are λ_1 and λ_2

Thus, $\lambda_1 + \lambda_2 = 16, \lambda_1\lambda_2 = 1$

Now, $\frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1}$

$$= \frac{(\lambda_1 + \lambda_2)^2 - 2\lambda_1\lambda_2}{\lambda_1\lambda_2}$$

$$= \frac{256 - 2}{1}$$

$$= 254$$

55. The given equation is

$x^2 - (x + 1)\,p - c = 0$

$x^2 - px - (p + c) = 0$

$\alpha + \beta = p,\ \alpha\beta = -(p + c)$

Now, $(1 + \alpha)(1 + \beta)$

$= 1 + (\alpha + \beta) + \alpha\beta$

$= 1 + p - p - c$

$= 1 - c$

Now, $\dfrac{\alpha^2 + 2\alpha + 1}{\alpha^2 + 2\alpha + c} + \dfrac{\beta^2 + 2\beta + 1}{\beta^2 + 2\beta + c}$

$$= \frac{(\alpha+1)^2}{(\alpha+1)^2 - (1-c)} + \frac{(\beta+1)}{(\beta+1)^2 - (1-c)}$$

$$= \frac{(\alpha+1)^2}{(\alpha+1)^2 - (1+\alpha)(1+\beta)} + \frac{(\beta+1)^2}{(\beta+1)^2 - (1+\alpha)(1+\beta)}$$

$$= \frac{(\alpha+1)}{(\alpha+1) - (1+\beta)} + \frac{(\beta+1)}{(\beta+1) - (1+\alpha)}$$

$$= \frac{(\alpha+1)}{(\alpha-\beta)} + \frac{(\beta+1)}{(\beta-\alpha)}$$

$$= \frac{(\alpha+1)}{(\alpha-\beta)} - \frac{(\beta+1)}{(\alpha-\beta)}$$

$$= \frac{(\alpha-\beta)}{(\alpha-\beta)}$$

$= 1$

56. Do yourself

57. $ax^2 + bx + c = 0$ …(i)

$cx^2 + bx + a = 0$ …(ii)

(i) - (ii) , we get, $(a - c)x^2 = (a - c)$

$x^2 = 1$

$x = \pm 1$

$x = -1$

Hence, the value of

$a - b + c = 0$

58. $x^2 + ax + b = 0$ …(i)

$x^2 + bx + a = 0$ ….(ii)

(i) - (ii), we get,

$(a - b)x = (a - b)$

$x = 1$

Hence, the value of

$a + b = -1$

59. $ax^2 + 2cx + b = 0$ …(i)

$ax^2 + 2bx + c = 0$ …(ii)

(i) - (ii), we get,

$2(c - b)x = (c - b)$

$x = \dfrac{1}{2}$

Put the value of $x = \dfrac{1}{2}$ in (i), we get,

$\dfrac{a}{4} + c + b = 0$

$a + 4b + 4c = 0$

60. Let S = sum of the co-efficients

$= a - b + b - c + c - a$

$= 0$

Thus, 1 is a root

Now , $1 + m + 1 = 0$

$m = -2$

Also, $\dfrac{c - a}{a - b} = 1$

$c - a = a - b$

$b + c = 2a$

$2a - b - c = 0$

61. $D = 4 - 12 = -8 < 0$

Clearly, both roots are common

Thus, $\dfrac{a}{1} = \dfrac{b}{2} = \dfrac{c}{3}$

$a : b : c = 1 : 2 : 3$

62. Do yourself

63. Do yourself

64. (i) $x^2 - 2x - 3 < 0$

$(x - 3)(x + 1) < 0$

$-1 < x < 3$

(ii) $x^2 - 3x + 2 > 0$

$(x - 1)(x - 2) > 0$

$x < 1$ and $x > 2$

65. The given inequation is

$x^2 - (k - 3)\,x - (k - 6) > 0$

Clearly, $D < 0$

$(k - 3)^2 + 4(k - 6) < 0$

$k^2 - 2k - 15 < 0$

$(k - 5)(k + 3) < 0$

$-3 < k < 5$

66. Clearly, $k - 2 < 0$

$k < 2$ …(i)

Also, $D < 0$

$64 - 4(k - 2)(k + 4) < 0$

$(k - 2)(k + 4) - 16 > 0$

$k^2 + 2k - 24 > 0$

$(k + 6)(k - 4) > 0$

$k < -6$ and $k > 4$ …(ii)

From (i) and (ii), we get,

$k < -6$

67. Clearly, $D < 0$

$(m + 1)^2 - 4m\,(2m - 1) < 0$

$7m^2 - 6m - 1 < 0$

$(m - 1)(7m + 1) < 0$

$m \in \left(-\dfrac{1}{7}, 1\right)$

68. Clearly, $2px + 1 = (p - 6)\,x^2 - 2$

$(p - 6)\,x^2 - 2px - 3 = 0$

Obviously, $D < 0$

$4p^2 + 12(p - 6) < 0$

$p^2 + 3\,(p - 6) < 0$

$p^2 + 3p - 18 < 0$

$(p + 6)(p - 3) < 0$

$-6 < p < 3$

69. Clearly, $D = 0$
$k^2 - 16 = 0$
$k = \pm 4$

70. Clearly, $D < 0$
$4(k-12)^2 - 8(k-12) < 0$
$(k-12)^2 - 2(k-12) < 0$
$(k-12)(k-14) < 0$
$12 < k < 14$

71. The given in equation is
$x^2 + bx + 1 > 0$
It is true only when $b^2 - 4 < 0$
$(b+2)(b-2) < 0$
$-2 < b < 2$

72. The given in equation is
$x^2 + 2ax + 10 - 3a > 0$
Clearly , $D < 0$
$4a^2 - 40 + 12a < 0$
$a^2 + 3a - 10 < 0$
$(a+5)(a-2) < 0$
$-5 < a < 2$

73. Clearly, $D < 0$
$4(4\lambda - 1)^2 - 4(15\lambda^2 - 2\lambda - 7) < 0$
$\lambda^2 - 6\lambda + 18 < 0$
$(\lambda - 2)(\lambda - 4) < 0$
$2 < \lambda < 4$

74. (i) $x^2 + 2x + 4 = (x+1)^2 + 3 \geq 3$
Min $V = 3$
(ii) $x^2 + 4x + 4 = (x+2)^2 \geq 0$
Min $V = 0$

75. (i) $x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4}$
Min $V = 3/4$
(ii) $x^2 + 3x + 2 = \left(x + \frac{3}{2}\right)^2 - \frac{1}{4} \geq -\frac{1}{4}$
Min $V = -1/4$

76. We have $f(x) = ax^2 + bx + 8$
$$= a\left\{\left(x + \frac{b}{2a}\right)^2 + \frac{8}{a} - \frac{b^2}{4a^2}\right\}$$
$$= a\left\{\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 32a}{4a^2}\right\}$$
Clearly, $\frac{b}{2a} = -2$
$b = -4a$
$$\frac{32a - b^2}{4a} = 6$$
$$\frac{32a - 16a^2}{4a} = 6$$
$8a - 4a^2 = 6a$
$4a^2 = 2a$
$a = \frac{1}{2}$
$b = -2$

77. We have $f(x) = 2x^2 - 3x + 2$
$$= 2\left(x^2 - \frac{3}{2}x + 1\right)$$
$$= 2\left(x - \frac{3}{2}\right)^2 + \frac{7}{8}$$
Min value is 7/8
Also, $f(0) = 2, f(2) = 4$
Thus, Range = $\left[\frac{7}{8}, 4\right]$

78. We have $f(x) = -x^2 + 6x - 1$
$= -(x-3)^2 + 8$
Max value = 8
Also, $f(0) = -1, f(4) = 7$
Thus, Range = $[-1, 8]$

79. Let $y = \frac{x^2 - 6x + 1}{x^2 + 6x + 1}$
$$y = \frac{\left(x + \frac{1}{x}\right) - 6}{\left(x + \frac{1}{x}\right) + 6}$$
Let $g(x) = \left(x + \frac{1}{x}\right)$
$$g'(x) = 1 - \frac{1}{x^2}$$
$g'(x) = 0$ gives $1 - \frac{1}{x^2} = 0$
$x = \pm 1$
Min $V = -2$ at $x = -1$
Max $V = -1/2$ at $x = 1$

80. Similiar to 79
81. Similiar to 79
82. Similiar to 79

83. Let $y = \frac{3x^2 + 9x + 17}{3x^2 + 9x + 7} = 1 + \frac{10}{3x^2 + 9x + 7}$
$y \leq 1 + 10.4 = 41$
Hence, the maximum value of y is 41.
where, minimum value of $3x^2 + 9x + 7$ is $\frac{1}{4}$
Thus, the maximum value of $\frac{1}{3x^2 + 9x + 7}$ is 4

84. Let $y = \frac{x^2 + 2x + a}{x^2 + 4x + 3a}$
$(y-1)x^2 + 2(2y-1)x + a(3y-1) = 0$
Clearly , $D \geq 0$
$(2y-1)^2 - a(y-1)(3y-1) \geq 0$
$(4-3a)y^2 + 4(a-1)y + (5-a) \geq 0$

Thus, $(4-3a) > 0,\ 16(a-1)^2 - 4(1-a)(4-3a) \geq 0$

$(4-3a) > 0,\ 16(a-1)^2 - 4(1-a)(4-3a) < 0$

$(4-3a) > 0,\ 4(a-1)^2 - (1-a)(4-3a) < 0$

$4(-3a) > 0,\ a(a-1) < 0$

$a < \frac{3}{4},\ 0 < a < 1$

Hence, the value of a is (0, 1)

85. We have

$$y = \frac{\tan x}{\tan 3x} = \frac{1-3t^2}{3-t^2},\ t = \tan x$$

$$\frac{3y+1}{y-3} = t^2 > 0$$

$$\frac{3y+1}{y-3} > 0$$

$$y < -\frac{1}{3},\ y > 3$$

86. Let $y = \dfrac{\tan^2\theta - 2\tan\theta - 8}{\tan^2\theta - 4\tan\theta - 5}$

$$y = \frac{t^2-2t-8}{t^2-4t-5}$$

$(y-1)t^2 + 2(1-2y)t + (8-5y) = 0$

Clearly, $D \geq 0$

$4(1-2y)^2 - 4(y-1)(8-5y) \geq 0$

$(1-2y)^2 - (y-1)(8-5y) \geq 0$

$9y^2 - 17y + 9 \geq 0$

Clearly, $D < 0$

Thus it is true for all y

Hence, the range = R

87. See the solution of Q-2 (Level-IV)
88. See the solution of Q-28 (Level-IV)
89. Do yourself
90. See the solution of Q-26 (Level-IV)
91. See the solution of Q-27 (Level-IV)
92. Here, $a = 2,\ b = 3,\ c = -2,\ h = \frac{m}{2},\ g = 0,\ f = \frac{5}{2}$

We have ,

$abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

$$-12 + 2\times 0 - 2\times\frac{25}{4} - 0 + 2\times\frac{m^2}{4} = 0$$

$$\frac{m^2}{2} - \frac{25}{2} - 12 = 0$$

$$\frac{m^2}{2} = \frac{49}{2}$$

$m^2 = 49$

$m = \pm 7$

93. Do yourself
94. Do yourself
95. We have $f(x, y, z)$

$= x^2 + 4y^2 + 9z^2 - 6yz - zx - 2xy$

$= x^2 + (2y)^2 + (3z)^2 - (2y\cdot 3z) - (x\cdot z) - (x\cdot 2y)$

$= \frac{1}{2}[(x-2y)^2 + (2y-3z)^2 + (3z-x)^2]$

≥ 0

96. We have $4y^2 + 4xy + x + 6 = 0$

$4y^2 + 4xy + (x+6) = 0$

Clearly, $D \geq 0$

$16x^2 - 16(x+6) \geq 0$

$x^2 - (x+6) \geq 0$

$(x-3)(x+2) \geq 0$

$x \leq -2,\ x \geq 3$

97. See the solution of Q-5 of Level - III
98. Let $f(x) = x^2 - mx + 1$

Apply

(i) $D \geq 0$

$\Rightarrow\ m^2 - 4 \geq 0$

$\Rightarrow\ (m+2)(m-2) \geq 0$

$\Rightarrow\ m \leq -2$ and $m \geq 2$

(ii) $af(1) > 0$

$\Rightarrow\ (1-m+1) < 0$

$\Rightarrow\ m < 2$

(iii) $\alpha + \beta < 2 \Rightarrow m < 2$

From (i), (ii) and (iii), we get

$m < -2$

99. Let $f(x) = x^2 - 6mx + 9m^2 - 2m + 2$

Apply

(i) $D \geq 0$

$36m^2 - 36m^2 + 8m - 8 \geq 0$

$m \geq 1$

(ii) $a\cdot f(3) > 0$

$1\cdot f(3) > 0$

$f(3) > 0$

$9 - 18m + 9m^2 - 2m + 2 > 0$

$9m^2 - 20m + 11 > 0$

$(m-1)(9m-11) > 0$

$m < 1$ and $m > \frac{11}{9}$

(iii) $\alpha + \beta > 6$

$6m > 6$

$m > 6$

From (i), (ii) and (iii), we get, $m \in \left(\frac{11}{9}, \infty\right)$

100 Let $f(x) = x^2 + 2(p-3)x + 9$

Apply

(i) $D > 0$

$4(p-3)^2 - 36 > 0$

$(p-3)^2 - 9 > 0$

$p(p-6) > 0$

$p < 0$ and $p > 6$

(ii) $af(6) < 0$

$f(6) < 0$

$36 + 12(p-3) + 9 < 0$

$12p + 9 < 0$

$4p + 3 < 0$

$$p < -\frac{3}{4}$$

From (i) and (ii), we get,

$$p \in \left(-\infty, -\frac{3}{4}\right)$$

101. Let $f(x) = x^2 - 2mx + m^2 - 1$

Apply

(i) $D \geq 0$

$4m^2 - 4(m^2 - 1) \geq 0$

$4 > 0$

It is true all values of m

(ii) $f(-2)f(4) < 0$

$(m^2 + 4m + 3)(m^2 - 8m + 15) < 0$

$(m + 1)(m + 3)(m - 3)(m - 5) < 0$

$m \in (-3, -1) \cup (3, 5)$

From (i) and (ii), we get

$m \in (-3, -1) \cup (3, 5)$

102. Let $f(x) = 4x^2 - 2x + a$

Apply

(i) $D \geq 0$

(ii) $af(-1) > 0$

(iii) $af(1) > 0$

(iv) $\alpha < 0 < \beta$

103. Let $f(x) = (a - 5)x^2 - 2ax + (a - 4)$

Apply

(i) $D > 0$

(ii) $af(1) < 0$

(iii) $af(2) < 0$

104. Let $f(x) = x^2 - 2x + a^2 + a - 3$

Apply

(i) $D \geq 0$

(ii) $af(3) > 0$

(iii) $\alpha + \beta < 6$

105. Let $f(x) = x^2 - 12kx + k^2 + k - 5$

Apply

(i) $D \geq 0$

(ii) $af(5) > 0$

(iii) $\alpha + \beta < 10$

106. Let $f(x) = x^2 - 6ax + 9a^2 - 2a + 2$

Apply

(i) $D \geq 0$

(ii) $af(3) > 0$

(iii) $\alpha + \beta > 6$

107. Let $f(x) = ax^2 + bx + c$

Apply

(i) $D > 0$

(ii) $af(-2) < 0$

(iii) $af(2) < 0$

108. Similiar to 107

109. Let $f(x) = 2x^2 - 2(2a + 1)x + a(a - 1)$

Apply

(i) $D > 0$

(ii) $2f(a) < 0$

110. We have

$m > 0$ and $D < 0$

$16 - 4m(3m + 1) < 0$

$3m^2 + m - 4 > 0$

$(3m + 4)(m - 1) > 0$

$m < -\frac{4}{3}$ and $m > 1$

Hence, the solution is $m \in (1, \infty)$

112. The given in equation is

$k \cdot 4^x + (k - 1)2^{x+2} + (k - 1) > 0$

$k \cdot (2^x)^2 + 4(k - 1)2^x + (k - 1) > 0$

$k \cdot (t)^2 + 4(k - 1)t + (k - 1) > 0, t = 2^x$

Let $f(t) = k \cdot (t)^2 + 4(k - 1)t + (k - 1)$

Apply

(i) $D < 0$

$4(k - 1)^2 - k(k - 1) < 0$

$3k^2 - 7k + 4 < 0$

$3k^2 - 3k - 4k + 4 < 0$

$3k(k - 1) - 4(k - 1) < 0$

$(3k - 4)(k - 1) < 0$

$1 < k < \frac{4}{3}$

(ii) $k > 0, f(0) \geq 0$

$k > 0, k - 1 \geq 0$

$k > 0, k \geq 1$

$k \geq 1$

(iii) $\alpha + \beta < 0$

$\frac{k-1}{k} > 0$

$k < 0$ and $k > 1$

From (i), (ii) and (iii), we get,

$1 < k < \frac{4}{3}$

114. Put $x^2 = t$

The given equation reduces to

$t^2 - 5t + 4 = 0$

$(t - 4)(t - 1) = 0$

$t = 1, 4$

$x^2 = 1, 4$

$x = \pm 1, \pm 2$

115. Put $x^2 = t$ and then try to solve it

116. Put $y = \frac{(x-1) + (x-5)}{2} = x - 3$

$x = y + 3$

The given equation reduces to

$(y + 2)^4 + (y - 2)^4 = 82$

$y^4 + 24y^2 - 25 = 0$

$(y^2 + 25)(y^2 - 1) = 0$

$y = \pm 5i, \pm 1$

$x - 3 = \pm 5i, \pm 1$

$x = 3 \pm 5i, 3 \pm 1$

$x = 3 \pm 5i, 4, 2$

117. The given equation is

$$2x^4 - x^3 - 11x^2 - x + 2 = 0$$

$$2\left(x^2 + \frac{1}{x^2}\right) - \left(x + \frac{1}{x}\right) - 11 = 0$$

Put $x + \frac{1}{x} = t$

Then the given equation reduces to

$$2(t^2 - 2) - t - 11 = 0$$
$$2t^2 - t - 15 = 0$$
$$(t - 3)(2t + 5) = 0$$
$$t = 3, -\frac{5}{2}$$
$$x + \frac{1}{x} = 3, -\frac{5}{2}$$
$$x^2 - 3x + 1 = 0,\ 2x^2 + 5x + 2 = 0$$
$$x = \frac{3 \pm \sqrt{5}}{2},\ x = -2, -\frac{1}{2}$$

Hence, the solution set is

$$\left\{-2, -\frac{1}{2}, \frac{3 \pm \sqrt{5}}{2}\right\}$$

118. The given equation is

$$(x - 1)(x - 1)(x - 1)(x - 1) = 8$$
$$\{(x - 1)(x - 4)\{((x - 2)(x - 3)\} = 8$$
$$\{(x^2 - 5x + 4)\}\{x^2 - 5x + 6)\} = 8$$
$$(a + 4)(a + 6) = 8,\ x^2 - 5x = a$$
$$(a + 4)(a + 6) = 8$$
$$a^2 + 10a + 16 = 0$$
$$(a + 2)(a + 8) = 0$$
$$a = -2, -8$$
$$x^2 - 5x = -2, -8$$
$$x^2 - 5x + 2 = 0,\ x^4 - 5x + 8 = 0$$
$$x = \frac{5 \pm \sqrt{17}}{2}, \frac{5 \pm i\sqrt{7}}{2}$$

119. The given equation is

$$x^2 + \frac{x^2}{(x+1)^2} = 3$$
$$x^4 + 2x^3 - x^2 - 6x - 3 = 0$$

Let $f(x) = x^4 + 2x^3 - x^2 - 6x - 3$

$f(x)$: + + – – –

$f(-x)$: + – – + –

In $f(x)$, there are one change, so $f(x)$ has one +ve root and in $f(-x)$, there are 3 changes, so $f(-x)$ has 3 negative roots

Thus, the number of real roots = 1 + 3 = 4

120. Replace x by $\frac{1}{x}$ we get

$$32x^3 - 48x^2 + 22x - 3 = 0$$

Let its roots are $\alpha - \beta, \alpha, \alpha + \beta$

Sum of the roots $= \frac{48}{32} = \frac{3}{2}$

$$3\alpha = \frac{3}{2}$$
$$\alpha = \frac{1}{2}$$

Also, $\alpha(\alpha + \beta)(\alpha - \beta) = \frac{3}{32}$

$$\alpha(\alpha^2 - \beta^2) = \frac{3}{32}$$
$$\frac{1}{2}\left(\frac{1}{4} - \beta^2\right) = \frac{3}{32}$$
$$\beta = \pm\frac{1}{4}$$

Hence, the roots are

$$\frac{1}{2} - \frac{1}{4}, \frac{1}{2}, \frac{1}{2} + \frac{1}{4} \text{ or } \frac{1}{2} + \frac{1}{4}, \frac{1}{2}, \frac{1}{2} - \frac{1}{4}$$
$$\frac{1}{4}, \frac{1}{2}, \frac{3}{4} \text{ or } \frac{3}{4}, \frac{1}{2}, \frac{1}{4}$$

Therefore, the roots of the main equation are

$$4, 2, \frac{4}{3} \text{ or } \frac{4}{3}, 2, 4$$

121. The given equation is

$$6 - 11x + 6x^2 - x^3 = 0$$

Replace x by $1/x$, we get

$$x^3 - 6x^2 + 11x - 6 = 0$$
$$(x - 1)(x - 2)(x - 3) = 0$$
$$x = 1, 2, 3$$

Hence, the solutions are $1, \frac{1}{2}, \frac{1}{3}$

122. Let $y = \alpha\beta + \alpha\gamma + \beta\gamma - \beta\gamma$

$$y = q - \beta\gamma$$
$$y = q - \frac{\alpha\beta\gamma}{\alpha} = q + \frac{r}{\alpha}$$
$$\alpha = \frac{r}{y - q}$$

Since α is a root of the given equation, so

$$\alpha^3 + p\alpha^2 + q\alpha + r = 0$$
$$\left(\frac{r}{y-q}\right)^3 + p\left(\frac{r}{y-q}\right)^2 + q\left(\frac{r}{y-q}\right) + r = 0$$

Hence, the required equation is

$$r^3 + pr(x - q) + q(x - q)^2 + r(x - q)^3 = 0$$

123. Let $y = \frac{\beta + \gamma}{\alpha} = \frac{\alpha + \beta + \gamma - \alpha}{\alpha} = \frac{p}{\alpha} - 1$

$$\alpha = \frac{p}{y + 1}$$

Since α is a root of the given equation, so

$\gamma^3, p\alpha^2 + r = 0$

$$\left(\frac{p}{y+1}\right)^3 - p\left(\frac{p}{y+1}\right)^2 + r = 0$$

Hence, the required equation is

$p^3 - q^3(x - 1) + (x + 1)^3 = 0$

$(x + 1)^3 - p^3(x + 1) = p^3 = 0$

124. Let $y = \beta + \gamma = \alpha + \beta + \gamma - \alpha = -1 - \alpha$

$\alpha = -(y + 1)$

Since α is a root of the given equation, so

$\gamma^3 + \gamma^2 + 4\alpha + 7 = 0$

$-(y + 1)^3 + (y + 1)^2 - 4(y + 1) + 7 = 0$

$(y + 1)^3 - (y + 1)^2 + 4(y + 1) - 7 = 0$

Hence, the equation is

$(x + 1)^3 - (x + 1)^2 + 4(x + 1) - 7 = 0$

125. Let $y = (\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\alpha$

$= (\alpha + \beta + \gamma - \gamma)^2 - 4\alpha\alpha$

$= (0 - \gamma)^2 - 4\alpha\alpha$

$= \gamma^2 - 4\alpha\beta$

$$= \gamma^2 + \frac{8}{\gamma}$$

$$= \frac{\gamma^3 + 8}{\gamma}$$

$$= \frac{-\gamma - 2 + 8}{\gamma}$$

$$= \frac{-\gamma + 6}{\gamma}$$

$$\gamma = \frac{6}{y+1}$$

Since γ is a root of the given equation, so $\gamma^3 + \gamma + 2 = 0$

$$\left(\frac{6}{y+1}\right)^3 + \left(\frac{6}{y+1}\right) + 2 = 0$$

$(y + 1)^3 + 3(y + 1)^2 + 108 = 0$

Hence, the required equation is

$(x + 1)^3 + 3(x + 1)^2 + 108 = 0$

126. Let $f(x) = x^5 - x^4 + x^3 + 8x^2 + 2x - 2$

$f(x)$: + – + + + –

$f(-x)$: – – – + – –

There are 3 changes in $f(x)$ and 2 changes in $f(-x)$, so $f(x)$ has 3 +ve root and 2 negative roots.

Thus, the number of real roots = 3 + 2 = 5

127. Let $f(x) = x^4 - 41x^2 + 40x + 126$

$f(x)$: + – + +

$f(-x)$: + – – +

There are 2 changes in $f(x)$ and 2 changes in $f(-x)$. So $f(x)$ has 2 +ve roots and 2 negative roots

Hence, the number of real roots = 2 + 2 = 4

128. $f(x) = x^4 - 14x^2 + 16x + 9$

$f(x)$: + – + +

$f(x)$: + – – +

There are 2 changes in $f(x)$ and 2 changes in $f(-x)$. So $f(x)$ has 2 +ve roots and 2 negative roots

Hence, the number of real roots = 2 + 2 = 4

129. Let $f(x) = 3x^4 + 12x^2 + 5x - 4$

$f(x)$: + + + –

$f(x)$: + + – –

There are 1 change in $f(x)$ and also 1 change in $f(-x)$

So $f(x)$ has i +ve root and 1 –ve root

Thus, the number of real roots = 2

130. Let $f(x) = x^9 - x^5 + x^4 + x^2 + 1$

$f(x)$: + – + + +

$f(x)$: – + + + +

There are 2 changes in $f(x)$ and 1 change in $f(-x)$. So $f(x)$ has 2 +ve real roots and 1 –ve real root

Thus, the number of real roots = 2 + 1 = 3

Hence, the number of imaginary roots = 9 – 3 = 6

131. Let $f(x) = x^7 - 3x^4 + 2x^3 - 1$

$f(x)$: + – + –

$f(x)$: – – – –

There are 3 changes in $f(x)$ and no change in $f(-x)$

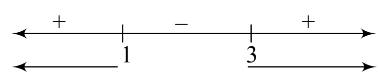
So $f(x)$ has 3 +ve real roots and 0 –ve root

Hence, the number of imaginary roots = 7 – 3 = 4.

132. Given in-equation is

$x^2 - 4x + 3 > 0.$

$\Rightarrow \quad (x - 1)(x - 3) > 0$



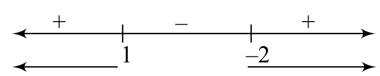
$\Rightarrow \quad x < 1$ and $x > 3$

$\Rightarrow \quad x \in (-\infty, 1) \cup (3, \infty)$

133. Given in-equation is

$x^2 + x - 2 \geq 0$

$\Rightarrow \quad (x + 2)(x - 1) \geq 0$



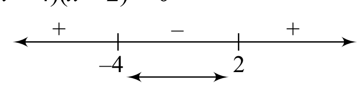
$\Rightarrow \quad x \leq -2$ and $x \geq 1$

$\Rightarrow \quad x \in (-\infty, -2) \cup (1, \infty)$

134. Given in-equation is

$x^2 + 2x - 8 < 0$

$\Rightarrow \quad (x + 4)(x - 2) < 0$



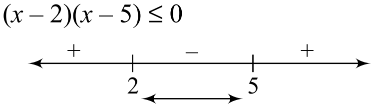
$\Rightarrow \quad -4 < x < 2$

$\Rightarrow \quad x \in (-4, 2)$

135. Given in-equation is

$x^2 - 7x + 10 \leq 0$

$\Rightarrow \quad (x - 2)(x - 5) \leq 0$



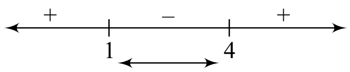
$\Rightarrow \quad 2 \leq x \leq 5$

$\Rightarrow \quad x \in [2, 5]$

143. Given in-equation is

$-x^2 + 5x - 4 > 0$

$\Rightarrow \quad x^2 - 5x + 4 < 0$



$\Rightarrow \quad (x-1)(x-4) < 0$

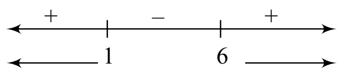
$\Rightarrow \quad 1 < x < 4$

144. Given in-equation is

$-x^2 + 7x - 6 \le 0$

$\Rightarrow \quad x^2 - 7x + 6 \ge 0$

$\Rightarrow \quad (x-1)(x-6) \ge 0$



$\Rightarrow \quad x \le 1, x \ge 6$

$\Rightarrow \quad x \in (-\infty, 1] \cup [6, \infty)$

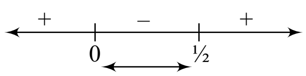
152. Given in-equation is

$\frac{1}{x} > 2$

$\Rightarrow \quad \frac{1}{x} - 2 > 0$

$\Rightarrow \quad \frac{1-2x}{x} > 0$

$\Rightarrow \quad \frac{2x-1}{x} > 0$



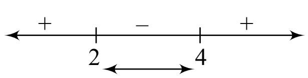
$\Rightarrow \quad 0 < x < \frac{1}{2}$

$\Rightarrow \quad x \in \left(0, \frac{1}{2}\right)$

153. Given in-equation is

$\frac{x-2}{4-x} \ge 0$

$\Rightarrow \quad \frac{x-2}{x-4} \le 0$

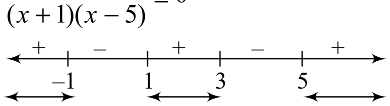


$\Rightarrow \quad 2 \le x \le 4$ and $x \ne 4$

$\Rightarrow \quad x \in [2, 4)$

154. Given in-equation is

$\frac{(x-1)(x-3)}{(x+1)(x-5)} \ge 0$



$\Rightarrow \quad x < -1, 1 \le x \le 3, x > 5$

$\Rightarrow \quad x \in (-\infty, -1) \cup [1, 3] \cup (5, \infty)$

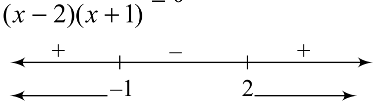
155. Given in-equation is

$\frac{1}{x-2} \ge \frac{1}{x+1}$

$\Rightarrow \quad \frac{1}{x-2} - \frac{1}{x+1} \ge 0$

$\Rightarrow \quad \frac{(x+1-x+2)}{(x-2)(x+1)} \ge 0$

$\Rightarrow \quad \frac{3}{(x-2)(x+1)} \ge 0$



$\Rightarrow \quad x < -1, x > 2$

$\Rightarrow \quad x \in (-\infty, -1) \cup (2, \infty)$

164. Given in-equation is

$x^2(x-2)^4(x-3) \ge 0$

$\Rightarrow \quad (x-3) \ge 0$

$\Rightarrow \quad x \ge 3$

$\Rightarrow \quad x \in [3, \infty)$

165. Given in-equation is

$(x-1)^4(x-3)^6(x-2)^5(x-4)^{10} > 0$

$\Rightarrow \quad (x-2)^5 > 0$

$\Rightarrow \quad (x-2) > 0$

$\Rightarrow \quad x > 2$, provided $x \ne 1, 3, 4$

Hence the solution set is $x \in (2, \infty) - \{3, 4\}$.

166. Given in-equation is

$\frac{(x-1)^{2013}(x-2)^{2014}}{(x-3)^{2016}(x-5)^{2010}} > 0$

$\Rightarrow \quad (x-1)^{2013} > 0$

$\Rightarrow \quad (x-1) > 0$

$\Rightarrow \quad x > 1$, provided $x \ne 2, 3, 5$

$\Rightarrow \quad x \in (1, \infty) \cup \{2, 3, 5\}$

174. Given in-equation is $x^2 + x + 2 > 0$.

This is true for every values of x.

Hence, $x \in R$

175. Given in-equation is $x^2 - x + 3 < 0$.

This statement is false for every values of x.

Hence, $x = \varphi$

176. Given in-equation is

$(x+2)(x^4 + x^2 + 1) > 0$

$\Rightarrow \quad (x+2)(x^2 - x + 1)(x^2 - x + 1) > 0$

$\Rightarrow \quad (x+2) > 0$

$\Rightarrow \quad x > -2$

$\Rightarrow \quad x \in (-2, \infty)$

177. Given in-equation is

$x^4 - 4 \le 0$

$\Rightarrow \quad (x^2 - 2)(x^2 + 2) \le 0$

$\Rightarrow \quad (x^2 - 2) \le 0$

$\Rightarrow \quad (x + \sqrt{2})(x - \sqrt{2}) \le 0$

$\Rightarrow \quad -\sqrt{2} \le x \le \sqrt{2}$

$\Rightarrow \quad x \in [-\sqrt{2}, \sqrt{2}]$

178. Given in-equation is

$(x^2 + 4x + 1)(x^2 + 1) \ge 0$

$\Rightarrow \quad (x^2 + 4x + 1) \ge 0$

$\Rightarrow \quad (x + 2 + \sqrt{3})(x + 2 - \sqrt{3}) \ge 0$

$\Rightarrow \quad (x + 2 + \sqrt{3}) \le 0, (x + 2 - \sqrt{3}) \ge 0$

$\Rightarrow \quad x \le -(2 + \sqrt{3}), x \ge -(2 - \sqrt{3})$

$\Rightarrow \quad x \in (-\infty, -(2 + \sqrt{3})] \cup [-(2 - \sqrt{3}), \infty)$

188. Now, $x^2 - 3x + 2 > 0$

$\Rightarrow \quad (x-1)(x-2) > 0$

$\Rightarrow \quad x < 1$ and $x > 2$

$\Rightarrow \quad x \in (-\infty, 1) \cup (2, \infty)$ …(i)

Also, $x^2 + 2x - 8 < 0$

$\Rightarrow \quad (x+4)(x-2) < 0$

$\Rightarrow \quad -4 < x < 2$

$\Rightarrow \quad x \in (4, 2)$ …(ii)

From Eqs (i) and (ii), we get

$x \in (-4, 1)$

189. Now, $x^2 - 9 \le 0$
$\Rightarrow \quad (x+3)(x-3) \le 0$
$\Rightarrow \quad -3 \le x \le 3$ …(i)
Also, $x^2 - 1 \ge 0$
$\Rightarrow \quad (x+1)(x-1) \ge 0$
$\Rightarrow \quad x \le -1, x \ge 1$
$\Rightarrow \quad x \in (-\infty, -1] \cup [1, \infty)$ …(ii)
From Eqs (i) and (ii), we get
$x \in [-3, -1] \cup [1, 3]$

190. Now, $x^3 - 9x \ge 0$
$\Rightarrow \quad x(x^2 - 9) \ge 0$
$\Rightarrow \quad x(x+3)(x-3) \ge 0$
$\Rightarrow \quad -3 \le x \le 0, x \ge 3$ …(i)
Also, $x^3 + 4x \le 0$
$\Rightarrow \quad x(x^2 + 4) \le 0$
$\Rightarrow \quad x \le 0$ …(ii)
From Eqs (i) and (ii), we get
$x \in [-3, 0]$

198. We have,
$|x|^3 - 3|x| + 2 = 0$
$\Rightarrow \quad (|x| - 1)(|x| - 2) = 0$
$\Rightarrow \quad (|x| - 1) = 0, (|x| - 2) = 0$
$\Rightarrow \quad |x| = 1, |x| = 2$
$\Rightarrow \quad x = \pm 1, x = \pm 2$

199. We have,
$x^2 - 4|x| + 3 = 0$
$\Rightarrow \quad |x|^3 - 4|x| + 3 = 0$
$\Rightarrow \quad (|x| - 1)(|x| - 3) = 0$
$\Rightarrow \quad (|x| - 1) = 0, (|x| - 3) = 0$
$\Rightarrow \quad |x| = 1, |x| = 3$
$\Rightarrow \quad x = \pm 1, x = \pm 3$
Hence the sum of the roots = 1 – 1 + 3 – 3 = 0

200. We have,
$|3x - 1| = |x + 5|$
$\Rightarrow \quad (3x - 1) = \pm(x + 5)$
Taking positive sign, we get
$3x - 1 = x + 5$
$\Rightarrow \quad 2x = 6$
$\Rightarrow \quad x = 3$
Taking negative sign, we get
$3x - 1 = -x - 5$
$\Rightarrow \quad 4x = -4$
$\Rightarrow \quad x = -1$
Hence, the solutions are $x = -1, 3$.

201. We have,
$|2x - 5| = x - 3$
$\Rightarrow \quad (x^2 - x - 6) = \pm(x + 2)$
Taking positive sign, we get
$2x - 5 = x - 3$
$\Rightarrow \quad 2x - x = 5 - 3$
$\Rightarrow \quad x = 2$
Taking negative sign, we get
$2x - 5 = -x + 3$
$\Rightarrow \quad 2x + x = 5 + 3$
$\Rightarrow \quad 3x = 8$
$\Rightarrow \quad x = 8/3$
Hence, the solution set is {2, 8/3}.

202. We have,
$|x^2 - x - 6| = x + 2$
$\Rightarrow \quad (x^2 - x - 6) = \pm(x + 2)$
Taking positive sign, we get
$(x^2 - x - 6) = (x + 2)$
$\Rightarrow \quad (x^2 - 2x - 8) = 0$
$\Rightarrow \quad (x - 4)(x + 2) = 0$
$\Rightarrow \quad x = -2, 4.$
Taking negative sign, we get
$(x^2 - x - 6) = -(x + 2)$
$\Rightarrow \quad (x^2 - 4) = 0$
$\Rightarrow \quad x = \pm 2$
Hence the solution set is {–2, 2, 4}.

203. We have $2|x - 2| + 3|x - 4| = 3$
Case I: When $x < 2$
$\Rightarrow \quad -2(x - 2) - 3(x - 4) = 3$
$\Rightarrow \quad -2x + 4 - 3x + 12 = 3$
$\Rightarrow \quad -5x = 3 - 16 = -13$
$\Rightarrow \quad x = 13/5$
It is rejected, since $x = 13/5 > 2$.
Case II: When $2 \le x \le 4$
$\Rightarrow \quad 2(x - 2) - 3(x - 4) = 3$
$\Rightarrow \quad 2x - 4 - 3x + 12 = 3$
$\Rightarrow \quad -x = 3 - 8$
$\Rightarrow \quad x = 5$
It is rejected, since $x = 5$ not lies in the given interval.
Case III: When $x \ge 4$
$\Rightarrow \quad 2(x - 2) + 3(x - 4) = 3$
$\Rightarrow \quad 2x - 4 + 3x - 12 = 3$
$\Rightarrow \quad 5x = 3 + 4 + 12 = 19$
$\Rightarrow \quad x = 19/5$
It is rejected, since $x = 19/5 < 4$.
Hence, the solution set is ϕ.

204. We have $|x| + |x - 2| = 4$
Case I: When $x < 0$.
$\Rightarrow \quad -x - (x - 2) = 4$
$\Rightarrow \quad -x - x + 2 = 4$
$\Rightarrow \quad -2x = 4 - 2 = 2$
$\Rightarrow \quad x = -1$
Case II: When $0 \le x \le 2$
$\Rightarrow \quad x - (x - 2) = 4$
$\Rightarrow \quad 2 = 4$
It is not possible.
Case III: When $x \ge 2$
$\Rightarrow \quad x + x - 2 = 4$
$\Rightarrow \quad 2x = 4 + 2$
$\Rightarrow \quad 2x = 6$
$\Rightarrow \quad x = 3$
Hence, the solution set is {–1, 3}

205. We have,
$|x - 1| + |x - 3| = 2$
$\Rightarrow \quad |x - 1| + |3 - x| = 2$
$\Rightarrow \quad (x - 1)(3 - x) \ge 0$
$\Rightarrow \quad (x - 1)(x - 3) \le 0$
$\Rightarrow \quad 1 \le x \le 3$
$\Rightarrow \quad x \in [1, 3]$

206. We have,
$|x^2 - 1| + |x^2 - 4| = 3$

$\Rightarrow \quad |x^2 - 1| + |4 - x^2| = 3$
$\Rightarrow \quad (x^2 - 1)(4 - x^2) \geq 0$
$\Rightarrow \quad (x^2 - 1)(x^2 - 4) \leq 0$
$\Rightarrow \quad 1 \leq x^2 \leq 4$
$\Rightarrow \quad 1 \leq \sqrt{x^2} \leq 2$
$\Rightarrow \quad 1 \leq |x| \leq 2$
$\Rightarrow \quad x \in [-2, -1] \cup [1, 2]$

207. We have,

$$\sqrt{x + 2\sqrt{x-1}} + \sqrt{x - 2\sqrt{x-1}} = 2$$

Put $x - 1 = t^2$

$\Rightarrow \quad \sqrt{t^2 + 1 + 2t} + \sqrt{t^2 + 1 - 2t} = 2$
$\Rightarrow \quad \sqrt{(t+1)^2} + \sqrt{(t-1)^2} = 2$
$\Rightarrow \quad |t + 1| + |t - 1| = 2$
$\Rightarrow \quad |t + 1| + |1 - t| = 2$
$\Rightarrow \quad (t + 1)(1 - t) \geq 0$
$\Rightarrow \quad (t + 1)(t - 1) \leq 0$
$\Rightarrow \quad -1 \leq t \leq 1$
$\Rightarrow \quad 0 \leq t^2 \leq 1$
$\Rightarrow \quad 0 \leq x - 1 \leq 1$
$\Rightarrow \quad 1 \leq x \leq 2$
$\Rightarrow \quad x \in [1, 2]$

208. We have,

$$\left|\frac{x}{x-1}\right| + |x| = \frac{x^2}{|x-1|}$$

$\Rightarrow \quad \left|\frac{x}{x-1}\right| + |x| = \left|\frac{x}{x-1} + x\right|$
$\Rightarrow \quad x \cdot \left(\frac{x}{x-1}\right) \geq 0$
$\Rightarrow \quad \left(\frac{x^2}{x-1}\right) \geq 0$
$\Rightarrow \quad \left(\frac{1}{x-1}\right) \geq 0$
$\Rightarrow \quad x > 1$
$\Rightarrow \quad x \in (1, \infty)$

209. We have,

$$2^{|x+1|} - 2^x = |2^x - 1| + 1$$

$\Rightarrow \quad |2^x - 1| + (2^x + 1) = 2^{|x+1|}$
$\Rightarrow \quad |2^x - 1| + |2^x + 1| = 2^{|x+1|}$
$\Rightarrow \quad (2^x - 1)(2^x + 1) \geq 0$
$\Rightarrow \quad (2^{2x} - 1) \geq 0$
$\Rightarrow \quad 2^{2x} \geq 1$
$\Rightarrow \quad 2^{2x} \geq 2^0$
$\Rightarrow \quad x \geq 0$
$\Rightarrow \quad x \in [0, \infty)$

210. We have,

$$|x^2 + x - 20| = -(x^2 + x - 20)$$

$\Rightarrow \quad (x^2 + x - 20) \leq 0$
$\Rightarrow \quad (x + 5)(x - 4) \leq 0$
$\Rightarrow \quad -5 \leq x \leq 4$

211. We have,

$$\left|\left(\frac{x^2 - 6x + 8}{x^2 - 4x + 3}\right)\right| = -\left(\frac{x^2 - 6x + 8}{x^2 - 4x + 3}\right)$$

$\Rightarrow \quad \left(\frac{x^2 - 6x + 8}{x^2 - 4x + 3}\right) \leq 0$
$\Rightarrow \quad \left(\frac{(x-2)(x-4)}{(x-1)(x-3)}\right) \leq 0$
$\Rightarrow \quad 1 < x \leq 2, 3 < x \leq 4$
$\Rightarrow \quad x \in (1, 2] \cup (3, 4]$

212. We have,

$$|x - 1| < 3$$

$\Rightarrow \quad -3 < (x - 1) < 3$
$\Rightarrow \quad -3 + 1 < x < 3 + 1$
$\Rightarrow \quad -2 < x < 4$
$\Rightarrow \quad x \in (-2, 4)$

213. We have,

$$|x - 4| \leq 3$$

$\Rightarrow \quad -3 \leq (x - 4) \leq 3$
$\Rightarrow \quad -3 + 4 \leq x \leq 3 + 4$
$\Rightarrow \quad 1 \leq x \leq 7$
$\Rightarrow \quad x \in [1, 7]$

214. We have,

$$|x| 2| > 5$$

$\Rightarrow \quad (x + 2) > 5, (x + 2) < -5$
$\Rightarrow \quad x > 3, x < -7$
$\Rightarrow \quad x \in (-\infty, -7] \cup (3, \infty)$

215. We have,

$$|x - 2| \geq 3$$

$\Rightarrow \quad (x - 2) \geq 3, (x - 2) \leq -3$
$\Rightarrow \quad x \geq 5, x \leq -1$
$\Rightarrow \quad x \in (-\infty, -1] \cup [5, \infty)$

216. We have,

$$\left|\frac{4x-2}{3}\right| \leq 2$$

$\Rightarrow \quad -2 \leq \frac{4x-2}{3} \leq 2$
$\Rightarrow \quad -6 \leq 4x - 2 \leq 6$
$\Rightarrow \quad -6 + 2 \leq 4x \leq 6 + 2$
$\Rightarrow \quad -4 \leq 4x \leq 8$
$\Rightarrow \quad -1 \leq x \leq 2$
$\Rightarrow \quad x \in [-1, 2]$

217. We have,

$$x^2 - |3x - 2| > 0$$

Case I: When $x \geq 3/2$

Then $x^2 - 3x + 2 > 0$

$\Rightarrow \quad (x - 1)(x - 2) > 0$
$\Rightarrow \quad x < 1, x > 2$
$\Rightarrow \quad x > 2$

Case II: When $x < 3/2$

Then $x^2 + 3x + 2 > 0$

$\Rightarrow \quad (x + 1)(x + 2) > 0$
$\Rightarrow \quad x < -2$ and $x > -1$

Hence the solution set is

$$(-\infty, -2) \cup (1, 2) \cup (2, \infty)$$

218. We have,

$$|x| + |x - 2| > 3$$

Case I: When $x < 0$

Then $-x - x + 2 > 3$

$\Rightarrow \quad -2x > 1$
$\Rightarrow \quad x < -1/2$

Case II: When $x \geq 2$

Then $x + x - 2 > 3$

$\Rightarrow \quad 2x > 5$

$\Rightarrow \quad x > 5/2$

Hence the solution set is $(-\infty, -1/2) \cup (5/2, \infty)$.

219. We have,

$$|x + 2| + |x| + |x - 2| > 6$$

Case I: When $x < -2$

Then $-x - 2 - x - x + 2 > 6$

$\Rightarrow \quad -3x > 6$

$\Rightarrow \quad x < -2$

Case II: When $x \geq 2$

Then $x + 2 + x + x - 2 > 6$

$\Rightarrow \quad 3x > 6$

$\Rightarrow \quad x > 2$

Hence the solution set is $(-\infty, -2) \cup (2, \infty)$.

220. We have,

$$\left|\frac{2}{x+3}\right| \leq 1$$

$\Rightarrow \quad \dfrac{2}{|x+3|} \leq 1$

$\Rightarrow \quad |x + 3| \geq 2$

$\Rightarrow \quad x + 3 \geq 2, x + 3 \leq -2$

$\Rightarrow \quad x \geq -1, x \leq -5$

$\Rightarrow \quad x \in (-\infty, -5] \cup [-1, \infty)$

Hence the solution set is $(-\infty, -5] \cup [-1, \infty)$.

221. We have,

$$|3x + 2| > |2x - 1|$$

$\Rightarrow \quad |3x + 2|^2 > |2x - 1|^2$

$\Rightarrow \quad (3x + 2)^2 > (2x - 1)^2$

$\Rightarrow \quad (3x + 2)^2 - (2x - 1)^2 > 0$

$\Rightarrow \quad ((3x + 2) + (2x - 1))((3x + 2) - (2x - 1)) > 0$

$\Rightarrow \quad (5x + 1)(x + 3) > 0$

$\Rightarrow \quad x < -3, x > -1/5$

$\Rightarrow \quad x \in (-\infty, -3) \cup (-1/5, \infty)$

Hence the solution set is $(-\infty, -3) \cup (-1/5, \infty)$.

222. We have,

$$\sqrt{(2x+7)} + \sqrt{(x+4)} = 0$$

The LHS of the equation is defined for all positive real numbers whereas RHS is zero.

So, the equation has no real roots.

223. We have,

$$\sqrt{(x-4)} = -5$$

The LHS of the equation is defined for all positive real numbers whereas RHS is negative.

So, the equation has no real roots.

224. We have,

$$\sqrt{(x-6)} - \sqrt{(8-x)} = 2$$

The LHS of the equation is defined when $x \geq 6$, $x \leq 8$ i.e. $6 \leq x \leq 8$ whereas RHS is 2.

So, the equation has no real roots.

225. We have,

$$\sqrt{(-2-x)} = \sqrt[5]{(x-7)}$$

The LHS of the equation is defined for $x \leq -2$ and its value is zero at $x = -2$ and positive $x < -2$ whereas RHS of the equation is defined for all real values and its value is zero at $x = 7$, positive at $x > 7$ and negative at $x < 7$.

So, the equation has no real roots.

226. The L.H.S of the given equation is ≥ 4 for $x \geq 0$ whereas the RHS is 3

So, the equation has no real roots.

227. We have,

$$7\sqrt{x} + 8\sqrt{-x} + \frac{15}{x^3} = 98.$$

The expression $7\sqrt{x} + 8\sqrt{-x}$ of the given equation is defined only when at $x = 0$ and the expression $\dfrac{15}{x^3}$ is undefined at $x = 0$.

So the LHS of the given equation is not satisfied any real value of x, whereas RHS. is 98.

Therefore, the given equation has no real roots.

228. We have,

$$\sqrt{x-2} + \sqrt{4-x} = \sqrt{6-x}$$

$\Rightarrow \quad x - 2 + 4 - x + 2\sqrt{x-2}\sqrt{4-x} = 6 - x$

$\Rightarrow \quad 2\sqrt{x-2}\sqrt{4-x} = 4 - x$

$\Rightarrow \quad 4(x - 2)(4 - x) = (4 - x)^2$

$\Rightarrow \quad (4 - x)(4x - 8 - 4 + x) = 0$

$\Rightarrow \quad (4 - x)(5x - 12) = 0$

$\Rightarrow \quad x = 4, 12/5$

Hence the solution is $\{4, 12/5\}$.

229. We have,

$$\sqrt{2x-4} - \sqrt{x+5} = 1$$

$\Rightarrow \quad \sqrt{2x-4} = 1 + \sqrt{x+5}$

$\Rightarrow \quad 2x - 4 = 1 + x + 5 + 2\sqrt{x+5}$

$\Rightarrow \quad x - 10 = 2\sqrt{x+5}$

$\Rightarrow \quad (x - 10)^2 = 4(x + 5)$

$\Rightarrow \quad x^2 - 20x + 100 = 4x + 20$

$\Rightarrow \quad x^2 - 24x + 80 = 0$

$\Rightarrow \quad (x - 4)(x - 20) = 0$

$\Rightarrow \quad x = 4, 20$

Here, $x = 4$ is not satisfied the given equation.

Hence the solution set is $x = 20$.

230. We have,

$$\sqrt{3x+4} + \sqrt{x-4} = 2\sqrt{x}$$

$\Rightarrow \quad 3x + 4 + x - 4 + 2\sqrt{3x+4}\sqrt{x-4} = 4x$

$\Rightarrow \quad 2\sqrt{3x+4}\sqrt{x-4} = 0$

$\Rightarrow \quad x = -4/3, 4$

Here, $x = 4/3$ does not satisfy the given equation. Hence the solution set is $x = 4$.

231. We have,

$$\sqrt{x-1} + \sqrt{2x+6} = 6$$

$\Rightarrow \quad x - 1 + 2x + 6 + 2\sqrt{x-1}\sqrt{2x+6} = 36$

$\Rightarrow \quad 2\sqrt{(x-1)(2x+6)} = -3x + 31$

$\Rightarrow \quad 4(2x^2 + 4x - 6) = (3x - 31)^2$

$\Rightarrow \quad 4(2x^2 + 4x - 6) = 9x^2 - 186x + 961$

$\Rightarrow \quad x^2 - 202x + 985 = 0$

$\Rightarrow \quad (x-5)(x-197)=0$

$\Rightarrow \quad x = 5, 197$

Here, $x = 197$ does not satisfy the given equation.

Hence, the solution set is $x = 5$.

232. We have,

$$\sqrt[5]{\left[\frac{3}{x+1}+\frac{7}{x+2}\right]} < \sqrt[5]{\frac{6}{x-1}}$$

$$\Rightarrow \quad \frac{3}{x+1}+\frac{7}{x+2} < \frac{6}{x-1}$$

$$\Rightarrow \quad \frac{3x+6+7x+7}{(x+1)(x+2)} - \frac{6}{x-1} < 0$$

$$\Rightarrow \quad \frac{10x+13}{(x+1)(x+2)} - \frac{6(x^2+3x+2)}{x-1} < 0$$

$$\Rightarrow \quad \frac{10x^2+3x-13-6x^2-18x-12}{(x+1)(x+2)(x-1)} < 0$$

$$\Rightarrow \quad \frac{4x^2-15x-25}{(x+1)(x+2)(x-1)} < 0$$

$$\Rightarrow \quad \frac{(x-5)(4x+5)}{(x+1)(x+2)(x-1)} < 0$$

$\Rightarrow \quad x < -2, -5/4 < x < -1, 1 < x < 5$

Hence the solution set is

$(-\infty, -2) \cup (-5/4, -1) \cup (1, 5)$.

233. We have,

$$\sqrt[3]{\frac{x-2}{x-1}} < \sqrt[3]{\frac{1}{x-1}}$$

$$\Rightarrow \quad \frac{x-2}{x-1} - \frac{1}{x-1} < 0$$

$$\Rightarrow \quad \frac{x-2-1}{x-1} < 0$$

$$\Rightarrow \quad \frac{x-3}{x-1} < 0$$

$\Rightarrow \quad 1 < x < 3$

Hence the solution set is $(1, 3)$.

234 We have,

$$\sqrt{3x-2} < \sqrt{x+4}$$

$$\Rightarrow \quad \begin{cases} 3x-2 \geq \\ x+4>0 \\ x+4>3x-2 \end{cases}$$

$$\Rightarrow \quad \begin{cases} x \geq 2/3 \\ x > -4 \\ x < 6 \end{cases}$$

Hence, the solution set is $\left[\frac{2}{3}, 6\right)$.

235. We have,

$$\sqrt{4x-3} < \sqrt{2x+5}$$

$$\Rightarrow \quad \begin{cases} 4x-3 \geq 0 \\ 2x+5>0 \\ 2x+5>4x+3 \end{cases}$$

$$\Rightarrow \quad \begin{cases} x \geq 3/4 \\ x > -5/2 \\ x < 1 \end{cases}$$

Hence the solution set is $\left[\frac{3}{4}, 1\right)$.

236. We have $\sqrt[3]{(3x-5)} < (x-1)$

$\Rightarrow \quad (3x-5) < (x-1)^3$

$\Rightarrow \quad x^3 - 3x^2 + 3x - 1 > 3x - 5$

$\Rightarrow \quad x^3 - 3x^2 + 4 > 0$

$\Rightarrow \quad x^3 + x^2 - 4x^2 - 4x + 4x + 4 > 0$

$\Rightarrow \quad x^2(x+1) - 4x(x+1) + 4(x+1) > 0$

$\Rightarrow \quad (x+1)(x^2 - 4x + 4) > 0$

$\Rightarrow \quad (x+1)(x-2)^2 > 0$

$\Rightarrow \quad x + 1 > 0$

$\Rightarrow \quad x > -1$

Hence the solution set is $(-1, \infty) - \{2\}$

237. We have,

$$\sqrt[3]{3x-2} < x$$

$\Rightarrow \quad 3x - 2 < x^3$

$\Rightarrow \quad x^3 - 3x + 2 > 0$

$\Rightarrow \quad x^3 - x^2 + x^2 - x - 2x + 2 > 0$

$\Rightarrow \quad x^2(x-1) + x(x-1) - 2(x-1) > 0$

$\Rightarrow \quad (x-1)((x^2 + x - 2) > 0$

$\Rightarrow \quad (x-1)(x-1)(x+2) > 0$

$\Rightarrow \quad (x-1)^2(x+2) > 0$

$\Rightarrow \quad x + 2 > 0$

$\Rightarrow \quad x > -2$

Hence the solution set is $(-2, 1) \cup (1, \infty)$.

238. We have,

$$\sqrt{(x+14)} < (x+2)$$

$$\Rightarrow \quad \begin{cases} x+14 \geq 0 \\ x+2>0 \\ (x+2)^2 > x+14 \end{cases}$$

$$\Rightarrow \quad \begin{cases} x \geq -14 \\ x > -2 \\ (x^2+3x-10) > 0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} x \geq -14 \\ x > -2 \\ (x+5)(x-2) > 0 \end{cases}$$

$$\Rightarrow \quad \begin{cases} x \geq -14 \\ x > -2 \\ x < -5, x > 2 \end{cases}$$

Hence the solution set is $(2, \infty)$.

239. We have,

$$\sqrt{2x-2} < x-1$$

$$\Rightarrow \quad \begin{cases} 2x-2 \geq 0 \\ x-1>0 \\ (2x-2) < (x-1)^2 \end{cases}$$

$$\Rightarrow \begin{cases} x \geq 1 \\ x > 1 \\ x^2 - 4x + 3 > 0 \end{cases}$$

$$\Rightarrow \begin{cases} x \geq 1 \\ x > 1 \\ (x-1)(x-3) > 0 \end{cases}$$

$$\Rightarrow \begin{cases} x \geq 1 \\ x > 1 \\ x < 1, x > 3 \end{cases}$$

Hence the solution set is $(3, \infty)$.

240. We have,

$$\sqrt[3]{5x-4} > x$$
$$\Rightarrow 5x - 4 > x^3$$
$$\Rightarrow 5x - 4 - x^3 > 0$$
$$\Rightarrow x^3 - 5x + 4 > 0$$
$$\Rightarrow x^3 - x^2 + x^2 - x - 4x + 4 > 0$$
$$\Rightarrow x^2(x-1) + x(x-1) - 4(x-1) > 0$$
$$\Rightarrow (x-1)(x^2 + x - 4) > 0$$
$$\Rightarrow \frac{-1-\sqrt{17}}{2} < x < 1, x > \frac{-1+\sqrt{17}}{2}$$

Hence the solution set is

$$\left(\frac{-1-\sqrt{17}}{2}, 1\right) \cup \left(\frac{-1+\sqrt{17}}{2}, \infty\right)$$

241. We have,

$$\sqrt[3]{x^3 - 7x^2} > (-36)^{1/3}$$
$$\Rightarrow x^3 - 7x^2 > -36$$
$$\Rightarrow x^3 - 7x^2 + 36 > 0$$
$$\Rightarrow x^3 + 2x^2 - 9x^2 - 18x + 18x + 36 > 0$$
$$\Rightarrow x^2(x+2) - 9x(x+2) + 18(x+2) > 0$$
$$\Rightarrow (x+2)(x^2 - 9x + 18) > 0$$
$$\Rightarrow (x+2)(x^2 - 9x + 18) > 0$$
$$\Rightarrow (x+2)(x-3)(x-6) > 0$$
$$\Rightarrow -2 < x < 3, x > 6$$

Hence the solution set is $(-2, 3) \cup (6, \infty)$.

242. We have,

$$\sqrt{-x^2 + 4x - 3} > (6 - 2x)$$

$$\Rightarrow \begin{cases} -x^2 + 4x - 3 > (6-2x)^2 \\ 6 - 2x \geq 0 \end{cases} \text{and} \begin{cases} -x^2 + 4x - 3 \geq 0 \\ 6 - 2x < 0 \end{cases}$$

$$\Rightarrow \begin{cases} 5x^2 - 28x + 39 < 0 \\ 6 - 2x \geq 0 \end{cases} \text{and} \begin{cases} x^2 - 4x + 3 \leq 0 \\ 3 - x < 0 \end{cases}$$

$$\Rightarrow \begin{cases} 5x^2 - 15x - 13x + 39 < 0 \\ 6 - 2x \geq 0 \end{cases} \text{and} \begin{cases} (x-1)(x-3) \leq 0 \\ x - 3 > 0 \end{cases}$$

$$\Rightarrow \begin{cases} (x-3)(5x-13) < 0 \\ 6 - 2x \geq 0 \end{cases} \text{and} \begin{cases} 1 \leq x \leq 3 \\ x > 3 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{13}{5} < x < 3 \\ x \leq 3 \end{cases} \text{and} \begin{cases} 1 \leq x \leq 3 \\ x > 3 \end{cases}$$

Hence the solution set is $\left(\frac{13}{5}, 3\right)$.

243. We have,

$$\sqrt{x-2} + \sqrt{x-1} > 2$$

$$\Rightarrow \begin{cases} x - 2 \geq 0 \\ x - 1 \geq 0 \\ x - 2 + x - 1 + 2\sqrt{x-2}\sqrt{x-1} > 4 \end{cases}$$

$$\Rightarrow \begin{cases} x \geq 2 \\ x \geq 1 \\ 2\sqrt{x-2}\sqrt{x-1} > (7-2x) \end{cases}$$

$$\Rightarrow \begin{cases} x \geq 2 \\ x \geq 1 \\ 4(x^2 - 3x + 2) > (7-2x)^2 \end{cases}$$

$$\Rightarrow \begin{cases} x \geq 2 \\ x \geq 1 \\ 4(x^2 - 3x + 2) > 4x^2 - 28x + 49 \end{cases}$$

$$\Rightarrow \begin{cases} x \geq 2 \\ x \geq 1 \\ 16x > 41 \end{cases}$$

$$\Rightarrow \begin{cases} x \geq 2 \\ x \geq 1 \\ x > 41/16 \end{cases}$$

Hence, the solution set is $\left(\frac{41}{16}, \infty\right)$.

244. We have,

$$\sqrt{x-6} - \sqrt{10-x} \geq 1$$

$$\Rightarrow \begin{cases} x - 6 \geq 0 \\ 10 - x \leq 0 \\ \sqrt{x-6} \geq 1 + \sqrt{10-x} \end{cases}$$

$$\Rightarrow \begin{cases} x \geq 6 \\ 10 \leq x \\ x - 6 \geq 1 + 10 - x + 2\sqrt{10-x} \end{cases}$$

$$\Rightarrow \begin{cases} x \geq 6 \\ 10 \leq x \\ 2x - 17 \geq 2\sqrt{10-x} \end{cases}$$

$$\Rightarrow \begin{cases} x \geq 6 \\ 10 \leq x \\ (2x-17)^2 \geq 4(10-x) \end{cases}$$

$$\Rightarrow \begin{cases} x \geq 6 \\ 10 \leq x \\ 4x^2 - 68x + 289 \geq 40 - 4x \end{cases}$$

$$\Rightarrow \begin{cases} x \geq 6 \\ 10 \leq x \\ 4x^2 - 64x + 249 \geq 0 \end{cases}$$

$$\Rightarrow \begin{cases} x \geq 6 \\ 10 \leq x \\ x \leq \left(8 - \frac{\sqrt{7}}{2}\right), x \geq \left(8 + \frac{\sqrt{7}}{2}\right) \end{cases}$$

Hence the solution set is $\left[8 + \frac{\sqrt{7}}{2}, 10\right]$

245. We have,

$5^{x^2+3x+2} = 5^0$

$\Rightarrow$ $x^2 + 3x + 2 = 0$

$\Rightarrow$ $(x + 1)(x + 2) = 0$

$\Rightarrow$ $x = -1, 2$

Hence the solution set is $\{-1, -2\}$.

246. We have,

$3^{x^2+5|x|+6} = 1$.

$\Rightarrow$ $3^{x^2+5|x|+6} = 3^0$

$\Rightarrow$ $x^2 + 5|x| + 6 = 0$

$\Rightarrow$ $|x|^2 + 5|x| + 6 = 0$

$\Rightarrow$ $(|x| + 2)(|x| + 3) = 0$

$\Rightarrow$ $|x| = -2, -3$

$\Rightarrow$ $x = \varphi$

Hence the solution set is ϕ.

247. We have,

$5^{2x} - 24.5^x - 25 = 0$

$\Rightarrow$ $(5^x)^2 - 24(5^x) - 25 = 0$

$\Rightarrow$ $a^2 - 24a - 25 = 0, a = 5^x$

$\Rightarrow$ $(a - 25)(a + 1) = 0$

$\Rightarrow$ $a = 25, -1$

When $a = 25 \Rightarrow 5x = 5^2 \Rightarrow x = 2$

When $a = -1 \Rightarrow 5x = -1 \Rightarrow x = \phi$

Hence the solution set is $\{2\}$.

248. We have,

$64.9^x - 84.12^x + 27.16^x = 0$

$\Rightarrow$ $64\left(\frac{9}{16}\right)^x - 84\left(\frac{12}{16}\right)^x + 27 = 0$

$\Rightarrow$ $64\left(\frac{3}{4}\right)^{2x} - 84\left(\frac{3}{4}\right)^x + 27 = 0$

$\Rightarrow$ $64a^2 - 84a + 27 = 0, a = \left(\frac{3}{4}\right)^x$

$\Rightarrow$ $64a^2 - 48a - 36a + 27 = 0$

$\Rightarrow$ $16a(4a - 3) - 9(4a - 3) = 0$

$\Rightarrow$ $(4a - 3)(16a - 9) = 0$

$\Rightarrow$ $a = 3/4, 9/16$

When $a = \frac{3}{4} \Rightarrow \left(\frac{3}{4}\right)^x = \left(\frac{3}{4}\right) \Rightarrow x = 1$

When $a = \frac{9}{16} \Rightarrow \left(\frac{3}{4}\right)^x = \left(\frac{3}{4}\right)^2 \Rightarrow x = 2$

Hence the solution set is $\{1, 2\}$.

249. We have,

$15.2^{x+1} + 15.2^{2-x} = 135$

$\Rightarrow$ $30.2^x + \frac{60}{2^x} = 135$

$\Rightarrow$ $30.a + \frac{60}{a} = 135, a = 2^x$

$\Rightarrow$ $30a^2 - 135a + 60 = 0$

$\Rightarrow$ $6a^2 - 27a + 12 = 0$

$\Rightarrow$ $2a^2 - 9a + 4 = 0$

$\Rightarrow$ $2a^2 - 8a = a + 4 = 0$

$\Rightarrow$ $2a(a - 4) - 1(a - 4) = 0$

$\Rightarrow$ $(a - 4)(2a - 1) = 0$

$\Rightarrow$ $a = 4, 1/2$

When $a = 4 \Rightarrow 2^x = 4 \Rightarrow 2^x = 2^2 \Rightarrow x = 2$

When $a = \frac{1}{2} \Rightarrow 2^x = 2^{-1} \Rightarrow x = -1$

250. We have,

$3^{x-4} + 5^{x-4} = 34$.

$\Rightarrow$ $3^{x-4} + 5^{x-4} = 3^2 + 5^2$

$\Rightarrow$ $x - 4 = 2$

$\Rightarrow$ $x = 6$

Hence the solution is $x = 6$

251. We have,

$1 + 3^{x/2} = 2^x$

$\Rightarrow$ $1^x + (\sqrt{3})^x = 2^x$

$\Rightarrow$ $x = 2$

Hence the solution is $x = 2$.

252. We have,

$3^x + 4^x + 5^x = 6^x$

$\Rightarrow$ $\left(\frac{3}{6}\right)^x + \left(\frac{4}{6}\right)^x + \left(\frac{5}{6}\right)^x = 1$

$\Rightarrow$ $x = 3$

Hence the solution is $x = 3$.

253. We have,

$1^x + 6^x + 8^x = 9^x$

$\Rightarrow$ $\left(\frac{1}{9}\right)^x + \left(\frac{6}{9}\right)^x + \left(\frac{8}{9}\right)^x = 1$

$\Rightarrow$ $x = 3$

Hence the solution set is $x = 3$

254. We have,

$16^{\sin^2 x} + 16^{\cos^2 x} = 10$

$\Rightarrow$ $16^{\sin^2 x} + 16^{1-\sin^2 x} = 10$

$\Rightarrow$ $16^{\sin^2 x} + \frac{16}{16^{\sin^2 x}} = 10$

$\Rightarrow$ $a + \frac{16}{a} = 10$, where $a = 16^{\sin^2 x}$

$\Rightarrow$ $a^2 - 10a + 16 = 0$

$\Rightarrow$ $(a - 2)(a - 8) = 0$

$\Rightarrow$ $a = 2, 8$

When $a = 2 \Rightarrow 16^{\sin^2 x} = 2$

$\Rightarrow$ $2^{4\sin^2 x} = 2$

$\Rightarrow$ $4\sin^2 x = 1$

$\Rightarrow$ $\sin^2 x = \left(\frac{1}{2}\right)^2 = \sin^2\left(\frac{\pi}{6}\right)$

$\Rightarrow$ $x = n\pi \pm \frac{\pi}{6}, n \in I$

When $a = 8 \Rightarrow 16^{\sin^2 x} = 8$

$\Rightarrow \quad 2^{4\sin^2 x} = 2^3$

$\Rightarrow \quad 4\sin^2 x = 3$

$\Rightarrow \quad \sin^2 x = \left(\frac{\sqrt{3}}{2}\right)^2 = \sin^2\left(\frac{\pi}{3}\right)$

$\Rightarrow \quad x = n\pi \pm \frac{\pi}{3}, n \in I$

Hence the solution set is $\left\{n\pi \pm \frac{\pi}{6}, n\pi \pm \frac{\pi}{3}\right\}, n \in I.$

255. We have,

$2^x + 2^{x-1} + 2^{x-2} = 5^x + 5^{x-1} + 5^{x-2}$

$\Rightarrow \quad 2^x\left(1 + \frac{1}{2} + \frac{1}{2^2}\right) = 5^x\left(1 + \frac{1}{5} + \frac{1}{5^2}\right)$

$\Rightarrow \quad 2^x\left(\frac{4+2+1}{4}\right) = 5^x\left(\frac{25+5+1}{25}\right)$

$\Rightarrow \quad 2^x\left(\frac{7}{4}\right) = 5^x\left(\frac{31}{25}\right)$

$\Rightarrow \quad \left(\frac{2}{5}\right)^x = \left(\frac{124}{175}\right)$

$\Rightarrow \quad x = \log_{\frac{2}{5}}\left(\frac{124}{175}\right)$

Hence the solution is $x = \log_{\frac{2}{5}}\left(\frac{124}{175}\right)$.

256. We have,

$(5+2\sqrt{6})^{x^2-3} + (5-2\sqrt{6})^{x^2-3} = 10.$

$\Rightarrow \quad (5+2\sqrt{5})^{x^2-3} + \left(\frac{1}{5+2\sqrt{6}}\right)^{x^2-3} = 10$

$\Rightarrow \quad (5+2\sqrt{5})^{x^2-3} + \frac{1}{(5+2\sqrt{6})^{x^2-3}} = 10$

$\Rightarrow \quad a + \frac{1}{a} = 10, a = (5+2\sqrt{6})^{x^2-3}$

$\Rightarrow \quad a^2 - 10a + 1 = 0$

$\Rightarrow \quad a = \frac{10 \pm \sqrt{100-4}}{2}$

$= \frac{10 \pm 4\sqrt{6}}{2} = 5 \pm 2\sqrt{6}$

When $a = 5 + 2\sqrt{6}$

$\Rightarrow \quad (5+2\sqrt{6})^{x^2-3} = (5+2\sqrt{6})$

$\Rightarrow \quad x^2 - 3 = 1$

$\Rightarrow \quad x^2 = 4$

$\Rightarrow \quad x = \pm 2$

When $a = 5 - 2\sqrt{6}$

$\Rightarrow \quad (5+2\sqrt{6})^{x^2-3} = (5-2\sqrt{6})$

$\Rightarrow \quad (5+2\sqrt{6})^{x^2-3} = (5+2\sqrt{6})^{-1}$

$\Rightarrow \quad x^2 - 3 = -1$

$\Rightarrow \quad x^2 = 2$

$\Rightarrow \quad x = \pm\sqrt{2}$

Hence the solution set is $\{-2, -\sqrt{2}, \sqrt{2}, 2\}$

257. We have,

$(15+4\sqrt{14})^x + (15-4\sqrt{14})^x = 30$

$\Rightarrow \quad (15+4\sqrt{14})^x + \frac{1}{(15+4\sqrt{14})^x} = 30$

$\Rightarrow \quad a + \frac{1}{a} = 30$, where $a = 15 + 4\sqrt{14}$

Also, $a^2 - 30a + 1 = 0$

$\Rightarrow \quad a = \frac{30 \pm \sqrt{900-4}}{2}$

$= \frac{30 \pm 8\sqrt{14}}{2} = 15 \pm 4\sqrt{14}$

When $a = 15 + 4\sqrt{14}$

$\Rightarrow \quad (15+4\sqrt{14})^x = (15+4\sqrt{14})$

$\Rightarrow \quad x = 1$

When $a = (15 - 4\sqrt{14})$

$\Rightarrow \quad (15+4\sqrt{14})^x = (15-4\sqrt{14})$

$\Rightarrow \quad (15+4\sqrt{14})^x = (15+4\sqrt{14})^{-1}$

$\Rightarrow \quad x = -1$

Hence the solution set is $\{-1, 1\}$.

258. We have $(5^x + 5^{-x}) = \log_{10}25, x \in R$

The value of $5^x + 5^{-x} \geq 2$, where as the RHS < 2.

Hence, the given equation has no solution.

259. The value of the LHS is $[-1, 1]$, whereas the

RHS $= \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4}$

Graphical approach.

Hence the given equation has no solution.

260. We have,

$p^x + p^{-x} = \log_q q^2 - 1$

$p^x + p^{-x} = \log_q q^2 - 1 = 2 - 1 = 1 = 2 - 1 = 1$

The value of the LHS is ≥ 2, whereas the RHS is 1

Hence the given equation has no solution.

261. We have,

$n^{-|x|}\ |m - x|| = 1$

$\Rightarrow \quad |m - |x|| = n^{|x|}$

$\Rightarrow \quad ||x| - m| = n^{|x|}$

Graphical approach:

Hence the number of solutions is 1.

262. Given equation is $2x + x^2 = 1$

$2x = 1 - x^2$

Graphical approach.

It is satisfies for $x = 0$

Hence the number of solution is 1.

263. We have,

$2^x > 1$

$\Rightarrow 2^x > 2^0$

$\Rightarrow x > 0$

Hence the solution set is $(0, \infty)$.

264. We have,

$\left(\frac{1}{3}\right)^x > 1$

$\Rightarrow \left(\frac{1}{3}\right)^x > \left(\frac{1}{3}\right)^0$

$\Rightarrow x < 0$

Hence the solution set is $(-\infty, 0)$.

265. We have,

$5^{x^2 3x+3} > 5$

$\Rightarrow 5^{x^2 3x+3} > 5^1$

$\Rightarrow x^2 - 3x + 3 > 1$

$\Rightarrow x^2 - 3x + 2 > 1$

$\Rightarrow (x-1)(x-2) > 0$

$\Rightarrow x < 1$ and $x > 2$

Hence the solution set is $(-\infty, 1) \cup (2, \infty)$.

266. We have,

$\left(\frac{1}{2}\right)^{x^2-5x+8} > \left(\frac{1}{4}\right)$

$\Rightarrow \left(\frac{1}{2}\right)^{x^2-5x+8} > \left(\frac{1}{2}\right)^2$

$\Rightarrow x^2 - 5x + 8 > 2$

$\Rightarrow x^2 - 5x + 6 > 0$

$\Rightarrow (x-2)(x-3) > 0$

$\Rightarrow x < 2$ and $x > 3$

Hence the solution set is $(-\infty, 2) \cup (3, \infty)$.

267. We have,

$4^x + 2^{x+1} - 8 \le 0$

$\Rightarrow (2x)^2 + 2.2^x - 8 \le 0$

$\Rightarrow a^2 + 2a - 8 \le 0$

$\Rightarrow (a+4)(a-2) \le 0$

$\Rightarrow -4 \le a \le 2$

$\Rightarrow -4 \le 2^x \le 2$

$\Rightarrow 0 < 2^x \le 2$

$\Rightarrow -\infty < x \le 1$

Hence the solution set is $(-\infty, 1)$.

268. We have,

$|x|^{x^2-x-2} \ge 1, -1 < x < 1$

$\Rightarrow |x|^{x^2-x-2} \ge x^0$

$\Rightarrow x^2 - x - 2 \le 0$

$\Rightarrow (x-2)(x+1) \le 0$

$\Rightarrow -1 \le x \le 2$

Hence the solution set is $[-1, 2]$.

269. We have,

$\left(\frac{1}{3}\right)^{\sqrt{x+4}} > \left(\frac{1}{3}\right)^{\sqrt{x^2+3x+4}}$

$\Rightarrow \sqrt{x+4} < \sqrt{x^2+3x+4}$

$\Rightarrow x^2 + 3x + 4 > x + 4, x \ge -4$

$\Rightarrow x^2 + 2x > 0, x \ge -4$

$\Rightarrow x(x+2) > 0, x \ge -4$

$\Rightarrow x \le -2, x \ge 0, x \ge -4$

Hence, the solution set is

$[-4, 2] \cup [0, \infty)$.

270. We have,

$\frac{2^{1-x} - 2^x + 1}{2^x - 1} \le 0$

$\Rightarrow \frac{2 - (2^x)^2 + 2^x}{2^x(2^x - 1)} \le 0$

$\Rightarrow \frac{(2^x)^2 - 2^x - 2}{2^x(2^x - 1)} \ge 0$

$\Rightarrow \frac{(2^x - 2)(2^x + 1)}{2^x(2^x - 1)} \ge 0$

$\Rightarrow \frac{(2^x - 2)}{(2^x - 1)} \ge 0$

$\Rightarrow 2^x \ge 2, 2^x < 1$

$\Rightarrow x \ge 1, x < 0$

Hence the solution set is $(-\infty, 0) \cup [1, \infty)$.

271. We have,

$2^{x+2} - 2^{x+3} - 2^{x+4} > 5^{x+1} - 5^{x+2}$

$\Rightarrow 2^{x+2}(1 - 2 - 2^2) > 5^{x+2}(5^{-1} - 1)$

$\Rightarrow 2^{x+2} \times (-5) > 5^{x+2}\left(-\frac{4}{5}\right)$

$\Rightarrow \frac{2^{x+2}}{5^{x+2}} < \frac{4}{25}$

$\Rightarrow \left(\frac{2}{5}\right)^{x+2} < \left(\frac{2}{5}\right)^2$

$\Rightarrow x + 2 > 2$

$\Rightarrow x > 0$

Hence the solution set is $(0, \infty)$.

272. We have,

$3^{x^2-2} < 2^{x^2-x}$

$\Rightarrow \frac{3^{x^2-x}}{2^{x^2-x}} < 1$

$\Rightarrow \left(\frac{3}{2}\right)^{x^2-x} < \left(\frac{3}{2}\right)^0$

$\Rightarrow x^2 - x < 0$

$\Rightarrow x(x-1) < 0$

$\Rightarrow 0 < x < 1$

Hence the solution set is $(0, 1)$.

Level III

1. We have

$(x-2)(x-3) + (x-3)(x-1) + (x+1)((x-2) = 0$

$\Rightarrow 3x^2 - 10x + 7 = 0$

$\Rightarrow x = 1, \frac{7}{3}$

Now $\frac{1}{(\alpha+1)(\beta+1)}+\frac{1}{(\alpha-2)(\beta-2)}+\frac{1}{(\alpha-3)(\beta-3)}$

$=\frac{3}{20}-3+\frac{3}{4}=\frac{18}{20}-3$

$=\frac{9}{10}-3=\frac{9-30}{10}=-\frac{21}{10}$

2. We have, $\alpha+\beta=3$ and $\alpha\beta=1$.
Now, $(\alpha^5+\beta^5)(\alpha^4+\beta^4)$
$=\{(\alpha^2+\beta^2)(\alpha^3+\beta^3)-\alpha^2\beta^2(\alpha+\beta)\}$
$=\{(\alpha+\alpha)^2-2\alpha\beta\}\{(\alpha+\beta)^3-3\alpha\beta(\alpha+\alpha)\}$
$-\alpha^2\beta^2(\alpha+\alpha)\times[\{(\alpha+\beta)^2-2\alpha\beta\}^2-2\alpha^2\beta^2]$
$=5781$

3. Since α is a root of $4x^2+2x-1=0$.
Then $4\alpha^2+2\alpha-1=0$.

$\Rightarrow\quad \alpha=\frac{-1\pm\sqrt{5}}{4}=\frac{\sqrt{5}-1}{4},\frac{-\sqrt{5}-1}{4}$

$\Rightarrow\quad \alpha=\sin 18°, -\sin 18°$

Now, $4\alpha^3-3\alpha=4\sin^3 18°-3\sin 18°$
$=-\sin 54°=-\cos 36°$
$=-\frac{\sqrt{5}+1}{4}$

Hence, $4\alpha^3-3\alpha$ is the other root.

4. We have $\alpha+\beta+\gamma=0$, $\alpha\beta+\beta\gamma+\gamma\alpha=3$ and $\alpha\beta\gamma=-2$.
Solving we get, $\alpha=1$, $\beta=1$, $\gamma=2$.
Therefore, $\alpha^2+2+\beta^2+2+\gamma^2+2$
$=1+2+1+2+4+2=12$
Also, $(\alpha^2+2)(\beta^2+2)+(\alpha^2+2)(\gamma^2+2)+(\gamma^2+2)(\beta^2+2)$
$=3.\,3+3.\,6+6.\,3=45$
and $(\alpha^2+2)(\beta^2+2)(\gamma^2+2)$
$=3\cdot 3.\,6=54.$
Hence the required equation is
$x^3-12x^2+45x-64=0.$

5. The given equations are
$3x^2+px+1=0$ …(i)
and $2x^2+qx+1=0$ …(ii)
Multiplying Eq. (i) by 2 and Eq. (ii) by 3 and subtracting, we get,
$2px+2-3qx-3=0.$
$\Rightarrow\quad (2p-3q)x=1$

$\Rightarrow\quad x=\frac{1}{2p-3q}$

Put the value of x in Eq. (i), we get

$$3\left(\frac{1}{2p-3q}\right)^2+p\left(\frac{1}{2p-3q}\right)+1=0$$

$\Rightarrow\quad 3+p(2p-3q)+(2p-3q)^2=0$
$\Rightarrow\quad 3+(2p^2-3pq)+(4p^2+9q^2-12pq)=0$
$\Rightarrow\quad (6p^2+9q^2-15pq+3)=0$
$\Rightarrow\quad (2p^2+3q^2-5pq+1)=0$

6. Since the roots are of opposite sign, so
$af(-2)>0$ and $af(2)>0$.
$\Rightarrow\quad a(4a-2b+c)>0$ and $a(4a+2b+c)>0$

$\Rightarrow\quad \left(a-\frac{b}{2}+\frac{c}{4}\right)>$ and $\left(a+\frac{b}{2}+\frac{c}{4}\right)>0$

$\Rightarrow\quad \left(1-\frac{b}{2a}+\frac{c}{4a}\right)$ and $\left(1+\frac{b}{2a}+\frac{c}{4a}\right)$

$\Rightarrow\quad \left(1+\frac{c}{4a}-\left|\frac{b}{2a}\right|\right)$

Hence, the result.

7. We have,

$\sqrt{x+\sqrt{x+11}}+\sqrt{x-\sqrt{x+11}}=4$

$\Rightarrow\quad \sqrt{x+\sqrt{x+11}}=4-\sqrt{x-\sqrt{x+11}}$

$\Rightarrow\quad x+\sqrt{x+11}=16+x-\sqrt{x+11}-8\sqrt{x-\sqrt{x+11}}$

$\Rightarrow\quad 2\sqrt{x+11}-16=-8\sqrt{x-\sqrt{x+11}}$

$\Rightarrow\quad \sqrt{x+11}-8=-4\sqrt{x-\sqrt{x+11}}$

$\Rightarrow\quad x+11+64-16\sqrt{x+11}=16(x-\sqrt{x+11})$

$\Rightarrow\quad x+75=16x$
$\Rightarrow\quad 15x=75$
$\Rightarrow\quad x=5$

Hence the solution is $x=5$.

8. Let $u=\sqrt{2x^2=5x-2}$ and $v=\sqrt{2x^2+5x-9}$
Thus $u-v=1$ …(i)
Now, $u^2-v^2=-2+9=7$
$\Rightarrow\quad u+v=7$ …(ii)
Adding Eqs (i) and (ii), we get,
$2u=8\Rightarrow u=4$ and $v=3$
When $u=4\Rightarrow u^2=16$
$\Rightarrow\quad 2x^2+5x-2=16$
$\Rightarrow\quad 2x^2+5x-18=0$
$\Rightarrow\quad 2x^2-4x+9x-18=0$
$\Rightarrow\quad 2x(x-2)+9(x-2)=0$
$\Rightarrow\quad (x-2)(2x+9)=0$
$\Rightarrow\quad x=2,-9/2$
When $v=3\Rightarrow v^2=9$
$\Rightarrow\quad 2x^2+5x-9=9$
$\Rightarrow\quad 2x^2+5x-18=0$
$\Rightarrow\quad x=2,-9/2$
Hence the solution set is $\{2,-9/2\}$.

9. We have,

$\sqrt{x}+\sqrt{x-\sqrt{1-x}}=1$

$\Rightarrow\quad \sqrt{x-\sqrt{1-x}}=1-\sqrt{x}$

$\Rightarrow\quad x-\sqrt{1-x}=1+x-2\sqrt{x}$

$\Rightarrow\quad -\sqrt{1-x}=1-2\sqrt{x}$

$\Rightarrow\quad 1-x=1+4x-4\sqrt{x}$

$\Rightarrow\quad 5x=4\sqrt{x}$

$\Rightarrow\quad 25x^2=16x$
$\Rightarrow\quad x(25x-16)=0$
$\Rightarrow\quad x=0, x=16/25$

Here, $x=0$ does not satisfy the given equation.
Hence the solution set is $x=16/25$.

10. We have,

$$\sqrt[3]{(2x-1)} + \sqrt[3]{(x-1)} = 1.$$

$\Rightarrow$ $2x - 1 + x - 1 + 3\sqrt[3]{(2x-1)(x-1)}\,(\sqrt[3]{(2x-1)} + \sqrt[3]{(x-1)}) = 1$

$\Rightarrow$ $3x - 2 + 3\sqrt[3]{(2x-1)(x-1)}\,(\sqrt[3]{(2x-1)} + \sqrt[3]{(x-1)}) = 1$

$\Rightarrow$ $3x - 2 + 3\sqrt[3]{(2x-1)(x-1)} = 1$

$\Rightarrow$ $3\sqrt[3]{(2x-1)(x-1)} = 3 - 3x$

$\Rightarrow$ $\sqrt[3]{(2x-1)(x-1)} = 1 - x$

$\Rightarrow$ $(2x - 1)(x - 1) = (1 - x)^3$

$\Rightarrow$ $(x - 1)(2x - 1) = -(x - 1)^3$

$\Rightarrow$ $(x - 1)\{(2x - 1) + (x - 1)^2\} = 0$

$\Rightarrow$ $(x - 1)\{(2x - 1) + x^2 - 2x + 1\} = 0$

$\Rightarrow$ $(x - 1)x^2 = 0$

$\therefore$ $x = 0, 1$

Here, $x = 0$ does not satisfy the given equation.

Hence the solution set is $x = 1$.

11. We have,

$$\sqrt{x^2 + 2x + 1} + \sqrt{x^2 - 4x + 4} = 3$$

$\Rightarrow$ $\sqrt{(x+1)^2} + \sqrt{(x-2)^2} = 3$

$\Rightarrow$ $|x + 1| + |x - 2| = 3$

$\Rightarrow$ $|x + 1| + |2 - x| = 3$

$\Rightarrow$ $|x + 1| + |2 - x| = |x + 1 + 2 - x|$

$\Rightarrow$ $(x + 1)(2 - x) \geq 0$

$\Rightarrow$ $(x + 1)(x - 2) \leq 0$

$\therefore$ $-1 \leq x \leq 2$

Hence the solution set is $[-1, 2]$.

12. We have,

$|x^2 - 1| + |x^2 - 2| = 1$

$\Rightarrow$ $|x^2 - 1| + |2 - x^2| = 1$ $\quad (\because |-x| = |x|)$

$\Rightarrow$ $(x^2 - 1)(2 - x^2) \geq 0$

$\Rightarrow$ $(x^2 - 1)(x^2 - 2) \leq 0$

$\Rightarrow$ $(x-1)(x+1)(x-\sqrt{2})(x+\sqrt{2}) \leq 0$

$\Rightarrow$ $-\sqrt{2} \leq x \leq -1, 1 \leq x \leq \sqrt{2}$

$\therefore$ $x \in [-\sqrt{2}, -1] \cup [\sqrt{2}, 1]$

Hence the solution set is $[-\sqrt{2}, -1] \cup [\sqrt{2}, 1]$.

13. We have,

$|x^2 - x| + 2|x - 1| = |x^2 + x - 2|$

$\Rightarrow$ $|x^2 - x| + 2|x - 1| = |(x^2 - x) + 2(x - 2)|$

$\Rightarrow$ $(x^2 - x)2(x - 1) \geq 0$

$\Rightarrow$ $x(x - 1)(x - 1) \geq 0$

$\Rightarrow$ $x(x - 1)^2 \geq 0$

$\Rightarrow$ $x \geq 0$

$\therefore$ $x \in [0, \infty)$

Hence the solution set is $[0, \infty)$.

14. We have

$$\sqrt{x + 3 - 4\sqrt{x-1}} + \sqrt{x + 8 - 6\sqrt{x-1}} = 1$$

Put $x - 1 = t^2 \Rightarrow x = 1 + t^2$

$\Rightarrow$ $\sqrt{t^2 + 4 - 4t} + \sqrt{t^2 + 9 - 6t} = 1$

$\Rightarrow$ $\sqrt{(t-2)^2} + \sqrt{(t-3)^2} = 1$

$\Rightarrow$ $|t - 2| + |t - 3| = 1$

$\Rightarrow$ $|t - 2| + |3 - t| = 1$

$\Rightarrow$ $|t - 2| + |3 - t| = |t - 2 + 3 - t|$

$\Rightarrow$ $(t - 2)(3 - t) \geq 0$

$\Rightarrow$ $(t - 2)(t - 3) \leq 0$

$\Rightarrow$ $2 \leq t \leq 3$

$\Rightarrow$ $4 \leq t^2 \leq 9$

$\Rightarrow$ $4 \leq x - 1 \leq 9$

$\Rightarrow$ $5 \leq x \leq 10$

$\therefore$ $x \in [1, 5]$

15. Let $y = \dfrac{x^2 + 34x - 71}{x^2 + 2x - 7}$

$\Rightarrow$ $x^2y + 2xy - 7y = x^2 + 34x - 71$

$\Rightarrow$ $(y - 1)x^2 + 2(y - 17)x + (71 - 7y) = 0$

Since x is real, so $D \geq 0$

$4(y - 17)^2 - 4(y - 1)(71 - 7y) \geq 0$

$\Rightarrow$ $(y - 17)^2 - (y - 1)(71 - 7y) \geq 0$

$\Rightarrow$ $(y^2 - 34y + 289) + (y - 1)(7y - 71) \geq 0$

$\Rightarrow$ $(y^2 - 34y + 289) + (7y^2 - 78y + 71) \geq 0$

$\Rightarrow$ $(8y^2 - 112y + 360) \geq 0$

$\Rightarrow$ $(y^2 - 14y + 45) \geq 0$

$\Rightarrow$ $(y - 5)(y - 9) \geq 0$

$\Rightarrow$ $y \leq 5$ and $y \geq 9$

$\therefore$ $y \in (-\infty, 5] \cup [9, \infty)$

Thus, $R_f = (-\infty, 5] \cup [9, \infty)$

Note. No questions asked in 1984.

16. Given equation is

$$\sqrt{5x^2 - 6x + 8} - \sqrt{5x^2 - 6x - 7} = 1$$

Let $u = \sqrt{5x^2 - 6x + 8}$

and $v = \sqrt{5x^2 - 6x - 7}$

Thus, $u - v = 1$ $\quad$ …(i)

Also, $u^2 - v^2 = 15$

$\Rightarrow$ $(u + v)(u - v) = 15$

$\Rightarrow$ $(u + v) = 15$ $\quad$ …(ii)

Adding Eqs (i) and (ii), we get

$u = 8$ and $v = 7$

When $u = 8$, then $u^2 = 64$

$\Rightarrow$ $5x^2 - 6x + 8 = 64$

$\Rightarrow$ $5x^2 - 6x - 56 = 0$

$\Rightarrow$ $5x^2 - 20x + 14x - 56 = 0$

$\Rightarrow$ $5x(x - 4) + 14(x - 4) = 0$

$\Rightarrow$ $(x - 4)(5x + 14) = 0$

$\Rightarrow$ $x = 4, -\dfrac{14}{5}$

When $v = 7$, then $v^2 = 49$.

$\Rightarrow$ $5x^2 - 6x - 7 = 49$

$\Rightarrow$ $5x^2 - 6x - 56 = 0$

$\Rightarrow \quad x = 4, -\frac{14}{5}$

Hence the solutions are

$$x = 4, -\frac{14}{5}$$

17. The given equation is

$$\frac{2\sqrt{x+1}}{3-\sqrt{x}} = \frac{11-3\sqrt{x}}{5\sqrt{x-9}}$$

$\Rightarrow \quad (11-3\sqrt{x})(3-\sqrt{x}) = 10\sqrt{(x+1)(x-9)}$

$\Rightarrow \quad (33-20\sqrt{x}+3x) = 10\sqrt{(x^2-8x-9)}$

which is satisfied for $x = 9$

But $x = 9$ does not satisfy the given equation

Hence the equation has no solution.

18. The given equation reduces to

$a^2 + 8x^2 - 6ax = 0$ where $a = (x^2 + 2)$

$\Rightarrow \quad 8x^2 - 6ax + a^2 = 0$

$\Rightarrow \quad 8x^2 - 4ax - 2ax + a^2 = 0$

$\Rightarrow \quad 4x(2x - a) - a(2x - a) = 0$

$\Rightarrow \quad (4x - a)(2x - a) = 0$

$\therefore \quad a = 2x, 4x$

When $a = 2x$, then $x^2 + 2 = 2x$

$\Rightarrow \quad x^2 - 2x + 2 = 0$

$\Rightarrow \quad x = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$

When $a = 4x$ then $x^2 + 2 = 4x$

$\Rightarrow \quad x^2 - 4x + 2 = 0$

$\Rightarrow \quad x = \frac{4 \pm \sqrt{16-8}}{2} = 2 \pm \sqrt{2}$

Hence the solutions are $\{1 \pm i, 2 \pm \sqrt{2}\}$.

19. It is a false statement.

$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ holds good only when at least one of a and b is non-negative.

20. The given equation is

$$3x^3 = (x^2 + x\sqrt{18} + \sqrt{32})(x^2 - x\sqrt{18} - \sqrt{32}) - 4x^2$$

$\Rightarrow \quad 3x^3 = x^4 - (x\sqrt{18} + \sqrt{32})^2 - 4x^2$

$\Rightarrow \quad 3x^3 + 4x^2 = x^4 - (3\sqrt{2}x + 4\sqrt{2})^2$

$\Rightarrow \quad 3x^3 + 4x^2 = x^4 - 2(3x + 4)^2$

$\Rightarrow \quad (3x + 4)x^2 = x^4 - 2(3x + 4)^2$

$\Rightarrow \quad x^2y = x^4 - 2y^2$

$\Rightarrow \quad 2y^2 + x^2y - x^4 = 0$, where $y = 3x + 4$

$\Rightarrow \quad 2y^2 + 2x^2y - x^2y - x^4 = 0$

$\Rightarrow \quad 2y(y + x^2) - x^2(y + x^2) = 0$

$\Rightarrow \quad (2y - x^2)(y + x^2) = 0$

$\therefore \quad y = -x^2, \frac{x^2}{2}$

When $y = -x^2$, then $3x + 4 = -x^2$

$\Rightarrow \quad x^2 + 3x + 4 = 0$

$\Rightarrow \quad x = \frac{-3 \pm i\sqrt{7}}{2}$

When $y = \frac{x^2}{2}$, then $x^2 - 6x - 8 = 0$

$\Rightarrow \quad x = 3 \pm \sqrt{17}$

Hence, the solutions are

$$\left\{\frac{-3 \pm i\sqrt{7}}{2}, 3 \pm \sqrt{7}\right\}.$$

21. The given equation is

$2^{|x+1|} - 2^x = |2^x - 1| + 1$

$\Rightarrow \quad (2^x + 1) + |2^x - 1| = 2^{|x+1|}$

$\Rightarrow \quad |(2^x + 1)| + |2^x - 1| = 2^{|x+1|}$

$\Rightarrow \quad (2x + 1)(2x - 1) \geq 0$

$\Rightarrow \quad (2^{2x} - 1) \geq 0$

$\Rightarrow \quad 2^{2x} \geq 1 = 2^0$

$\Rightarrow \quad x \geq 0$

Hence the solution is $x \in [0, \infty)$.

22. Let the quotient be $\frac{n}{n^2-1}$, where n is natural number.

According to the question, we get

$\Rightarrow \quad \frac{n+2}{n^2-1+2} > \frac{1}{3}$

$\quad \frac{n+2}{n^2+1} > \frac{1}{3}$

$\Rightarrow \quad \frac{n+2}{n^2+1} - \frac{1}{3} > 0$

$\Rightarrow \quad \frac{3n+6-n^2-1}{3(n^2+1)} > 0$

$\Rightarrow \quad 3n + 5 - n^2 > 0$

$\Rightarrow \quad n^2 - 3n - 5 < 0$

$\Rightarrow \quad \left(n - \left(\frac{3}{2} - \frac{\sqrt{29}}{2}\right)\right)\left(n - \left(\frac{3}{2} + \frac{\sqrt{29}}{2}\right)\right) < 0$

$\Rightarrow \quad \left(\frac{3}{2} - \frac{\sqrt{29}}{2}\right) < n < \left(\frac{3}{2} + \frac{\sqrt{29}}{2}\right) \quad \ldots\text{(i)}$

Also, $0 < \frac{n-3}{n^2-1-3} < \frac{1}{10}$

$\Rightarrow \quad 0 < \frac{n-3}{n^2-4} < \frac{1}{10}$

When $\frac{n-3}{n^2-4} > 0$

$\Rightarrow \quad \frac{(n-3)}{(n-2)(n+2)} > 0$

By the sign scheme, we get

$n \in (-2, 2) \cup (3, \infty) \quad \ldots\text{(ii)}$

When $\frac{n-3}{n^2-4} < \frac{1}{10}$

$\Rightarrow \quad \frac{n-3}{n^2-4} - \frac{1}{10} < 0$

$\Rightarrow \quad \frac{10n-30-n^2+4}{10(n^2-4)} < 0$

$\Rightarrow \quad \frac{10n-26-n^2}{(n^2-4)} < 0$

$\Rightarrow \quad \frac{n^2-10n+26}{(n^2-4)} > 0$

$\Rightarrow \quad \frac{n^2-10n+26}{(n+2)(n-2)} > 0$

$\Rightarrow \quad \frac{1}{(n+2)(n-2)} > 0$, since D is negative.

By the sign scheme, we get

$n \in (-\infty, -2) \cup (2, \infty)$ …(iii)

From Eqs (i), (ii) and (iii), we get

$3 < n < \frac{3}{2} + \frac{\sqrt{29}}{2}$

Hence, $n = 4$ (since n is a natural number) is the required solution.

23. We have,

$(15+4\sqrt{14})^x + (15-4\sqrt{14})^x = 30$

$\Rightarrow \quad (15+4\sqrt{14})^t + \frac{1}{(15+4\sqrt{14})^t} = 30$

$\Rightarrow \quad a + \frac{1}{a} = 30$, where $a = 15+4\sqrt{14}$

Also, $a^2 - 30a + 1 = 0$

$\Rightarrow \quad a = \frac{30 \pm \sqrt{900-4}}{2}$

$= \frac{30 \pm 8\sqrt{14}}{2} = 15 \pm 4\sqrt{14}$

When $a = 15+4\sqrt{14}$

$\Rightarrow \quad (15+4\sqrt{14})^t = (15+4\sqrt{14})$

$\Rightarrow \quad t = 1$

$\Rightarrow \quad x^2 - 2|x| = 1$

$\Rightarrow \quad |x|^2 - 2|x| - 1 = 0$

$\Rightarrow \quad |x| = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$

$\Rightarrow \quad x = \pm(1 \pm \sqrt{2})$

When $a = (15-4\sqrt{14})$

$\Rightarrow \quad (15+4\sqrt{14})^t = (15-4\sqrt{14})$

$\Rightarrow \quad (15+4\sqrt{14})^t = (15+4\sqrt{14})^{-1}$

$\Rightarrow \quad t = -1$

$\Rightarrow \quad x^2 - 2|x| = -1$

$\Rightarrow \quad |x|^2 - 2|x| + 1 = 0$

$\Rightarrow \quad (|x| - 1)^2 = 0$

$\Rightarrow \quad (|x| - 1) = 0$

$\Rightarrow \quad |x| = 1$

$\Rightarrow \quad x = \pm 1$

Hence the solution set is $\{\pm(1 \pm \sqrt{2}), -1, 1\}$.

24. Given $x^{x+y} = y^n$ …(i)

$y^{x+y} = x^{2n}y^n$ …(ii)

Multiplying Eqs (i) and (ii), we get

$(xy)^{x+y} = (xy)^{2n}$

$\Rightarrow \quad x + y = 2n$, when $xy \neq 1$

From Eqs (i) and (iii), we get

$\Rightarrow \quad (x)^{2n} = y^n$

$\Rightarrow \quad x^2 = y$

From Eqs (iii), we get

$x + x^2 = 2n$

$\Rightarrow \quad x^2 + x - 2n = 0$

$\Rightarrow \quad x = \frac{-1 \pm \sqrt{1+8n}}{2}$

$\Rightarrow \quad x = \frac{-1 + \sqrt{1+8n}}{2}$

and $\quad y = x^2 = \frac{1+1+8n-2\sqrt{1+8n}}{4}$

$= \frac{1}{2}(1 + 4n - \sqrt{1+8n})$

25. Given equations are

$xy + 3y^2 - x + 4y - 7 = 0$ …(i)

$2xy + y^2 - 2x - 2y + 1 = 0$ …(ii)

Multiplying Eq. (i) by 2 and subtract it from Eq. (ii), we get

$-5y^2 - 10y + 15 = 0$

$\Rightarrow \quad y^2 + 2y - 3 = 0$

$\Rightarrow \quad (y+3)(y-1) = 0$

$\Rightarrow \quad y = -3, 1$

When $y = -3$, then $-3x + 27 - x - 12 - 7 = 0$

$\Rightarrow \quad -4x + 8 = 0$

$\Rightarrow \quad x = 2$

Hence the solutions are $x = 2, y = -3; y = 1, x \in R$

26. We have,

$\alpha + \beta = p$ and $\alpha\beta = q$

Now, sum of the roots

$= (\alpha^2 - \beta^2)(\alpha^3 - \beta^3) + \alpha^3\beta^3 + \alpha^2\beta^3$

$= (\alpha - \beta)^2(\alpha + \beta)(\alpha^2 + \alpha\beta + \beta^2) + (\alpha\beta)^2(\alpha + \alpha)$

$= \{(\alpha + \alpha)^2 - 4\alpha\beta\}p(p^2 - q) + q^2p$

$= (p^2 - 4q)p(p^2 - q) + q^2p$

$= p((p^2 - 4q)(p^2 - q) + q^2)$

$= p(p^4 - 5p^2q + 4q^2)$

and product of the roots

$= (\alpha^2 - \beta^2)(\alpha^3 - \beta^3)(\alpha\beta)^2(\alpha + \alpha)$

$= (p^2 - 4q^2)p(p^2 - q)pq^2$
$= p^2q^2(p^2 - 4q^2)(p^2 - q)$

Hence, the required equation is

$$x^2 - p(p^4 - 5p^2q + 5q^2)x + p^2q^2(p^2 - 5p^2q + 4q^2) = 0$$

27. Let the roots be α, β, γ.

Now, $\alpha + \beta + \gamma = \frac{c}{10}$,

$\alpha\beta + \alpha\gamma + \beta\gamma = -\frac{54}{10}$,

$\alpha\beta\gamma = \frac{27}{10}$

Since α, β, γ are in HP, so $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ are in AP.

Thus, $\frac{1}{\alpha} + \frac{1}{\gamma} = \frac{2}{\beta}$

$\Rightarrow \quad \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{3}{\beta}$

$\Rightarrow \quad \frac{3}{\beta} = \frac{\beta\gamma + \alpha\beta + \alpha\gamma}{\alpha\beta\gamma} = -\frac{54}{27} = -2$

$\Rightarrow \quad \beta = -\frac{3}{2}$.

Since β is a root of the given equation, so

$\Rightarrow \quad 10\left(-\frac{3}{2}\right)^3 - c\left(-\frac{3}{2}\right)^2 - 54\left(-\frac{3}{2}\right) - 27 = 0$

$-\frac{135}{4} - \frac{9c}{4} + 81 - 27 = 0$

$\Rightarrow \quad \frac{9c}{4} = -\frac{135}{4} + 81 - 27$

$\Rightarrow \quad \frac{9c}{4} = -\frac{135}{4} + 54 = \frac{-135 + 216}{4}$

$\Rightarrow \quad \frac{9c}{4} = \frac{81}{4}$

$\Rightarrow \quad c = 9$

Solving, we get

$\alpha = 3, \beta = -\frac{3}{2}, \gamma = -\frac{3}{5}$

28. Let the roots be $\alpha + i\beta, \alpha - i\beta, \gamma$.

Sum of the roots = 0

$\Rightarrow \quad \alpha + i\beta + \alpha - i\beta + \gamma = 0$

$\Rightarrow \quad 2\alpha + \gamma = 0$

$\Rightarrow \quad 2\alpha = -\gamma$

Since 2α is a root of a new equation,

$(2\alpha)^3 + p(2\alpha) + r = 0$

$\Rightarrow \quad (-\gamma)^3 + p(-\gamma) + r = 0$

$\Rightarrow \quad (\gamma)^3 + p(\gamma) - r = 0$

Hence the new equation is $x^3 + px - r = 0$ which is independent of α and β.

29. Since α and β are the roots of $(x - a)(x - b) + c = 0$,

$(x - a)(x - b) + c = (x - \alpha)(x - \beta)$

$\Rightarrow \quad (x - \alpha)(x - \beta) = (x - a)(x - b) + c$

$\Rightarrow \quad (x - \alpha)(x - \beta) - c = (x - a)(x - b)$

Thus the roots of $(x - \alpha)(x - \beta) - c = 0$ are a and b.

30. Given that α, γ are the roots of $Ax^2 - 4x + 1 = 0$

$\Rightarrow \quad \alpha + \gamma = \frac{4}{A}$ and $\alpha\gamma = \frac{1}{A}$ …(i)

$\frac{1}{\alpha} + \frac{1}{\gamma} = 4$

Also, β and δ are the roots of $Bx^2 - 6x + 1 = 0$.

$\Rightarrow \quad \beta + \delta = \frac{6}{B}$ and $\beta\delta = \frac{1}{B}$ …(ii)

$\frac{1}{\beta} + \frac{1}{\delta} = 6$

Also, it is given that $\alpha, \beta, \gamma, \delta \in$ HP

$\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta} \in$ AP

Now, from Eq. (i), we get

$\frac{1}{\alpha} + \frac{1}{\gamma} = 4$

$\Rightarrow \quad \frac{1}{\alpha} + \frac{1}{\alpha} + 2d = 4$

$\Rightarrow \quad \frac{2}{\alpha} + 2d = 4$

$\Rightarrow \quad \frac{1}{\alpha} + d = 2$ …(iii)

Again, $\frac{1}{\beta} + \frac{1}{\delta} = 6$

$\Rightarrow \quad \frac{1}{\alpha} + d + \frac{1}{\alpha} + 3d = 6$

$\Rightarrow \quad \frac{2}{\alpha} + 4d = 6$

$\Rightarrow \quad \frac{1}{\alpha} + 2d = 3$ …(iv)

Solving Eqs (iii) and (iv), we get

$\alpha = 1, \delta = 1$

Now, $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}, \frac{1}{\delta} = \frac{1}{\alpha}, \frac{1}{\alpha} + d, \frac{1}{\alpha} + 2d, \frac{1}{\alpha} + 3d$

$= \frac{1}{1}, \frac{1}{1} + 1, \frac{1}{1} + 2.1, \frac{1}{1} + 3.1$

$= 1, 2, 3, 4$

Thus, $\alpha = 1, \beta = \frac{1}{2}, \gamma = \frac{1}{3}, \delta = \frac{1}{4}$

Therefore, $A = \frac{1}{\alpha\gamma} = 3, B = \frac{1}{\beta\delta} = 8$

31. Let α and β be its roots.

It is given that $\alpha + \beta = \frac{1}{\alpha^2} + \frac{1}{\beta^2}$

$\Rightarrow \quad \alpha+\beta=\dfrac{\alpha^2+\beta^2}{(\alpha\beta)^2}$

$\Rightarrow \quad \alpha+\beta=\dfrac{(\alpha+\beta)^2-2\alpha\beta}{(\alpha\beta)^2}$

$\Rightarrow \quad -\dfrac{b}{a}=\dfrac{\dfrac{b^2}{a^2}-2\dfrac{c}{a}}{\dfrac{c^2}{\alpha^2}}=\dfrac{b^2-2ac}{c^2}$

$\Rightarrow \quad -\dfrac{b}{a}=\dfrac{b^2-2ac}{c^2}$

$\Rightarrow \quad ab^2-2a^2c=-bc^2$

$\Rightarrow \quad ab^2+bc^2=2a^2c$

$\Rightarrow \quad ab^2, a^2c, bc^2$ are in AP.

32. Let the other root be β.

Thus, $\alpha+\beta=-\dfrac{1}{2}$ and $\alpha\beta=-\dfrac{1}{4}$

Clearly, $4\alpha^2+2\alpha-1=0$

$\Rightarrow \quad 4\alpha^2=1-2\alpha$

Now, $4\alpha^3-3\alpha=\alpha(4\alpha^2-3)$

$=\alpha(1-2\alpha-3)$

$=-\alpha(2\alpha+2)$

$=-(2\alpha^2+2\alpha)$

$=-\dfrac{1}{2}(4\alpha^2+4\alpha)$

$=-\dfrac{1}{2}(1-2\alpha+4\alpha)$

$=-\dfrac{1}{2}(1+2\alpha)$

$=-\dfrac{1}{2}-\alpha=\beta$

Hence, the result.

33. We have,

$(x^4+6x^2+25)=(x^4+25)+6x^2$

$=((x^2)^2+(5)^2)+6x^2$

$=(x^2+5)^2-10x^2+6x^2$

$=(x^2+5)^2-4x^2$

$=(x^2+2x+5)(x^2-2x+5)$

Also, $\dfrac{3x^4+4x^2+28x+5}{(x^2-2x+5)}=(3x^2+6x+1)$

Thus, $P(x)=x^2-2x+5$

$\Rightarrow \quad P(1)=1-2+5=4$

Hence, the value of $P(1)$ is 4.

34. We have,

$$\alpha+\beta=-\frac{b_1}{\alpha_1}, \alpha\beta=\frac{c_1}{\alpha_1}$$

$$\beta+\gamma=-\frac{b_2}{a_2}, \beta\gamma=\frac{c_2}{a_2}$$

and $\quad \gamma+\alpha=-\dfrac{b_3}{a_3}, \gamma\alpha=\dfrac{c_3}{\alpha_3}$

Now,

$(1+\alpha)(1+\beta)=1+(\alpha+\beta)+\alpha\beta$

$=1-\dfrac{b_1}{a_1}+\dfrac{c_1}{a_1}$

$=\dfrac{a_1-b_1+c_1}{a_1}$ …(i)

Similarly, $(1+\beta)(1+\gamma)=\dfrac{a_2-b_2+c_2}{a_2}$ …(ii)

$(1+\gamma)(1+\alpha)=\dfrac{a_3-b_3+c_3}{a_3}$ …(iii)

Multiplying Eqs (i), (ii) and (iii), we get

$\{(1+\alpha)(1+\beta)(1+\gamma)\}^2$

$=\left(\dfrac{a_1-b_1+c_1}{a_1}\right)\left(\dfrac{a_2-b_2+c_2}{a_2}\right)\left(\dfrac{a_3-b_3+c_3}{a_3}\right)$

$=\prod_{i=1}^{3}\left(\dfrac{a_i-b_i+c_i}{a_i}\right)$

Thus, $(1+\alpha)(1+\beta)(1+\gamma)=\left(\prod_{i=1}^{3}\left(\dfrac{a_i-b_i+c_i}{a_i}\right)\right)^{1/2}$

35. Given

$x^3+4x=8$

$\Rightarrow \quad (x^3+4x)^2=64$

$\Rightarrow \quad x^6+8x^4+16x^2=64$

$\Rightarrow \quad x^6+8x(x^3+2x)=64$

$\Rightarrow \quad x^6+8x(8-4x+2x)=64$

$\Rightarrow \quad x^6+8x(8-2x)=64$

$\Rightarrow \quad x^6+64x-16x^2=64$

$\Rightarrow \quad x^6+64x=64+16x^2$

$\Rightarrow \quad x^7+64x^2=64x+16x^3$

$=64x+16(8-4x)$

$=64x+128-64x$

$=128$

$\Rightarrow \quad x^7+6x^2+2=128+2=130$

36. Given equation is $x^3-x^2-672=0$

Thus, $a+b+c=1$ and $ab+bc+ca=0$

Now,

$a^2+b^2+c^2=(a+b+c)^2-2(ab+bc+ca)$

$=1-2.\,0$

$=1$

Thus, $\Sigma a^2=1$

Now, $x^3-x^2-672=0$

$\Rightarrow \quad x^3=x^2+672$

$\Rightarrow \quad \Sigma a^3=\Sigma a^2+\Sigma 672$

$\Rightarrow \quad \Sigma a^3=1+3\cdot(672)=1+2016=2017.$

Hence the value of $(a^3+b^3+c^3)$ is 2017.

37. Given equation is $x^3-10x+11=0$

Thus, $a+b+c=0$, $ab+bc+ca=11$, $abc=-11$

Now,

$m=\tan^{-1}(a)+\tan^{-1}(b)+\tan^{-1}(c)$

$$= \tan^{-1}\left(\frac{a+b+c-abc}{1-(ab+bc+ca)}\right)$$

$$= \tan^{-1}\left(\frac{0+11}{1+10}\right)$$

$$= \tan^{-1}\left(\frac{11}{11}\right)$$

$$= \frac{\pi}{4}$$

Thus, the value of $\tan\left(\frac{m}{2}\right) = \tan\left(\frac{\pi}{8}\right) = (\sqrt{2}-1)$

38. Let $y = \frac{x}{x^2+x+4}$.

$\Rightarrow \quad y = \frac{1}{x+\frac{4}{x}+1}$

Let $g(x) = x + \frac{4}{x}$

$\Rightarrow \quad g'(x) = 1 - \frac{4}{x^2}$

For max or min $g'(x) = 0$ gives

$\Rightarrow \quad 1 - \frac{4}{x^2} = 0$

$\Rightarrow \quad x^2 = 4$

$\Rightarrow \quad x = \pm 2$

Thus, the maximum value of y is 1/5 and minimum is –1/5

Therefore, $a = -1/5$ and $b = 1/5$

Now,

$(5a + 10b + 2) = -1 + 2 + 2$

$= 3.$

39 Given,

$x^2 - x - 1 = 0$

$\Rightarrow \quad x^2 - 1 = x$

$\Rightarrow \quad x - \frac{1}{x} = 1$

$\Rightarrow \quad \left(x - \frac{1}{x}\right)^2 = 1$

$\Rightarrow \quad x^2 + \frac{1}{x^2} = 1 + 2 = 3$

$\Rightarrow \quad \left(x^2 + \frac{1}{x^2}\right)^2 = 9$

$\Rightarrow \quad \left(x^4 + \frac{1}{x^4}\right) = 9 - 2 = 7$

$\Rightarrow \quad \left(x^4 + \frac{1}{x^4}\right)^2 = 49$

$\Rightarrow \quad \left(x^8 + \frac{1}{x^8}\right) = 49 - 2 = 47$

$\Rightarrow \quad \left(x^8 + \frac{1}{x^8} + 3\right) = 47 + 3 = 50$

40. Given,

$x^2 - 2x - 1 = 0$

$\Rightarrow \quad x^2 - 1 = 2x$

$\Rightarrow \quad x - \frac{1}{x} = 2$

$\Rightarrow \quad \left(x - \frac{1}{x}\right)^2 = 4$

$\Rightarrow \quad \left(x^2 + \frac{1}{x^2}\right) = 4 + 2 = 6$

Also,

$$x + \frac{1}{x} = \sqrt{\left(x - \frac{1}{x}\right)^2 + 4} = \sqrt{4+4} = 2\sqrt{2}$$

$\Rightarrow \quad \left(x + \frac{1}{x}\right)^3 = (2\sqrt{2})^3 = 16\sqrt{2}$

$\Rightarrow \quad x^3 + \frac{1}{x^3} = 16\sqrt{2} - 6\sqrt{2} = 10\sqrt{2}$

Now,

$$x^5 + \frac{1}{x^5} = \left(x^3 + \frac{1}{x^3}\right)\left(x^2 + \frac{1}{x^2}\right) - \left(x + \frac{1}{x}\right)$$

$= 10\sqrt{2} \times 6 - 2\sqrt{2}$

$= 60\sqrt{2} - 2\sqrt{2}$

$= 58\sqrt{2}$

Thus, $\left[\sqrt{2}\left(x^5 + \frac{1}{x^5}\right) + 42\right] = 58 + 42$

$= 100.$

41. Given,

$x^3 + \frac{1}{x^3} = 18$

$\Rightarrow \quad \left(x + \frac{1}{x}\right)^3 - 3\left(x + \frac{1}{x}\right) = 18$

$\Rightarrow \quad a^3 - 3a = 18$, where $a = \left(x + \frac{1}{x}\right)$

$\Rightarrow \quad a^3 - 3a - 18 = 0$

$\Rightarrow \quad a^3 - 3a^2 + 3a^2 - 9a + 6a - 18 = 0$

$\Rightarrow \quad a^2(a-3) + 3a(a-3) + 6(a-3) = 0$

$\Rightarrow \quad (a-3)(a^2 + 3a + 6) = 0$

$\Rightarrow \quad (a-3) = 0, (a^2 + 3a + 6) = 0$

$\Rightarrow \quad a = 3$, since a is real

$\Rightarrow \quad \left(x + \frac{1}{x}\right) = 3$

$\Rightarrow \quad \left(x^2 + \frac{1}{x^2}\right) = 9 - 2 = 7$

$$\Rightarrow \left(x^2+\frac{1}{x^2}\right)^2 = 49$$

$$\Rightarrow \left(x^4+\frac{1}{x^4}\right) = 47$$

Now,

$$\left(x^7+\frac{1}{x^7}\right) = \left(x^4+\frac{1}{x^4}\right)\left(x^3+\frac{1}{x^3}\right) - \left(x+\frac{1}{x}\right)$$
$$= 47 \times 18 - 3$$
$$= 826 - 3$$
$$= 823$$

42. Let a, b and c be the roots of $x^3 + px + q = 0$

Here, $s_1 = (a + b + c) = 0$

$s_2 = a^2 + b^2 + c^2 = (a + b + c)^2 - 2(ab + bc + ca)$

$\Rightarrow$ $1 = 0 - 2(ab + bc + ca)$

$\Rightarrow$ $(ab + bc + ca) = -1/2$

$\Rightarrow$ $p = -1/2$

Now,

$x^3 + px + q = 0$

$\Rightarrow$ $x^3 = -px - q$

$\Rightarrow$ $x^4 = -px^2 - qx$

$\Rightarrow$ $\Sigma a^4 = -p\Sigma a^2 - q\Sigma a$

$\Rightarrow$ $\Sigma a^4 = -p(1) - q(0)$

$$\Rightarrow \sum a^4 = -p(1) = -\left(-\frac{1}{2}\right) = \frac{1}{2}$$

$$\Rightarrow (a^4 + b^4 + c^4) = \frac{1}{2}$$

Hence the value of $(a^4 + b^4 + c^4)$ is $\frac{1}{2}$.

43. Given equation is $x^3 + px^2 + qx + r = 0$

Here, $a + b + c = -p$, $ab + bc + ca = q$, $abc = -r$

Now, $(b + c - a)(c + a - b)(a + b - c)$

$= (b + c - a)(c + a - b)(a + b - c)$

$= (a + b + c - 2a)(a + b + c - 2b)(a + b + c - 2c)$

$= (p - 2a)(p - 2b)(p - 2c)$

$= p^3 - 2(a + b + c)p + 2(ab + bc + ca)q - 8abc$

$= p^3 - 2(-p)p + 2(q)q - 8(-r)$

$= p^3 + 2p^2 + 2q^2 + 8r$

44. We have,

$x^5 - x^3 + x^2 - 1 = 0$

$\Rightarrow$ $x^3(x^2 - 1) + 1(x^2 - 1) = 0$

$\Rightarrow$ $(x^2 - 1)(x^3 + 1) = 0$

$\Rightarrow$ $(x^2 - 1) = 0, (x^3 + 1) = 0$

$\Rightarrow$ $x = \pm 1$ and $x = -1, \omega, \omega^2$

Also, $x^4 - 1 = 0$

$\Rightarrow$ $(x^2 + 1)(x^2 - 1) = 0$

$\Rightarrow$ $(x^2 + 1) = 0, (x^2 - 1) = 0$

$\Rightarrow$ $x = \pm i, x = \pm 1$

Hence the common roots of the above two equations are $\{-1, 1\}$.

45. Let $g(x) = \left(x+\frac{1}{x}\right)^4 - \left(x^4+\frac{1}{x^4}\right) - 1$

$$= \left\{\left(x+\frac{1}{x}\right)^2\right\}^2 - \left(x^4+\frac{1}{x^4}\right) - 1$$

$$= \left(x^2+\frac{1}{x^2}+2\right)^2 - \left(x^4+\frac{1}{x^4}\right) - 1$$

$$= \left(x^2+\frac{1}{x^2}\right)^2 + 2\left(x^2+\frac{1}{x^2}\right) - \left(x^4+\frac{1}{x^4}\right) + 3$$

$$= 2\left(x^2+\frac{1}{x^2}\right) + 5$$

and $h(x) = \left(x+\frac{1}{x}\right)^2 + \left(x^2+\frac{1}{x^2}\right)$

$$= 2\left(x^2+\frac{1}{x^2}\right) + 3$$

Thus, $$f(x) = \frac{g(x)}{h(x)} = \frac{2\left(x^2+\frac{1}{x^2}\right)+5}{2\left(x^2+\frac{1}{x^2}\right)+3}$$

$$= 1 + \frac{2}{2\left(x^2+\frac{1}{x^2}\right)+3}$$

$$= 1 + \frac{2}{2\left(x+\frac{1}{x}\right)^2 - 1}$$

It will be maximum when $x = -1$

Hence the maximum value of $f(x)$ is $\frac{9}{7}$.

46. Given equation is $x^2 - 2x - a + 1 = 0$.

$\Rightarrow$ $(x^2 - 2x + 1) = a^2$

$\Rightarrow$ $(x - 1)^2 = a^2$

$\Rightarrow$ $(x - 1) = \pm a$

$\Rightarrow$ $x = 1 \pm$ a

$\Rightarrow$ $\alpha = 1 + a, \beta = 1 - a$

Let $f(x) = x^2 - 2(a + 1)x + a(a - 1)$

Since α and β lie in (γ, δ), so

$f(1 + a) < 0, f(1 - a) < 0$

When $f(1 + a) < 0$

$\Rightarrow$ $(1 + a)^2 - 2(1 + a)(1 + a) + a(a - 1) < 0$

$\Rightarrow$ $a(a - 1) - (1 + a)^2 < 0$

$\Rightarrow$ $(a^2 - 1) - (a^2 + 2a + 1) < 0$

$\Rightarrow$ $-2a - 2 < 0$

$\Rightarrow$ $a + 1 > 0$

$\Rightarrow$ $a > -1$ …(i)

Also, $f(1 - a) < 0$

$\Rightarrow$ $(1 - a)^2 - 2(1 + a)(1 - a) + a(a - 1) < 0$

$\Rightarrow$ $(1 - a)((1 - a) - 2(1 + a) - a) < 0$

$\Rightarrow$ $(1 - a)(1 - a - 2 - 2a - a) < 0$

$\Rightarrow$ $(1 - a)(-1 - 4a) < 0$

$\Rightarrow\quad (a-1)(4a+1) < 0$

$\Rightarrow\quad -\frac{1}{4} < a < 1$ ...(ii)

From Relations (i) and (ii), we get

$-\frac{1}{4} < a < 1$

47. Given equation is

$x^3 - x^2 - x - 1 = 0$ ...(i)

Thus, $\alpha + \beta + \gamma = 1$

$\alpha\beta + \beta\gamma + \gamma\alpha = -1$

and $\alpha\beta\gamma = 1$

Put $x = \alpha, \beta, \gamma$ in Eq.(i) successively and adding, we get

$\Sigma\alpha^3 - \Sigma\alpha^2 - \Sigma\alpha - \Sigma 1 = 0$

$\Rightarrow\quad \Sigma\alpha^3 = \Sigma\alpha^2 + \Sigma\alpha + \Sigma 1$

$= (\Sigma\alpha)^2 - 2\Sigma\alpha\beta + \Sigma\alpha + 3$

$= (1)1 - 2(-1) + (1) + 3$

$= 1 + 2 + 1 + 3$

$= 7$

Hence the value of $(\alpha^3 + \beta^3 + \gamma^3) = 7$

48. Given equation is

$x^3 + 3x + 9 = 0$ ...(i)

Clearly, $\alpha + \beta + \gamma = 0$, $\alpha\beta + \beta\gamma + \gamma\alpha = 3$, $\alpha\beta\gamma = -9$

Now, $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$

$= 0 - 2 \cdot 3 = -6$

Put $x = \alpha, \beta, \gamma$ in Eq. (i) successively and adding, we get

$\Sigma\alpha^3 + 3\Sigma\alpha + \Sigma 9 = 0$

$\Rightarrow\quad \Sigma\alpha^3 + 3(0) + 3(9) = 0$

$\Rightarrow\quad \Sigma\alpha^3 = -27$

Again, $x^3 + 3x + 9 = 0$

$\Rightarrow\quad x^3 = -3(x + 3)$

$\Rightarrow\quad (x^3)^3 = -27(x + 3)^3$

$\Rightarrow\quad x^9 = -27(x^3 + 9x^2 + 27x + 27)$ ...(ii)

Put $x = \alpha, \beta, \gamma$ in Eq. (ii) successively and adding, we get

$\Sigma\alpha^9 = -27(\Sigma\alpha^3 + 9\Sigma\alpha^2 + 27\Sigma\alpha + 27\,\Sigma 1)$

$= -27(-27 + 9(-6) + 27(0) + 27(3))$

$= -27(-27 - 54 + 81)$

$= -27(-81 + 81)$

$= 0$

Hence the value of $(\alpha^9 + \beta^9 + \gamma^9)$ is 0.

49. Let α be the repeated roots.

Thus, $x^4 + px^3 + qx^2 + rx + s = (x - \alpha)^4$

$= x^4 - 4x^3\alpha + 6x^2\alpha^2 - 4x\alpha^3 + \alpha^4$

Comparing the co-efficients of x, x^2, x^3 and x^4, we get

$p = -4\alpha,\ q = 6\alpha^2,\ r = -4\alpha^3,\ s = \alpha^4$

Eliminating α, we get,

$p^3 = 16r$ and $p^4 = 256s$

$\Rightarrow\quad p^3 = 2^4 r$ and $p^4 = 2^8 s$

Comparing with $p^3 = 2^m r$ and $p^4 = 2^n s$, we have

$\Rightarrow\quad m = 4$ and $n = 8$

Hence, the value of $(m + n - 4)$ is 8 .

50. Let $\alpha, \beta, \gamma, \delta$ be the roots of

$x^4 + x^3 - 16x^2 - 4x + 48 = 0$, where $\alpha\beta = 6$

Clearly, $\alpha + \beta + \gamma + \delta = -1$, $\alpha\beta\gamma\delta = 48$

Then $\gamma\delta = 8$

Let $\alpha + \beta = p$, $\gamma + \delta = q$

Thus $x^4 + x^3 - 16x^2 - 4x + 48$

$= (x^2 - px + 6)(x^2 - qx + 8)$

$= x^4 - (p + q)x^3 + (pq + 14)x^2 - (8p + 6q)x + 48$

Comparing the co-efficients, we get,

$p + q = -1,\ pq + 14 = -16$

$\Rightarrow\quad p + q = -1,\ pq = -30$

$\Rightarrow\quad p = 5$ and $q = -6$

Hence, the roots are

$x^2 - 5x + 6 = 0,\ x^2 + 6x + 8 = 0$

$\Rightarrow\quad x = 2, 3, -2, -4$

Level IV

1. Given $y = \dfrac{\tan^2\theta - 2\tan\theta - 8}{\tan^2\theta - 4\tan\theta - 5}$.

Let $\tan\theta = x$

Then $y = \dfrac{x^2 - 2x - 8}{x^2 - 4x - 5}$

$\Rightarrow\quad y(x^2 - 4x - 5) = (x^2 - 2x - 8)$

$\Rightarrow\quad (y - 1)x^2 + 2(1 - 2y)x + (8 - 5y) = 0$

For every $y \in R$, $D \geq 0$

$\Rightarrow\quad 4(1 - 2y)^2 - 4(y - 1)(8 - 5y) \geq 0$

$\Rightarrow\quad (1 - 2y)^2 - (y - 1)(8 - 5y) \geq 0$

$\Rightarrow\quad (1 - 4y + 4y^2) - (8y - 5y^2 - 8 + 5y) \geq 0$

$\Rightarrow\quad (1 - 4y + 4y^2) - (13y - 5y^2 - 8) \geq 0$

$\Rightarrow\quad 9y^2 - 17y + 9 \geq 0$

Since $D < 0$, so it is true for every values of R.

Hence the range of y is R.

2. Given $y = \dfrac{(\cot^2\theta + 5)(\cot^2\theta + 10)}{(\cot^2\theta + 1)}$.

Let $\cot^2\theta = x$

Then $y = \dfrac{(x + 5)(x + 10)}{(x + 1)}$

$\Rightarrow\quad y = \dfrac{x^2 + 15x + 50}{x + 1}$

$\Rightarrow\quad x^2 + 15x + 50 - xy - y = 0$

$\Rightarrow\quad x^2 + (15 - y)x + (50 - y) = 0$

$\therefore\quad D \geq 0$

$\Rightarrow\quad (15 - y)^2 - 4(50 - y) \geq 0$

$\Rightarrow\quad 225 - 30y + y^2 - 200 + 4y \geq 0$

$\Rightarrow\quad y^2 - 26y + 25 \geq 0$

$\Rightarrow\quad (y - 1)(y - 25) \geq 0$

$\Rightarrow\quad y \leq 1$ and $y \geq 25$

Thus $y \geq 25$.

3. Given $y = \dfrac{2x^2 - 3x + 2}{2x^2 + 3x + 2}$

$\Rightarrow\quad y = \dfrac{(2x^2 + 3x + 2) - 6x}{2x^2 + 3x + 2}$

$\Rightarrow\quad y = 1 - \dfrac{6x}{2x^2 + 3x + 2}$

$\Rightarrow \quad y = 1 - \dfrac{6}{2x + 3 + \dfrac{2}{x}}$

$\Rightarrow \quad y = 1 - \dfrac{6}{2\left(x + \dfrac{1}{x}\right) + 3}$

Clearly, range of y is $\left[\dfrac{1}{7}, 1\right]$.

4. Given expression is

$f(x, y) = 12x^2 - 10xy = 2y^2 + 11x - 5y + k \quad \ldots\text{(i)}$

The expression (i) represents a pair of linear factors if

$$\begin{vmatrix} 12 & -5 & 11/2 \\ -5 & 2 & -5/2 \\ 11/2 & -5/2 & k \end{vmatrix} = 0$$

Solving, we get,

$k = 2.$

5. Given equation can be written as

$y^2 + 8y + x^2 - 6x = 0$

For every y in R, $D \geq 0$

$64 - 4(x^2 - 6x) \geq 0$

$\Rightarrow \quad 16 - (x^2 - 6x) \geq 0$

$\Rightarrow \quad (x^2 - 6x - 16) \leq 0$

$\Rightarrow \quad (x + 2)(x - 8) \leq 0$

$\Rightarrow \quad -2 \leq x \leq 8$

Again, the given equation can be written as

$x^2 - 6x + (y^2 + 8y) = 0$

For every x in R,

$D \geq 0$

$36 - 4(y^2 + 8y) \geq 0$

$\Rightarrow \quad 9 - (y^2 + 8y) \geq 0$

$\Rightarrow \quad (y^2 + 8y - 9) \leq 0$

$\Rightarrow \quad (y + 1)(y - 9) \leq 0$

$\Rightarrow \quad -1 \leq y \leq 9$

Hence $-2 \leq x \leq 8$ and $-1 \leq y \leq 9$

6. Given a, b, c, d are in GP.

$\Rightarrow \quad \dfrac{a}{b} = \dfrac{b}{c} = \dfrac{c}{d} = k \text{ (say)}$

$\Rightarrow \quad c = dk, b = dk^2, a = dk^3$

Now, $ax^2 + c = dk^3x^2 + dk = dk(k^2x^2 + 1) \quad \ldots\text{(i)}$

Also, $ax^3 + bx^2 + cx + d$

$= dk^3x^3 + dk^2x^2 + dk + d$

$= d(k^3x^3 + k^2x^2 + k + 1)$

$= d(k^2x^2(k + 1) + (k + 1))$

$= d(k^2x^2 + 1)(kx + 1) \quad \ldots\text{(ii)}$

Clearly Eq. (ii) is divisible by Eq. (i).

7. Since $ax^3 + bx + c$ is divisible by $(x^2 + px + 1)$, so

$ax^3 + bx + c = (x^2 + px + 1)(ax + q)$

$= ax^3 + apx^2 + ax + qx^2 + pqx + q$

$= ax^3 + (ap + q)x^2 + (a + pq)x + q$

Comparing the co-efficients, we get

$(ap + q) = 0, (a + pq) = b, q = c$

$\Rightarrow \quad p = -\dfrac{c}{a}, (a + pq) = b, q = c$

$\Rightarrow \quad \left(a - \dfrac{c}{a} \cdot c\right) = b$

$\Rightarrow \quad a^2 - c^2 = ab$

Hence, the result.

8. We have,

$x^3 + px^2 + qx + r = (x + \alpha)^3$

$= x^3 + 3x^2\alpha + 3x\alpha^2 + \alpha^3$

Comparing the co-efficients, we get,

$\Rightarrow \quad p = 3a, q = 3\alpha^2, r = \alpha^3$

Hence, $p^3 = 27\alpha^3 = 27r$

and $\quad 3pr = 9\alpha^4 = (3\alpha^2)^2 = q^2$

9. Given equation is

$x^4 - x^3 - 19x^2 + 49x - 30 = 0$

$\Rightarrow \quad x^3(x - 1) - 19x(x - 1) + 30(x - 1) = 0$

$\Rightarrow \quad (x - 1)(x^3 - 19x + 30) = 0$

$\Rightarrow \quad (x - 1) = 0, (x^3 - 19x + 30) = 0$

Also, $(x^3 - 19x + 30) = 0$

$\Rightarrow \quad x^2(x - 2) + 2x(x - 2) - 15(x - 2) = 0$

$\Rightarrow \quad (x - 2)(x^2 + 2x - 15) = 0$

$\Rightarrow \quad (x - 2)((x + 5)(x - 3) = 0$

$\Rightarrow \quad x = 2, 3, -5$

Hence, the integral roots are 1, 2, 3, and –5.

10. Given equation is

$(6 - x)^4 + (8 - x)^4 = 16 \quad \ldots\text{(i)}$

put $\quad y = \dfrac{(6 - x) + (8 - x)}{2} = \dfrac{14 - 2x}{2} = 7 - x$

$\Rightarrow \quad x = 7 - y$

Equation (i) reduces to

$(6 - 7 + y)^4 + (8 - 7 + y)^4 = 16$

$\Rightarrow \quad (y - 1)^4 + (y + 1)^4 = 16$

$\Rightarrow \quad (y^4 - 4y^3 + 6y^2 - 4y + 1)$

$+ (y^4 + 4y^3 + 6y^2 + 4y + 1) = 16$

$\Rightarrow \quad 2(y^4 + 6y^2 + 1) = 16$

$\Rightarrow \quad (y^4 + 6y^2 + 1) = 8$

$\Rightarrow \quad (y^4 + 6y^2 - 7) = 0$

$\Rightarrow \quad (y^2 + 7)(y^2 - 1) = 0$

$\Rightarrow \quad (y^2 + 7) = 0, (y^2 - 1) = 0$

$\Rightarrow \quad y^2 = 1, -7$

$\Rightarrow \quad y = \pm 1, \pm i\sqrt{7}$

$\Rightarrow \quad 7 - x = \pm 1, \pm i\sqrt{7}$

$\Rightarrow \quad x = 7 \pm 1, 7 \pm i\sqrt{7}$

Hence, the roots are $\{8, 6, 7 \pm i\sqrt{7}\}$.

11. Let the third root be γ.

Thus, $\alpha + \beta + \gamma = -\alpha, \alpha\beta + \alpha\gamma + \beta\gamma = b, \alpha\beta\gamma = -c$

Given condition is $\alpha\beta + 1 = 0$

$\Rightarrow \quad (-1)\gamma = -c$

$\Rightarrow \quad \gamma = c$

Also, $\alpha\beta + \alpha\gamma + \beta\gamma = b$

$\Rightarrow \quad \alpha\beta + (\alpha + \beta)\gamma = b$

$\Rightarrow \quad -1 + \gamma(-\alpha - \gamma) = b$

$\Rightarrow \quad 1 + \gamma(\alpha + \gamma) = -b$

$\Rightarrow \quad \gamma^2 + a\gamma + b + 1 = 0$

$\Rightarrow \quad c^2 + ac + b + 1 = 0$

Hence, the result.

12. Given equation is $x^3 + x + 2 = 0$

Thus, $\alpha + \beta + \gamma = 0$, $\alpha\beta + \alpha\gamma + \beta\gamma = 1$, $\alpha\beta\gamma = -2$

Let $y = (\alpha - \beta)^2 = \alpha^2 + \beta^2 - 2\alpha\beta$

$= (\alpha + \beta)^2 - 2\alpha\beta - 2\alpha\beta$

$= \gamma^2 - 4\alpha\beta$

$$= \gamma^2 - \frac{4\alpha\beta\gamma}{\gamma}$$

$$= \gamma^2 + \frac{8}{\gamma}$$

$$= \frac{\gamma^3 + 8}{\gamma} = \frac{6 - \gamma}{\gamma} = \frac{6}{\gamma} - 1$$

$$\Rightarrow \quad y = \frac{6}{\gamma} - 1$$

$$\Rightarrow \quad \gamma = \frac{6}{y+1}$$

Since γ is a root of the given equation, so

$\gamma^3 + \gamma + 2 = 0$

$$\Rightarrow \quad \left(\frac{6}{y+1}\right)^3 + \left(\frac{6}{y+1}\right) + 2 = 0$$

Hence, the required equation is

$$\Rightarrow \quad \left(\frac{6}{y+1}\right)^3 + \left(\frac{6}{y+1}\right) + 2 = 0$$

$\Rightarrow \quad 2(x + 1)^3 + 6(x + 1)^2 + 216 = 0$

$\Rightarrow \quad x^3 + 6x^2 + 9x + 112 = 0$

13. Given equation is

$x^3 + qx + r = 0$ …(i)

Since α, β, γ be the roots of Eq. (i), we get

$\alpha^3 + q\alpha + r = 0$

$\beta^3 + q\beta + r = 0$

$\gamma^3 + q\gamma + r = 0$

Adding all, we get

$(\alpha^3 + \beta^3 + \gamma^3) + q(\alpha + \beta + \gamma) + 3r = 0$

$\Rightarrow \quad (\alpha^3 + \beta^3 + \gamma^3) + q.0 + 3r = 0$

$\Rightarrow \quad (\alpha^3 + \beta^3 + \gamma^3) = -3r$ …(ii)

Also,

$(\alpha^2 + \beta^2 + \gamma^2) = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$

$= 0 - 2q = -2q$ …(iii)

Again, $x^3 + qx + r = 0$

$\Rightarrow \quad x^5 + qx^3 + rx^2 = 0$

put $x = \alpha, \beta, \gamma$ successively and adding, we get

$\Sigma\alpha^5 + q\Sigma\alpha^3 + r\Sigma\alpha^2 = 0$

$\Rightarrow \quad \Sigma\alpha^5 + q(-3r) + r(-2q) = 0$

$\Rightarrow \quad \Sigma\alpha^5 = 5qr$

$$\Rightarrow \quad \left(\frac{\sum\alpha^5}{5}\right) = \left(\frac{\sum\alpha^3}{3}\right)\left(\frac{\sum\alpha^2}{2}\right)$$

$$\Rightarrow \quad \left(\frac{\alpha^5 + \beta^5 + \gamma^5}{5}\right) = \left(\frac{\alpha^3 + \beta^3 + \gamma^3}{3}\right)\left(\frac{\alpha^2 + \beta^2 + \gamma^2}{2}\right)$$

14. Given curve is

$y^2 = 4x$ …(i)

Put $y = mx + 1$ in Eq. (i), we get

$(mx + 1)^2 = 4x$

$\Rightarrow \quad m^2x^2 + 2mx + 1 = 4x$

$\Rightarrow \quad m^2x^2 + 2(m - 2)x + 1 = 0$ …(ii)

Since $y = mx + 1$ is tangent, so the Eq. (ii) has equal roots.

Thus, $4(m - 2)^2 - 4m^2 = 0$

$\Rightarrow \quad (m - 2)^2 - m^2 = 0$

$\Rightarrow \quad m^2 - 4m + 4 - m^2 = 0$

$\Rightarrow \quad -4m + 4 = 0$

$\Rightarrow \quad 4m = 4$

$\Rightarrow \quad m = 1$

Hence, the value of $(m^2 + m + 1) = 1 + 1 + 1 = 3$.

15. We have,

$x = 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots$

$\Rightarrow \quad x = (1 + x)^4$

$$\Rightarrow \quad x = \frac{1}{(1+x)}$$

$\Rightarrow \quad x + x^2 - 1 = 0$

$\Rightarrow \quad x^2 + x - 1 = 0$

$$\Rightarrow \quad x = \frac{-1 \pm \sqrt{1+4}}{2}$$

$$\Rightarrow \quad x = \frac{-1+\sqrt{5}}{2} (\because -1 < x < 1))$$

$$\Rightarrow \quad x = \frac{\sqrt{5}-1}{2} = 2\left(\frac{\sqrt{5}-1}{4}\right) = 2\sin(18°)$$

Hence the solution is $x = 2\sin(18°)$.

16. Given a, b are the roots of $x^2 - 10cx - 11d = 0$ and c, d are the roots of $x^2 - 10ax - 11b = 0$.

Thus, $a + b = 10c$, $c + d = 10a$

and $ab = -11d$, $cd = -11b$

So, $a + b + c + d = 10(c + a)$ …(i)

and $(a + b) - (c + d) = 10(c - a)$

$\Rightarrow \quad (b - d) = 11(c - a)$ …(ii)

Also, a is a root of $x^2 - 10cx - 11d = 0$ and c is a root of $x^2 - 10ax - 11b = 0$.

Thus, $c^2 - 10ac = 11b$, $a^2 - 10ca = 11d$

$\Rightarrow \quad c^2 - a^2 = 11(b - d)$

$\Rightarrow \quad c^2 - a^2 = 11(b - d) = 11 \times 11(c - a)$

$\Rightarrow \quad c + a = 11 \times 11 = 121$

From Eq. (i), we get

$a + b + c + d = 121 \times 10 == 1210$.

17. Given $\alpha + \beta = -p$, $\alpha^3 + \beta^3 = q$

Now, $\alpha^3 + \beta^3 = q$

$\Rightarrow \quad (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) = q$

$\Rightarrow \quad (-p)^3 - 3\alpha\beta(-p) = q$

$\Rightarrow \quad -p^3 + 3\alpha\beta.p = q$

$$\Rightarrow \quad \alpha\beta = \frac{p^3 + q}{3p}$$

Now, $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}$

$$=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$$

$$=\frac{(\alpha+\beta)^2}{\alpha\beta}-2$$

$$=\frac{p^2}{\left(\frac{p^3+q}{3p}\right)}-2$$

$$=\frac{3p^3}{p^3+q}-2$$

$$=\frac{p^3-2q}{p^3+q}$$

Hence, the required equation is

$\Rightarrow \quad x^2-\left(\frac{\alpha}{\beta}+\frac{\beta}{\alpha}\right)x+\left(\frac{\alpha}{\beta}\cdot\frac{\beta}{\alpha}\right)=0$

$\Rightarrow \quad x^2-\left(\frac{p^3-2q}{p^3+q}\right)x+1=0$

$\Rightarrow \quad (p^3+q)x^2-(p^3-2q)x+p^3+q)=0$

18. Given equation is

$$(\sqrt[3]{a+1}-1)x^2+(\sqrt{1+a}-1)x+(\sqrt[6]{a+1}-1)=0$$

Let $1+a=y$.

$\Rightarrow \quad (y^{1/3}-1)x^2+(y^{1/2}-1)x+(y^{1/6}-1)=0$

$\Rightarrow \quad \left(\frac{y^{1/3}-1}{y-1}\right)x^2+\left(\frac{y^{1/2}-1}{y-1}\right)x+\left(\frac{y^{1/6}-1}{y-1}\right)=0$

Taking limit $y\to 1$, we get,

$\Rightarrow \quad \frac{1}{3}x^2+\frac{1}{2}x+\frac{1}{6}=0$

$\Rightarrow \quad 2x^2+3x+1=0$

$\Rightarrow \quad 2x^2+2x+x+1=0$

$\Rightarrow \quad 2x(x+1)+1(x+1)=0$

$\Rightarrow \quad (2x+1)(x+1)=0$

$\Rightarrow \quad x=-1,-\frac{1}{2}$

Thus, $L=-1$, $M=-1/2$

Hence, the value of $L+2M+3=-1-1+3=1$.

19. Given equation is $x^2-9kx+16(k^2-k+1)=0$

Since the roots are real and distinct, so

$64k^2-64(k^2-k+1)>0$

$\Rightarrow \quad k^2-(k^2-k+1)>0$

$\Rightarrow \quad k-1>0$

$\Rightarrow \quad k>1$

Thus, $k=2, 3, 4, 5$

Hence the smallest integral value of k is 2.

20. Given equation is $x^3+qx+r=0$

$$\alpha+\beta+\gamma=0,\ \alpha\beta+\beta\gamma+\gamma\alpha=\alpha\beta\gamma=-r$$

Put $\quad y=\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}$

$\Rightarrow \quad y=\frac{\gamma^2-2\alpha\beta}{\alpha\beta}=\frac{\gamma^3-2\alpha\beta\gamma}{\alpha\beta\gamma}=\frac{\gamma^3+2r}{-r}$

$\Rightarrow \quad y=\frac{\gamma^3+2r}{-r}$

$\Rightarrow \quad \gamma^3=-(ry+2r)=-r(y+2)$

$\Rightarrow \quad \gamma^3=yr+2r=0$

Also, γ is a root of $x^3+qx+r=0$, so

$\gamma^3+q\gamma+r=0$

Subtracting Eq. (i) and (ii), we get

$-q\gamma+r(y+1)=0$

$\Rightarrow \quad \gamma=\frac{r(y+1)}{q}$

But γ is a root of $x^3+qx+r=0$, so

$\Rightarrow \quad \left(\frac{r(y+1)}{q}\right)^3+q\left(\frac{r(y+1)}{q}\right)^2+r=0$

$\Rightarrow \quad r^2(y+1)^3+q^3(y+1)+q^3=0$

Hence, the required equation is

$r^2(x+1)^3+q^3(x+1)+q^3=0$

21. Given equation is

$(x^2+x+1)+(x^2+2x+3)+(x^2+3x+5)$
$+\ldots+(x^2+20x+39)=4500$

$\Rightarrow \quad 20x^2+(1+2+3+\ldots+20)x$
$+(1+3+\ldots+39)=4500$

$\Rightarrow \quad 20x^2+\left(\frac{20\times 21}{2}\right)x+(20)^2=4500$

$\Rightarrow \quad x^2+\frac{21}{2}x+(20)=225$

$\Rightarrow \quad 2x^2+21x+40=450$

$\Rightarrow \quad 2x^2+21x-410=0$

$\Rightarrow \quad 2x^2-20x+41x-410=0$

$\Rightarrow \quad 2x(x-10)+41(x-10)=0$

$\Rightarrow \quad (x-10)(2x+41)=0$

$\therefore \quad x=10, 20.5$

Hence the solution set is $\{10, 20.5\}$.

22. Given equation is

$x^3+2x^2-x-3=0$ …(i)

Put $\quad y=\left(\frac{\alpha+3}{\alpha-2}\right)$

$\Rightarrow \quad \alpha=\frac{2y+3}{y-1}$

Since α is a root of Eq. (i), so

$$\left(\frac{2y+3}{y-1}\right)^3+2\left(\frac{2y+3}{y-1}\right)^2-\left(\frac{2y+3}{y-1}\right)-3=0$$

$\Rightarrow \quad (2y+3)^3+2(2y+3)(y-1)^2$
$-(2y+3)(y-1)^2-3(y-1)^3=0$

$\Rightarrow$ $(8y^3 + 36y^2 + 54y + 27) + (-3y^2 + 9y^2 - 9y + 3) + (-2y^3 + y^2 + 4y - 3) + (8y^3 + 20y^2 - 6y - 18) = 0$

$\Rightarrow$ $(11y^3 + 66y^2 + 43y + 9) = 0$

Hence the equation is $(11x^3 + 66x^2 + 43x + 9) = 0$ whose roots are $\left(\frac{\alpha+3}{\alpha-2}\right), \left(\frac{\beta+3}{\beta-2}\right), \left(\frac{\gamma+3}{\gamma-2}\right)$.

So, $$\left|\left(\frac{\alpha+3}{\alpha-2}\right)+\left(\frac{\beta+3}{\beta-2}\right)+\left(\frac{\gamma+3}{\gamma-2}\right)\right| = \left|-\frac{66}{11}\right|$$
$$= 6 = \frac{6}{1}$$

Thus, $p = 6$ and $q = 1$

Hence the value of $(p + q - 2)$ is 5.

23. Since the given curve always lies above x-axis, so its $D < 0$.

$\Rightarrow$ $(8(a + 5))^2 + 64(7a + 5) < 0$

$\Rightarrow$ $(a + 5)^2 + (7a + 5) < 0$

$\Rightarrow$ $a^2 + 10a + 25 + 7a + 5 < 0$

$\Rightarrow$ $a^2 + 17a + 30 < 0$

$\Rightarrow$ $(a + 2)(a + 5) < 0$

$\therefore$ $-15 < a < -2$

Hence the number of integral values of a is 12.

24. Since the given bi-quadratic expression $x^4 + 4x^3 + 6px^2 + 4qx + r$ is divisible by the expression $x^3 + 3x^2 + 9x + 3$, so we can write it as

$x^4 + 4x^3 + 6px^2 + 4qx + r$
$= (x^3 + 3x^2 + 9x + 3)(x - a)$
$= x^4 + (3 - \alpha)x^3 + 3(3 + \alpha)x^2 + 3(1 + 3\alpha)x - 3\alpha$

Comparing the co-efficients of x^3, x^2, x and constant terms, we get

$(3-\alpha)=4, 3(3+\alpha)=6p, 3(1+3\alpha)=4q$ and $-3\alpha=r$

Thus, $\alpha = -1,\ p = 1,\ q = -\frac{3}{2},\ r = 3$

Hence, the value of

$$2(p + q)r = 2\times\left(1-\frac{3}{2}\right)\times 3 = -3$$

25. Given equation is

$2016x^3 + 2x^2 + 1 = 0$

Replacing x by $1/x$, we get

$$\frac{2016}{x^3}+\frac{2}{x^2}+1=0$$

$x^3 + 2x + 2016 = 0$

whose roots are a, b and c, where

$$a=\frac{1}{\alpha}, b=\frac{1}{\beta}, c=\frac{1}{\gamma}$$

Thus, $a + b + c = 0$

Now, $$\left[\frac{1}{12}\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\beta^2}\right)\left(\frac{1}{\alpha^3}+\frac{1}{\beta^3}+\frac{1}{\beta^3}\right)+4\right]$$

$$=\left(\frac{1}{12}(a^2+b^2+c^2)(a^3+b^3+c^3)+4\right)$$
$$=\left(\frac{1}{12}\{(a+b+c)^2-2(ab+bc+ca)\}\times 3abc+4\right)$$
$$=\left(\frac{1}{12}\{0-2.2\}\times 3abc+4\right)$$
$= -abc + 4$
$= 2016 + 4$
$= 2020.$

26. We have,

Numerator

$$=\left(x+\frac{1}{x}\right)^6-\left(x^6+\frac{1}{x^6}\right)-2$$
$$=\left\{\left(x+\frac{1}{x}\right)^3\right\}^2-\left(x^6+\frac{1}{x^6}\right)-2$$
$$=\left\{\left(x^3+\frac{1}{x^3}\right)+3\left(x+\frac{1}{x}\right)\right\}^2-\left(x^6+\frac{1}{x^6}\right)-2$$
$$=\left\{\left(x^3+\frac{1}{x^3}\right)^2+9\left(x+\frac{1}{x}\right)^2+6\left(x^3+\frac{1}{x^3}\right)\left(x+\frac{1}{x}\right)\right\}-\left(x^6+\frac{1}{x^6}\right)-2$$
$$=\left(x^6+\frac{1}{x^6}+2\right)+9\left(x^2+\frac{1}{x^2}\right)+18+6\left(x^3+\frac{1}{x^3}\right)\left(x+\frac{1}{x}\right)-\left(x^6+\frac{1}{x^6}\right)-2$$
$$=9\left(x^2+\frac{1}{x^2}\right)+18+6\left(x^3+\frac{1}{x^3}\right)\left(x+\frac{1}{x}\right)$$
$$=9\left(x+\frac{1}{x}\right)^2+6\left(x^3+\frac{1}{x^3}\right)\left(x+\frac{1}{x}\right)$$
$$=3\left(x+\frac{1}{x}\right)\left[3\left(x+\frac{1}{x}\right)+2\left(x^3+\frac{1}{x^3}\right)\right]$$
$$=3\left(x+\frac{1}{x}\right)\left[\left(x^3+\frac{1}{x^3}\right)+3\left(x+\frac{1}{x}\right)+\left(x^3+\frac{1}{x^3}\right)\right]$$
$$=3\left(x+\frac{1}{x}\right)\left[\left(x+\frac{1}{x}\right)^3+\left(x^3+\frac{1}{x^3}\right)\right]$$

Therefore,

$$f(x)=\frac{\left(x+\frac{1}{x}\right)^6-\left(x^6+\frac{1}{x^6}\right)-2}{\left(x+\frac{1}{x}\right)^3+\left(x^3+\frac{1}{x^3}\right)}$$
$$=3\left(x+\frac{1}{x}\right)\geq 3.2=6$$

Hence the minimum value of $f(x)$ is 6.

27. We have $y = \dfrac{x^2 + 2 - \sqrt{x^4+4}}{x}$

$$= \left(x + \frac{2}{x}\right) - \sqrt{x^2 + \frac{4}{x^2}}$$

$$= \left(x + \frac{2}{x}\right) - \sqrt{\left(x + \frac{2}{x}\right) - 4}$$

Let $g(x) = \left(x + \dfrac{2}{x}\right)$

$\Rightarrow \quad g'(x) = \left(1 - \dfrac{2}{x^2}\right)$

For maximum or minimum $g'(x) = 0$ gives

$$\left(1 - \frac{2}{x^2}\right) = 0$$

$\Rightarrow \quad x^2 = 2$

$\Rightarrow \quad x = \sqrt{2}$, since x is positive.

Hence, the maximum value of

$$y = \sqrt{2} + \frac{2}{\sqrt{2}} - \sqrt{2 + \frac{4}{2}}$$

$$= \sqrt{2} + \sqrt{2} - 2$$

$$= 2(\sqrt{2} - 1)$$

28. We have $f(x) < 0$, for all real x.

$\Rightarrow \quad \dfrac{ax^2 + 2(a+1)x + (9a-4)}{(x^2 - 8x + 32)} < 0$

$\Rightarrow \quad \dfrac{ax^2 + 2(a+1)x + (9a-4)}{((x-4)^2 + 16)} < 0$

$\Rightarrow \quad ax^2 + 2(a+1)x + (9a-4) < 0$

It is possible only when $a < 0, D < 0$

Thus, $D < 0$ gives

$4(a+1)^2 - 4a(9a-4) < 0$

$\Rightarrow \quad (a+1)^2 - a(9a-4) < 0$

$\Rightarrow \quad a(9a-4) - (a+1)^2 > 0$

$\Rightarrow \quad 9a^2 - 4a - a^2 - 2a - 1 > 0$

$\Rightarrow \quad 8a^2 - 6a - 1 > 0$

$\Rightarrow \quad a < \dfrac{3 - \sqrt{17}}{8}$ and $a > \dfrac{3 + \sqrt{17}}{8}$

Hence, the range of a is $\left(-\infty, \dfrac{3 - \sqrt{17}}{8}\right)$.

29. Let α, β, γ be the root of $x^3 + px^2 + qx + r = 0$

Here $s_1 = \alpha + \beta + \gamma = 2$

$\Rightarrow \quad p = 2$

Also $s_2 = \alpha^2 + \beta^2 + \gamma^2$

$\Rightarrow \quad 6 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$\Rightarrow \quad 6 = 4 - 2q$

$\Rightarrow \quad q = -1$

and $\quad -r = \Sigma\alpha^3 + p\Sigma\alpha^2 + q\Sigma\alpha$

$\Rightarrow \quad -r = 8 + 2.6 + (-1).2$

$\Rightarrow \quad r = -14$

Hence, the equation is

$x^3 + 2x^2 - x - 14 = 0$

$\Rightarrow \quad x^3 = -2x^2 + x + 14$

$\Rightarrow \quad x^4 = -2x^3 + x^2 + 14x$

$\Rightarrow \quad \Sigma\alpha^4 = -2\Sigma\alpha^3 + \Sigma\alpha^2 + 14\Sigma\alpha$

$\Rightarrow \quad \Sigma\alpha^4 = -2(8) + (6) + 14(2)$

$\Rightarrow \quad \Sigma\alpha^4 = 34 - 16 = 18$

Hence the value of $(\alpha^4 + \beta^4 + \gamma^4)$ is 18.

30. Given equation is $x^3 - (a+1)x^2 + (b-a)x - b = 0$

Now, $\tan\alpha + \tan\beta + \tan\gamma = (a+1)$

$\tan\alpha \cdot \tan\beta + \tan\alpha \cdot \tan\gamma + \tan\beta \cdot \tan\gamma = (b-a)$

$\tan\alpha \cdot \tan\beta \cdot \tan\gamma = b$

We have

$\tan(\alpha + \beta + \gamma)$

$$= \frac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}{1 - (\tan\alpha\tan\beta + \tan\beta\tan\gamma + \tan\gamma\tan\alpha)}$$

$$= \frac{(a+1) - b}{1 - (b-a)} = \frac{a+1-b}{a+1-b} = 1 = \tan\left(\frac{\pi}{4}\right)$$

Thus, $\tan(\alpha + \beta + \gamma) = \tan\left(\dfrac{\pi}{4}\right)$

$\Rightarrow \quad (\alpha + \beta + \gamma) = \dfrac{\pi}{4}$

Integer Type Questions

1. Here $\alpha + \beta + \gamma + \delta = 0$

and $\quad \sum \alpha\beta = \dfrac{4}{16} = \dfrac{1}{4}$

$\sum \alpha\beta\gamma = 0$

$\sum \alpha\beta\gamma\delta = \dfrac{1}{16}$

Now, $\quad \Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$

$$= 0 - 2 \cdot \frac{1}{4} = -\frac{1}{2}$$

We have $16x^4 + 4x^2 + 1 = 0$

$\Rightarrow \quad 16\Sigma\alpha^4 + 4\Sigma\alpha^2 + \Sigma 1 = 0$

$\Rightarrow \quad -16\Sigma\alpha^4 = 4\Sigma\alpha^2 + \Sigma 1$

$\Rightarrow \quad -16\sum\alpha^4 = 4\left(-\dfrac{1}{2}\right) + 4 = 2$

$\Rightarrow \quad \sum\alpha^4 = -\dfrac{2}{16} = -\dfrac{1}{8}$

$\Rightarrow \quad (8\Sigma\alpha^4) + 4 = -1 + 4 = 3$

$\Rightarrow \quad 8(\alpha^4 + \beta^4 + \gamma^4 + \delta^4) + 4 = 3$

2. Here $\alpha + \beta + \gamma + \delta = 0$

$\Sigma\alpha\beta = 1$

$\Sigma\alpha\beta\gamma = 0$

and $\quad \Sigma\alpha\beta\gamma\delta = 1$

3. Given a, b are the roots of $x^2 - 10cx - 11d = 0$ and c, d are the roots of $x^2 - 10ax - 11b = 0$.
Thus, $a + b = 10c$, $c + d = 10a$
and $ab = -11d$, $cd = -11b$
So, $a + b + c + d = 10(c + a)$ …(i)
and $(a + b) - (c + d) = 10(c - a)$
$\Rightarrow (b - d) = 11(c - a)$ …(ii)
Also, a is a root of $x^2 - 10cx - 11d = 0$ and c is a root of $x^2 - 10ax - 11b = 0$.
Thus, $c^2 - 10ac = 11b$, $a^2 - 10ca = 11d$
$\Rightarrow c^2 - a^2 = 11(b - d)$
$\Rightarrow c^2 - a^2 = 11(b - d) = 11 \times 11(c - a)$
$\Rightarrow c + a = 11 \times 11 = 121$
Hence, the value of $(a + b + c + d - 1208)$
$= 121 \times 10 - 1208$
$= 1210 - 1208$
$= 2.$

4. We have $\alpha + \beta = -p$, $\alpha\beta = -q$
and $\gamma + \delta = -p$, $\gamma\delta = r$
Now
$(\alpha - \gamma)(\alpha - \delta) = \alpha^2 - (\gamma + \delta)\alpha + \gamma\delta$
$= \alpha^2 + p\alpha + r$
$= q + r$
Also,
$(\beta - \gamma)(\beta - \delta) = \beta^2 - (\gamma + \delta)\beta + \gamma\delta$
$= \beta^2 + p\beta + r$
$= (q + r)$
Hence, the value of
$$\frac{(\alpha-\gamma)(\alpha-\delta)}{(\beta-\gamma)(\beta-\delta)} = \frac{(q+r)}{(q+r)} = 1$$

5. Let $f(x) = x^2 - 2ax + a^2 - 1$
Case I: $D \geq 0$
$\Rightarrow 4a^2 - 4(a^2 - 1) \geq 0$
$\Rightarrow 4 > 0$
$\Rightarrow a \in R$
Case II: $af(-3) > 0$
$\Rightarrow 1.f(-3) > 0$
$\Rightarrow f(-3) > 0$
$\Rightarrow 9 + 6a + a^2 - 1 > 0$
$\Rightarrow a^2 + 6a + 8 > 0$
$\Rightarrow (a + 2)(a + 4) > 0$
$\Rightarrow a < -4$ and $a > -2$
Case III: $af(4) > 0$
$\Rightarrow f(4) > 0$
$\Rightarrow 16 - 8a + a^2 - 1 > 0$
$\Rightarrow a^2 - 8a + 15 > 0$
$\Rightarrow (a - 3)(a - 5) > 0$
$\Rightarrow a < 3$ and $a > 5$
Case IV: $-3 < \dfrac{\alpha+\beta}{2} < 4$
$\Rightarrow -3 < \dfrac{2a}{2} < 4$
$\Rightarrow -3 < a < 4$
Hence, the value of $[a]$ cannot be 4.

6. Given $\log_3 2$, $\log_3(2^x - 5)$, $\log_3\left(2^x - \frac{7}{2}\right)$ are in AP
$$\Rightarrow 2\log_3(2^x - 5) = \log_3 2 + \log_3\left(2^x - \frac{7}{2}\right)$$
$$\Rightarrow \log_3(2^x - 5)^2 = \log_3\left\{2\cdot\left(2^x - \frac{7}{2}\right)\right\}$$
$$\Rightarrow (2^x - 5)^2 = \left\{2\cdot\left(2^x - \frac{7}{2}\right)\right\}$$
$$\Rightarrow (a - 5)^2 = \left\{2\cdot\left(a - \frac{7}{2}\right)\right\}, a = 2^x$$
$\Rightarrow a^2 - 10a + 25 = 2a - 7$
$\Rightarrow a^2 - 12a + 32 = 0$
$\Rightarrow (a - 8)(a - 4) = 0$
$\therefore a = 4, 8$
When $a = 4$
$\Rightarrow 2^x = 4 = 2^2$
$\Rightarrow x = 2$
When $a = 8$
$\Rightarrow 2^x = 2^3$
$\Rightarrow x = 3$
But $x = 2$ does not satisfy the given equation.
Hence, the value of x is 3.

7. We have
$n^{-x}|m - |x|| = 1$
$\Rightarrow ||x| - m| = n^{|x|}$
Put $m = 2$ and $n = 3$
Thus, $||x| - 2| = 3^x$
So, the number of solutions is 2.
Thus, $p = 2$
2nd part: Given equation is $x^2 + ax + a + 1 = 0$
Now, $D = a^2 - 4(a + 1$
$= a^2 - 4a - 4 = \lambda$ (say)
$\Rightarrow (a - 2)^2 - 8 = \lambda^2$
$\Rightarrow (a - 2)^2 - \lambda^2 = 8$
$\Rightarrow (a - 2 + \lambda)(a - 2 - \lambda) = 8$
Case I: $(a - 2 + \lambda) = 4$, $(a - 2 - \lambda) = 2$
$\Rightarrow a = 5, \lambda = 1$
Case II: $(a - 2 + \lambda) = 2$, $(a - 2 - \lambda) = 4$
$\Rightarrow a = 5, \lambda = -1$
Case III: $(a - 2 + \lambda) = -4$, $(a - 2 - \lambda) = -2$
$\Rightarrow a = -1, \lambda = -1$
Case IV: $(a - 2 + \lambda) = -2$, $(a - 2 - \lambda) = -4$
$\Rightarrow a = -1, \lambda = 1$
Thus, $a = -1, 5$
Therefore the number of integral values of a is 2.
So, $q = 2$
Hence the value of $p + q + 2$ is 6.

8. Given equation is $x^2 + 3x + 5 = 0$
Now, $D = 9 - 20 = -11 < 0$
It has imaginary root common.
Thus, $\dfrac{a}{1} = \dfrac{b}{3} = \dfrac{c}{5} = \lambda$ (say)

Now, $a + b + c = 9\lambda$

Thus, the minimum value of $(a + b + c)$ is 9.

9. Clearly $m = 7$ and $n = 6$

Thus, $(m - n + 1)$ is 2.

10. Given equation is

$x^2 - (x + 1)p - q = 0$

$\Rightarrow \quad x^2 - px - (p + q) = 0$

$\alpha + \beta = p,\ \alpha\beta = -(p + q)$

Now, $1 + (\alpha + \beta) + \alpha\beta = 1 + p - p - q = 1 - q$

$\Rightarrow \quad (1 + \alpha)(1 + \beta) = 1 - q$

$$\text{Now, } \frac{\alpha^2 + 2\alpha + 1}{\alpha^2 + 2\alpha + q} + \frac{\beta^2 + 2\beta + 1}{\beta^2 + 2\beta + q}$$

$$= \frac{(\alpha + 1)^2}{(\alpha + 1)^2 - (1 - q)} + \frac{(\beta + 1)^2}{(\beta + 1)^2 - (1 - q)}$$

$$= \frac{(\alpha + 1)^2}{(\alpha + 1)^2 - (\alpha + 1)(\beta + 1)} + \frac{(\beta + 1)^2}{(\beta + 1)^2 - (\alpha + 1)(\beta + 1)}$$

$$= \frac{(\alpha + 1)}{(\alpha + 1) - (\beta + 1)} + \frac{(\beta + 1)}{(\beta + 1) - (\alpha + 1)}$$

$$= \frac{(\alpha + 1)}{(\alpha - \beta)} - \frac{(\beta + 1)}{(\alpha - \beta)}$$

$$= \frac{(\alpha - \beta)}{(\alpha - \beta)}$$

$= 1.$

11. We have

$6x^6 - 25x^5 + 31x^4 - 31x^2 + 25x - 6 = 0$

$\Rightarrow \quad 6(x^6 - 1) - 25x(x^4 - 1) + 31x^2(x^2 - 1) = 0$

$\Rightarrow \quad (x^2 - 1)\{6(x^4 + x^2 + 1) - 25x(x^2 + 1) + 31x^2\} = 0$

$\Rightarrow \quad (x^2 - 1) = 0$

and $\quad \{6(x^4 + x^2 + 1) - 25x(x^2 + 1) + 31x^2\} = 0$

$$\Rightarrow \quad \left\{6\left(x^2 + \frac{1}{x^2} + 1\right) - 25\left(x + \frac{1}{x}\right) + 31\right\} = 0$$

$$\Rightarrow \quad 6\left(x^2 + \frac{1}{x^2}\right) - 25\left(x + \frac{1}{x}\right) + 37 = 0$$

$$\Rightarrow \quad 6(a^2 - 2) - 25a + 37 = 0,\ a = x + \frac{1}{x}$$

$\Rightarrow \quad 6a^2 - 25a + 25 = 0$

$\Rightarrow \quad (3a - 5)(2a - 5) = 0$

$$\Rightarrow \quad a = \frac{5}{3}, \frac{5}{2}$$

$$\Rightarrow \quad \left(x + \frac{1}{x}\right) = \frac{5}{3}, \frac{5}{2}$$

$\Rightarrow \quad 3x^2 - 5x + 3 = 0,\ 2x^2 - 5x + 2 = 0$

$$\Rightarrow \quad x = \frac{5 \pm i\sqrt{11}}{6}, 2, \frac{1}{2}$$

Hence the solutions are $\left\{\pm 1, \dfrac{5 \pm i\sqrt{11}}{6}, 2, \dfrac{1}{2}\right\}$.

Thus, the number of solutions is 6.

12. Given equation is

$x^4 - 2ax^2 + x + a^2 - a = 0$

$\Rightarrow \quad a^2 - (2x^2 + 1)a + (x^4 + x) = 0$

$$\Rightarrow \quad a = \frac{(2x^2 + 1) \pm \sqrt{(2x^2 + 1)^2 - 4(x^4 + x)}}{2}$$

$$\Rightarrow \quad 2a = (2x^2 + 1) \pm \sqrt{4x^2 - 4x + 1}$$

$\Rightarrow \quad 2a = (2x^2 + 1) \pm (2x - 1)$

$\Rightarrow \quad a = x^2 + x,\ a = x^2 - x + 1$

When $a = x^2 + x$,

$$x = \frac{-1 \pm \sqrt{1 + 4a}}{2}$$

When $a = x^2 - x + 1$

$$x = \frac{1 \pm \sqrt{4a - 3}}{2}$$

For x to be real, $a \ge -\dfrac{1}{4}$ and $a \ge \dfrac{3}{4}$

$$\Rightarrow \quad a \ge \frac{3}{4}$$

$$\Rightarrow \quad a \in \left[\frac{3}{4}, \infty\right)$$

Thus, $p = 3$ and $q = 4$

Hence the value of $(p + q + 1)$ is 8.

13. Given equation is $x^3 - 3x - 1 = 0$

$$\text{Put} \quad y = \frac{\alpha + 1}{\alpha - 1}$$

$\Rightarrow \quad y\alpha - y = \alpha + 1$

$\Rightarrow \quad (y - 1)\alpha = y + 1$

$$\Rightarrow \quad \alpha = \frac{y + 1}{y - 1}$$

Since α is a root of $x^3 - 3x - = 0$, so

$$\Rightarrow \quad \left(\frac{y + 1}{y - 1}\right)^3 - 3\left(\frac{y + 1}{y - 1}\right) - 1 = 0$$

$\Rightarrow \quad (y + 1)^3 - 3(y + 1)(y - 1)^2 - (y - 1)^3 = 0$

$\Rightarrow \quad 3y^3 - 9y^2 - 3y + 1 = 0$

Thus,

$$\left(\frac{\alpha + 1}{\alpha - 1} + \frac{\beta + 1}{\beta - 1} + \frac{\gamma + 1}{\gamma - 1}\right) = \frac{9}{3} = 3$$

14. Clearly $m = 1$ at $x = \sqrt{3}$ and $n = 1$

Thus, $(m + n + 2) = 4$.

15. Given $\alpha + \beta = 3,\ \alpha\beta = A$

and $\quad \alpha + \delta = 12,\ \gamma\delta = B$

Also, it is given that, $\alpha, \beta, \gamma, \delta$ are in GP

$$\frac{\alpha}{\beta} = \frac{\beta}{\gamma} = \frac{\gamma}{\delta} = k \text{ (say)}$$

$\Rightarrow \quad \gamma = \delta k,\ \beta = \delta k^2,\ \alpha = \delta k^3$

$$\text{Now, } \frac{\alpha + \beta}{\gamma + \delta} = \frac{3}{12} = \frac{1}{4}$$

$\Rightarrow \quad \dfrac{\delta k^3+\delta k^2}{\delta k+\delta}=\dfrac{1}{4}$

$\Rightarrow \quad k^2=\dfrac{1}{4}$

$\Rightarrow \quad k=\dfrac{1}{2}$

Again $\gamma+\delta=12$

$\Rightarrow \quad \alpha k+\delta=12$

$\Rightarrow \quad \delta(k+1)=12$

$\Rightarrow \quad \delta\left(\dfrac{3}{2}\right)=12$

$\Rightarrow \quad \delta=8$

Thus, $\gamma=8\times\dfrac{1}{2}=4, \beta=8\times\dfrac{1}{4}=2, \alpha=8\times\dfrac{1}{8}=1$

Thus, $A=\alpha\beta=2$ and $B=\gamma\delta=32$

Hence, the value of

$$\left(\frac{B}{A^2+\text{A}+2}+1\right)=\left(\frac{32}{4+2+2}+1\right)=4+1=5$$

Previous Years' JEE-Advanced Examinations

1. Now, $(5\sqrt{2}-\sqrt{38+5\sqrt{3}})^2$

$= 50-10\sqrt{76+10\sqrt{3}}+(38+5\sqrt{3})$

$= 88+5\sqrt{3}-10\sqrt{76+10\sqrt{3}}$

$= 88+5\sqrt{3}-10\sqrt{76+2\sqrt{75}}$

$= 88+5\sqrt{3}-10\sqrt{(\sqrt{75}+1)^2}$

$= 88+5\sqrt{3}-10(\sqrt{75}+1)$

$= 88+\sqrt{75}-10(\sqrt{75}+1)$

$= 78-9\sqrt{75}$

$= 3(26-3\sqrt{75})$

$= 3(26-15\sqrt{3})$

Thus, the square of $\dfrac{\sqrt{26-15\sqrt{3}}}{5\sqrt{2}-\sqrt{38+5\sqrt{3}}}$

$= \dfrac{1}{3}$ = Rational number.

2. Given $\alpha+\beta=-p, \alpha\beta=1$

and $\gamma+\delta=-q, \gamma\delta=1$

Now, $(\alpha-\gamma)(\beta-\gamma)(\alpha+\delta)(\beta+\delta)$

$= (\gamma-\alpha)(\gamma-\beta)(\delta+\alpha)(\delta+\beta)$

$= (\gamma^2-(\alpha+\beta)\gamma+\alpha\beta)(\delta^2+(\alpha+\beta)\delta+\alpha\beta)$

$= (\gamma^2+p\gamma+1)(\delta^2-p\delta+1)$

$= (\gamma^2+1+p\gamma)(\delta^2+1-p\delta)$

$= (-q\gamma+p\gamma)(-q\delta-p\delta)$

$= (q-p)(q+p)\gamma\delta$

$= (q-p)(q+p).1$

$= (q^2-p^2)$

3. Given $\alpha+\beta=-p, \alpha\beta=q$

$\gamma+\delta=-r, \quad \gamma\delta=s$

Now, $(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)$

$= (\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)$

$= (\alpha^2-(\gamma+\delta)\alpha+\gamma\delta)(\beta^2-(\gamma+\delta)\beta+\gamma\delta)$

$= (\alpha^2+r\alpha+s)(\beta^2+r\beta+s)$

$= (-p\alpha-q+r\alpha+s)(-p\beta-q+r\beta+s)$

$= ((r-p)\alpha+(s-q))((r-p)\beta+(s-q)$

$= (r-p)^2\alpha\beta+(s-q)^2+(r-p)(s-q)(\alpha+\beta)$

$= (r-p)^2q+(s-q)^2-(r-q)p$

4. Given a, b, c are the sides of a triangle.

So, $\dfrac{b^2+c^2-a^2}{2bc}=\cos A\le 1$

$\Rightarrow \quad (b^2+c^2-a^2)\le 2bc$

Similarly, $(a^2+c^2-b^2)\le 2ac$

and $(a^2+b^2+c^2)\le 2ab$

Thus, $(a^2+b^2+c^2)\le 2(ab+bc+ca)$

$\Rightarrow \quad (a^2+b^2+c^2)+2(ab+bc+ca)\le 4(ab+bc+ca)$

$\Rightarrow \quad (a+b+c)^2\le 4(ab+bc+ca)$ …(i)

Again, $a^2+b^2+c^2-ab-bc-ca$

$= \dfrac{1}{2}\left[(a-b)^2+(b-c)^2+(c-a)^2\right]\ge 0$

$(a^2+b^2+c^2)\ge(ab+bc+ca)$

$\Rightarrow \quad (a^2+b^2+c^2)+2(ab+bc+ca)\ge 3(ab+bc+ca)$

$\Rightarrow \quad (a+b+c)^2\ge 3(ab+bc+ca)$ …(ii)

From Relations (i) and (ii), we get,

$3(ab+bc+ca)\le(a+b+c)^2\le 4(ab+bc+ca)$

5. (i) $x=3$; (ii) no integral solution

6. We have

$U=x^2+4y^2+9z^2-6yz-3zx-2xy$

$= (x)^2+(2y)^2+(3z)^2-(2y)(3z)-(x)(3z)-(x)(2y)$

$= \dfrac{1}{2}[(x-2y)^2+(2y-3z)^2+(3z-x)^2]$

≥ 0

Thus, U always non-negative.

7. Now, $D=b^2-4ac$

If $D\ge 0$, then $D-b^2<0$

So, $D<b^2$ and the roots of

$\dfrac{(-b\pm\sqrt{D})}{2a}$ are negative

When $D<0$, the roots are $\dfrac{(-b\pm i\sqrt{D})}{2a}$, which have negative real parts.

8. Solving, we get,

$x=\dfrac{25m}{2m+15}$

and $y=\dfrac{2m-60}{2m+15}$

Since x and y both are positive, so

$$x \in \left(-\infty, -\frac{15}{2}\right) \cup (30, \infty).$$

9. Given equation is $e^{\sin x} - e^{-\sin x} - 4 = 0$

$\Rightarrow \quad e^{\sin x} - \dfrac{1}{e^{\sin x}} - 4 = 0$

$\Rightarrow \quad (e^{\sin x})^2 - 4 \cdot e^{\sin x} - 1 = 0$

$\Rightarrow \quad (e^{\sin x} - 2)^2 = 4 + 1 = 5$

$\Rightarrow \quad (e^{\sin x} - 2)^2 = (\sqrt{5})^2$

$\Rightarrow \quad (e^{\sin x} - 2) = (\pm\sqrt{5})$

$\Rightarrow \quad e^{\sin x} = (2 \pm \sqrt{5})$

$\Rightarrow \quad \sin x = \log_e(2 \pm \sqrt{5})$

No real value of x satisfies the above equation.

10. We have

$$(x - b)(x - c) + (x - c)(x - a) + (x - a)(x - b) = 0$$

When $b = c$

$(x - b)^2 + 2(x - b)(x - a) = 0$

$\Rightarrow \quad (x - b)((x - b) + 2(x - a)) = 0$

$\Rightarrow \quad (x - b)(3x - 2a - b) = 0$

$\Rightarrow \quad x = b, \dfrac{2a + b}{3}$

When $a < b < c$

Let $f(x) = (x - b)(x - c) + (x - c)(x - a) + (x - a)(x - b)$

Now $f(a) = (a - b)(a - c) > 0$

and $f(b) = (b - c)(b - a) < 0$

Thus, by the intermediate value theorem, $f(x) = 0$ has a real root between a and b.

The other root also real.

12. Ans. (a)

We have $|x|^2 - 3|x| + 2 = 0$

$x^2 - 3x + 2 = 0, x^2 + 3x + 2 = 0$

$(x - 1)(x - 2) = 0, (x + 1)(x + 2) = 0$

$x = 1, 2, -1, -2$

Thus, the number of real roots = 4

15. Given $x^{12} - x^9 + x^4 - x + 1 > 0$

$\Rightarrow \quad x^9(x^3 - 1) + x(x^3 - 1) + 1 > 0$

$\Rightarrow \quad (x^3 - 1)(x^9 + x) + 1 > 0$

$\Rightarrow \quad x(x^3 - 1)(x^8 + 1) + 1 > 0$

$\Rightarrow \quad x(x - 1)(x^2 + x - 1)(x^8 + 1) + 1 > 0$

which is true for all real values of x.

18. Ans. (a)

Let $f(x) = ax^3 + bx^2 + cx$

Then $f(0) = 0, f(1) = a + b + c = 0$

By the Rolles theorem, between any two roots of a polynomial there is at least one root of its derivative. So, $f(x)$ has at least one root in (0, 1).

19. Let the roots are α, α^n.

Thus, $\alpha + \alpha^n = -\dfrac{\mathrm{b}}{\mathrm{a}}, \alpha \cdot \alpha^n = \dfrac{c}{a}$

$\Rightarrow \quad \alpha^{n+1} = \dfrac{c}{a}$

$\Rightarrow \quad \alpha = \left(\dfrac{c}{a}\right)^{\frac{1}{n+1}}$

Now, $\alpha + \alpha^n = -\dfrac{b}{a}$

$\Rightarrow \quad \left(\dfrac{c}{a}\right)^{\frac{1}{n+1}} + \left(\dfrac{c}{a}\right)^{\frac{n}{n+1}} = -\dfrac{b}{a}$

$\Rightarrow \quad (ac^n)^{\frac{1}{n+1}} + (a^n c)^{\frac{1}{n+1}} + b = 0$

20. Given inequations are

$x^2 - 3x + 2 \geq 0$ and $x^2 - 3x - 4 \leq 0$

$\Rightarrow \quad (x - 1)(x - 2) \geq 0$ and $(x - 4)(x + 1) \leq 0$

$\Rightarrow \quad x \leq 1, x \geq 2$ and $-1 \leq x \leq 4$

$\Rightarrow \quad x \in [-1, 1] \cup [2, 4]$

21. Since $(2 + i\sqrt{3})$ is a root of $x^2 + px + q = 0$, so the other root will be its conjugate, i.e. $(2 - i\sqrt{3})$.

Now,

Sum of the roots = 4 and product of the roots

$= (2 + i\sqrt{3})(2 - i\sqrt{3})$

$= 4 + 3 = 7$

Thus, the required equation is $x^2 - 4x + 7 = 0$

Therefore, $p = -4$ and $q = 7$.

22. Given equation is $2x^2 + 3x + 1 = 0$

Now, $D = 9 - 8 = 1 > 0$

If $D > 0$ and a perfect square, the roots are rational.

So, the statement is false.

23. Given equation is

$f(x) = (x - a)(x - c) + 2(x - b)(x - d)$

$f(a) = 2(a - b)(a - d) > 0$

and $f(b) = (b - a)(b - c) < 0$

By the intermediate value theorem, $f(x) = 0$ has a real root between (a, b) and the other root lies between (c, d).

24. No real value of x satisfies the given equation.

25. We have

$(a^2 + b^2 + c^2) + 2(ab + bc + ca)$

$= (a + b + c)^2 \geq 0$

$\Rightarrow \quad 1 + 2(ab + bc + ca) \geq 0$

$\Rightarrow \quad (ab + bc + ca) \geq -\dfrac{1}{2}$ …(i)

Also, $a^2 + b^2 + c^2 - ab - bc - ca$

$= \dfrac{1}{2}[(a - b)^2 + (b - c)^2 + (c - a)^2] \geq 0$

$\Rightarrow \quad a^2 + b^2 + c^2 - ab - bc - ca \geq 0$

$\Rightarrow \quad a^2 + b^2 + c^2 \geq ab + bc + ca$

$\Rightarrow \quad (ab + bc + ca) \leq 1$ …(ii)

From Relations (i) and (ii), we get

$$-\frac{1}{2} \leq (ab + bc + ca) \leq 1$$

26. Let $\quad y = \dfrac{(x - a)(x - b)}{(x - c)}$

$\Rightarrow \quad y = \dfrac{x^2 - (a + b)x + ab}{(x - c)}$

$\Rightarrow \quad x^2 - (a+b)x + ab = y(x-c)$

$\Rightarrow \quad x^2 - (a+b+y)x + ab + cy = 0$

Since x is real, so $D \geq 0$

$\Rightarrow \quad (a+b+y)^2 - 4(ab+cy) \geq 0$

$\Rightarrow \quad y^2 + (a+b)^2 + 2y(a+b) - 4(ab+cy) \geq 0$

$\Rightarrow \quad y^2 + 2(a+b-2c)y + (a^2+b^2-2ab) \geq 0$

$\Rightarrow \quad y^2 + 2(a+b-2c)y + (a-b)^2 \geq 0$

Also, y is real, so $D \geq 0$

$\Rightarrow \quad 4(a+b+c)^2 - (a-b)^2 \geq 0$

$\Rightarrow \quad (a+b-2c)^2 - (a-b)^2 \geq 0$

$\Rightarrow \quad (a+b-2c+a-b)(a+b-2c-a+b) \geq 0$

$\Rightarrow \quad (2a-2c)(2b-2c) \geq 0$

$\Rightarrow \quad (a-b)(b-c) \geq 0$

$\Rightarrow \quad (a-c)(c-b) \leq 0$

$\therefore \quad a \leq c \leq b$

27. Given equation is $x^2 - 2kx + 2e^{2\ln k} - 1 = 0$

$\Rightarrow \quad x^2 - 2kx + 2k^2 - 1 = 0$

Given the product of the roots = 7

$\Rightarrow \quad 2k^2 - 1 = 7$

$\Rightarrow \quad 2k^2 = 8$

$\Rightarrow \quad k^2 = 4$

$\Rightarrow \quad k = \pm 2$

Hence, the values of k are –2, 2.

28. Given equation is

$$(5+2\sqrt{6})^{x^2-3} + (5-2\sqrt{6})^{x^2-3} = 10$$

$\Rightarrow \quad a + \frac{1}{a} = 10$, where $a = (5+2\sqrt{6})$

$\Rightarrow \quad a^2 - 10a + 1 = 0$

$\Rightarrow \quad (a-5)^2 = (2\sqrt{6})^2$

$\Rightarrow \quad (a-5) = \pm 2\sqrt{6}$

$\Rightarrow \quad a = 5 \pm 2\sqrt{6}$

When $a = 5 + 2\sqrt{6}$,

$$(5+2\sqrt{6})^{x^2-3} = (5+2\sqrt{6})$$

$\Rightarrow \quad x^2 - 3 = 1$

$\Rightarrow \quad x^2 = 4$

$\Rightarrow \quad x = \pm 2$

When $a = (5 - 2\sqrt{6})$,

$$(5+2\sqrt{6})^{x^2-3} = (5-2\sqrt{6})$$

$\Rightarrow \quad (5+2\sqrt{6})^{x^2-3} = (5+2\sqrt{6})^{-1}$

$\Rightarrow \quad x^2 - 3 = -1$

$\Rightarrow \quad x^2 = 2$

$\Rightarrow \quad x = \pm\sqrt{2}$

Hence, the solutions set are $\{-2, 2, -\sqrt{2}, \sqrt{2}\}$

29. True

30. Given equation is

$\Rightarrow \quad \log_7\left[\log_5(\sqrt{x+5} + \sqrt{x})\right] = 0$

$\Rightarrow \quad \log_5(\sqrt{x+5} + \sqrt{x}) = 7^0 = 1$

$\Rightarrow \quad \sqrt{x+5} + \sqrt{x} = 5^1 = 5$

$\Rightarrow \quad \sqrt{x+5} = 5 - \sqrt{x}$

$\Rightarrow \quad x + 5 = 25 + x - 10\sqrt{x}$

$\Rightarrow \quad 10\sqrt{x} = 20$

$\Rightarrow \quad \sqrt{x} = 2$

$\therefore \quad x = 4$

Hence the solution is $x = 4$.

31. Given equations are

$x^2 + ax + b = 0$ …(i)

and $x^2 + bx + a = 0$ …(ii)

Subtracting Eqs. (i) and (ii), we get

$(a-b)x = (a-b)$

$\Rightarrow \quad x = 1$

Put $x = 1$, in Eq. (i), we get

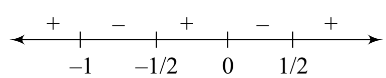
$a + b + 1 = 0$

$\Rightarrow \quad a + b = -1.$

33. Given $S = \left(\frac{2x-1}{2x^3 + 3x^2 + x}\right) > 0$

$\Rightarrow \quad \frac{(2x-1)}{x(2x^2+3x+1)} > 0$

$\Rightarrow \quad \frac{(2x-1)}{x(x+1)(2x+1)} > 0$



Thus, S contains

$$x \in (-\infty, -1) \cup \left(-\frac{1}{2}, 0\right) \cup \left(\frac{1}{2}, \infty\right)$$

35. Given in-equation is

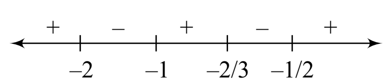
$$\frac{2x}{2x^2+5x+2} > \frac{1}{x+1}$$

$\Rightarrow \quad \frac{2x}{2x^2+5x+2} - \frac{1}{x+1} > 0$

$\Rightarrow \quad \frac{2x^2+2x-2x^2-5x-2}{(2x^2+5x+2)(x+1)} > 0$

$\Rightarrow \quad \frac{-3x-2}{(2x^2+5x+2)(x+1)} > 0$

$\Rightarrow \quad \frac{(3x+2)}{(x+2)(2x+1)(x+1)} < 0$



Thus, $x \in (-2, -1) \cup \left(-\frac{2}{3}, -\frac{1}{2}\right)$

36. Given $\alpha_1+\alpha_2=-\frac{b}{a}, \alpha_1\alpha_2=\frac{c}{a}$

and $\beta_1+\beta_2=-\frac{q}{p}, \beta_1\beta_2=\frac{r}{p}$

The given system of equations has non-trivial solutions if and only if

$$\begin{vmatrix}\alpha_1 & \beta_1\\ \alpha_2 & \beta_2\end{vmatrix}=0$$

$\Rightarrow \quad \alpha_1\beta_2=\alpha_2\beta_1$

$$\Rightarrow \quad \frac{\alpha_1}{\alpha_2}=\frac{\beta_1}{\beta_2}$$

$$\Rightarrow \quad \frac{\alpha_1-\alpha_2}{\alpha_1+\alpha_2}=\frac{\beta_1-\beta_2}{\beta_1+\beta_2}$$

$$\Rightarrow \quad \left(\frac{\alpha_1-\alpha_2}{\alpha_1+\alpha_2}\right)^2=\left(\frac{\beta_1-\beta_2}{\beta_1+\beta_2}\right)^2$$

$$\Rightarrow \quad \left(\frac{(\alpha_1-\alpha_2)^2}{(\alpha_1+\alpha_2)^2}\right)=\left(\frac{(\beta_1-\beta_2)^2}{(\beta_1+\beta_2)^2}\right)$$

$$\Rightarrow \quad \left(\frac{(\alpha_1+\alpha_2)^2-4\alpha_1\alpha_2}{(\alpha_1+\alpha_2)^2}\right)=\left(\frac{(\beta_1+\beta_2)^2-4\beta_1\beta_2}{(\beta_1+\beta_2)^2}\right)$$

$$\Rightarrow \quad \left(1-\frac{4\alpha_1\alpha_2}{(\alpha_1+\alpha_2)^2}\right)=\left(1-\frac{4\beta_1\beta_2}{(\beta_1+\beta_2)^2}\right)$$

$$\Rightarrow \quad \left(\frac{\alpha_1\alpha_2}{(\alpha_1+\alpha_2)^2}\right)=\left(\frac{\beta_1\beta_2}{(\beta_1+\beta_2)^2}\right)$$

$$\Rightarrow \quad \frac{c/a}{(-b/a)^2}=\frac{r/p}{(-q/p)^2}$$

$$\Rightarrow \quad \frac{b^2}{q^2}=\frac{ac}{pr}.$$

37. The given equation is

$$\log_{(2x+3)}(6x^2+23x+21)$$
$$=4-\log_{(3x+7)}(4x^2+12x+9)$$

$$\Rightarrow \quad \log_{(2x+3)}(2x+3)(3x+7)$$
$$=4-\log_{(3x+7)}(2x+3)^2$$

$$\Rightarrow \quad 1+\log_{(2x+3)}(3x+7)=4-2\log_{(3x+7)}(2x+3)$$

$$\Rightarrow \quad \log_{(2x+3)}(3x+7)=3-\frac{2}{\log_{(2x+3)}(3x+7)}$$

$$\Rightarrow \quad y=3-\frac{2}{y}, \text{ where } y=\log_{(2x+3)}(3x+7)$$

$\Rightarrow \quad y^2-3y+2=0$

$\Rightarrow \quad (y-1)(y-2)=0$

$\Rightarrow \quad y=1, 2$

When $y=1$,

$\log_{(2x+3)}(3x+7)=1$

$\Rightarrow \quad (3x+7)=(2x+3)$

$\therefore \quad x=-4$

When $y=2$,

$\log_{(2x+3)}(3x+7)=2$

$\Rightarrow \quad (3x+7)=(2x+3)^2$

$\Rightarrow \quad (3x+7)=4x^2+12x+9$

$\Rightarrow \quad 4x^2+9x+2=0$

$\Rightarrow \quad 4x^2+8x+x+2=0$

$\Rightarrow \quad 4x(x+2)+1(x+2)=0$

$\Rightarrow \quad (x+2)(4x+1)=0$

$$\therefore \quad x=-2, -\frac{1}{4}$$

As $x>-\frac{3}{2}$, so $x=-\frac{1}{4}$

Hence the solution is $x=-\frac{1}{4}$

39. We have

$$x^{(3/4)(\log_2 x)^2+\log_2 x-\frac{5}{4}}=\sqrt{2}$$

$$\Rightarrow \quad \left((3/4)(\log_2 x)^2+\log_2 x-\frac{5}{4}\right)\log x=\log(\sqrt{2})$$

$$\Rightarrow \quad \left((3/4)b^2+b-\frac{5}{4}\right)b=\frac{1}{2}, b=\log_2 x$$

$\Rightarrow \quad 3b^3+4b^2-5b-2=0$

$\Rightarrow \quad 3b^3-3b^2+7b^2-7b+2b-2=0$

$\Rightarrow \quad 3b^2(b-1)+7b(b-1)+2(b-1)=0$

$\Rightarrow \quad (b-1)(3b^2+7b+2)=0$

$\Rightarrow \quad (b-1)(3b^2+6b+b+2)=0$

$\Rightarrow \quad (b-1)(3b(b+2)+1(b+2))=0$

$\Rightarrow \quad (b-1)(b+2)(3b+1)=0$

$$\therefore \quad b=1, -2, -\frac{1}{3}$$

$$\Rightarrow \quad \log_2 x=1, -2, -\frac{1}{3}$$

$$\Rightarrow \quad x=2, 2^{-2}, 2^{-\frac{1}{3}}$$

Thus, the equation has exactly three real solutions out of which exactly one is irrational.

40. Given

$$\frac{x^n}{(1+x^{2n})}\frac{y^m}{(1+y^{2m})}$$
$$=\frac{1}{\left(x^n+\frac{1}{x^n}\right)}\times\frac{1}{\left(y^m+\frac{1}{y^m}\right)}$$
$$\le\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$$

42. Given $\alpha + \beta = -p$, $\alpha\beta = q$

$\alpha^4 + \beta^4 = r$, $\alpha^4\alpha^4 = s$

Let the roots of $x^2 - 4qx + 2q^2 - r = 0$ be γ, δ.

Thus, $\gamma, \delta = (2q^2 - r)$

$= 2(\alpha, \beta)^2 - (\alpha^4 + \beta^4)$

$= -((\alpha)^4 + (\beta)^4 - 2\alpha^2\beta^2)$

$= -(\alpha^2 - \beta^2)^2 < 0$

So the roots are real and of opposite sign.

43. Since α is a root of $a^2x^2 + bx + c = 0$

So, $a^2\alpha^2 + b\alpha + c = 0$

Also β is a root of $a^2x^2 - bx - 2c = 0$, so $a^2\beta^2 - b\beta - 2c = 0$

Let $f(x) = a^2x^2 + 2bx + 2c$

Thus, $f(\alpha) = a^2\alpha^2 + 2b\alpha + 2c$

$= a^2\alpha^2 - 2a^2\alpha^2 = -a^2\alpha^2 < 0$

Also, $f(\beta) = a^2\beta^2 + 2b\beta + 2c$

$= a^2\beta^2 - 2a^2\beta^2 = 3a^2\beta^2 < 0$

Therefore, $f(\alpha)f(\beta) < 0$

Thus $f(x)$ has one root which lies in (α, β)

i.e. $\alpha < \gamma < \beta$.

46. Given $(x - a)(x - b) - c = (x - \alpha)(x - \beta)$

$\Rightarrow (x - \alpha)(x - \beta) = (x - a)(x - b) - c$

$\Rightarrow (x - \alpha)(x - \beta) + c = (x - a)(x - b)$

Hence the roots of $(x - \alpha)(x - \beta) + c = 0$ are a and b.

49. Given $p, q, r \in$ AP.

$2q = p + r$

Since roots are real, so

$D \geq 0$

$\Rightarrow q^2 - 4pr \geq 0$

$\Rightarrow \left(\frac{p+r}{2}\right)^2 - 4pr \geq 0$

$\Rightarrow (p + r)^2 - 16pr \geq 0$

$\Rightarrow p^2 + r^2 - 14pr \geq 0.$

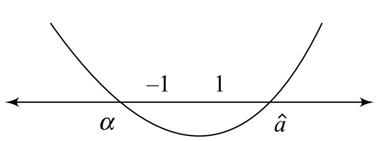
$\Rightarrow \left(\frac{p}{r}\right)^2 - 14\left(\frac{p}{r}\right) + 1 \geq 0$

$\Rightarrow \left(\frac{p}{r} - 7\right)^2 \geq 49 - 1 = 48$

$\Rightarrow \left|\left(\frac{p}{r} - 7\right)\right| \geq \sqrt{48}$

$\Rightarrow \left|\left(\frac{p}{r} - 7\right)\right| \geq 4\sqrt{3}$

51. We have



$af(-1) < 0$ and $af(1) < 0$

$\Rightarrow a(a - b + c) < 0$ and $a(a + b + c) < 0$

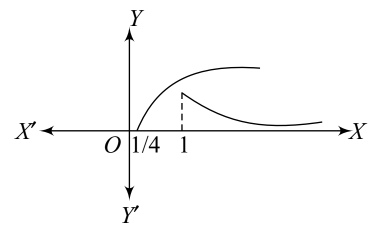
$\Rightarrow a^2\left(1 - \frac{b}{a} + \frac{c}{a}\right) < 0$ and $a^2\left(1 + \frac{b}{a} + \frac{c}{a}\right) < 0$

$\Rightarrow \left(1 - \frac{b}{a} + \frac{c}{a}\right) < 0$ and $\left(1 + \frac{b}{a} + \frac{c}{a}\right) < 0$

$\Rightarrow \left(1 + \left|\frac{b}{a}\right| + \frac{c}{a}\right) < 0$

Hence, the result.

53. Clearly, it has no solution.



54. We have $2^{|y|} - |2^{y-1} - 1| = 2^{y-1} + 1$

$\Rightarrow 2^{|y|} - |2^{y-1} + 1| + |2^{y-1} - 1|$

$\Rightarrow |2^{y-1} + 1| = |2^{y-1} - 1| = 2^{|y|}$

$\Rightarrow (2^{y-1} + 1)(2^{y-1} - 1) \geq 0$

$\Rightarrow (2^{2(y-1)} - 1^2) \geq 0$

$\Rightarrow (2^{2(y-1)}) \geq 2^0$

$\Rightarrow y - 1 \geq 0$

$\Rightarrow y \geq 1$

When $y < 0$, the given equation reduces to

$2^{-y} - (1 - 2^{y-1}) = 2^{y-1} + 1$

$\Rightarrow 2^{-y} - 1 + 2^{y-1} = 2^{y-1} + 1$

$\Rightarrow 2^{-y} = 2$

$\Rightarrow -y = 1$

$\Rightarrow y = -1$

Hence the solution set is $x \in [1, \infty) \cup \{-1\}$.

55. Given equation is

$|x - 2|^2 + |x - 2| - 2 = 0$

$\Rightarrow a^2 + a - 2 = 0, a = |x - 2|$

$\Rightarrow (a + 2)(a - 1) = 0$

$\Rightarrow (a + 2) = 0, (a - 1) = 0$

$\Rightarrow a = 1, a = -2$

$\Rightarrow |x - 2| = 1$ and $|x - 2| = -2$

Thus, $|x - 2| = 1$

$\Rightarrow x - 2 = \pm 1$

$\Rightarrow x = 2 \pm 1 = 3, 1$

57. On solving we get,

$u = -\frac{1}{3}, v = \frac{2}{3}, w = \frac{5}{3}$

It is given that a, b, c, d are in G.P

So, let $b = ar$, $c = ar^2$, $d = \text{ar}^3$

We have $\lfloor(b - c)^2 + (c - a)^2 + (d - b)^2\rfloor$

$= (ar - ar^2)^2 + (ar^2 - a)^2 + (ar^3 - ar)^2$

$= (a^2r^2 - 2a^2r^3 + a^2r^4) + (a^2r^4 - 2a^2r^2 + a^2)$

$+ (a^2r^6 - 2a^2r^4 + a^2r^2)$

$= a^2(r^2 - 2r^3 + r^4 + r^4 - r^2 + 1 + r^6 - 2r^4 + r^2)$

$= a^2(r^6 - 2r^3 + 1)$

$= a^2(r^3 - 1)^2$
$= (ar^3 - a)^2$
$= (d - a)^2 = (a - d)^2$

Thus, the given quadratic equation reduces to

$$-\frac{9}{10}x^2 + (a-d)^2 x + 2 = 0$$

$9x^2 - 10(a - d)^2 x - 20 = 0$

Replace x by $1/x$, we get,

$$9\left(\frac{1}{x}\right)^2 - 10(a-d)^2\left(\frac{1}{x}\right) - 20 = 0$$

$9 - 10(a - d)^2 x - 20x^2 = 0$

$20x^2 + 10(a - d)^2 x - 9 = 0.$

Hence, the result

58. Let $f(x) = x^2 - 2x + a^2 + a - 3$

(i) $D \geq 0$

$\Rightarrow \quad 4a^2 - 4(a^2 + a - 3) \geq 0$
$\Rightarrow \quad a^2 - (a^2 + a - 3) \geq 0$
$\Rightarrow \quad -(a - 3) \geq 0$
$\Rightarrow \quad (a - 3) \leq 0$
$\Rightarrow \quad a \leq 3$

(ii) $af(3) > 0$

$\Rightarrow \quad 1.f(3) > 0$
$\Rightarrow \quad f(3) > 0$
$\Rightarrow \quad 9 - 6a + a^2 + a - 3 > 0$
$\Rightarrow \quad a^2 - 5a + 6 > 0$
$\Rightarrow \quad (a - 2)(a - 3) > 0$
$\Rightarrow \quad a < 2, a > 3$

(iii) $a + b < 2.3 = 6$

$\Rightarrow \quad 2a < 6$
$\Rightarrow \quad a < 3.$

Hence, the solution set is $a < 2$

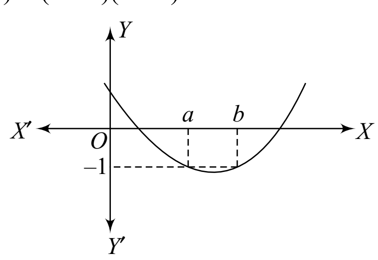
59. Given $\alpha + \beta = -b$, $\alpha\beta = c$

Since $b > 0$ and $c < 0$, so $\alpha < 0 < \beta$

Also, $\beta = -b - \alpha < -\alpha = |\alpha|$

Therefore, $\alpha < 0 < \beta < |\alpha|$

60. Let $f(x) = (x - a)(x - b) - 1$



Now, $f(a) = -1 < 0, f(b) = -1 < 0$

Thus $f(x)$ has one root in $(-\infty, a)$ and the other root in (b, ∞)

61. Given equation is $3x^2 + px + 3 = 0, p > 3$

Let the roots be α, α^2.

Thus $\alpha, \alpha^2 = 1 \Rightarrow \alpha^3 = 1$

$\Rightarrow \quad \alpha = 1, \omega, \omega^2$

When $\alpha = \omega$,

$$\omega + \omega^2 = -\frac{p}{3}$$

$$\Rightarrow \quad -1 = -\frac{p}{3}$$

$\Rightarrow \quad p = 3$

62. Given α, β are the roots of $ax^2 + bx + c = 0$.

Thus, $\alpha + \beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}$

Also, $\alpha + \delta, \beta + \delta$ are the roots of $Ax^2 + Bx + C = 0$.

Thus, $\alpha + \delta + \beta + \delta = -\frac{B}{A}$

and $\quad (\alpha + \delta)\cdot(\beta + \delta) = -\frac{C}{A}$

Now, $\alpha - \beta = (\alpha + \delta) - (\beta + \delta)$

$\Rightarrow \quad (\alpha - \beta)^2 = ((\alpha + \delta) - (\beta + \delta))^2$

$\Rightarrow \quad (\alpha+\beta)^2 - 4\alpha\beta = ((\alpha+\delta)+(\beta+\delta))^2 - 4(\alpha+\delta)(\beta+\delta)$

$$\Rightarrow \quad \frac{b^2}{a^2} - 4\frac{c}{a} = \frac{B^2}{A^2} - 4\frac{C}{A}$$

$$\Rightarrow \quad \frac{b^2 - 4ac}{a^2} = \frac{B^2 - 4AC}{A^2}$$

63. Given $\alpha + \beta = -\frac{b}{a}, \alpha\beta = \frac{c}{a}$

It is also given that $a^2x^2 + abc\,x + c^2 = 0$

$$\Rightarrow \quad x^2 + \frac{abc}{a^3}x + \frac{c^3}{a^3} = 0$$

$$\Rightarrow \quad x^2 + \frac{bc}{a^2}x + \frac{c^3}{a^3} = 0$$

$$\Rightarrow \quad x^2 + \left(\frac{b}{a}\right)\left(\frac{c}{a}\right)x + \left(\frac{c}{a}\right)^3 = 0$$

$\Rightarrow \quad x^2 - (\alpha + \beta)\alpha\beta x + (\alpha\beta)^3 = 0$

$\Rightarrow \quad (x - \alpha^2\beta)(x - \alpha\beta^2) = 0$

Hence, the roots of the given equations are

$\alpha^2\beta, \alpha\beta^2$

64. Here, $\alpha + \beta = 1$, $\alpha\beta = p$

and $\quad \gamma + \delta = 4$, $\gamma\delta = q$

Also, $\alpha, \beta, \gamma, \delta$ are in G.P .

Thus, $\beta = \alpha, r, \gamma = \alpha r^2, \delta = \alpha r^3$

Now, $\dfrac{\gamma + \delta}{\alpha + \beta} = \dfrac{4}{1}$

$$\Rightarrow \quad \frac{\alpha r^2 + \alpha r^3}{\alpha + \alpha r} = 4$$

$$\Rightarrow \quad \frac{\alpha r^2(1 + r)}{\alpha(1 + r)} = 4$$

$\Rightarrow \quad r^2 = 4$

$\Rightarrow \quad r = \pm 2$

When $r = 2$, $\alpha + \beta = 1$

$\Rightarrow \quad \alpha + 2\alpha = 1$

$\Rightarrow \quad \alpha = \frac{1}{3}$

It is not an integer, so $r = -2$

In this case, $\alpha + \beta = 1$

$\Rightarrow \quad \alpha - 2\alpha = 1$

$\Rightarrow \quad \alpha = -1$

Thus, $\beta = \alpha r = 2$, $\gamma = \alpha r^2 = -4$, $\delta = \alpha r^3 = 8$

Therefore, $p = \alpha\beta = -1.2 = -2$

and $q = \gamma\delta = -4.8 = -32$

65. Given $f(x) = (1 + b^2)x^2 + 2bx + 1$

$\Rightarrow \quad f'(x) = 2(1 + b^2)x + 2b$

For extrem a, $f'(x) = 0$ gives $2(1 + b^2)x + 2b = 0$

$\Rightarrow \quad x = -\frac{b}{1+b^2}$

Thus, the minimum value of $f(x)$ is $= m(b)$

$$= (1+b^2) \times \frac{b^2}{(1+b^2)^2} - \frac{2b^2}{1+b^2} + 1$$

$$= \frac{b^2}{(1+b^2)} - \frac{2b^2}{(1+b^2)} + 1$$

$$= 1 - \frac{b^2}{(1+b^2)}$$

$$= \frac{1}{(1+b^2)}.$$

Hence the range of $m(b)$ is (0, 1]

66. The given equation is

$$\log_4(x - 1) = \log_2(x - 3)$$

$\Rightarrow \quad \frac{1}{2}\log_2(x-1) = \log_2(x-3)$

$\Rightarrow \quad \log_2(x - 1) = \log_2(x - 3)^2$

$\Rightarrow \quad (x - 3)^2 = (x - 1)$

$\Rightarrow \quad x^2 - 6x + 9 = x - 1$

$\Rightarrow \quad x^2 - 7x + 10 = 0$

$\Rightarrow \quad (x - 2)(x - 5) = 0$

$\Rightarrow \quad x = 2, 5$

$\Rightarrow \quad x = 5$, since $x = 2$ does not satisfy the equation.

Thus, the number of solution is one.

67. Given system of equations are

$$(k + 1)x + 8y = 4k$$

and $\quad kx + (k + 3)y = 3k - 1$

It will provide us infinitely many solutions if

$$\frac{k+1}{k} = \frac{8}{k+3} = \frac{4k}{3k-1}$$

we have $\frac{k+1}{k} = \frac{8}{k+3}$

$\Rightarrow \quad (k + 1)(k + 3) = 8k$

$\Rightarrow \quad k^2 + 4k + 3 = 8k$

$\Rightarrow \quad k^2 - 4k + 3 = 0$

$\Rightarrow \quad (k - 1)(k - 3) = 0$

$\Rightarrow \quad k = 1, 3$

68. Given $x^2 - |x + 2| + x > 0$

$\Rightarrow \quad x^2 + x > |x + 2|$

Case I: When $x \geq -2$

$\Rightarrow \quad x^2 + x > x + 2$

$\Rightarrow \quad x^2 - 2 > 0$

$\Rightarrow \quad (x+\sqrt{2})(x-\sqrt{2}) > 0$

$\Rightarrow \quad x < -\sqrt{2}, x > \sqrt{2}$

Case II: When $x < -2$

$\Rightarrow \quad x^2 + x > -x - 2$

$\Rightarrow \quad x^2 + 2x + 2 > 0$

$\Rightarrow \quad (x + 1)^2 + 1 > 0$

Thus, it is true for all values of x.

Hence the solution set is

$$x \in (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$$

69. Since roots are real and unequal, so

$D > 0$

$\Rightarrow \quad (a - b)^2 - 4(1 - a - b) > 0$

$\Rightarrow \quad (2 - a)^2 - (a^2 + 4a - 4) < 0$

$\Rightarrow \quad a^2 - 4a + 4 - (a^2 + 4a - 4) < 0$

Clearly, $D < 0$

$\Rightarrow \quad (4 - 2a)^2 - 4(a^2 + 4a - 4) < 0$

$\Rightarrow \quad (2 - a)^2 - (a^2 + 4a - 4) < 0$

$\Rightarrow \quad a^2 - 4a + 4 - (a^2 + 4a - 4) < 0$

$\Rightarrow \quad -4a + 4 - 4a + 4 < 0$

$\Rightarrow \quad -8a + 8 < 0$

$\Rightarrow \quad a - 1 > 0$

$\Rightarrow \quad a > 1$

70. Given $x^2 + 2ax + 10 - 3a > 0$

So $\quad D < 0$

$\Rightarrow \quad 4a^2 - 4(10 - 3a) < 0$

$\Rightarrow \quad a^2 - (10 - 3a) < 0$

$\Rightarrow \quad a^2 + 3a - 10 < 0$

$\Rightarrow \quad (a + 5)(a - 2) < 0$

$\Rightarrow \quad -5 < a < 2$

71. Let the roots are α, α^2.

Given $\alpha + \alpha^2 = -p$, $\alpha, \alpha^2 = q$

Now, $\alpha + \alpha^2 = -p$

$\Rightarrow \quad (\alpha + \alpha^2)^3 = (-p)^3 = -p^3$

$\Rightarrow \quad \alpha^3 + \alpha^6 + 3\alpha^3(\alpha + \alpha^2) = -p^3$

$\Rightarrow \quad q + q^2 - 3pq = -p^3$

$\Rightarrow \quad p^3 - q(3p - 1) + q^2 = 0$

72. Let $f(x) = bx^2 + ax$

Clearly, $a + b = 1$.

Thus, $f(x) = (1 - a)x^2 + ax$

Also, $f'(x) = 2(1 - a)x + a > 0$

$\Rightarrow \quad 2(1 - a)x + a > 0$

When $x = 0$, then $a > 0$

When $x = 1$, then $2 - 2a + a > 0$

$\Rightarrow \quad 2 - a > 0$

$\Rightarrow \quad a < 2$

Thus, $a \in (0, 2)$

73. Ans. (c)

74. Ans. (a)

Since roots are real, so

$D > 0$

$\Rightarrow \quad 4(a+b+c)^2 - 12\lambda(ab+bc+ca) > 0$

$\Rightarrow \quad (a+b+c)^2 - 3\lambda(ab+bc+ca) > 0$

$\Rightarrow \quad (a+b+c)^2 + 3\lambda(ab+bc+ca)$

$\Rightarrow \quad (a^2+b^2+c^2) > (3\lambda - 2)(ab+bc+ca)$

$\Rightarrow \quad (3\lambda - 2) < \dfrac{(a^2+b^2+c^2)}{(ab+bc+ca)}$

$\Rightarrow \quad (3\lambda - 2) < 2$

$\Rightarrow \quad \lambda < \dfrac{4}{3}$ $\quad\left[\text{since } \cos A = \dfrac{b^2+c^2-a^2}{2bc} \le 1\right.$

$\Rightarrow \quad b^2 + c^2 - a^2 \le 2bc$

Similarly, $c^2 + a^2 - b^2 \le 2ac$

and $a^2 + b^2 - c^2 \le 2ab$

Thus, $a^2 + b^2 - c^2 \le 2(ab + bc + ca)$

$\Rightarrow \quad \left.\dfrac{(a^2+b^2+c^2)}{(ab+bc+ca)} \le 2\right]$

75. Given a, b are the roots of $x^2 - 10cx - 11d = 0$ and c, d are the roots of $x^2 - 10ax - 11b = 0$.

Thus, $a + b = 10c$, $c + d = 10a$

and $ab = -11d$, $cd = -11b$

So, $a + b + c + d = 10(c + a)$ …(i)

and $(a + b) - (c + d) = 10(c - a)$

$\Rightarrow \quad (b - d) = 11(c - a)$ …(ii)

Also, a is a root of $x^2 - 10cx - 11d = 0$ and c is a root of $x^2 - 10ax - 11b = 0$.

Thus, $c^2 - 10ac = 11b$, $a^2 - 10ca = 11d$

$\Rightarrow \quad c^2 - a^2 = 11(b - d)$

$\Rightarrow \quad c^2 - a^2 = 11(b - d) = 11 \times 11(c - d)$

$\Rightarrow \quad c + a = 11 \times 11 = 121$

From Eq. (i), we get,

$a + b + c + d = 121 \times 10 = 1210.$

76. Given α, β be the roots of $x^2 - px + r = 0$

Thus, $\alpha + \beta = p$, $\alpha\beta = r$

Also, $\dfrac{\alpha}{2}$, 2β be the roots of $x^2 - qx + r = 0$

Thus, $\dfrac{\alpha}{2} + 2\beta = q$ and $\dfrac{\alpha}{2} \cdot 2\beta = r$

$\Rightarrow \quad \dfrac{\alpha}{2} + 2\beta = q, \alpha\beta = r.$

Since β is a roots of $x^2 - px + r = 0$ and 2β is a root of $x^2 - qx + r = 0$, so we can write

$p\beta - \beta^2 = 2q\beta - 4\beta^2$

$\Rightarrow \quad 3\beta^2 = (2q - p)\beta$

$\Rightarrow \quad 3\beta = (2q - p)$

$\Rightarrow \quad \beta = \dfrac{(2q-p)}{3}$

Therefore, $r = p\beta - \beta^2$

$$= \frac{p(2q-p)}{3} - \frac{(2q-p)^2}{9}$$

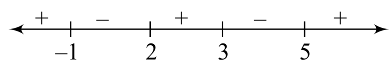
$$= \frac{(2q-p)}{9}(3p - 2q + p)$$

$$= \frac{2(2q-p)(2p-q)}{9}$$

77. (i) → A, C, D ; (ii) → B, D ; (iii) → B , D ; (iv) → A, C, D.

Given $f(x) = \dfrac{x^2 - 6x + 5}{x^2 - 5x + 6}$

$$= \frac{(x-1)(x-5)}{(x-2)(x-3)}$$



Also , Range of $f(x)$ is $(-\infty, 1)$

(i) when $-1 < x < 1$, clearly $0 < f(x) < 1$

(ii) when $1 < x < 2$, clearly $f(x) < 0$

(iii) when $3 < x < 5$, then also $f(x) < 0$

(iv) when $x > 5$, then $f(x) > 0$.

79. Given equation is $x^2 - 8kx + 16(k^2 - k + 1) = 0$

Since the roots are real and distinct, so

$64k^2 - 64(k^2 - k + 1) > 0$

$\Rightarrow \quad k^2 - (k^2 - k + 1) > 0$

$\Rightarrow \quad k - 1 > 0$

$\Rightarrow \quad k > 1$

Thus, $k = 2, 3$

Hence the smallest integral value of k is 2.

80. Given $\alpha + \beta = -p$, $\alpha^3 + \beta^3 = q$

Now, $\alpha^3 + \alpha^3 = q$

$\Rightarrow \quad (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \alpha) = q$

$\Rightarrow \quad (-p)^3 - 3\alpha\beta(-p) = q$

$\Rightarrow \quad -p^3 + 3\alpha\beta . p = q$

$\Rightarrow \quad \alpha\beta = \dfrac{p^3 + q}{3p}$

Now, $\dfrac{\alpha}{\beta} + \dfrac{\beta}{\alpha} = \dfrac{\alpha^2 + \beta^2}{\alpha\beta}$

$$= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{\alpha\beta}$$

$$= \frac{(\alpha+\beta)^2}{\alpha\beta} - 2$$

$$= \frac{p^2}{\left(\dfrac{p^3+q}{3p}\right)} - 2$$

$$= \frac{3p^3}{p^3+q} - 2$$

$$= \frac{p^3 - 2q}{p^3 + q}$$

Hence the required equation is

$\Rightarrow \quad x^2 - \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right)x + \left(\frac{\alpha}{\beta}\cdot\frac{\beta}{\alpha}\right) = 0$

$\Rightarrow \quad x^2 - \left(\frac{p^3 - 2q}{p^3 + q}\right)x + 1 = 0$

$\Rightarrow \quad (p^3 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0$

81. Given equation is $x^2 - 6x - 2 = 0$

$\alpha + \beta = 6, \ \alpha\beta = -2$

We have $(a_{10} - 2a_8)$

$= \alpha^{10} - \beta^{10} - 2(\alpha^8 - \beta^8)$

$= \alpha^{10} - \beta^{10} - 2\alpha^8 + 2\beta^8$

$= \alpha^{10} - 2\alpha^8 - \beta^{10} + 2\beta^8$

$= \alpha^8(\alpha^2 - 2) - \beta^8(\beta^2 - 2)$

$= \alpha^8.6\alpha - \alpha^2.6\beta$

$= 6(\alpha^9 - \beta^9)$

Thus, $\frac{a_{10} - 2a_8}{2a_9} = \frac{6(\alpha^9 - \beta^9)}{2(\alpha^9 - \beta^9)} = 3.$

82. Given equations are

$x^2 + bx - 1 = 0$ …(i)

and $x^2 + x + b = 0$ …(ii)

Subtracting Eq. (ii) from Eq. (i), we get,

$(b - 1)x = (b + 1)$

$\Rightarrow \quad x = \left(\frac{b+1}{b-1}\right)$

Put the value of x in Eq. (i), we get,

$\left(\frac{b+1}{b-1}\right)^2 + b\left(\frac{b+1}{b-1}\right) - 1 = 0$

$\Rightarrow \quad (b + 1)^2 + b(b + 1)(b - 1) - (b - 1)^2 = 0$

$\Rightarrow \quad (b + 1)^2 + b(b^2 - 1) - (b - 1)^2 = 0$

$\Rightarrow \quad (b^2 + 2b + 1 - b^2 + 2b - 1) + b(b^2 - 1) = 0$

$\Rightarrow \quad 4b + b(b^2 - 1) = 0$

$\Rightarrow \quad b(4 + b^2 - 1) = 0$

$\Rightarrow \quad b(b^2 + 3) = 0$

$\Rightarrow \quad b = 0, \pm i\sqrt{3}$

83. Let

$x = \sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}} \ldots \text{to } \infty}}}$

$\Rightarrow \quad x = \sqrt{4 - \frac{1}{3\sqrt{2}}x}$

$\Rightarrow \quad x^2 = \left(4 - \frac{1}{3\sqrt{2}}x\right)$

$\Rightarrow \quad 3\sqrt{2}x^2 + x - 12\sqrt{2} = 0$

$\Rightarrow \quad 3\sqrt{2}x^2 + 9x - 8x - 12\sqrt{2} = 0$

$\Rightarrow \quad 3x(\sqrt{2}x + 3) - 4\sqrt{2}(\sqrt{2}x + 3) = 0$

$\Rightarrow \quad (\sqrt{2}x + 3)(3x - 4\sqrt{2}) = 0$

$\Rightarrow \quad x = -\frac{3}{\sqrt{2}}, \frac{4\sqrt{2}}{3}$

$\Rightarrow \quad x = \frac{4\sqrt{2}}{3}$

Thus,

$6 + \log_{3/2}\left(\frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}} \ldots \text{to } \infty}}}\right)$

$= 6 + \log_{3/2}\left(\frac{1}{3\sqrt{2}} \times \frac{4\sqrt{2}}{3}\right)$

$= 6 + \log_{3/2}\left(\frac{4}{9}\right)$

$= 6 + \log_{\left(\frac{2}{3}\right)^{-1}}\left(\frac{2}{3}\right)^2$

$= 6 - 2\log_{\left(\frac{2}{3}\right)}\left(\frac{2}{3}\right)$

$= 6 - 2 = 4$

85. Given

$(\sqrt[3]{a+1} - 1)x^2 + (\sqrt{1+a} - 1)x + (\sqrt[6]{a+1} - 1) = 0$

Let $1 + a = y$

$\Rightarrow \quad (y^{1/3} - 1)x^2 + (y^{1/2} - 1)x + (y^{1/6} - 1) = 0$

$\Rightarrow \quad \left(\frac{y^{1/3} - 1}{y - 1}\right)x^2 + \left(\frac{y^{1/2} - 1}{y - 1}\right)x + \left(\frac{y^{1/6} - 1}{y - 1}\right) = 0$

Taking limit $y \to 1$, we get

$\Rightarrow \quad \frac{1}{3}x^2 + \frac{1}{2}x + \frac{1}{6} = 0$

$\Rightarrow \quad 2x^2 + 3x + 1 = 0$

$\Rightarrow \quad 2x^2 + 2x + x + 1 = 0$

$\Rightarrow \quad 2x(x + 1) + 1(x + 1) = 0$

$\Rightarrow \quad (2x + 1)(x + 1) = 0$

$\Rightarrow \quad x = -1, -\frac{1}{2}$

86. Let $P(x) = ax^2 + b$ with a, b are of same sign.

Now,

$P(P(x)) = a(ax^2 + b)^2 + b$

$= a(ax^4 + 2abx^2 + b^2) + b$

$= a^2x^4 + 2a^2bx^2 + ab^2 + b$

Clearly, $P(P(x)) \neq 0$

Thus, real and purely imaginary number cannot satisfy the equation $P(P(x)) = 0$

87. Let $f(x) = ax^2 + bx$

Given $\int_0^1 f(x)\,dx = 1$

$$\Rightarrow \quad \int_0^1 (ax^2 + bx)\,dx = 1$$

$$\Rightarrow \quad \left(\frac{ax^3}{3} + \frac{bx^2}{2}\right)_0^1 = 1$$

$$\Rightarrow \quad \left(\frac{a}{3} + \frac{b}{2}\right) = 1$$

$\Rightarrow \quad 2a + 3b = 6$

$\Rightarrow \quad (0, 2), (3, 0)$

Hence the require number of polynomials is 2.

CHAPTER

3

Logarithm

CONCEPT BOOSTER

1. Introduction

The first mention of the natural logarithm was by Nicholas Mercator in his work *Logarithmotechnia* published in 1668, although the teacher of mathematics John Speidell had already, in 1619, compiled a table of what, in fact, were effectively natural logarithms. It was formerly called *hyperbolic logarithm*, as it corresponds to the area under a hyperbola. It is also sometimes referred to as the Napierian logarithm, named after John Napier, although Napier's original 'logarithms' (from which Speidell's numbers were derived) were slightly different (see *Logarithm: from Napier to Euler*).

Notational Conventions

The notations '$\ln x$' and '$\log_e x$' both refer unambiguously to the natural logarithm of x. $\log x$ without an explicit base may also refer to the natural logarithm. This usage is common in mathematics and some scientific contexts as well as in many programming languages.

2. Basic Formulae on Logarithm

F-1

1. If $a_x = n$, then $x = \log_a n$, where $n > 0$, $a > 0$ and $a \neq 1$.
2. Exponential function is always positive.
3. $\log_a a = 1$
4. $\log_a 1 = 0$, $a \neq 1$, $a > 0$
5. $\log_a 0 = \pm\infty$
6. $\log_a \infty = \infty$

F-2

1. $\log_m a + \log_m b = \log_m (ab)$
2. $\log_m a - \log_m b = \log_m \left(\frac{a}{b}\right)$
3. $\log_m a^n = n \log_m a$
4. $a^{\log_a n} = n$

F-3

1. $\log_a b \times \log_b a = 1$

   Note. (i) $\log_a b \times \log_b c \times \log_c d \times \log_d a = 1$

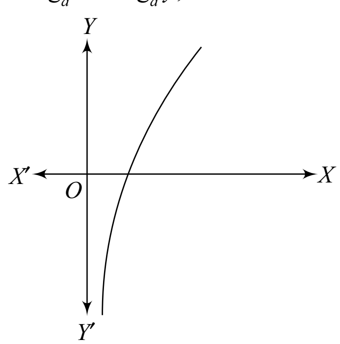
   (ii) $\log_a b \times \log_b c \times \log_c d \times \ldots \times \log_d a = 1$
2. $\log_a b = \dfrac{1}{\log_b a}$
3. $\log_b a = \dfrac{\log_m a}{\log_m b}$
4. $a^{\log_n b} = b^{\log_n a}$

F-4

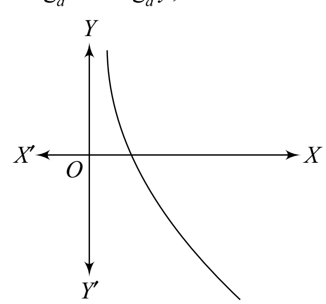
1. $\log_{a^\alpha} b = \dfrac{1}{\alpha} \log_a b$
2. $\log_{a^\alpha} (b^\beta) = \dfrac{\beta}{\alpha} \log_a b$

F-5

1. If $x > y \Rightarrow \log_a x > \log_a y$, when $a > 1$



2. If $x > y \Rightarrow \log_a x > \log_a y$, when $0 < a < 1$



3. Logarithmic Equation

Type 1: A logarithmic equation is of the form

$\log_{g(x)} f(x) = b$

$\Rightarrow \quad f(x) = g(x)^b,\ g(x) > 0,\ g(x) \neq 1$

Type 2: A logarithmic equation is of the form

$\log_{f_1(x)}\{\log_{f_2(x)} f(x)\} = 0$

$\Rightarrow \quad f_2(x) = f(x), \begin{cases} f_1(x) > 0, & f_1(x) \neq 1 \\ f_2(x) > 0, & f_2(x) \neq 1 \end{cases}$

Type 3: A logarithmic equation is of the form

$\log_a f_1(x) = \log_a f_2(x),\ a > 0,\ a \neq 1$

$\Rightarrow \quad f_1(x) = f_2(x), f_1(x) > 0 \text{ or } f_2(x) > 0$

Type 4: A logarithmic equation is of the form

$\log_{f_1(x)} A = \log_{f_2(x)} A$

$\Rightarrow \quad f_1(x) = f_2(x), \begin{cases} f_1(x) > 0, & f_1(x) \neq 1 \\ & \text{or} \\ f_2(x) > 0, & f_2(x) \neq 1 \end{cases}$

Type 5: A logarithmic equation is of the form

$\log_{f(x)} g_1(x) = \log_{f(x)} g_2(x)$

$\Rightarrow \quad g_1(x) = g_2(x), \begin{cases} g_1(x) > 0, & f(x) > 0, \neq 1 \\ & \text{or} \\ g_2(x) > 0, & f(x) > 0, \neq 1 \end{cases}$

Type 6: A logarithmic equation is of the form

$\log_{g_1(x)} f(x) = \log_{g_2(x)} f(x)$

$\Rightarrow \quad g_1(x) = g_2(x), \begin{cases} g_1(x) > 0, & \neq 1, f(x) > 0 \\ & \text{or} \\ g_2(x) > 0, & \neq 1, f(x) > 0 \end{cases}$

Type 7: A logarithmic equation is of the form

$2n \log_a f_1(x) = \log_a f_2(x),\ a > 0,\ a \neq 1,\ n \in N$

$\Rightarrow \quad f_1^{\ 2n+1}(x) = f_2(x), f_1(x) > 0,$

Type 8: A logarithmic equation is of the form

$(2n + 1)\log_a f_1(x) = \log_a f_2(x),\ a > 0,\ a \neq 1,\ n \in N$

$\Rightarrow \quad f_1^{\ (2n+1)}(x) = f_2(x), \quad f_1(x) > 0$

Type 9: A logarithmic equation is of the form

$\log_a f(x) + \log_a g(x) = \log_a m(x),\ a > 0,\ a \neq 1$

$\Rightarrow \quad \begin{cases} f(x) > 0 \\ g(x) > 0 \\ f(x)\, g(x) = m(x) \end{cases}$

Type 10: A logarithmic equation is of the form

$\log_a f(x) - \log_a g(x) = \log_a h(x) - \log_a t(x)$

where $a > 0,\ a \neq 1$

$\Rightarrow \quad \log_a f(x) + \log_a t(x) = \log_a g(x) + \log_a h(x)$

$\Rightarrow \quad \begin{cases} f(x) > 0, t(x) > 0, g(x) > 0, h(x) > 0 \\ f(x) \cdot t(x) = g(x) \cdot h(x) \end{cases}$

4. Logarithmic In-equation

Type I: A logarithmic in-equation is of the form

$\begin{cases} \log_a f(x) > \log_a g(x) \\ a > 1 \end{cases} \Rightarrow \begin{cases} g(x) > 0 \\ a > 1 \\ f(x) > g(x) \end{cases}$

Type II: A logarithmic in-equation is of the form

$\begin{cases} \log_a f(x) > \log_a g(x) \\ 0 < a < 1 \end{cases} \Rightarrow \begin{cases} f(x) > 0 \\ 0 < a < 1 \\ f(x) < g(x) \end{cases}$

Type III: A logarithmic in-equation is of the form

1. $\begin{cases} \log_a x > 0 \\ a > 1 \end{cases} \Rightarrow \begin{cases} x > 0 \\ a > 1 \end{cases}$
2. $\begin{cases} \log_a x > 0 \\ 0 < a < 1 \end{cases} \Rightarrow \begin{cases} 0 < x < 1 \\ 0 < a < 1 \end{cases}$
3. $\begin{cases} \log_a x < 0 \\ a > 1 \end{cases} \Rightarrow \begin{cases} 0 < x < 1 \\ a > 1 \end{cases}$
4. $\begin{cases} \log_a x < 0 \\ 0 < a < 1 \end{cases} \Rightarrow \begin{cases} x > 1 \\ 0 < a < 1 \end{cases}$

Logarithm

Conceptual Problem

Example 1: Find the value of $\log_2 32$.

Solution: We have

$$\log_2 (32) = \log_2 (2^5) = 5 \times \log_2 (2) = 5$$

Example 2: Find the value of $\log_3\left(\frac{1}{243}\right)$.

Solution: We have

$$\log_3\left(\frac{1}{243}\right) = \log_3 (3^{-5}) = -5 \times \log_3 (3) = -5$$

Example 3: Find the value of $\log_{5\sqrt{5}}(5)$.

Solution: We have

$$\log_{5\sqrt{5}}(5) = \log_{5^{3/2}}(5) = \frac{3}{2}\log_5(5) = \frac{3}{2}$$

Example 4: Find the value of $\log_8 64$.

Solution: We have

$$\log_8 64 = \log_{2^3}(4^3)$$

$= \log_2 (4)$
$= \log_2 (2^2)$
$= 2$

Example 5: Find the value of $\log_{100} (0.1)$.

Solution: We have

$$\log_{100}(0.1) = \log_{10^2}\left(\frac{1}{10}\right)$$
$$= \log_{10^2}(10^{-1})$$
$$= -\frac{1}{2}\times \log_{10}(10)$$
$$= -\frac{1}{2}\times 1 = -\frac{1}{2}$$

Example 6: Find the value of $10^{\log_{10} m + \log_{10} n}$.

Solution: We have

$$10^{\log_{10} m+\log_{10} n} = 10^{\log_{10}(mn)} = 10$$

Example 7: Find the value of $\log_2 (64) + \log_4 (256)$.

Solution: We have

$$\log_2 (64) + \log_4 (256) = \log_2 (2^6) + \log_4 (4^4)$$
$$= 6\log_2 (2) + 4\log_4 (4)$$
$$= 6 + 4$$
$$= 10$$

Example 8: Find the value of $\log_2 \log_3 \log_4 (64)$.

Solution: We have

$$\log_2 \log_3 \log_4 (64) = \log_2 \log_3 \log_4 (4^3)$$
$$= \log_2 \log_3 \{3 \log_4 (4)\}$$
$$= \log_2 [(\log_3 3)\{\log_4 (4)\}]$$
$$= \log_2 \{1 \times 1\} = \log_2 (1) = 0$$

Example 9: Find the value of $3^{-\frac{1}{3}\log_3 7}$.

Solution: We have

$$3^{-\frac{1}{3}\log_3 7} = 3^{\log_3 (7)^{-1/3}}$$
$$= (7)^{-1/3}$$

Example 10: Find the value of $2^{\log_{2\sqrt{2}}(125)}$.

Solution: We have

$$2^{\log_{2\sqrt{2}}(125)} = 2^{\log_{2^{3/2}}(125)}$$
$$= 2^{\frac{2}{3}\log_2 (125)}$$
$$= 2^{\log_2 (125)^{2/3}}$$
$$= (125)^{2/3}$$
$$= (5^3)^{2/3}$$
$$= 25$$

Example 11: Find the value of $2^{\log_3 5} - 5^{\log_3 2}$.

Solution: We have

$$2^{\log_3 5} - 5^{\log_3 2} = 5^{\log_3 2} - 5^{\log_3 2} = 0$$

Example 12: Find the value of $3^{\log_5 2} - 2^{\log_5 3}$.

Solution: We have

$$3^{\log_5 2} - 2^{\log_5 3} = 2^{\log_5 3} - 2^{\log_5 3}$$
$$= 0$$

Example 13: Find the value of

$$\log_{10} \tan 1° + \log_{10} \tan 2° + \log_{10} \tan 3° + \ldots + \log_{10} \tan (89°)$$

Solution: We have,

$$\log_{10} \tan 1° + \log_{10} \tan 2° + \log_{10} \tan 3° + \ldots + \log_{10} \tan (89°)$$
$$= \log_{10} (\tan 1° \cdot \tan 2° \ldots \tan 44° \cdot \tan 45° \ldots \tan 46° \ldots \tan 89°)$$
$$= \log_{10} (\tan 45°) = \log_{10} (1) = 0$$

Example 14: Find the value of

$$\log_3 4 \cdot \log_4 5 \cdot \log_5 6 \cdot \log_6 7 \cdot \log_7 8 \cdot \log_8 9.$$

Solution: We have,

$$\log_3 4 \cdot \log_4 5 \cdot \log_5 6 \cdot \log_6 7 \cdot \log_7 8 \cdot \log_8 9$$
$$= \log_3 4 \cdot \log_4 5 \cdot \log_5 6 \cdot \log_6 7 \cdot \log_7 8 \cdot \log_8 9$$
$$= \log_3 9 = \log_3 (3^2) = 2 \log_3 (3) = 2$$

Example 15: Find the value of $\log_3 4 \cdot \log_4 5 \cdot \log_5 10 \cdot \log_{10} 32$.

Solution: We have

$$\log_3 4 \cdot \log_4 5 \cdot \log_5 10 \cdot \log_{10} 32$$
$$= \log_2 (32)$$
$$= \log_2 (2^5)$$
$$= 5$$

Example 16: Find the value of $\dfrac{1}{\log_2(100)} + \dfrac{1}{\log_5(100)}$.

Solution: We have,

$$\frac{1}{\log_2(100)} + \frac{1}{\log_5(100)} = \log_{100} (2) + \log_{100} (5)$$
$$= \log_{100} (2 \times 5)$$
$$= \log_{10^2} (10)$$
$$= \frac{1}{2}\times \log_{10}(10) = \frac{1}{2}$$

Example 17: Find the value of x, if $\log_5 a \cdot \log_a x = 2$.

Solution: We have,

$$\log_5 a \cdot \log_a x = 2$$
$$\Rightarrow \quad \frac{\log a}{\log 5}\times\frac{\log x}{\log a} = 2$$
$$\Rightarrow \quad \frac{\log x}{\log 5} = 2$$
$$\Rightarrow \quad \log_5 x = 2$$
$$\Rightarrow \quad x = 5^2 = 25$$

Example 18: Find the value of x, if $\log_k x \cdot \log_5 k = \log_x 5$, where $k \neq 1, k > 0$.

Solution: We have,

$$\log_k x \cdot \log_5 k = \log_x 5$$
$$\Rightarrow \quad \frac{\log x}{\log k}\times\frac{\log k}{\log 5} = \frac{\log 5}{\log x}$$
$$\Rightarrow \quad (\log x)^2 = (\log 5)^2$$
$$\Rightarrow \quad (\log x)^2 - (\log 5)^2 = 0$$
$$\Rightarrow \quad \{(\log x) + (\log 5)\}\{(\log x) - (\log 5)\} = 0$$
$$\Rightarrow \quad \text{Either } (\log x) + (\log 5)) = 0 \text{ or } (\log x) - (\log 5) = 0$$

$\Rightarrow \quad \log(5x) = 0 \text{ or } \left(\log\left(\frac{x}{5}\right)\right) = 0$

$\Rightarrow \quad \log(5x) = \log 1 \text{ or } \left(\log\left(\frac{x}{5}\right)\right) = \log 1$

$\Rightarrow \quad 5x = 1 \text{ or } \left(\frac{x}{5}\right) = 1$

$\Rightarrow \quad x = \frac{1}{5} \text{ or } 5$

Therefore, the solutions are $\left\{5, \frac{1}{5}\right\}$.

Example 19: Find the value of $A + B + 10$, if

$A = \log_2\log_2\log_4 256$ and $B = 2\log_{\sqrt{2}} 2$.

Solution: Now,

$A = \log_2\log_2\log_4 256$

$= \log_2\log_2\log_4 256$

$= \log_2\log_2\log_4 (4^4)$

$= \log_2\{(\log_2 4)(\log_4 4)\}$

$= \log_2(\log_2 4)$

$= \log_2(\log_2 2^2)$

$= (\log_2 2)(\log_2 2) = 1$

and $\quad B = 2\log_{\sqrt{2}} 2 = 2\log_{2^{1/2}}(2)$

$= 2 \times 2(\log_2 2) = 4$

Therefore, the value of

$A + B + 10 = 1 + 4 + 10 = 15$

Logarithmic Equation

Conceptual Problem

Example 1: Solve for x: $\log_x (3x^2 + 10x) = 3$.

Solution: The given equation is

$\log_x (3x^2 + 10x) = 3$

$\Rightarrow \quad (3x^2 + 10x) = x^3, \quad x > 0, x \neq 1$

$\Rightarrow \quad x^3 - 3x^{10} - 10x = 0$

$\Rightarrow \quad x(x^2 - 3x - 10)$

$\Rightarrow \quad x(x - 5)(x + 2)$

$\Rightarrow \quad x = 0, -2, 5$

$\Rightarrow \quad x = 5$

Hence, the solution of x is 5.

Example 2: Solve for x: $\log_{x+1} (x^2 - 3x + 5) = 2$.

Solution: The given equation is $\log_{x+1} (x^2 - 3x + 5) = 2$

$\Rightarrow \quad (x^2 - 3x + 5) = (x + 1)^2, \quad x + 1 > 0, x \neq 0$

$\Rightarrow \quad (x^2 - 3x + 5) = (x + 1)^2$

$\Rightarrow \quad (x^2 - 3x + 5) = x^2 + 2x + 1$

$\Rightarrow \quad 5x = 4$

$\Rightarrow \quad x = 4/5$

Thus, the solution of x is 4/5.

Example 3: Solve for x: $\log_{x^2+6x+6}\{\log_{2x^2+2x+3}(x^2 - 2x)\} = 0$.

Solution: Given equation is

$\log_{x^2+6x+6}\{\log_{2x^2+2x+3}(x^2 - 2x)\} = 0$

$\Rightarrow \quad x^2 - 2x = 2x^2 + 2x + 3$

$\Rightarrow \quad x^2 + 4x + 3 = 0$

$\Rightarrow \quad (x + 1)(x + 3) = 0$

$\Rightarrow \quad x = -1, -3$

Also, $x^2 + 6x + 6 > 0, \quad x^2 + 6x + 6 \neq 1$

$\Rightarrow \quad x \in (-\infty, -3 - \sqrt{3}) \cup (-3 + \sqrt{3}, \infty), x \neq -1, -5$

and, $2x^2 + 2x + 3 > 0, \quad x^2 + x + 1 \neq 0$

$\Rightarrow \quad x \in R$

So, the solution is $x = \varphi$.

Example 4: Solve for x: $\log_{x^2+x+1}\{\log_{2x^2+3x+5}(x^2 + 3)\} = 0$.

Solution: Given equation is

$\log_{x^2+x+1}\{\log_{2x^2+3x+5}(x^2 + 3)\} = 0$

$\Rightarrow \quad 2x^2 + 3x + 5 = x^2 + 3$

$\Rightarrow \quad x^2 + 3x + 2 = 0$

$\Rightarrow \quad (x + 1)(x + 2) = 0$

$\Rightarrow \quad x = -1, -2$

Also, $x^2 + x + 1 > 0, \quad x^2 + 3 \neq 1$

$\Rightarrow \quad x \in R$

and, $2x^2 + 3x + 5 > 1, \quad x^2 + 3 \neq 1$

$\Rightarrow \quad x \in R$

Thus, the solution is $x = \varphi$.

Example 5: Solve for x: $\log_5(x^2 - 4x + 3) = \log_5(3x + 21)$.

Solution: Given equation is

$\log_5(x^2 - 4x + 3) = \log_5(3x + 21)$

$\Rightarrow \quad x^2 - 4x + 3 = 3x + 21, \quad 3x + 21 > 0$

$\Rightarrow \quad x^2 - 7x - 18 = 0, \quad x > -7$

$\Rightarrow \quad (x - 9)(x + 2) = 0, \quad x > -7$

$\Rightarrow \quad x = -2, 9 \text{ and } x > -7$

Hence, the solution of x is $\{-2, 9\}$.

Example 6: Solve for x: $\log_{1/3}\left(2\left(\frac{1}{2}\right)^x - 1\right) = \log_{1/3}\left(\left(\frac{1}{4}\right)^x - 4\right)$

Solution: Given equation is

$$\log_{1/3}\left(2\left(\frac{1}{2}\right)^x - 1\right) = \log_{1/3}\left(\left(\frac{1}{4}\right)^x - 4\right)$$

$\Rightarrow \quad 2\left(\frac{1}{2}\right)^x - 1 = \left(\frac{1}{4}\right)^x - 4$

$\Rightarrow \quad \left(\frac{1}{2}\right)^{2x} - 2\left(\frac{1}{2}\right)^x - 3 = 0$

$\Rightarrow \quad a^2 - 2a - 3 = 0, \quad \text{where } a = \left(\frac{1}{2}\right)^x$

$\Rightarrow \quad (a - 3)(a + 1) = 0$

$\Rightarrow \quad a = 3, -1$

$\therefore a = 3$, since the exponential function is always positive

$\Rightarrow \quad \left(\frac{1}{2}\right)^x = 3$

$\Rightarrow \quad 2^{-x} = 3$

$\Rightarrow \quad -x = \log_2 3$

$\Rightarrow \quad x = -\log_2 3 = \log_2\left(\frac{1}{3}\right)$

Therefore, the solution is $x = \log_2\left(\frac{1}{3}\right)$

Example 7: Solve for x: $\log_{\left(\frac{x+5}{3}\right)} 3 = \log_{\left(\frac{-1}{x+1}\right)} 3$.

Solution: Given equation is

$\log_{\left(\frac{x+5}{3}\right)} 3 = \log_{\left(\frac{-1}{x+1}\right)} 3$

$\Rightarrow \quad \left(\frac{x+5}{3}\right) = \left(\frac{-1}{x+1}\right), \quad x+5>0, \frac{x+5}{3} \neq 1$

$\Rightarrow \quad (x+1)(x+5) = -3, \quad x > -5, x \neq -2$

$\Rightarrow \quad x^2 + 6x + 8 = 0, \quad x > -5, x \neq -2$

$\Rightarrow \quad (x+2)(x+4) = 0, \quad x > -5, x \neq -2$

$\Rightarrow \quad x = -2, -4, \quad x > -5, x \neq -2$

Hence, the solution is $x = -4$.

Example 8: Solve for x: $\log_{\left(\frac{x+2}{3}\right)} 5 = \log_{\left(\frac{4}{x-3}\right)} 5$.

Solution: Given equation is

$\log_{\left(\frac{x+2}{3}\right)} 5 = \log_{\left(\frac{4}{x-3}\right)} 5$

$\Rightarrow \quad \left(\frac{x+2}{3}\right) = \left(\frac{4}{x-3}\right), \quad x > -2, x+2 \neq 3$

$\Rightarrow \quad (x+2)(x-3) = 12, \quad x > -2, x \neq 1$

$\Rightarrow \quad x^2 + 5x - 6 = 0, \quad x > -2, x \neq 1$

$\Rightarrow \quad (x+6)(x-1) = 0, \quad x > -2, x \neq 1$

$\Rightarrow \quad x = -6, 1, x > -2, \quad x \neq 1$

Hence, the solution is $x = 1$.

Example 9: Solve for x: $\log_{x^2-1}(x^3+6) = \log_{x^2-1}(4x^2 - x)$.

Solution: Given equation is

$\log_{x^2-1}(x^3+6) = \log_{x^2-1}(4x^2 - x)$

$\Rightarrow \quad x^3 + 6, 4x^2 - x, \quad x^3 + 6 > 0, x^2 - 1 > 0, x \neq 1$

$\Rightarrow \quad x^3 - 4x^2 + x + 6 = 0, \quad x^3 > -6, x^3 > 1, x \neq \pm\sqrt{2}$

Now, $x^3 - 4x^2 + x + 6 = 0$

$\Rightarrow \quad x^3 + x^2 - 5x^2 - 5x + 6x + 6 = 0$

$\Rightarrow \quad x^2(x+1) - 5x(x+1) + 6(x+1) = 0$

$\Rightarrow \quad (x+1)(x^2 - 5x + 6) = 0$

$\Rightarrow \quad (x+1)(x-2)(x-3) = 0$

$\Rightarrow \quad x = -1, 2, 3$

Also, $x^3 > -6 \Rightarrow x > -\sqrt[3]{6}$

and, $x^2 > 1 \Rightarrow x \in (-\infty, -1) \cup (1, \infty)$

Hence, the solution is $x = 2, 3$.

Example 10: Solve for x: $\log_{x^3+x}(x^2-4) = \log_{4x^2-6}(x^2-4)$.

Solution: Given equation is

$\log_{x^3+x}(x^2-4) = \log_{4x^2-6}(x^2-4)$

$\Rightarrow \quad x^3 + x = 4x^2 - 6, \quad x^3 + x > 0, x^3 + x \neq 1, x^2 - 4 > 0$

$\Rightarrow \quad x^3 - 4x^2 + x + 6 = 0, \quad x > 0, x^3 + x \neq 1, x^2 > 4$

$\Rightarrow \quad (x+1)(x-2)(x-3) = 0$

$\Rightarrow \quad x = -1, 2, 3$

Also, $x > 0$

and, $x^2 > 4 > 0 \Rightarrow x \in (-\infty, -2) \cup (2, \infty)$

Hence, the solution is $x = 3$.

Example 11: Solve for x: $\log_3 2x = 2 \log_3 (4x - 15)$

Solution: Given equation is

$\log_3 2x = 2 \log_3 (4x - 15)$

$\Rightarrow \quad (4x-15)^2 = 2x, \quad x > \frac{15}{4}$

$\Rightarrow \quad 16x^2 - 122x + 225 = 0$

$$\Rightarrow \quad x = \frac{122 \pm \sqrt{(122)^2 - 64 \times 225}}{32}$$

$$= \frac{122 \pm \sqrt{14884 - 14400}}{32}$$

$$= \frac{122 \pm \sqrt{484}}{32}$$

$$= \frac{122 \pm 22}{32}$$

$$= \frac{144}{32}, \frac{100}{32}$$

$$= \frac{71}{16}, \frac{25}{4}$$

Also, $x > 15/4$

Hence, the solution set is $x = 25/4$

Example 12: Solve for x: $2 \log 2x = \log(7x - 2 - 2x^2)$.

Solution: Given equation is

$2 \log 2x = \log(7x - 2 - 2x^2)$

$4x^2 = 7x - 2 - 2x^2, \quad x > 0$

$6x^2 - 7x + 2 = 0, \quad x > 0$

$\Rightarrow \quad 6x^2 - 3x - 4x + 2 = 0$

$\Rightarrow \quad 3x(2x-1) - 2(2x-1) = 0$

$\Rightarrow \quad (2x-1)(3x-2) = 0$

$\Rightarrow \quad x = \frac{1}{2}, \frac{2}{3}$

Also, $x > 0$

Hence, the solution set is $x = \frac{1}{2}, \frac{2}{3}$.

Example 13: Solve for x: $\log (8 - 10x - 12x^2) = 3 \log (2x - 1)$.

Solution: Given equation is

$\log (8 - 10x - 12x^2) = 3 \log (2x - 1)$

$\Rightarrow \quad 8 - 10x - 12x^2 = (2x-1)^3, \quad x > 1/2$

$\Rightarrow \quad 8x^3 + 16x - 9 = 0$

$\Rightarrow \quad (2x-1)(4x^2 + 2x + 9) = 0$

$\Rightarrow \quad x = 1/2$

Also, $x > 1/2$

Hence, the solution set is $x = \varphi$.

EXAMPLE 14: Solve for x:

$$2\log_3 x + \log_3 (x^2 - 3) = \log_3 (0.5) + \log_3 8$$

Solution: Given equation is

$$2\log_3 x + \log_3 (x^2 - 3) = \log_3 (0.5) + \log_3 8$$

$\Rightarrow$ $x^2(x^2 - 3) = 4, \quad x > 0, x^2 - 3 > 0$

$\Rightarrow$ $x^2(x^2 - 3) = 4$

$\Rightarrow$ $x^4 - 3x^2 - 4 = 0$

$\Rightarrow$ $(x^2 - 1)(x^2 + 4) = 0$

$\Rightarrow$ $x = \pm 1$

Also, $x > 0$

and, $x^2 - 3 > 0 \Rightarrow x \in (-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$

Hence, the solution set is $x = 1$.

EXAMPLE 15: Solve for x:

$$\log_2(3 - x) - \log_2\left(\frac{\sin\left(\frac{3\pi}{4}\right)}{5 - x}\right) = \frac{1}{2} + \log_2(x + 7).$$

Solution: Given equation is

$$\log_2(3 - x) - \log_2\left(\frac{\sin\left(\frac{3\pi}{4}\right)}{5 - x}\right) = \frac{1}{2} + \log_2(x + 7)$$

$\Rightarrow$ $\log_2(3 - x) - \log_2\left(\frac{1}{\sqrt{2}(5 - x)}\right) = \frac{1}{2} + \log_2(x + 7)$

$\Rightarrow$ $\log_2(3 - x) + \log_2 \sqrt{2}(5 - x) = \frac{1}{2} + \log_2(x + 7)$

$\Rightarrow$ $\log_2(3 - x) + \frac{1}{2} + \log_2(5 - x) = \frac{1}{2} + \log_2(x + 7)$

$\Rightarrow$ $\log_2\{(3 - x)(5 - x)\} = \log_2(x + 7)$

$\Rightarrow$ $(x - 3)(x - 5) = x + 7$

$\Rightarrow$ $x^2 - 8x + 15 - x - 7 = 0$

$\Rightarrow$ $x^2 - 9x + 8 = 0$

$\Rightarrow$ $(x - 1)(x - 8) = 0$

$\Rightarrow$ $x = 1, 8$

$\Rightarrow$ $x = 1$ and $x = 8$ is rejected.

Hence, the solution set is $x = 1$.

LOGARITHMIC IN-EQUATION

CONCEPTUAL PROBLEM

EXAMPLE 1: Solve for x:

$$\log_{(2x+1)} x^2 < \log_{(2x+1)} (2x + 3), x > -1.$$

Solution: Given in-equation is

$$\log_{(2x+1)} x^2 < \log_{(2x+1)} (2x + 3)$$

$\Rightarrow$ $x^2 < 2x + 3,\ 2x + 1 > 1, 2x + 3 > 0$

$\Rightarrow$ $x^2 - 2x - 3 < 0, \quad x > -1/2, \quad x > -3/2$

$\Rightarrow$ $(x - 3)(x + 1) < 0, \quad x > -1/2, \quad x > -3/2$

$\Rightarrow$ $-1 < x < 3, \quad x > -1/2, \quad x > -3/2$

Hence, the solution set is $x \in \left(-\frac{1}{2}, 3\right)$.

EXAMPLE 2: Solve for x: $\log_{0.3}(x - 1) < \log_{0.09}(x - 1)$

Solution: Given in-equation is

$$\log_{0.3}(x - 1) < \log_{0.09}(x - 1)$$

$\Rightarrow$ $\log_{0.3}(x - 1) < \log_{(0.3)^2}(x - 1)$

$\Rightarrow$ $\log_{0.3}(x - 1) < \frac{1}{2}\log_{(0.3)}(x - 1)$

$\Rightarrow$ $2\log_{0.3}(x - 1) < \log_{(0.3)}(x - 1)$

$\Rightarrow$ $\log_{0.3}(x - 1)^2 < \log_{(0.3)}(x - 1)$

$\Rightarrow$ $(x - 1)^2 > (x - 1)$

$\Rightarrow$ $(x - 1)(x - 2) > 0$

$\Rightarrow$ $x < 1$ and $x > 2$

$\Rightarrow$ $x > 2$

$\Rightarrow$ $x \in (2, \infty)$

EXAMPLE 3: Solve for x: $\log_{12}(2x + 3) > 0$.

Solution: Given in-equation is

$$\log_{12}(2x + 3) > 0$$

$\Rightarrow$ $(2x + 3) < \left(\frac{1}{2}\right)^0$

$\Rightarrow$ $(2x + 3) < 1$

$\Rightarrow$ $2x + 2 < 0$

$\Rightarrow$ $x + 1 < 0$

$\Rightarrow$ $x < -1$

Also, $2x + 3 > 0 \Rightarrow x > -3/2$

Hence, the solution set is $x \in \left(-\frac{3}{2}, -1\right)$.

EXAMPLE 4: Solve for x: $\log_{1/2} x > \log_{1/3} x$.

Solution: Given in-equation is

$$\log_{1/2} x > \log_{1/3} x$$

$\Rightarrow$ $\log_{1/2} x > \left(\frac{\log_{1/2} x}{\log_{1/2}(1/3)}\right)$

$\Rightarrow$ $\log_{1/2} x\left(1 - \frac{1}{\log_{1/2}(1/3)}\right) > 0$

$\Rightarrow$ $\log_{1/2} x > 0 \qquad \left(\because \left(1 - \frac{1}{\log_{1/2}(1/3)}\right) > 0\right)$

$\Rightarrow$ $x > 0$ and $x < \left(\frac{1}{2}\right)^0$

$\Rightarrow$ $0 < x < 1$

EXAMPLE 5: Solve for x: $\log_{1/2} x + \log_3 x > 1$.

Solution: Given in-equation is

$$\log_{1/2} x + \log_3 x > 1$$

$\Rightarrow$ $-\log_2 x + \log_3 x > 1$

$\Rightarrow$ $\log_3 x - \log_2 x > 1$

$$\Rightarrow \quad \frac{\log_{10} x}{\log_{10} 3} - \frac{\log_{10} x}{\log_{10} 2} > 1$$

$$\Rightarrow \quad \log_{10} x\left(\frac{1}{\log_{10} 3} - \frac{1}{\log_{10} 2}\right) > 1$$

$$\Rightarrow \quad \log_{10} x \times M > 1 \qquad \left[\because M = \left(\frac{1}{\log_{10} 3} - \frac{1}{\log_{10} 2}\right)\right]$$

$$\Rightarrow \quad \log_{10} x > \frac{1}{M}$$

$$\Rightarrow \quad x > 10^{1/M}$$

Hence, the solution set is $x \in (10^{1/M}, \infty)$

Example 6: Solve for x: $\frac{1}{\log_2 x} - \frac{1}{\log_2 x - 1} < 1$.

Solution: Given in-equation is

$$\frac{1}{\log_2 x} - \frac{1}{\log_2 x - 1} < 1$$

$$\Rightarrow \quad \frac{1}{a} - \frac{1}{a-1} < 1 \text{ where } a = \log_2 x$$

$$\Rightarrow \quad \frac{1}{a} - 1 - \frac{1}{a-1} < 0$$

$$\Rightarrow \quad \frac{1-a}{a} - \frac{1}{a-1} < 0$$

$$\Rightarrow \quad \frac{-(1-a)^2 - a}{a(a-1)} < 0$$

$$\Rightarrow \quad \frac{(1-a)^2 + a}{a(a-1)} > 0$$

$$\Rightarrow \quad \frac{a^2 - a + 1}{a(a-1)} > 0$$

$$\Rightarrow \quad \frac{1}{a(a-1)} > 0$$

$\Rightarrow \quad a > 1$ and $a < 0$

$\Rightarrow \quad \log_2 x > 1$ and $\log_2 x < 0$

$\Rightarrow \quad x > 2$ and $x < 1$

Also, $\log_2 x$ is defined only when $x > 0$.

Hence, the solution set is $0 < x < 1$ and $x > 2$, i.e.

$$x \in (0, 1) \cup (2, \infty).$$

Example 7: Solve for x: $\log_{(2x+3)} x^2 < 1$.

Solution: Given in-equation is

$$\log_{(2x+3)} x^2 < 1$$

It is defined only when $x \neq 0$, $2x + 3 > 0$, $2x + 3 \neq 1$

$\Rightarrow \quad x \neq 0, x > -3/2, x \neq -1$

Case I: When $0 < 2x + 3 < 1$

$x^2 > 2x + 3$

$\Rightarrow \quad x^2 - 2x - 3 > 0$

$\Rightarrow \quad (x-3)(x+1) > 0$

$\Rightarrow \quad x < -1$ and $x > 3$ …(i)

Also, $0 < 2x + 3 < 1$

$\Rightarrow \quad -\frac{3}{2} < x < -1$ …(ii)

Thus, from Relations (i) and (ii), we get

$x \in \left(-\frac{3}{2}, -1\right)$ …(iii)

Case II: When $2x + 3 > 1$

$x^2 < 2x + 3$

$\Rightarrow \quad x^2 - 2x - 3 < 0$

$\Rightarrow \quad (x-3)(x+1) < 0$

$\Rightarrow \quad -1 < x < 3$

$\Rightarrow \quad x \in (-1, 3)$ …(iv)

Hence, the solution set [from Relations (iii) and (iv)] is

$$x \in \left(-\frac{3}{2} - 1\right) \cup (-1, 3) - \{0\}$$

Example 8: Solve for x: $\frac{\log^2 x - 3\log x + 3}{\log x - 1} < 1$.

Solution: Given in-equation is

$$\frac{\log^2 x - 3\log x + 3}{\log x - 1} < 1$$

$$\Rightarrow \quad \frac{a^2 - 3a + 3}{a-1} < 1 \text{, where } a = \log x$$

$$\Rightarrow \quad \frac{a^2 - 3a + 3}{a-1} - 1 < 0$$

$$\Rightarrow \quad \frac{a^2 - 3a + 3 - a + 1}{a-1} < 0$$

$$\Rightarrow \quad \frac{a^2 - 4a + 4}{a-1} < 0$$

$$\Rightarrow \quad \frac{1}{a-1} < 0$$

$\Rightarrow \quad 0 < a < 1$

$\Rightarrow \quad 0 < \log x < 1$

$\Rightarrow \quad 0 < \log_{10} x < 1$

$\Rightarrow \quad 1 < x < 10$

Hence, the solution set is $x \in (1, 10)$.

Example 9: Solve for x: $\frac{\log_2(x+1)}{(x-1)} > 0$.

Solution: Given in-equation is

$$\frac{\log_2(x+1)}{(x-1)} > 0$$

It is possible only when $x > 1$, $\log_2(x+1) > 0$ and $x > -1$.

Now, $\quad \log_2(x+1) > 0$

$\Rightarrow \quad (x+1) > 2^0 = 1$

$\Rightarrow \quad x > 0$

Hence, the solution set is $x \in (1, 10)$.

Example 10: Solve for x: $\dfrac{1}{\log_4\left(\dfrac{x+1}{x+2}\right)} \le \dfrac{1}{\log_4(x+3)}$.

Solution: Given in-equation is

$$\frac{1}{\log_4\left(\dfrac{x+1}{x+2}\right)} \le \frac{1}{\log_4(x+3)}$$

$$\Rightarrow \quad \log_4\left(\frac{x+1}{x+2}\right) \ge \log_4(x+3)$$

$$\Rightarrow \quad \log_4\left(\frac{x+1}{x+2}\right) - \log_4(x+3) \ge 0$$

$$\Rightarrow \quad \log_4\left(\frac{x+1}{(x+2)(x+3)}\right) \ge 0$$

$$\Rightarrow \quad \left(\frac{x+1}{(x+2)(x+3)}\right) \ge 1$$

$$\Rightarrow \quad \left(\frac{x+1}{(x+2)(x+3)}\right) - 1 \ge 0$$

$$\Rightarrow \quad \left(\frac{(x+1)-(x+2)(x+3)}{(x+2)(x+3)}\right) \ge 0$$

$$\Rightarrow \quad \left(\frac{(x+2)(x+3)-(x+1)}{(x+2)(x+3)}\right) \le 0$$

$$\Rightarrow \quad \left(\frac{x^2+4x+5}{(x+2)(x+3)}\right) \le 0$$

$$\Rightarrow \quad \left(\frac{1}{(x+2)(x+3)}\right) \le 0$$

$$\Rightarrow \quad -2 < x < 3$$

Hence, the solution set is $x \in (-2, 3)$.

Example 11: Solve for x: $\dfrac{(x^2-4)}{\log_2(x^2-1)} < 0$.

Solution: Given in-equation is

$$\frac{(x^2-4)}{\log_2(x^2-1)} < 0$$

It is possible only when $x^2 - 4 > 0$ and $\dfrac{1}{\log_{1/2}(x^2-1)} < 0$

When $\quad x^2 - 4 > 0$

$\quad (x+2)(x-2) > 0$

$\Rightarrow \quad x \in (-\infty, -2) \cup (2, \infty)$ …(i)

When $\dfrac{1}{\log_{1/2}(x^2-1)} < 0$

$\quad \log^{1/2}(x^2 - 1) > 0$

$\Rightarrow \quad (x^2 - 1) < 1$

$\Rightarrow \quad x^2 - 2 < 0$

$\Rightarrow \quad -\sqrt{2} < x < \sqrt{2}$ …(ii)

When $x^2 - 1 > 0$

$\quad (x+1)(x-1) > 0$

$\Rightarrow \quad x \in (-\infty, -1) \cup (1, \infty)$ …(iii)

From Relations (i), (ii) and (iii), we get

$$x \in (-\infty, -2) \cup (-\sqrt{2}, -1) \cup (1, \sqrt{2}) \cup (2, \infty)$$

which is the required solution set.

Exercises

Level I

(Questions based on Fundamentals)

1. Find the value of $\log_3 \log_5 \log_3 (243)$.
2. Find the value of $\log_9 (27) - \log_{27} (9)$.
3. Find the value of
   $\log_{10} \tan 40° + \log_{10} \tan 41° + \log_{10} \tan 42° + \ldots + \log_{10} \tan 50°$.
4. Find the value of
   $\log_{\sqrt{3}} 300$, if $a = \log_{\sqrt{3}} 5$, $b = \log_{\sqrt{3}} 2$.
5. Find the minimum value of
   (i) $\log_b a + \log_a b$
   (ii) $\log_b a + \log_c b + \log_a c$
6. Prove that $\dfrac{1}{\log_2 n} + \dfrac{1}{\log_3 n} + \dfrac{1}{\log_4 n} + \ldots + \dfrac{1}{\log_{43} n} = \dfrac{1}{\log_{43!} n}$
7. If $n = 1983!$, prove that
   $\dfrac{1}{\log_2 n} + \dfrac{1}{\log_3 n} + \dfrac{1}{\log_4 n} + \ldots + \dfrac{1}{\log_{1983} n} = 1$
8. Determine b satisfying
   $\log_a 2 \cdot \log_b 625 = \log_{10} 16 \cdot \log_a 10$
9. If $\log_a ab = x$, find the value of $\log_b ab$.
10. If $\log_{10} 2 = x$, find the value of $\log_{10} 5$.
11. If $a = \log_4 5$ and $b = \log_5 6$, find $a = \log_3 2$.
12. Find the value of $\log_{12} 54$, where $b = a = \log_{12} 24$.
13. Find the value of $\dfrac{1}{\log_2 36} + \dfrac{1}{\log_3 36}$.
14. Prove that $\dfrac{1}{\log_3 \pi} + \dfrac{1}{\log_4 \pi} > 2$.
15. Simplify: $7 \cdot \log\dfrac{16}{15} + 5 \cdot \log\dfrac{25}{24} + 3 \cdot \log\dfrac{81}{80}$.

16. If $a^2 + b^2 = 7ab$, prove that
$$\log\frac{1}{3}(a+b) = \frac{1}{2}(\log a + \log b)$$
17. If $a^2 + b^2 = 11ab$, prove that
$$\log\left(\frac{a-b}{3}\right) = \frac{1}{2}(\log a + \log b).$$
18. Prove that $\dfrac{\log_a n}{\log_{ab} n} = 1 + \log_a b$.
19. If $\log 25 = a$, $\log 225 = b$, prove that $\log\left(\left(\frac{1}{9}\right)^2\right) + \log\left(\frac{1}{2250}\right) = 2a - 3b - 1$
20. If a, b, c are in GP, prove that, $\log_a n$, $\log_b n$, $\log_c n$ are in HP.
21. If $\log_3 2, \log_3(2^x - 5), \log_3\left(2^x - \frac{7}{2}\right)$ are in AP, find the value of x.
22. If $y = a^{\frac{1}{1-\log_a x}}$, $z = a^{\frac{1}{1-\log_a y}}$, prove that, $x = a^{\frac{1}{1-\log_a z}}$.
23. If $x = \log_a c + \log_c a$, $y = \log_a c + \log_c a$ and $z = \log_b a + \log_a b$, find the minimum value of $x^2 + y^2 + z^2 - xyz$.
24. If $\dfrac{\log a}{b-c} = \dfrac{\log b}{c-a} = \dfrac{\log c}{a-b}$, prove that $a^a \cdot b^b \cdot c^c = 1$.
25. Find the value of $81^{\frac{1}{\log_5 3}} + 27^{\log_9 36} + 3^{\frac{4}{\log_7 9}}$.
26. Prove that
$$\frac{1}{1+\log_b a+\log_b c} + \frac{1}{1+\log_c a+\log_c b} + \frac{1}{1+\log_a b+\log_a c} = 1$$
27. Find x, if $\log_2 x + \log_4 x + \log_8 x = 11$.
28. Find x, if $\log_2 x + \log_4 x + \log_8 x + \log_{16} x = \dfrac{25}{4}$.
29. If $\log_a x = \alpha$, $\log_b x = \beta$, $\log_c x = \gamma$ and $\log_d x = \delta$, find the value of $\log_{abcd} x$.
30. If $x = \log_a bc$, $y = \log_b ca$ and $z = \log_c ab$, find the value of $\dfrac{1}{1+x} + \dfrac{1}{1+y} + \dfrac{1}{1+z}$.
31. If $y = 2^{\frac{1}{\log_x 4}}$, find x.
32. If $N = 6^{\log_{10} 40} \cdot 5^{\log_{10} 36}$, find the value of $N + 10$.
33. Find the value of $(0.5)^{\log_{\sqrt{2}}\left(\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots \text{to} \infty\right)}$.
34. If $x = 2^{\log_{10} 3}$ and $y = 3^{\log_{10} 2}$, find a relation between x and y.
35. Find the value of
$$2^{\log_{10} 3 - \log_{10} 5} \times 3^{\log_{10} 5 - \log_{10} 2} \times 5^{\log_{10} 2 - \log_{10} 3}.$$
36. If $a = \log_{30} 3$ and $b = \log_{30} 5$, find the value of $\log_{10} 8$.
37. If $a = \log_{12} 18$, $b = \log_{24} 54$, prove that $ab + 5(a - b) = 1$.
38. If $a = \log_7 12$, $b = \log_{12} 24$, find the value of $\log_{54} 168$.
39. If $a = \log_6 30$, $b = \log_{15} 24$, prove that $\log_{12} 60 = \left(\dfrac{2ab + 2a - 1}{ab + b + 1}\right)$.
40. Find x, if $\log_7(\log_5(\sqrt{x+5} + \sqrt{x})) = 0$.
41. If $\log_2 x + \log_2 y \geq 6$, find the least value of $x + y$.
42. Solve for x and y:
$$4^{\log x} = 3^{\log y}, (3x)^{\log 3} = (4y)^{\log 4}$$
43. If $x^{18} = y^{21} = z^{28} = k$, prove that, 3, $3 \log_y x$, $3 \log_z y$, $7 \log_x z$ are in AP.
44. If $\dfrac{1}{\log_3 \pi} + \dfrac{1}{\log_4 \pi} > x$, find the value of x.
45. If $\log_{0.3}(x - 1) < \log_{0.09}(x - 1)$, find x.
46. If $\log_e \log_5(\sqrt{2x-2} + 3) = 0$, find the value of x.
47. Find the least value of $2.\log_{10} x - \log_x(0.01)$ for $x > 1$.
48. Find x, if $4^{\log_9 3} + 9^{\log_2 4} = 10^{\log_x 83}$
49. Find x, if $3^{4\log_9(x+1)} = 2^{2\log_2 x} + 3$
50. If $a = \log_{24} 12$, $b = \log_{36} 24$ and $c = \log_{48} 36$, prove that $\left(\dfrac{abc+1}{bc}\right) = 2$
51. If $\log_{10}\left(\sin\left(x + \frac{\pi}{4}\right)\right) = \frac{1}{2}(\log_{10} 6 - 1)$, find the value of $\log_{10} \sin x + \log_{10} \cos x$.
52. If a, b, c are in GP, prove that
$$\frac{1}{1+\log_e a}, \frac{1}{1+\log_e b}, \frac{1}{1+\log_e c}$$ are in HP.
53. Find x, if $5^{\log_{10} x} = 50 - x^{\log_{10} 5}$.
54. Find x, if $\log_5 [2 + \log (3 + x)] = 0$.

Find the number of real solutions

55. $\log_4 (x - 1) = \log_2 (x - 3)$
56. $\log_4 (x - 2) = \log_2 (x - 2)$
57. $\log_9 (x - 1) = \log_3 (x - 1)$
58. $\log_2 x + \log_2 (x + 3) = 1/4$
59. $\log_4 (x^2 + x) - \log_4 (x + 1) = 2$
60. $1 + 2 \log_{(x+2)} 5 = \log_5 (x + 2)$
61. $\log_2 x + \log_4 (x + 2) = 2$
62. $\log_{10} (x - 1)^3 - 3 \log_{10} (x - 3) = \log_{10} 8$

Solve for x:

63. $\log_5 (x^2 - 3x + 3) > 0$
64. $\log_7 [\log_5 (x^2 - 7x + 15)] > 0$
65. $\log_{(1/2)} [\log_5 (x^2 - 7x + 17)] > 0$
66. $\log_{(1/2)} (\log_5 (\log_2 (x^2 - 6x + 40))) > 0$
67. $\log_3 [\log_5 \log_2 (x^2 - 9x + 50)] > 0$

68. $\log_6\left(\frac{x-2}{6-x}\right) > 0$
69. $\log_{(1/2)} x > \log_{(1/3)} x$
70. $\log_{0.5}(x^2 - 5x + 6) > -1$
71. $\log_8(x^2 - 4x + 3) < 1$
72. $\log_{(1/4)}\left(\frac{35-x^2}{x}\right) \geq -\frac{1}{2}$
73. $\log_{(x^3+6)}(x^2-1) = \log_{(2x^2+5x)}(x^2-1)$
74. $\log(3x^2 + x - 3) = 3\log(3x - 2)$
75. $\log_{(x^2-1)}(x^3+6) = \log_{(x^2-1)}(2x^2+5x)$
76. $\log_3(x^2 - 3x - 5) = \log_3(7 - 2x)$
77. $\log(\sqrt{x-1}) + \frac{1}{2}\log(2x+15) = 1$
78. $\log_{(3x+4)}(4x^2 + 4x + 1) + \log_{(2x+1)}(6x^2 + 11x + 4) = 4$
79. $5^{\log_{10} x} = 50 - x^{\log_{10} 5}$
80. $4^{\log_9 x} - 6.2^{\log_9 x} + 2^{\log_3 27} = 0$

Solve the in-equality wherever base is not given, take it as 10:

81. $(\log_2 x)^4 - \left(\log_{1/2}\left(\frac{x^5}{4}\right)\right)^2 - 20\log_2 x + 148 < 0$
82. $(\log 100x)^2 + (\log 10x)^2 + \log x \leq 14$
83. $\log_{1/2}(x+1) > \log_2(2-x)$
84. $\log_{1/5}(2x^2 + 5x + 1) < 0$
85. $\log_{1/2} x + \log_3 x > 1$
86. $\log_x\left(\frac{4x+5}{6-5x}\right) < -1$
87. $\log_{0.2}\left(\frac{x+2}{x}\right) \leq 1$
88. $\log_{10}(x^2 - 16) \leq \log_{10}(4x - 11)$

Level II

(Mixed Problems)

1. If a, b, c are in GP, $\log_{2016} a, \log_{2016} b, \log_{2016} c$ are in
(a) GP (b) AP (c) HP (d) AGP

2. If $y = 3^{\frac{1}{\log_x 9}}$, x is
(a) y (b) $\sqrt{y}$ (c) y^2 (d) y^3

3. If $\frac{1}{\log_a x} + \frac{1}{\log_c x} = \frac{2}{\log_b x}$, a, b, c are in
(a) AP (b) GP (c) HP (d) AGP

4. If $x = \log_3 5$ and $y = \log_{27} 25$,
(a) $x > y$ (b) $x = y$ (c) $x < y$ (d) $x^2 = y$

5. If $\log_{10} 2, \log_{10}(2^x + 1), \log_{10}(2^x + 3)$ are in AP,
(a) $x = 0$ (b) $x = 1$
(c) $x = \log_{10} 2$ (d) $x = \frac{1}{2}\log_2 5$

6. If $\log_a(ab) = x$, $\log_b(ab)$ is
(a) $\frac{x}{1-x}$ (b) $\frac{x}{1+x}$ (c) $\frac{x}{x-2}$ (d) none

7. The value of $4^{2\log_9 3}$ is
(a) 9 (b) 2 (c) 4 (d) 3

8. If $\log_7 2 = x$, $\log_{49}(28)$ is
(a) $\left(x+\frac{1}{2}\right)$ (b) $\left(x-\frac{1}{2}\right)$
(c) $-\left(x-\frac{1}{2}\right)$ (d) $-\left(x+\frac{1}{2}\right)$

9. If $\log_{2016}(\log_5(\sqrt{2x-2}+3)) = 0$, x is
(a) 1/3 (b) 1/2 (c) 3 (d) 2

10. If $\frac{1}{\log_2 \pi} + \frac{1}{\log_6 \pi} > x$, x is
(a) 2 (b) 3 (c) 4 (d) 5

11. The value of $5^{\log_2 7} - 7^{\log_2 5}$ is
(a) 5 (b) 0 (c) 7 (d) 2

12. If $\log_{10} 2 = x$, $\log_{10} 5$ is
(a) 1 (b) $1 - x$ (c) $x + 1$ (d) $2x$

13. The number of real solutions of $\log_2 x + \log_4(x + 2) = 2$ is
(a) 1 (b) 2 (c) 3 (d) 0

14. The number of real solutions of $1 + \log_2(x - 1) = \log_{(x-1)} 4$ is
(a) 1 (b) 2 (c) 3 (d) 0

15. The number of real solutions of $x^{\log_{\sqrt{x}}(x-2)} = 9$ is
(a) 4 (b) 3 (c) 2 (d) 1

16. The value of $\left(\frac{1}{\log_3 \pi} + \frac{1}{\log_4 \pi}\right)$ lies in between
(a) (1, 2) (b) (2, 3) (c) (3, 4) (d) (0, 1)

17. The number of real roots of $x \ln x - 1 = 0$ is
(a) 2 (b) 1 (c) 3 (d) infinite

18. The number of real roots of $2 - x \ln x = 0$ is
(a) 1 (b) 2 (c) 0 (d) infinite

19. If $3^x = 10 - \log_2 x$, x is
(a) 0 (b) 1 (c) 2 (d) 3

20. If $|1 - \log_{1/5} x| + 2 = |3 - \log_{1/5} x|$, x is
(a) 2 (b) 5 (c) 1 (d) 3

Level III

(Problems for JEE-Advanced)

1. If $a^4b^5 = 1$, find the value of $\log_a(a^5b^4)$.
2. If $x = \log_{10} 5 \times \log_{10} 20 + \log_{10^2} 2$ and

$$y = \frac{2\log 2 + \log 3}{\log(48) - \log 4},$$

prove that $x = y$

3. Find the sum of all the equations $2\log x - \log(2x - 75) = 2$
4. If $\log_x(\log_{18}(\sqrt{2} + \sqrt{8})) = -\frac{1}{2}$, find x.
5. If $\log_6 9 - \log_9 27 + \log_8 x = \log_{64} x - \log_6 4$, find the value of x.
6. If $x = \sqrt{12 + 6\sqrt{3}} + \sqrt{12 - 6\sqrt{3}}$, then find value of $\log_{36} x$.
7. If $\log_a b = 2$, $\log_b c = 2$ and $\log_3 c = 3 + \log_3 a$, find the value of $(a + b + c) + 7$.
8. If $\log_9 x + \log_4 y = \frac{7}{2}$ and $\log_9 x - \log_8 y = -\frac{3}{2}$, find the value of $\log_4 (x + y - 3)$.
9. If $a = \log_{10} 2$, $b = \log_{10} 3$ such that $3^{x+2} = 5$, find x (in terms of a and b).
10. Let the number $N = 6\log_{10} 2 + \log_{10} 31$. If N lies between two successive integers, find their sum.
11. Let $M = \log^2_{\sqrt{2}}\left(\frac{1}{4}\right)$, $N = \log^3_{2\sqrt{2}}(8)$ and
$$P = \log_5(\log_3(\sqrt{\sqrt[5]{9}})),$$
find the value of $\left(\frac{M}{N} + P + 3\right)$.
12. If $\log_3(x) = a$ and $\log_7(x) = b$, then the value of $\log_{21}(x)$.
13. If x and y are satisfying the relations $\log_8 x + \log_4 y^2 = 5$ and $\log_8 y + \log_4 x^2 = 7$ find the value of $2xy$.
14. If $\log_{10}(x^2 + x) = \log_{10}(x^3 - x)$, find the product of all the solutions.
15. If $\log_{10}(x - 2) + \log_{10} y = 0$ and
$$\sqrt{x} + \sqrt{y - 2} = \sqrt{x + y},$$
find the value of $(x + y - 2\sqrt{2})$.
16. If $a, b \in R^+$ such that $\log_{27} a + \log_9 b = \frac{7}{2}$ and $\log_{27} b + \log_9 a = \frac{2}{3}$, find ab.
17. Find the number of values of x satisfying the equation $\log_{\tan x}(2 + 4\cos^2 x) = 2$ in $[0, 2\pi]$.
18. If $\cos(\ln x) = 0$, find $\left(\frac{2}{\pi} \times \log(x) + 10\right)$.
19. If $c(a - b) = a(b - c)$ such that $a \neq b \neq c$, find the value of $\frac{\log(a + c) + \log(a - 2b + c)}{\log(a - c)}$.
20. If $x = \log_{10}(A + \sqrt{B})$ is a solution of $10^x + 10^{-x} = 4$ find the value of $(A + B + 3)$.
21. Solve for x:
$$\log_{10}(98 + \sqrt{x^3 - x^2 - 12x + 36}) = 2.$$
[Roorkee, 1975]
22. Solve for x and y:
$$\log_{10} x + \log_{10} x^{1/2} + \log_{10} x^{1/4} + \ldots = y$$
and $\frac{1 + 3 + 5 + \ldots + (2y - 1)}{4 + 7 + 10 + \ldots + (3y + 1)} = \frac{20}{7\log_{10} x}$
[Roorkee, 1987]
23. Solve for x:
$$\frac{6}{5} a^{\log_a x \log_{10} a \log_a 5 - 3\log_{10}\left(\frac{x}{10}\right)} = 9^{\log_{100} x + \log_4 2}.$$
[Roorkee, 1988]
24. No questions asked in between 1990-1997
25. Solve for x and y:
$$\log_{100}|x + y| = \frac{1}{2}$$
$$\log_{100}\left(\frac{y}{|x|}\right) = \log_{100} 4$$
[Roorkee-JEE-1998]
26. Find all real number x which satisfy the equation $2\log_2 \log_2 x + \log_{1/2} \log_2(2\sqrt{2}x) = 1$. **[Roorkee, 1999]**
27. Solve for x:
$$\log_{3/4} \log_8 (x^2 + 7) + \log_{1/2} \log_{1/4} (x^2 + 7)^{-1} = -2.$$
[Roorkee, 2000]
28. Solve for x and y:
$$\log_2 x + \log_4 x + \log_8 x + \ldots = y$$
$$\frac{5 + 9 + 13 + \ldots + (4y + 1)}{1 + 3 + 5 + \ldots + (2y - 1)} = 4\log_4 x$$
[Roorkee, 2001]
29. If (a, b) and (c, d) are the solutions of the system of equations $\begin{cases} \log_{225}(x) + \log_{64}(y) = 4 \\ \log_x(225) - \log_y(64) = 1 \end{cases}$, find the value of $\frac{1}{2}\log_{30}(abcd)$.
30. If $x = 1 + \log_a(bc)$, $y = 1 + \log_b(ca)$ and $z = 1 + \log_c(ab)$, find the value of $\frac{xyz}{xy + yz + zx}$.

Level IV

(Tougher Problems for JEE-Advanced)

1. Solve for x: $x + \log_{10}(1 + 2x) = x\log_{10} 5 + \log_{10} 6$.
2. Solve for x: $\log\left|\frac{x^2 - x - 1}{x^2 + x - 2}\right| = 0$.
3. Solve for x: $|4 + \log_{17} x| = 2 + |2 + \log_{1/7} x|$.
4. Solve for x:
$$\log^2\left(1 + \frac{4}{x}\right) + \log_2\left(1 - \frac{4}{x + 4}\right) = 2\log^2\left(\frac{2}{x - 1} - 1\right).$$
5. Solve the system of equations:
$$\begin{cases} \log_y x - \log_x y = \frac{8}{3}. \\ xy = 16 \end{cases}$$

6. Solve for x:
$\log(3x^2 + 12x + 19) - \log(3x + 4) + \log_{32} 4$
$= 1 - \log_{1/16}(\sqrt[5]{256})$.
7. Solve for x:
$$\log^2(4-x)+\log(4-x)\cdot\log\left(x+\frac{1}{2}\right)-2\log^2\left(x+\frac{1}{2}\right)=0.$$
8. Solve for x:
$\log_{3/4}[\log^8(x^2+7)] + \log_{1/2}(\log_{1/4}(x^2+7)^{-1}) = -2$.
9. Solve for x:
$\log_{10}(x_2 - x - 6) - x = \log_{10}(x+2) - 4$.
10. Solve for x:
$$\frac{1}{2}\log_5(x+5)+\log_5(\sqrt{x-3})=\frac{1}{2}\log_5(2x+1).$$
11. Solve for x:
$$\frac{3}{2}\log_4(x+2)^2+3=\log_4(4-x)^3+\log_4(6+x)^3.$$
12. Solve for x:
$$\frac{1+\log_2(x-4)}{2\log_2(\sqrt{x+3}-\sqrt{x-3})}=1.$$
13. Solve for x:
$$\left(1+\frac{1}{2x}\right)\log 3=\log\left(\frac{\sqrt[x]{3}+27}{4}\right).$$
14. Solve for x:
$$4^{\log_{10}x+1}-6^{\log_{10}x}-2.3^{\log_{10}x^2+2}=0.$$
15. Solve for x:
$$\log_3(\sqrt{x}+|\sqrt{x}-1|)^2=\log_3(4\sqrt{3}-3+4|\sqrt{x}-1|).$$

Integer Type Questions

1. If $\sum_{r=0}^{n-1}\log_2\left(\frac{r+2}{r+1}\right)=\prod_{r=10}^{99}\log_r(r+1)$,
find the value of n.
2. Solve for x: $7^{\log_2 x} = 98 - x^{\log_2 7}$.
3. Solve for x: $4^{\log_3 x} = 32 - x^{\log_3 4}$.
4. If α and β be the roots of
$3\log_x 4 + 2\log_{4x} 4 + 3\log_{16x} 4 = 0$,
find the value of $\frac{1}{2}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)$.
5. Solve for x:
$x + \log_{10}(2x+1) = \log_{10} 6 + x\log_{10} 5$.
6. Find the integral value of x for which
$\log_{(x+1)}(x^2+x-6)^2 = 4$.
7. If α and β be the solutions of
$|x-2|^{\log_2(x^3)-3\log_x 4}=(x-2)^3$,
find the value of $(\alpha+2\beta+3)$.
8. If α is the integral solutions of $6(\log_x 2 - \log_4 x) + 7 = 0$, find the value of $\left(\frac{2\alpha-1}{5}\right)$.

9. Let the number $N = 6\log_{10} 2 + \log_{10} 31$. If N lies in between two successive integers, find their sum.
10. Find the value of the expression
$$(0.16)^{\log_{2.5}\left(\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots\right)}.$$
11. If $a = \log_{12} 18$, $b = \log_{24} 54$, find the value of $5(a-b) + ab$.

Comprehensive Link Passages

Passage I

Let A be the sum of the roots of
$$\frac{1}{5-4\log_4 x}+\frac{4}{1+\log_4 x}=3,$$
B be the product of m and n, where $2^m = 3$ and $3^n = 4$, and C be the sum of the integral roots of
$$\log_{3x}\left(\frac{3}{x}\right)+(\log_3 x)^2=1.$$

1. The value of $A + B$ is
(a) 10 (b) 6 (c) 8 (d) 4
2. The value of $B + C$ is
(a) 6 (b) 2 (c) 4 (d) 8
3. The value of $(A + C \div B)$ is
(a) 5 (b) 8 (c) 7 (d) 4

Passage II

A function $f: R^+ \to R$ is defined as
$f(x) = \log_a x, \quad x > 0, a > 0, a \neq 1$
Then $D_f = R^+$ and $R_f = R$.

1. If $f(x)=\log\left(\frac{x-3}{5-x}\right)$, the domain of the function $f(x)$ is
(a) $(3, 5)$ (b) $(-\infty, 5)$ (c) $(5, \infty)$ (d) none
2. Let $f(x) = (-x^2 + 3x - 2)$. The domain of the function $f(x)$ is
(a) $(-1, 2)$ (b) $(1, 2)$ (c) $(-\infty, 1]$ (d) $[2, \infty)$
3. Let $f(x)=\sqrt{x-2}+\sqrt{4-x}$. The range of the function $f(x)$ is
(a) $[\sqrt{2}, 2]$ (b) $[1, 2]$
(c) $(2, 4)$ (d) $[2, 4]$

Matching List Type (Only One Option is Correct)

This section contains four questions, each having two matching list. Choices for the correct combination of elements from List I and List II are given as options (A), (B), (C) and (D), out of which ONE is correct.

1. Match the following lists

List I		List II	
(P)	The value of $\left(\frac{\log_2 32}{\log_3\sqrt{243}}\right)$ is	(1)	2/7

(Q)	The value of $\left(\dfrac{2\log_{2016}6}{\log_{2016}12+\log_{2016}3}\right)$ is	(2)	–2
(R)	The value of $\left(\log_{1/4}\left(\dfrac{1}{16}\right)^{-2}\right)$ is	(3)	1
(S)	The value of $\left(\dfrac{\log_5 16-\log_5 4}{\log_5 128}\right)$ is	(4)	2

Codes:

	P	Q	R	S
(A)	2	3	1	4
(B)	4	2	1	3
(C)	4	3	2	1
(D)	3	1	4	2

2. Match the following lists.

List I		List II	
(P)	The value of $\left(\dfrac{2\log 2+\log 3}{\log 48-\log 4}\right)$ is	(1)	3
(Q)	The value of $\dfrac{1}{6}\log_{\frac{\sqrt{3}}{2}}\left(\dfrac{64}{27}\right)$ is	(2)	0
(R)	The value of $\log^2{}_{10}5+\log_{10}5\cdot\log_{10}20+\log^2{}_{10}2-1$ is	(3)	1
(S)	The value of a for which $\dfrac{\log_a 7}{\log_6 7}=\log_9 36$ holds good is	(4)	–1

Codes

	P	Q	R	S
(A)	2	3	1	4
(B)	4	3	1	2
(C)	4	3	2	1
(D)	3	4	2	1

3. Match Matrix

Column I		Column II	
(A)	If α be the root of $3x^{\log_3 4}+4^{\log_3 x}=64,\ \sqrt{\alpha}+1$ is	(P)	2
(B)	The integral value of of x in $\log^2{}_2 x-\log_2 x-2=0$ is	(Q)	4
(C)	The value of $4\log_2 x\ 4\log_2 x$ is where $x=\sqrt{3+2\sqrt{2}}+\sqrt{3-2\sqrt{2}}$	(R)	3
(D)	If $a^2+b^2=1$, the value of $\log_{ab}(a^3b^5+a^5b^3)$ is	(S)	1

Questions asked in Previous Years' JEE-Advanced Examinations

1. For $a > 0$, solve for x, the equation
$$2\log_x a+\log_{ax}a+3\log_{a^2x}a=0$$
[IIT-JEE, 1978]

2. The least value of the expression $2\log_{10}x-\log_x(0.01)$, $x > 1$ is
(a) 10 (b) 2
(c) –0.01 (d) none of these
[IIT-JEE, 1980]

3. $y = 10^x$ is the reflection of $y = \log_{10} x$ in the line whose equation is . . . **[IIT-JEE, 1982]**

4. For $0 < a < x$, the minimum value of $\log_x a+\log_a x$ is . . . **[IIT-JEE, 1984]**

5. If $\log_{0.3}(x-1) < \log_{0.09}(x-1)$, x lies in
(a) $(2, \infty)$ (b) $(1, 2)$
(c) $(-2, -1)$ (d) none of these
[IIT-JEE, 1985]

6. The solution of the equation
$$\log_7(\log_5(\sqrt{x+5}+\sqrt{x}))=0 \text{ is ...}$$
[IIT-JEE, 1986]

7. Solve for x:
$\log_{(2x+3)}(6x^2+23x+21)=4-\log_{(3x+7)}(4x^2+12x+9)$
[IIT-JEE, 1987]

8. The equation $x^{(3/4)(\log_2 x)^2+\log_2 x-\frac{5}{4}}=\sqrt{2}$ has
(a) at least one real solution
(b) exactly three real solutions
(c) exactly one irrational solutions
(d) complex roots **[IIT-JEE, 1989]**

9. If $\log_3 2, \log_3(2^x-5), \log_3\left(2^x-\dfrac{7}{2}\right)$ are in AP, find x. **[IIT-JEE, 1990]**

10. The number $\log_2 7$ is
(a) an integer (b) a rational number
(c) an irrational number (d) a prime number
[IIT-JEE, 1990]

No questions asked in between 1991 to 2000.

11. The number of solutions of $\log_4(x-1)=\log_2(x-3)$ is
(a) 3 (b) 1 (c) 2 (d) 0
[IIT-JEE, 2001]

12. Let (x_0, y_0) be the solution of the following equations $(2x)^{\ln 2}=(3y)^{\ln 3}$, $3^{\ln x}=2^{\ln 3}$. Then x_0 is
(a) $\dfrac{1}{6}$ (b) $\dfrac{1}{3}$ (c) $\dfrac{1}{2}$ (d) 6
[IIT-JEE, 2011]

13. The value of
$$6+\log_{3/2}\left(\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\sqrt{4-\frac{1}{3\sqrt{2}}\ldots\text{to}\infty}}}\right)$$
is . . . **[IIT-JEE, 2012]**

Answers

Level I

1. 0
2. 5/6
3. 0
4. 12
5. (i) 2 (ii) 3
8. 21
9. $\left(\dfrac{x}{x-1}\right)$
10. $1-x$
11. $\dfrac{1}{2ab-1}$
13. 1/2
15. log 2
21. $x = 3$
23. 4
25. 216
27. 64
28. 25/4
29. $\left(\dfrac{1}{\alpha^{-1}+\beta^{-1}+\gamma^{-1}+\delta^{-1}}\right)$
30. 1
31. $x = y^2$
32. 226
33. 4
34. $x = y$
36. $\dfrac{3\{1-(a+b)\}}{(1-a)}$
40. $x = 4$
41. 16
44. $x = 2$
45. $(2, \infty)$
46. $x = 3$
47. 4
48. $x = 10$
49. $x = 1$
51. –1
53. $x = 100$
54. $x = 1$
55. 1
56. 1
57. 1
58. 2
59. 1
60. 1
61. 1
62. 1
63. $(-\infty, 1) \cup (2, \infty)$
64. $(-\infty, 2) \cup (5, \infty)$
65. (3, 4)
66. (2, 4)
67. $(-\infty, 3) \cup (6, \infty)$
68. (4, 6)
69. $690 < x < 1$
70. (1, 4)
71. (–1, 5)
72. $(-1, 0) \cup (5, \infty)$
73. $x = 3$
74. $x = \dfrac{5 \pm \sqrt{10}}{9}$
75. $x = 3$
76. $x = -3$
77. $x = 5$
78. $x = 3/4$
79. 100
80. {9, 81}
81. $x \in \left(\dfrac{1}{16}, \dfrac{1}{8}\right) \cup (8, 16)$
82. $x \in \left[\dfrac{1}{\sqrt{10}}, 10\right]$.
83. $-1 < x < \dfrac{1-\sqrt{5}}{2}$ or $\dfrac{1+\sqrt{5}}{2} < x < 2$.
84. $x \in (-\infty, -2.5) \cup (0, \infty)$
85. $0 < x < 3^{\frac{1}{1-\log 3}}$ (where the base is 2)
86. $1/2 < x < 1$
87. $x \in \left(-\infty, -\dfrac{5}{2}\right] \cup (0, \infty)$
88. $x \in (4, 5)$

Level II

1. (b)	2. (c)	3. (a)	4. (c)	5. (b)
6. (a)	7. (c)	8. (a)	9. (c)	10. (a)
11. (b)	12. (b)	13. (a)	14. (b)	15. (d)
16. (b)	17. (b)	18. (a)	19. (c)	20. (b)

Level IV

1. 1
2. $\left\{-\sqrt{\dfrac{3}{2}}, \dfrac{1}{2}, \sqrt{\dfrac{3}{2}}\right\}$
3. (0, 49]
4. $\{-\sqrt{6}, \sqrt{6}\}$
5. $x = 8, 1/4; y = 2, 64$
6. {–1, 7}
7. $\left\{0, \dfrac{7}{4}, \dfrac{3+\sqrt{24}}{2}\right\}$
8. $x = 3$ or -3
9. $x = 4$
10. $x = 4$
11. $x = 2$
12. $x = 5$

13. $\left\{\frac{1}{2}, \frac{1}{4}\right\}$
14. $x = 1/100$
15. $x \in (0, 1) \cup \{4\}$

INTEGER TYPE QUESTIONS

1. 3
2. $x = 4$
3. $x = 9$
4. 5
5. 1
6. $x = 1$
7. 8
8. 3
9. 7
10. 4
11. 1

COMPREHENSIVE LINK PASSAGES

Passage I: 1. (c) 2. (a) 3. (b)
Passage II: 1. (a) 2. (b) 3. (a)

MATCHING LIST

1. (C)
2. (D)
3. (A) → (Q);(B) → (P); (C) → (Q); (D) → (R)

Hints and Solutions

Level I

9. Given $\log_a (ab) = x$

$\Rightarrow \log_a a + \log_a b = x$

$\Rightarrow 1 + \log_a b = x$

$\Rightarrow \log_a b = x - 1$

Now, $\log_b (ab) = \log_b a + \log_b b$

$= \log_b a + 1$

$= \frac{1}{x-1} + 1$

$= \frac{x}{x-1}$

10. $\log_{10} 5 = \log_{10}\left(\frac{10}{2}\right)$

$= \log_{10} 10 - \log_{10} 2 = (1 - x)$

11. Now, $ab = \log_4 5 \cdot \log_5 6 = \log_4 6$

$= \log_{2^2}(6)$

$= \frac{1}{2}\log_2(6)$

$= \frac{1}{2}\log_2(2 \times 3)$

$= \frac{1}{2}[\log_2(2) + \log_2 3]$

$= \frac{1}{2}(1 + \log_2 3)$

$\Rightarrow 2ab = 1 + \log_2 3$

$\Rightarrow \log_2 3 = 2ab - 1$

$\Rightarrow \frac{1}{\log_2 3} = \frac{1}{2ab - 1}$

$\Rightarrow \log_3 2 = \frac{1}{(2ab - 1)}$

13. We have

$\frac{1}{\log_2(36)} + \frac{1}{\log_3(36)} = \log_{36}(2) + \log_{36}(3)$

$= \log_{36}(2 \times 3)$

$= \log_{36}(6)$

$= \log_{6^2}(6)$

$= 1/2$

14. We have

$\frac{1}{\log_3 \pi} + \frac{1}{\log_4 \pi} = \log_\pi 3 + \log_\pi 4$

$= \log_\pi (3 \times 4)$

$= \log_\pi (12) > \log_\pi (\pi^2) = 2$

Hence, the value of x is 2.

21. Given $\log_3 2, \log_3(2^x - 5), \log_3\left(2^x - \frac{7}{2}\right)$ are in AP.

$\Rightarrow 2\log_3(2^x - 5) = \log_3 2 + \log_3\left(2^x - \frac{7}{2}\right)$

$\Rightarrow \log_3(2^x - 5)^2 = \log_3 2 \cdot \left(2^x - \frac{7}{2}\right)$

$\Rightarrow (2^x - 5)^2 = 2 \times 2^x - 7$

$\Rightarrow (2^x)^2 - 12 \times 2^x + 32 = 0$

$\Rightarrow a^2 - 12a + 32 = 0$ where $a = 2^x$

$\Rightarrow (a - 4)(a - 8) = 0$

$\Rightarrow a = 4, 8$

When $a = 4 \Rightarrow 2^x = 4 = 2^2 \Rightarrow x = 2$

when $a = 8 \Rightarrow 2^x = 8 = 2^3 \Rightarrow x = 3$

But $x = 2$ does not satisfy the terms.

Hence, the solution of x is 3.

22. We have

$$y = a^{\frac{1}{1-\log_a x}}, \qquad z = a^{\frac{1}{1-\log_a y}}$$

$$\Rightarrow \log_a y = \frac{1}{1-\log_a x}, \qquad \log_a z = \frac{1}{1-\log_a y}$$

$$\Rightarrow \log_a y = \frac{1}{1-\log_a x}, \qquad \frac{1}{\log_a z} = 1-\log_a y$$

$$\Rightarrow \log_a y = \frac{1}{1-\log_a x}, \qquad \log_a y = 1-\frac{1}{\log_a z}$$

Hence, $1-\frac{1}{\log_a z} = \frac{1}{1-\log_a x}$

$$\Rightarrow \frac{1}{\log_a z} = 1-\frac{1}{1-\log_a x}$$

$$\Rightarrow \frac{1}{\log_a z} = \frac{-\log_a x}{1-\log_a x}$$

$$\Rightarrow \log_a z = \frac{1-\log_a x}{-\log_a x} = 1-\frac{1}{\log_a x}$$

$$\Rightarrow \frac{1}{\log_a x} = 1-\log_a z$$

$$\Rightarrow \log_a x = \frac{1}{1-\log_a z}$$

$$\Rightarrow x = a^{\frac{1}{1-\log_a z}}$$

Hence, the result.

24. Let $\frac{\log a}{b-c} = \frac{\log b}{c-a} = \frac{\log c}{a-b} = k$

$$\Rightarrow \frac{a\log a}{a(b-c)} = \frac{b\log b}{b(c-a)} = \frac{c\log c}{c(a-b)} = k$$

$$\Rightarrow \frac{\log a^a}{a(b-c)} = \frac{\log b^b}{b(c-a)} = \frac{\log c^c}{c(a-b)} = k$$

$\Rightarrow$ $\log(a^a) + \log(b^b) + \log(c^c)$

$= k(ab - bc + bc - ba + ca - cb)$

$= 0$

$\Rightarrow$ $\log(a^a b^b c^c) = 0$

$\Rightarrow$ $(a^a b^b c^c) = c^0 = 1$

27. We have,

$\log_2 x + \log_4 x + \log_8 x = 11$

$\Rightarrow$ $\log_2 x + \log_4 x + \log_8 x = 11$

$$\Rightarrow \left(1+\frac{1}{2}+\frac{1}{3}\right)\log_2 x = 11$$

$$\Rightarrow \left(\frac{11}{6}\right)\log_2 x = 11$$

$\Rightarrow$ $\log_2 x = 6$

$\Rightarrow$ $x = 2^6 = 32$

Hence, the value of x is 32.

30. Given

$x = \log_a bc$

$\Rightarrow$ $a_x = bc$

$\Rightarrow$ $a \times a^x = abc$

$\Rightarrow$ $a^{x+1} = abc$

$\Rightarrow$ $x + 1 = \log_a(abc)$

$$\Rightarrow \frac{1}{x+1} = \frac{1}{\log_a(abc)}$$

Similarly, $\frac{1}{y+1} = \frac{1}{\log_b(abc)}$,

and $\frac{1}{z+1} = \frac{1}{\log_c(abc)}$

Thus, $\frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1}$

$$= \frac{1}{\log_a(abc)} + \frac{1}{\log_b(abc)} + \frac{1}{\log_c(abc)}$$

$= \log_{abc}(a) + \log_{abc}(b) + \log_{abc}(c)$

$= \log_{abc}(abc)$

$= 1$

31. We have $y = 2^{\frac{1}{\log_x 4}}$

$$y = 2^{\log_4 x} = 2^{\frac{1}{2}\log_2 x} = 2^{\log_2 (x)^{1/2}} = x^{1/2} = \sqrt{x}$$

36. As we know that

$\log_{30}(30) = 1$

$\Rightarrow$ $\log_{30}(2 \times 3 \times 5) = 1$

$\Rightarrow$ $\log_{30}(2) + \log_{30}(3) + \log_{30}(5) = 1$

$\Rightarrow$ $\log_{30}(2) + a + b = 1$

$\Rightarrow$ $\log_{30}(2) = 1 - (a + b)$

Now, $\log_{30}(10) = \log_{30}(2 \times 5)$

$= \log_{30}(2) + \log_{30}(5)$

$= 1 - (a + b) + b = 1 - a$

Thus,

$\log_{10} 8 = \log_{10}(23) = 3\log_{10}(2)$

$$= 3 \times \frac{\log_{30}(2)}{\log_{30}(10)}$$

$$= \frac{3\{1-(a+b)\}}{(1-a)}$$

37. We have

$ab = \log_{12} 18 \times \log_{24} 54$

$$= \frac{\log 18}{\log 12} \times \frac{\log 54}{\log 24}$$

$$= \frac{\log(2\times 3^2)}{\log(3\times 2^2)} \times \frac{\log(2\times 3^3)}{\log(3\times 2^3)}$$

$$= \frac{\log 2 + 2\log 3}{\log 3 + 2\log 2} \times \frac{\log 2 + 3\log 3}{\log 3 + 3\log 2}$$

$$= \frac{1+2\log_2 3}{\log_2 3 + 2} \times \frac{1+3\log_2 3}{\log_2 3 + 3}$$

$$= \frac{1+2x}{x+2} \times \frac{1+3x}{x+3} \text{ where } x = \log_2 3$$

$$= \frac{1+5x+6x^2}{x^2+5x+6}$$

Also,

$$(a-b) = \log_{12} 18 - \log_{24} 54$$

$$= \frac{\log(2\times 3^2)}{\log(3\times 2^2)} - \frac{\log(2\times 3^3)}{\log(3\times 2^3)}$$

$$= \frac{\log 2 + 2\log 3}{\log 3 + 2\log 2} - \frac{\log 2 + 3\log 3}{\log 3 + 3\log 2}$$

$$= \frac{1+2\log_2 3}{\log_2 3 + 2} - \frac{1+3\log_2 3}{\log_2 3 + 3}$$

$$= \frac{1+2x}{x+2} - \frac{1+3x}{x+3} \text{ where } x = \log_2 3$$

$$= \frac{(1-x^2)}{(2+x)(3+x)}$$

Hence, the value of

$$5(a-b) + ab = \frac{5(1-x^2)}{(2+x)(3+x)} + \frac{(1+5x+6x^2)}{(2+x)(3+x)}$$

$$= \frac{5(1-x^2) + (1+5x+6x^2)}{(2+x)(3+x)}$$

$$= \frac{x^2+5x+6}{(2+x)(3+x)}$$

$$= \frac{(x+2)(x+3)}{(2+x)(3+x)}$$

$$= 1$$

41. Given

$\log_2 x + \log_2 y \geq 6$

$\Rightarrow \quad \log_2 (xy) \geq 6$

$\Rightarrow \quad xy \geq 2^6 = 64$

As we know that, $\frac{x+y}{2} \geq \sqrt{xy}$

$\Rightarrow \quad \frac{x+y}{2} \geq \sqrt{64} = 8$

$\Rightarrow \quad xy \geq 2.8 = 16$

Hence, the least value of $x + y$ is 16.

43. Given,

$x^{18} = y^{21} = z^{28}$

$\Rightarrow \quad \log(x^{18}) = \log(y^{21}) = \log(z^{28})$

$\Rightarrow \quad 18\log x = 21\log y = 28\log z = k$ (say)

Now, $3\log_y x = 3\cdot\frac{\log x}{\log y} = 3\cdot\frac{21}{18} = \frac{7}{2}$

$3\log_z y = 3.\frac{\log y}{\log z} = 3\cdot\frac{28}{21} = 4$

and $\quad 7\log_x z = 7\cdot\frac{\log z}{\log x} = 7\cdot\frac{18}{28} = \frac{9}{2}$

Thus, 3, 7/2, 4, 9/2 are in AP.

Hence, the result.

48. Given equation is

$$4^{\log_9 3} + 9^{\log_2 4} = 10^{\log_x 83}$$

$\Rightarrow \quad 4^{\frac{1}{2}\log_3 3} + 9^{2\log_2 2} = 10^{\log_x 83}$

$\Rightarrow \quad 2 + 81 = 10^{\log_x 83}$

$\Rightarrow \quad 83 = 83^{\log_x 10}$

$\Rightarrow \quad \log_x 10 = 1$

$\Rightarrow \quad x = 10$

Hence, the solution is $x = 10$.

50. We have,

$$abc = \log_{24} 12 \times \log_{36} 24 \times \log_{48} 12$$
$$= \log_{48} 36 \times \log_{36} 24 \times \log_{24} 12$$
$$= \log_{48} 12$$

Now,

$$abc + 1 = \log_{48} 12 + 1 = \log_{48} 12 + \log_{48} 48$$
$$= \log_{48}(12\times 48)$$

Also, $bc = \log_{36} 24 \times \log_{48} 36$
$= \log_{48} 36 \times \log_{36} 24$
$= \log_{48} 24$

Thus, $\left(\frac{abc+1}{bc}\right) = \frac{\log_{48}(12\times 48)}{\log_{48} 24}$

$$= \log_{24}(12\times 48)$$
$$= \log_{24}(24\times 48)$$
$$= \log_{24}(24^2)$$
$$= 2$$

Hence, the result.

51. Given $\log_{10}\left(\sin\left(x+\frac{\pi}{4}\right)\right) = \frac{1}{2}(\log_{10} 6 - 1)$

$\Rightarrow \quad 2\log_{10}\left(\sin\left(x+\frac{\pi}{4}\right)\right) = (\log_{10} 6 - 1)$

$\Rightarrow \quad 2\log_{10}\left(\frac{1}{\sqrt{2}}(\sin x + \cos x)\right) = (\log_{10} 6 - 1)$

$\Rightarrow \quad 2\log_{10}\left(\frac{1}{\sqrt{2}}\right) + 2\log_{10}(\sin x + \cos x)$

$= \log_{10} 6 - 1$

$\Rightarrow \quad 2\log_{10}(\sin x + \cos x) = \log_{10} 6 + 2\log_{10} 2 - 1$

$\Rightarrow \quad \log_{10}(\sin x + \cos x)^2 = \log_{10}\left(\frac{24}{10}\right)$

$\Rightarrow \quad (\sin x + \cos x)^2 = \left(\frac{24}{10}\right)$

$\Rightarrow \quad 1 + \sin 2x = \frac{24}{10}$

$\Rightarrow \quad \sin 2x = \frac{24}{10} - 1 = \frac{14}{10}$

$\Rightarrow \quad \sin x \cdot \cos x = \frac{7}{10}$

$\Rightarrow \quad \log_{10}(\sin x \cos x) = \log_{10}\left(\frac{7}{10}\right)$

$\Rightarrow \quad \log_{10}(\sin x) + \log_{10}(\cos x) = \log_{10} 7 - 1$

52. Given a, b, c are in GP.

$\therefore \quad b^2 = ac$

$\Rightarrow \log(b^2) = \log(ac)$

$\Rightarrow 2\log(b) = \log(a) + \log(c)$

$\Rightarrow$ $\log a, \log b, \log c$ are in AP

$\Rightarrow$ $1 + \log a, 1 + \log b, 1 + \log c$ are in AP

$\Rightarrow \dfrac{1}{1+\log a}, \dfrac{1}{1+\log b}, \dfrac{1}{1+\log c}$ are in HP.

Level III

1. Given

$a^4b^5 = 1$

$\Rightarrow \log(a^4b^5) = \log(1) = 0$

$\Rightarrow 4\log a + 5\log b = 0$

$\Rightarrow 4\log a = -5\log b$

$\Rightarrow \dfrac{\log a}{\log b} = -\dfrac{5}{4}$

$\Rightarrow \log_b a = -\dfrac{5}{4}$

Now,

$\log_a(a^5b^4) = 5\log_a a + 4\log_a b$

$= 5 + 4\log_a b$

$= 5 + \dfrac{4}{\log_b a}$

$= 5 - \dfrac{4\times 4}{5} = \dfrac{25-16}{5} = \dfrac{9}{5}$

2. We have $x = \log_{10}5 \times \log_{10}20 + \log_{10^2}2$

3. The given expression is

$2\log x - \log(2x-75) = 2$

$\Rightarrow \log(x^2) - \log(2x-75) = 2$

$\Rightarrow \log\left(\dfrac{x^2}{2x-75}\right) = 2$

$\Rightarrow \log_{10}\left(\dfrac{x^2}{2x-75}\right) = \log_{10}(10^2)$

$\Rightarrow \dfrac{x^2}{2x-75} = 100$

$\Rightarrow x^2 = 200x - 7500$

$\Rightarrow x^2 - 200x + 7500 = 0$

Hence, the sum of the roots = 200

4. Given

$\log_x(\log_{18}(\sqrt{2}+\sqrt{8})) = -\dfrac{1}{2}$

$\Rightarrow \log_x\left[\log_{18}(\sqrt{2}+2\sqrt{2})\right] = -\dfrac{1}{2}$

$\Rightarrow \log_x\left[\log_{(3\sqrt{2})^2}(3\sqrt{2})\right] = -\dfrac{1}{2}$

$\Rightarrow \log_x\left(\dfrac{1}{2}\right)\left[\log_{(3\sqrt{2})}(3\sqrt{2})\right] = -\dfrac{1}{2}$

$\Rightarrow \log_x\left(\dfrac{1}{2}\right) = -\dfrac{1}{2}$

$\Rightarrow -\log_x 2 = -\dfrac{1}{2}$

$\Rightarrow \log_x 2 = \dfrac{1}{2}$

$\Rightarrow \sqrt{x} = 2$

$\Rightarrow x = 4$

5. Given,

$\log_6 9 - \log_9 27 + \log^8 x = \log_{64} x - \log_6 4$

$\Rightarrow \log_8 x - \log_{64} x = \log_9 27 - \log_6 9 - \log_6 4$

$\Rightarrow \log_8 x - \dfrac{1}{2}\log_8 x = \log_{3^2}(3)^3 - \log_6 36$

$\Rightarrow \dfrac{1}{2}\log_8 x = \dfrac{3}{2} - \log_6(6^2)$

$\Rightarrow \dfrac{1}{2}\log_8 x = \dfrac{3}{2} - 2 = -\dfrac{1}{2}$

$\Rightarrow \dfrac{1}{2}\log_8 x = -\dfrac{1}{2}$

$\Rightarrow \log_8 x = 4$

$\Rightarrow x = \dfrac{1}{8}$

6. We have

$12 + 6\sqrt{3} = 12 + 2\times 3\times\sqrt{3}$

$= 3^2 + (\sqrt{3})^2 + 2\times 3\times\sqrt{3}$

$= (3+\sqrt{3})^2$

Similarly, $12 - 6\sqrt{3} = (3-\sqrt{3})^2$

Now, $x = \sqrt{12+6\sqrt{3}} + \sqrt{12-6\sqrt{3}}$

$= (3+\sqrt{3}) + (3-\sqrt{3})$

$= 6$

Thus, $\log_{36} x = \log_{36} 6$

$= \log_{6^2} 6 = \dfrac{1}{2}\log_6 6 = \dfrac{1}{2}$

7. We have

$\log_a b \times \log_b c = 2 \times 2 = 4$

$\Rightarrow \log_a c = 4$

$\Rightarrow \dfrac{\log_3 c}{\log_3 a} = 4$ …(i)

Also, $\log_3 c = 3 + \log_3 a$

$\Rightarrow 4\log_3 a = 3 + \log_3 a$ [from (i)]

$\Rightarrow 3\log_3 a = 3$

$\Rightarrow \log_3 a = 3$

$\Rightarrow a = 3$

Again, $\log_a b = 2$

$\Rightarrow \quad \log_3 b = 2$

$\Rightarrow \quad b = 3^2 = 9$

Further, $\log_b c = 2$

$\Rightarrow \quad \log_9 c = 2$

$\Rightarrow \quad c = 9^2 = 81$

Thus, $(a + b + c) + 7 = 3 + 9 + 81 + 7 = 100$

8. We have,

$$\log_9 x + \log_4 y = \frac{7}{2}$$

$$\Rightarrow \quad \frac{1}{2}\log_3 x + \frac{1}{2}\log_2 y = \frac{7}{2}$$

$$\Rightarrow \quad \log_3 x + \log_2 y = 7 \qquad \ldots\text{(i)}$$

Also, $\log_9 x - \log_8 y = -\frac{3}{2}$

$$\Rightarrow \quad \log_3 x - \log_2 y = -3 \qquad \ldots\text{(ii)}$$

Adding Eqs (i) and (ii), we get

$2 \log_3 x = 4$

$\Rightarrow \quad \log_3 x = 2$

$\Rightarrow \quad x = 3^2 = 9$

Subtracting Eqs (i) and (ii), we get

$2 \log_2 y = 10$

$\Rightarrow \quad \log_2 y = 5$

$\Rightarrow \quad y = 2^5 = 32$

Hence, the solutions are $x = 9$ and $y = 32$.

9. Given $3^{x+2} = 45$

$$\Rightarrow \quad x + 2 = \log_3 (45) = \log_3 (5 \times 9) = \log_3 5 + \log_3 9 = \log_3 5 + 2$$

$\Rightarrow \quad x = \log_3 5$

$$\Rightarrow \quad x = \frac{\log_{10} 5}{\log_{10} 3}$$

$$\Rightarrow \quad x = \frac{\log_{10}\left(\frac{10}{2}\right)}{\log_{10} 3} = \frac{\log_{10} 10 - \log_{10} 2}{\log_{10} 3}$$

$$\Rightarrow \quad x = \frac{1-a}{b}$$

10. We have,

$$N = 6 \log_{10} 2 + \log_{10} 31 = \log_{10} 2^6 + \log_{10} 31 = \log_{10} (64 \times 31) = \log_{10} (1984) < \log_{10} (1000) = 3$$

Also, $N = \log_{10} (1984) > \log_{10} (10000) = 4$

Thus, the sum of successive integers = 3 + 4 = 7.

11. We have

$$M = \log^2_{\sqrt{2}}\left(\frac{1}{4}\right) = (\log_{\sqrt{2}}(2^{-2}))^2$$

$$= \left(-\frac{2}{1/2}\log_2 2\right)^2 = (-4)^2 = 16$$

and $\quad N = \log^3_{2\sqrt{2}}(8) = [\log_{2\sqrt{2}}(8)]^3$

$$= [\log_{2\sqrt{2}}(2\sqrt{2})^3]^3 = [3 \log_{2\sqrt{2}}(2\sqrt{2})]^3$$

$$= (3)^3 = 27$$

Also, $P = \log_5(\log_3(\sqrt{\sqrt[5]{9}}))$

$$= \log_5 [\log_3 (9^{1/10})]$$

$$= \log_5 [\log_3 (3^{2/10})]$$

$$= \log_5\left(\frac{1}{5}\right)[\log_3(3)]$$

$$= \log_5 (5^{-1}) = -1$$

Thus, $\left(\frac{M}{N} + P + 3\right) = \frac{16}{27} - 1 + 3 = \frac{16}{27} + 2 = \frac{70}{27}$

12. We have,

$$\frac{1}{a} + \frac{1}{b} = \log_x 3 + \log_x 7 = \log_x (21)$$

Thus, $\log_{21}(x) = \frac{1}{\log_x(21)}$

$$= \frac{1}{\frac{1}{a} + \frac{1}{b}} = \frac{ab}{a + \text{b}}$$

13. We have,

$$\log_8 x + \log_4 y^2 = 5$$

$$\Rightarrow \quad \frac{1}{3}\log_2 x + \frac{2}{2}\log_2 y = 5$$

$\Rightarrow \quad \log_2 x^{1/3} + \log_2 y = 5$

$\Rightarrow \quad \log_2 (x^{1/3} y) = 5$

$$\Rightarrow \quad (x^{1/3} y) = 2^5 = 32 \qquad \ldots\text{(i)}$$

Also, $\log_8 y + \log_4 x^2 = 7$

$$\Rightarrow \quad \frac{1}{3}\log_2 y + \frac{2}{2}\log_2 x = 7$$

$\Rightarrow \quad \log_2 y^{1/3} + \log_2 x = 7$

$\Rightarrow \quad \log_2 (y^{1/3} x) = 7$

$$\Rightarrow \quad (y^{1/3} x) = 2^7 = 128 \qquad \ldots\text{(ii)}$$

Multiplying Eqs (i) and (ii), we get

$(x^{4/3} y^{4/3}) = 27 = 128$

$\Rightarrow \quad (xy)^{4/3} = 2^{12}$

$\Rightarrow \quad (xy) = 2^{12\times\frac{3}{4}} = 2^9$

$\Rightarrow \quad 2xy = 2^{10} = 1024$

14. We have,

$\log_{10}(x^2 + x) = \log_{10}(x^3 - x)$

$\Rightarrow \quad x + 1 = x^2 - 1$

$\Rightarrow \quad x + 1 = (x + 1)(x - 1)$

$\Rightarrow \quad x - 1 = 1$

$\Rightarrow \quad x = 2$

Hence, the solution is 2.

Thus, the product of all the solutions = 2.

15. We have $\log_{10}(x - 2) + \log_{10} y = 0$

$\Rightarrow \quad \log_{10}[y(x - 2)] = \log_{10} 1$

$\Rightarrow \quad y(x - 2) = 1 \quad \ldots\text{(i)}$

Also, $\sqrt{x} + \sqrt{y-2} = \sqrt{x+y}$

$\Rightarrow \quad x + y - 2 + 2\sqrt{x(y-2)} = x + y$

$\Rightarrow \quad -2 + 2\sqrt{x(y-2)} = 0$

$\Rightarrow \quad -2 = -2\sqrt{x(y-2)}$

$\Rightarrow \quad x(y - 2) = 1 \quad \ldots\text{(ii)}$

From Eqs (i) and (ii), we get

$x = y$

Put $x = y$ in Eq. (ii), we get

$x(x - 2) = 1$

$\Rightarrow \quad x^2 - 2x - 1 = 0$

$\Rightarrow \quad x^2 - 2x + 1 = 2$

$\Rightarrow \quad (x - 1)^2 = 2$

$\Rightarrow \quad x - 1 = \pm\sqrt{2}$

$\Rightarrow \quad x = 1 \pm \sqrt{2}$

$\Rightarrow \quad x = 1 + \sqrt{2} = y \quad (\because\ x = y)$

Thus, the value of

$x + y - 2\sqrt{2}$

$= 1 + \sqrt{2} + 1 + \sqrt{2} - 2\sqrt{2}$

$= 2$

16. We have,

$\log_{27} a + \log_9 b = \frac{7}{2}$

$\Rightarrow \quad \log_{3^3} a + \log_{3^2} b = \frac{7}{2}$

$\Rightarrow \quad \frac{1}{3}\log_3 a + \frac{1}{2}\log_3 b = \frac{7}{2}$

$\Rightarrow \quad 2\log_3 a + 3\log_3 b = 21 \quad \ldots\text{(i)}$

Also, $\log_{27} b + \log_9 a = \frac{2}{3}$

$\Rightarrow \quad \frac{1}{3}\log_3 b + \frac{1}{2}\log_3 a = \frac{2}{3}$

$\Rightarrow \quad 2\log_3 b + 3\log_3 a = 4 \quad \ldots\text{(ii)}$

Solving Eqs (i) and (ii), we get

$4\log_3 a - 9\log_3 a = 42 - 12$

$\Rightarrow \quad -5\log_3 a = 30$

$\Rightarrow \quad \log_3 a = -6$

$\Rightarrow \quad a = 3^{-6}$

From Eq. (ii), we get

$2\log_3 b - 18 = 5$

$\Rightarrow \quad 2\log_3 b = 2$

$\Rightarrow \quad \log_3 b = 1$

$\Rightarrow \quad b = 3^{11}$

Hence, the value of $ab = 3^{-6} \times 3^{11} = 243$.

17. Given equation is

$\log_{\tan x}(2 + 4\cos^2 x) = 2$

$\Rightarrow \quad 2 + 4\cos^2 x = \tan^2 x$

$\Rightarrow \quad 2 + 4\cos^2 x = \frac{\sin^2 x}{\cos^2 x}$

$\Rightarrow \quad 4\cos^4 x + 2\cos^2 x - \sin^2 x = 0$

$\Rightarrow \quad 4\cos^4 x + 3\cos^2 x - 1 = 0$

$\Rightarrow \quad 4\cos^4 x + 2\cos^2 x - \cos^2 x - 1 = 0$

$\Rightarrow \quad 4\cos^2 x(\cos^2 x + 1) - (\cos^2 x + 1) = 0$

$\Rightarrow \quad (4\cos^2 x - 1)(\cos^2 x + 1) = 0$

$\Rightarrow \quad (4\cos^2 x - 1) = 0 \quad (\because\ x \in [0, 2\pi)$

$\Rightarrow \quad \cos^2 x = \left(\frac{1}{2}\right)^2 = \cos^2\left(\frac{\pi}{3}\right)$

$\Rightarrow \quad x = n\pi \pm \frac{\pi}{3}, n = 0, 1, 2$

$\Rightarrow \quad x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

Hence, the number of values of x is 4.

18. Given,

$\cos(\ln x) = 0$

$\Rightarrow \quad \ln x = \frac{\pi}{2} \quad \Rightarrow \quad x = e^{\frac{\pi}{2}}$

Thus, the value of

$\left(\frac{2}{\pi} \times \log(x) + 10\right)$

$= \left(\frac{2}{\pi} \times \log(e^{\frac{\pi}{2}}) + 10\right)$

$= \left(\frac{2}{\pi} \times \frac{\pi}{2}\log(e) + 10\right)$

$= 1 + 10 = 11$

19. Given, $c(a - b) = a(b - c)$

$\Rightarrow \quad ac - bc = ab - ac$

$\Rightarrow \quad 2ac = ab + bc = b(a + c)$

$\Rightarrow \quad b = \frac{2ac}{(a + c)}$

Now, $\dfrac{\log(a+c)+\log(a-2b+c)}{\log(a-c)}$

$$= \frac{\log\{(a+c)(a+c-2b)\}}{\log(a-c)}$$

$$= \frac{\log\left\{(a+c)\left(a+c-\dfrac{4ac}{(a+c)}\right)\right\}}{\log(a-c)}$$

$$= \frac{\log\{(a+c)^2-4ac\}}{\log(a-c)}$$

$$= \frac{\log(a-c)^2}{\log(a-c)}$$

$$= \frac{2\log(a-c)}{\log(a-c)} = 2$$

20. Given $10^x + 10^{-x} = 4$

$\Rightarrow \quad 10^x + \dfrac{1}{10^x} = 4$

$\Rightarrow \quad (10^x)^2 - 4(10^x) + 1 = 0$

$\Rightarrow \quad 10^x = \dfrac{4 \pm \sqrt{16-4}}{2}$

$\Rightarrow \quad 10^x = \dfrac{4 \pm \sqrt{12}}{2}$

$\Rightarrow \quad 10^x = 2 + \sqrt{3}$

$\Rightarrow \quad x = \log_{10}(2+\sqrt{3})$

Thus $A = 2$, $B = 3$

Hence, the value of $A + B + 3 = 2 + 3 + 3 = 8$.

21. The given equation is

$$\log_{10}(98 + \sqrt{x^3 - x^2 - 12x + 36}) = 2$$

$\Rightarrow \quad 98 + \sqrt{x^3 - x^2 - 12x + 36} = 10^2 = 100$

$\Rightarrow \quad x^3 - x^2 - 12x + 36 = 4$

$\Rightarrow \quad x^3 - x^2 - 12x + 32 = 0$

$\Rightarrow \quad x^3 + 4x^2 - 5x^2 - 20x + 8x + 32 = 0$

$\Rightarrow \quad x^2(x+4) - 5x(x+4) + 8(x+4) = 0$

$\Rightarrow \quad (x+4)(x^2 - 5x + 8) = 0$

$\Rightarrow \quad x + 4 = 0$

$\Rightarrow \quad x = -4$

Hence, the solution set is $\{-4\}$.

22. We have

$$\log_{10} x + \log x^{1/2} + \log_{10} x^{1/4} + \ldots = y$$

$\Rightarrow \quad \log_{10}x + \dfrac{1}{2}\log_{10}x + \dfrac{1}{4}\log_{10}x + \ldots = y$

$\Rightarrow \quad y = \left(1 + \dfrac{1}{2} + \dfrac{1}{4} + \ldots\right)\log_{10}x$

$$= \left(\frac{1}{1-\frac{1}{2}}\right)\log_{10}x = 2\log_{10}x$$

Also, $\dfrac{1+3+5+\ldots+(2y-1)}{4+7+10+\ldots+(3y+1)} = \dfrac{20}{7\log_{10}x}$

$\Rightarrow \quad \dfrac{\frac{y}{2}(1+2y-1)}{\frac{y}{2}(4+3y+1)} = \dfrac{20}{7\left(\frac{y}{2}\right)}$

$\Rightarrow \quad \dfrac{2y}{(3y+5)} = \dfrac{40}{7y}$

$\Rightarrow \quad \dfrac{y}{(3y+5)} = \dfrac{20}{7y}$

$\Rightarrow \quad 7y^2 = 60y + 100$

$\Rightarrow \quad 7y^2 - 60y - 100 = 0$

$\Rightarrow \quad (y-10)(10y+7) = 0$

$\Rightarrow \quad y = 10, -10/7$

When $y = 10$,

$2\log_{10} x = 10$

$\Rightarrow \quad \log_{10} x = 5$

$\Rightarrow \quad x = 10^5$

When $y = -10/7$,

$2\log_{10} x = -10/7$

$\Rightarrow \quad \log_{10} x = -5/7$

$\Rightarrow \quad x = 10^{-5/7}$

23. We have

$$\frac{6}{5}a^{\log_a x \log_{10} a \log_a 5 - 3\log_{10}\left(\frac{x}{10}\right)} = 9^{\log_{100}x + \log_4 2}$$

$\Rightarrow \quad \dfrac{6}{5}a^{\frac{\log x}{\log a}\cdot\frac{\log 5}{\log 10} - 3(\log_{10}x - 1)} = 9^{\log_{100}x + \log_{2^2} 2}$

$\Rightarrow \quad \dfrac{6}{5}a^{\log_a x \log_{10} 5. - 3(\log_{10}x - 1)} = 9^{\log_{10^2}x + \log_{2^2} 2}$

$\Rightarrow \quad \dfrac{6}{5}a^{\log_a x \log_{10} 5. - 3(\log_{10}x - 1)} = 9^{\frac{1}{2}(\log_{10}x + 1)}$

$\Rightarrow \quad \dfrac{6}{5}a^{\log_a x \log_{10} 5. - 3(\log_{10}x - 1)} = 3^{(\log_{10}x + 1)}$

$\Rightarrow \quad \dfrac{2}{5}a^{\log_a x \log_{10} 5} \cdot a^{3 - 3\log_{10}x} = 3^{\log_{10}x}$

$\Rightarrow \quad \dfrac{2}{5}a^{\log_{10} x \log_a 5} \cdot a^{3 - 3\log_{10}x} = 3^{\log_{10}x}$

$\Rightarrow \quad \dfrac{2}{5}a^{b\log_a 5} \cdot a^{3-3b} = 3^b, b = \log_{10}x$

It is possible only when, $b = 1$

i.e., $\log_{10} x = 1$

$\Rightarrow \quad x = 10^1 = 10$

Hence, the solution is $x = 10$.

24. We have

$$|x-1|^{\log_3 x^2 - 2\log_x 9} = (x-1)^7$$

When $x > 1$,

$$(x-1)^{\log_3 x^2 - 2\log_x 9} = (x-1)^7$$

$\Rightarrow \log_3 x^2 - 2\log_x 9 = 7$

$\Rightarrow \log_3 x^2 - 2\log_x 3^2 = 7$

$\Rightarrow 2\log_3 x - 4\log_x 3 = 7$

$\Rightarrow 2\log_3 x - \dfrac{4}{\log_3 x} = 7$

$\Rightarrow 2(\log_3 x)^2 - 7\log_3 x - 4 = 0$

$\Rightarrow 2a^2 - 7a - 4 = 0,\ a = \log_3 x$

$\Rightarrow 2a^2 - 8a + a - 4 = 0$

$\Rightarrow 2a(a-4) + 1(a-4) = 0$

$\Rightarrow (a-4)(2a+1) = 0$

$\Rightarrow a = 4, -\dfrac{1}{2}$

When $a = 4$,

$\log_3 x = 4$

$\Rightarrow x = 3^4 = 81$

When $a = -\dfrac{1}{2}$,

$\log_3 x = -\dfrac{1}{2}$

$\Rightarrow x = 3^{-1/2} = \dfrac{1}{\sqrt{3}} < 1$

It is not possible, since $x > 1$

Hence, the solution is $\{2, 81\}$

25. We have

$\log_{100}|x+y| = \dfrac{1}{2}$

$\Rightarrow \log_{10^2}|x+y| = \dfrac{1}{2}$

$\Rightarrow \dfrac{1}{2}\log_{10}|x+y| = \dfrac{1}{2}$

$\Rightarrow \log_{10}|x+y| = 1$

$\Rightarrow |x+y| = 10$

$\Rightarrow x + y = \pm 10$ …(i)

Also, $\log_{10}\left(\dfrac{y}{|x|}\right) = \log_{100} 4$

$\Rightarrow \log_{10}\left(\dfrac{y}{|x|}\right) = \log_{10^2} 2^2$

$\Rightarrow \log_{10}\left(\dfrac{y}{|x|}\right) = \log_{10} 2$

$\Rightarrow \dfrac{y}{|x|} = 2$

$\Rightarrow |x| = \dfrac{y}{2}$

$\Rightarrow x = \pm\dfrac{y}{2}$ …(ii)

From Eqs (i) and (ii), we get

$y = \pm\dfrac{20}{3}, \pm 20$

and $x = \pm\dfrac{10}{3}, \mp 10$

Hence, the solutions are

$$\left\{x = \frac{10}{3},\ y = \frac{20}{3};\ x = 10,\ y = -20\right\}.$$

26. We have

$2\log_2\log_2 x + \log_{1/2}\log_2(2\sqrt{2}x) = 1$

$\Rightarrow 2\log_2\log_2 x - \log_2\log_2(2\sqrt{2}x) = 1$

$\Rightarrow \log_2(\log_2 x)^2 - \log_2(\log_2(2\sqrt{2}x)) = 1$

$\Rightarrow \log_2\left(\dfrac{(\log_2 x)^2}{(\log_2(2\sqrt{2}x))}\right) = 1$

$\Rightarrow \left(\dfrac{(\log_2 x)^2}{(\log_2(2\sqrt{2}x))}\right) = 2$

$\Rightarrow \left(\dfrac{(\log_2 x)^2}{(\log_2(2\sqrt{2}) + \log_2 x)}\right) = 2$

$\Rightarrow \left(\dfrac{(\log_2 x)^2}{\left(\log_2(2^{3/2}) + \log_2 x\right)}\right) = 2$

$\Rightarrow \left(\dfrac{(\log_2 x)^2}{\left(\frac{3}{2} + \log_2 x\right)}\right) = 2$

$\Rightarrow \left(\dfrac{a^2}{\frac{3}{2} + a}\right) = 2$, where $a = \log_2 x$

$\Rightarrow \dfrac{2a^2}{3 + 2a} = 2$

$\Rightarrow 2a^2 - 4a - 6 = 0$

$\Rightarrow a^2 - 2a - 3 = 0$

$\Rightarrow (a-3)(a+1) = 0$

$\Rightarrow a = -1, 3$

When $a = -1 \Rightarrow \log_2 x = -1 \Rightarrow x = 2^{-1} = \dfrac{1}{2}$

When $a = 3 \Rightarrow \log_2 x \Rightarrow x = 2^3 = 8$

Hence, the solution is $x = 8$.

27. We have

$\log_{3/4}\log_8(x^2+7) + \log_{1/2}\log_{1/4}(x^2+7)^{-1} = -2$

$\Rightarrow \log_{3/4}\log_{2^3}(x^2+7) + \log_{2^{-1}}\log_{2^{-2}}(x^2+7)^{-1} = -2$

$$\Rightarrow \quad \log_{3/4}\left(\frac{1}{3}(\log_2(x^2+7))\right) - \log_2\left(\frac{1}{2}\log_2(x^2+7)\right) = -2$$

$$\Rightarrow \quad \log_{3/4}\left(\frac{1}{3}y\right) - \log_2\left(\frac{1}{2}y\right) = -2,$$

where $y = (\log_2(x^2+7))$

$$\Rightarrow \quad \log_{3/4}\left(\frac{y}{3}\right) - \log_2\left(\frac{y}{2}\right) = -2$$

$\Rightarrow \quad y = 4$, by trial

$\Rightarrow \quad \log_2(x^2+7) = 4$

$\Rightarrow \quad x^2 + 7 = 2^4 = 16$

$\Rightarrow \quad x^2 = 9$

$\Rightarrow \quad x = \pm 3$

Hence, the solutions are $\{-3, 3\}$.

28. We have,

$$y = \log_2 x + \log_4 x + \log_8 x + \ldots$$

$$= \log_2 x + \frac{1}{2}\log_2 x + \frac{1}{4}\log_2 x + \ldots$$

$$= \left(1 + \frac{1}{2} + \frac{1}{4} + \ldots\right)\log_2 x$$

$$= \left(\frac{1}{1-\frac{1}{2}}\right)\log_2 x = 2\log_2 x$$

Also, $\dfrac{5+9+13+\ldots+(4y+1)}{1+3+5+\ldots+(2y-1)} = 4\log_4 x$

$$\Rightarrow \quad \frac{\frac{y}{2}(2.5+(y-1)4)}{\frac{y}{2}(2.+(y-1)2)} = 4\log_4 x$$

$$\Rightarrow \quad \frac{(10+4y-4)}{(2+2y-2)} = 4\log_4 x$$

$$\Rightarrow \quad \frac{(6+4y)}{(2y)} = 4\log_4 x$$

$$\Rightarrow \quad \frac{(3+2y)}{y} = 4\log_{2^2} x = 2\log_2 x = y$$

$\Rightarrow \quad y^2 = 2y + 3$

$\Rightarrow \quad y^2 - 2y - 3 = 0$

$\Rightarrow \quad (y-3)(y+1) = 0$

$\Rightarrow \quad y = -1, 3$

$\Rightarrow \quad y = 3$ ($y = -1$ is not possible)

When $y = 3$,

$$2\log_2 x = 3$$

$$\Rightarrow \quad \log_2 x = \frac{3}{2}$$

$\Rightarrow \quad x = 2^{3/2}$

Hence, the solutions are $x = 2^{3/2}$ and $y = 3$.

29. Given,

$$\log_{225} x + \log_{64} y = 4$$

$$\Rightarrow \quad \log_{15^2} x + \log_{8^2} y = 4$$

$$\Rightarrow \quad \log_{15} x + \log_8 y = 4 \qquad \ldots\text{(i)}$$

Also, $\log_x(225) - \log_y(64) = 1$

$$\Rightarrow \quad \log_x(15) - \log_y(8) = \frac{1}{2} \qquad \ldots\text{(ii)}$$

From Eqs (i) and (ii), we get

$$\log_x(15) - \frac{1}{8-\log_{15}(x)} = \frac{1}{2}$$

$$\Rightarrow \quad \frac{1}{\log_{15}(x)} - \frac{1}{8-\log_{15}(x)} = \frac{1}{2}$$

$$\Rightarrow \quad \frac{1}{p} - \frac{1}{8-p} = \frac{1}{2} \text{ where } p = \log_{15}(x)$$

$$\Rightarrow \quad \frac{1}{p} - \frac{1}{8-p} = \frac{1}{2}$$

$$\Rightarrow \quad \frac{8-p-p}{p(8-p)} = \frac{1}{2}$$

$$\Rightarrow \quad \frac{8-2p}{p(8-p)} = \frac{1}{2}$$

$\Rightarrow \quad 16 - 4p = 8p - p^2$

$\Rightarrow \quad p^2 - 12p + 16 = 0$

$\Rightarrow \quad p = 6 \pm 2\sqrt{5}$

$\Rightarrow \quad \log_{15} x = 6 \pm 2\sqrt{5}$

$\Rightarrow \quad x = 15^{(6\pm 2\sqrt{5})}$

and $\quad \log_8 y = 8 - \log_{15} x$

$\Rightarrow \quad \log_8 y = 8 - (6 \pm 2\sqrt{5})$

$\Rightarrow \quad \log_8 y = 2 \pm 2\sqrt{5}$

$\Rightarrow \quad y = 8^{2\pm 2\sqrt{5}}$

Hence, the solutions are

$$(15^{6+2\sqrt{5}}, 8^{2-2\sqrt{5}}), (15^{6-2\sqrt{5}}, 8^{2+2\sqrt{5}})$$

Now, $\dfrac{1}{2}\log_{30}(abcd)$

$$= \frac{1}{2}\log_{30}(15^{6+2\sqrt{5}+6-2\sqrt{5}} \times 8^{2-2\sqrt{5}+2+2\sqrt{5}})$$

$$= \frac{1}{2}\log_{30}(15^{12} \times 8^4)$$

$$= \frac{1}{2}\log_{30}(15^{12} \times 2^{12})$$

$$= \frac{1}{2}\log_{30}(15 \times 2)^{12}$$

$$= \frac{12}{2}\log_{30}(30)$$

$$= 6$$

30. We have,

$$x = 1 + \log_a$$
$$= \log_a a + \log_a bc = \log_a abc$$
$$\Rightarrow \quad \frac{1}{x} = \frac{1}{\log_a abc}$$
$$\Rightarrow \quad \frac{1}{x} = \log_{abc} a$$

Similarly, $\frac{1}{y} = \log_{abc} b$

and $\frac{1}{z} = \log_{abc} c$

Now, $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \log_{abc} a + \log_{abc} b + \log_{abc} c$

$$= \log_{abc}(abc)$$
$$= 1$$

Hence, $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$

$$\Rightarrow \quad \frac{xy + yz + zx}{xyz} = 1$$
$$\Rightarrow \quad \frac{xyz}{xy + yz + zx} = 1$$

Level IV

1. Given equation is

$$x + \log_{10}(1 + 2x) = x\log_{10} 5 + \log_{10} 6$$
$$\Rightarrow \quad x(1 - \log_{10} 5) + \log_{10}\left(\frac{1 + 2^x}{6}\right) = 0$$
$$\Rightarrow \quad x(\log_{10} 2) + \log_{10}\left(\frac{1 + 2^x}{6}\right) = 0$$
$$\Rightarrow \quad \log_{10}\left(\frac{1 + 2^x}{6}\right) = -x(\log_{10} 2)$$
$$\Rightarrow \quad \log_{10}\left(\frac{1 + 2^x}{6}\right) = \log_{10} 2^{-x}$$
$$\Rightarrow \quad \left(\frac{1 + 2^x}{6}\right) = 2^{-x} = \frac{1}{2^x}$$
$$\Rightarrow \quad (2^x)^2 + 2^x - 6 = 0$$
$$\Rightarrow \quad a^2 + a - 6 = 0 \text{ where } a = 2^x$$
$$\Rightarrow \quad (a + 3)(a - 2) = 0$$
$$\Rightarrow \quad a = 2 - 3$$
$$\Rightarrow \quad 2^x = 2, -3$$
$$\Rightarrow \quad 2^x = 2$$
$$\Rightarrow \quad x = 1$$

Hence, the solution is $x = 1$.

2. Given equation is

$$\log\left|\frac{x^2 - x - 1}{x^2 + x - 2}\right| = 0$$
$$\Rightarrow \quad \left|\frac{x^2 - x - 1}{x^2 + x - 2}\right| = 1$$
$$\Rightarrow \quad |x^2 - x - 1| = |x^2 + x - 2|$$
$$\Rightarrow \quad x^2 - x - 1 = \pm(x^2 + x - 2)$$

Taking positive sign, we get

$$\Rightarrow \quad (x^2 - x - 1) = (x^2 + x - 2)$$
$$\Rightarrow \quad 2x = 1 \quad \Rightarrow \quad x = 1/2$$

Taking negative sign, we get

$$(x^2 - x - 1) = (x^2 + x - 2)$$
$$\Rightarrow \quad 2x^2 = 3 \quad \Rightarrow \quad x = \pm\sqrt{\frac{3}{2}}$$

Hence, the solutions are

$$\left\{-\sqrt{\frac{3}{2}}, \frac{1}{2}, \sqrt{\frac{3}{2}}\right\}$$

3. Given equation is

$$|4 + \log_{1/7} x| = 2 + |2 + \log_{1/7} x|$$
$$\Rightarrow \quad |4 + \log_{1/7} x| = |2| + |2 + \log_{1/7} x|$$
$$\Rightarrow \quad 2(2 + \log_{1/7} x) \geq 0$$
$$\Rightarrow \quad (2 + \log_{1/7} x) \geq 0$$
$$\Rightarrow \quad \log_{1/7} \geq -2$$
$$\Rightarrow \quad x \leq \left(\frac{1}{7}\right)^{-2}$$
$$\Rightarrow \quad x \leq 49$$

Hence, the value of x is (0, 49].

4. Given equation is

$$\log^2\left(1 + \frac{4}{x}\right) + \log^2\left(1 - \frac{4}{x+4}\right) = 2\log^2\left(\frac{2}{x-1} - 1\right)$$
$$\Rightarrow \quad \log^2\left(\frac{x+4}{x}\right) + \log^2\left(\frac{x}{x+4}\right) = 2\log^2\left(\frac{2}{x-1} - 1\right)$$
$$\Rightarrow \quad \log^2\left(\frac{x+4}{x}\right) + \log^2\left(\frac{x}{x+4}\right) = 2\log^2\left(\frac{3-x}{x-1}\right)$$
$$\Rightarrow \quad -2\log\left(\frac{x}{x+4}\right)\log\left(\frac{x+4}{x}\right) = 2\log^2\left(\frac{3-x}{x-1}\right)$$
$$\Rightarrow \quad -\log\left(\frac{x}{x+4}\right)\log\left(\frac{x+4}{x}\right) = \log^2\left(\frac{3-x}{x-1}\right)$$
$$\Rightarrow \quad \log^2\left(\frac{x}{x+4}\right) = \log^2\left(\frac{3-x}{x-1}\right)$$
$$\Rightarrow \quad \left(\frac{x}{x+4}\right) = \left(\frac{3-x}{x-1}\right)$$
$$\Rightarrow \quad x^2 - x = 3x + 12 - x^2 - 4x$$

$\Rightarrow$ $2x^2 = 12$

$\Rightarrow$ $x^2 = 6$

$\Rightarrow$ $x = \pm\sqrt{6}$

Hence, the solution is $\{-\sqrt{6}, \sqrt{6}\}$.

5. We have,

$$\log_y x - \log_x y = \frac{8}{3}$$

$\Rightarrow$ $\frac{\log x}{\log y} - \frac{\log y}{\log x} = \frac{8}{3}$

$\Rightarrow$ $(\log x)^2 - (\log y)^2 = \frac{8}{3}\log x \log y$

$\Rightarrow$ $3(\log x)^2 - 8\log x \log y - 3(\log y)^2 = 0$

$\Rightarrow$ $3a^2 - 8ab - 3b^2 = 0$

Where $a = \log x$, $b = \log y$

$\Rightarrow$ $3a^2 - 9ab + ab - 3b^2 = 0$

$\Rightarrow$ $3a(a - 3b) + b(a - 3b) = 0$

$\Rightarrow$ $(a - 3b)(3a + b) = 0$

$\Rightarrow$ $(a - 3b) = 0, (3a + b) = 0$

$\Rightarrow$ $a = 3b, b = -3a$

$\Rightarrow$ $\log x = 3\log y, \log y = -3\log x$

$\Rightarrow$ $\log\left(\frac{x}{y^3}\right) = 0, \log(yx^3) = 0$

$\Rightarrow$ $\left(\frac{x}{y^3}\right) = 1, (yx^3) = 1$

Now, $x^3y = 1$

$\Rightarrow$ $x^2(xy) = 1$

$\Rightarrow$ $x^2 = \frac{1}{xy} = \frac{1}{16}$

$\Rightarrow$ $x = \frac{1}{4}$

When $x = \frac{1}{4}$, $y = \frac{1}{x^3} = 64$

Also, $\frac{x}{y^3} = 1$

$\Rightarrow$ $y^4 = xy = 16$

$\Rightarrow$ $y = 2$

When $y = 2$, $x = 8$

Hence, the solutions set are

$$\left(\frac{1}{4}, 64\right), (8, 2)$$

6. Given equation is

$$\log\left(\frac{3x^2+12x+19}{3x+4}\right) + \log_{25} 4 = 1 + \frac{1}{4}\log_2(\sqrt[5]{256})$$

$\Rightarrow$ $\log\left(\frac{3x^2+12x+19}{3x+4}\right) + \frac{2}{5} = 1 + \frac{1}{4}\log_2(\sqrt[5]{256})$

$\Rightarrow$ $\log\left(\frac{3x^2+12x+19}{3x+4}\right) = \frac{3}{5} + \frac{1}{4}\log_2(2^{8/5})$

$\Rightarrow$ $\log\left(\frac{3x^2+12x+19}{3x+4}\right) = \frac{3}{5} + \frac{2}{5}\log_2(2)$

$\Rightarrow$ $\log\left(\frac{3x^2+12x+19}{3x+4}\right) = 1$

$\Rightarrow$ $\frac{3x^2+12x+19}{3x+4} = 10$

$\Rightarrow$ $3x^2 + 12x + 19 = 30x + 40$

$\Rightarrow$ $3x^2 - 18x - 21 = 0$

$\Rightarrow$ $(x - 7)(3x + 3) = 0$

$\Rightarrow$ $x = -1, 7$

Hence, the solution set is $\{-1, 7\}$

7. Given equation is

$$\log^2(4-x) + \log(4-x)\cdot\log\left(x+\frac{1}{2}\right) - 2\log^2\left(x+\frac{1}{2}\right) = 0$$

$\Rightarrow$ $a^2 + ab - 2b^2 = 0,$

where $a = \log(4 - x), b = \log\left(x + \frac{1}{2}\right)$

$\Rightarrow$ $(a - b)(a + 2b) = 0$

$\Rightarrow$ $a = b, -2b$

When $a = b$

$\Rightarrow$ $\log(4 - x) = \log\left(x + \frac{1}{2}\right)$

$\Rightarrow$ $4 - x = x + \frac{1}{2}$

$\Rightarrow$ $2x = \frac{7}{2}$

$\Rightarrow$ $x = \frac{7}{4}$

When $a = -2b$

$\Rightarrow$ $\log(4 - x) = -2\log\left(x + \frac{1}{2}\right)$

$\Rightarrow$ $4 - x = \left(x + \frac{1}{2}\right)^{-2}$

$\Rightarrow$ $= \frac{1}{\left(x + \frac{1}{2}\right)^2}$

$\Rightarrow$ $(4 - x)\left(x + \frac{1}{2}\right)^2 = 1$

$\Rightarrow$ $(4 - x)\left(x^2 + x + \frac{1}{4}\right) = 1$

$\Rightarrow$ $(4 - x)(4x^2 + 4x + 1) = 4$

$\Rightarrow$ $16x^2 + 16x + 4 - 4x^3 - 4x^2 - x = 4$

$\Rightarrow$ $12x^2 + 15x - 4x^3 = 0$

$\Rightarrow$ $x(4x^2 - 12x - 15) = 0$

$\Rightarrow$ $x = 0, 4x^2 - 12x - 15 = 0$

$\Rightarrow \quad x = 0, x = \frac{12 \pm \sqrt{144 + 240}}{8}$

$\Rightarrow \quad x = 0, x = \frac{12 \pm \sqrt{384}}{8}$

$\Rightarrow \quad x = 0, x = \frac{12 \pm 4\sqrt{24}}{8}$

$\Rightarrow \quad x = 0, x = \frac{3 \pm \sqrt{24}}{2}$

$\Rightarrow \quad x = 0, x = \frac{3 + \sqrt{24}}{2}$

Hence, the solutions are

$$\left\{0, \frac{7}{4}, \frac{3 + \sqrt{24}}{2}\right\}$$

8. $\log_{3/4} \log_8 (x^2 + 7) + \log_{1/2} \log_{1/4} (x^2 + 7)–1 = –2$

$\Rightarrow \quad \log_{3/4}\log_{2^3}(x^2+7) + \log_{2^{-1}}\log_{2^{-2}}(x^2+7)^{-1} = -2$

$\log_{3/4}\left(\frac{1}{3}(\log_2(x^2+7))\right)$

$-\log_2\left(\frac{1}{2}(\log_2(x^2+7))\right) = -2$

$\Rightarrow \quad \log_{3/4}\left(\frac{1}{3}y\right) - \log_2\left(\frac{1}{2}y\right) = -2, y = (\log_2(x^2+7))$

$\Rightarrow \quad \log_{3/4}\left(\frac{y}{3}\right) - \log_2\left(\frac{y}{2}\right) = -2$

$\Rightarrow \quad y = 4$, by trial.

$\Rightarrow \quad \log_2 (x^2 + 7) = 4$

$\Rightarrow \quad x^2 + 7 = 2^2 = 16$

$\Rightarrow \quad x^2 = 9$

$\Rightarrow \quad x = \pm 13$

Hence, the solutions are $\{–3, 3\}$.

9. Given equation is

$\log_{10} (x^2 – x – 6) – x = \log_{10} (x + 2) – 4$

$\Rightarrow \quad \log_{10}\left(\frac{x^2 - x - 6}{x+2}\right) = (x-4)$

$\Rightarrow \quad \log_{10}\left(\frac{(x-3)(x+2)}{x+2}\right) = (x-4)$

$\Rightarrow \quad \log_{10} (x – 3) = (x – 4)$

$\Rightarrow \quad (x – 3) = 10^{(x–4)}$

Clearly $x = 4$ is the required solution by trial.

10. Given equation is

$\frac{1}{2}\log_5(x+5) + \log_5(\sqrt{x-3}) = \frac{1}{2}\log_5(2x+1)$

$\Rightarrow \quad \frac{1}{2}\log_5(x+5) + \frac{1}{2}\log_5(x-3) = \frac{1}{2}\log_5(2x+1)$

$\Rightarrow \quad \log_5 (x + 5) + \log_5 (x – 3) = \log_5 (2x + 1)$

$\Rightarrow \quad \log_5 \{(x + 5)(x – 3)\} = \log_5 (2x + 1)$

$\Rightarrow \quad (x + 5)(x – 3) = 2x + 1$

$\Rightarrow \quad x^2 + 2x – 15 = 2x + 1$

$\Rightarrow \quad x^2 = 16$

$\Rightarrow \quad x = \pm 4$

$\Rightarrow \quad x = 4$ is the required solution.

11. Given equation is

$\frac{3}{2}\log_4(x+2)^2 + 3 = \log_4(4-x)^3 + \log_4(6+x)^3$

$\Rightarrow \quad 3 \log_4 (x + 2) + 3 = 3 \log_4 (4 – x) + 3 \log_4 (6 + x)$

$\Rightarrow \quad \log_4 (x + 2) + 1 = \log_4 (4 – x) + \log_4 (6 + x)$

$\Rightarrow \quad \log_4 \{4(x + 2)\} = \log_4 (4 – x)(6 + x)$

$\Rightarrow \quad 4(x + 2) = (4 – x)(6 + x)$

$\Rightarrow \quad 4x + 8 = 24 – 2x – x^2$

$\Rightarrow \quad x^2 + 6x – 16 = 0$

$\Rightarrow \quad (x + 8)(x – 2) = 0$

$\Rightarrow \quad x = 2, –8$

Hence, the solution is $x = 2$.

12. The given equation is

$\frac{1 + \log_2(x-4)}{2\log_2(\sqrt{x+3} - \sqrt{x-3})} = 1$

$\Rightarrow \quad 1 + \log_2(x-4) = 2\log_2(\sqrt{x+3} - \sqrt{x-3})$

$\Rightarrow \quad \log_2\{2(x-4)\} = \log_2(\sqrt{x+3} - \sqrt{x-3})^2$

$\Rightarrow \quad 2(x-4) = (\sqrt{x+3} - \sqrt{x-3})^2$

$\Rightarrow \quad 2(x-4) = x + 3 + x - 3 - 2\sqrt{x^2 - 9}$

$\Rightarrow \quad 2\sqrt{x^2 - 9} = -8$

$\Rightarrow \quad \sqrt{x^2 - 9} = -4$

$\Rightarrow \quad x^2 – 9 = 16$

$\Rightarrow \quad x^2 = 25$

$\Rightarrow \quad x = \pm 5$

$\Rightarrow \quad x = 5$

Hence, the solution is $x = 5$.

13. Given equation is

$\left(1 + \frac{1}{2x}\right)\log 3 = \log\left(\frac{\sqrt[x]{3} + 27}{4}\right)$

$\Rightarrow \quad \log 3^{\left(1+\frac{1}{2x}\right)} = \log\left(\frac{\sqrt[x]{3} + 27}{4}\right)$

$\Rightarrow \quad 3^{\left(1+\frac{1}{2x}\right)} = \left(\frac{\sqrt[x]{3} + 27}{4}\right)$

$\Rightarrow \quad 4 \times 3^{\left(1+\frac{1}{2x}\right)} = 3^{\frac{1}{x}} + 27$

$\Rightarrow \quad 4 \times 3 \times 3^{\frac{1}{2x}} = 3^{\frac{1}{x}} + 27$

$\Rightarrow \quad 12 \times a = a^2 + 27, a = 3^{\frac{1}{2x}}$

$\Rightarrow \quad 12 \times a = a^2 + 27$

$\Rightarrow \quad a^2 – 12a + 27 = 0$

$\Rightarrow \quad (a – 3)(a – 9) = 0$

$\Rightarrow \quad a = 3, 9$

$\Rightarrow \quad 3^{\frac{1}{2x}} = 3, 3^2$

$\Rightarrow \quad x = \frac{1}{2}, \frac{1}{4}$

Hence, the solutions are $\left\{\frac{1}{2}, \frac{1}{4}\right\}$.

14. Given equation is

$$4^{\log_{10}x+1} - 6^{\log_{10}x} - 2\times 3^{\log_{10}x^2+2} = 0$$

$\Rightarrow \quad 4\times 4^{\log_{10}x} - 6^{\log_{10}x} - 2\times 3^2 \times 3^{\log_{10}x^2} = 0$

$\Rightarrow \quad 4\times 4^a - 6^a - 18\times 32^a = 0, a = \log_{10} x$

$\Rightarrow \quad 4\times 4^a - 6^a - 18\times 32^a = 0$

$\Rightarrow \quad 4(2^a)^2 - 2^a \times 3^a - 18(3^a)^2 = 0$

$\Rightarrow \quad 4b^2 - bc - 18c^2 = 0$, where $b = (2^a)$, $c = 3^a$

$\Rightarrow \quad 4b^2 - 9bc + 8bc - 18c^2 = 0$

$\Rightarrow \quad b(4b - 9c) + 2c(4b - 9c) = 0$

$\Rightarrow \quad (4b - 9c)(b + 2c) = 0$

$\Rightarrow \quad 4b - 9c = 0, b + 2c = 0$

$\Rightarrow \quad 4b = 9c, b = -2c$

$\Rightarrow \quad 4\times 2^a = 9\times 3^a, 2^a = -2.3^a$

$\Rightarrow \quad 4\times 2^a = 9\times 3^a$

$\Rightarrow \quad 2^{a+2} = 3^{a+2}$

$\Rightarrow \quad \left(\frac{2}{3}\right)^{a+2} = 1 = \left(\frac{2}{3}\right)^0$

$\Rightarrow \quad a + 2 = 0$

$\Rightarrow \quad a = -2$

$\Rightarrow \quad \log_{10} x = -2$

$\Rightarrow \quad x = 10^{-2} = \frac{1}{100}$

Hence, the solution is $x = 10^{-2} = \frac{1}{100}$

15. Given equation is

$$\log_3(\sqrt{x} + |\sqrt{x} - 1|)^2 = \log_3(4\sqrt{3} - 3 + 4|\sqrt{x} - 1|)$$

$\Rightarrow \quad (\sqrt{x} + |\sqrt{x} - 1|)^2 = 4\sqrt{3} - 3 + 4|\sqrt{x} - 1|$

$\Rightarrow \quad \sqrt{x} + 2\sqrt{x}|\sqrt{x} - 1| + x - 2\sqrt{x} + 1$
$= (4\sqrt{3} - 3 + 4|\sqrt{x} - 1|)$

$\Rightarrow \quad 2\sqrt{x}(\sqrt{x} - 1) + 2\sqrt{x}(|\sqrt{x} - 1|)$
$= 4(\sqrt{x} - 1) + 4|\sqrt{x} - 1|$

$\Rightarrow \quad 2\sqrt{x}(\sqrt{x} - 1) + 2|\sqrt{x} - 1| = 4[(\sqrt{x} - 1) + |\sqrt{x} - 1|]$

$\Rightarrow \quad 2\sqrt{x}[(\sqrt{x} - 1) + |\sqrt{x} - 1|] = 4[(\sqrt{x} - 1) + |\sqrt{x} - 1|]$

$\Rightarrow \quad (2\sqrt{x} - 4)[(\sqrt{x} - 1) + |\sqrt{x} - 1|] = 0$

$\Rightarrow \quad (2\sqrt{x} - 4) = 0, \left|(\sqrt{x} - 1) + |\sqrt{x} - 1|\right| = 0$

$\Rightarrow \quad \sqrt{x} = 2, \left|(\sqrt{x} - 1) + |\sqrt{x} - 1|\right| = 0$

$\Rightarrow \quad \sqrt{x} = 2, (\sqrt{x} - 1) < 0$

$\Rightarrow \quad x = 4, \sqrt{x} < 1$

$\Rightarrow \quad x = 4, 0 < x < 1$

$\Rightarrow \quad x \in (0, 1) \cup \{4\}$

Hence, the solution set is $(0, 1) \cup \{4\}$.

Integer Type Questions

1. We have

$$\sum_{r=0}^{n-1} \log_2\left(\frac{r+2}{r+1}\right) = \prod_{r=10}^{99} \log_r(r+1)$$

$$\log_2\left(\frac{2}{1}\right) + \log_2\left(\frac{3}{2}\right) + \log_2\left(\frac{4}{3}\right) + \ldots + \log_2\left(\frac{n+1}{n}\right)$$

$= \log_{10}(11)\log_{11}(12)\log_{12}(13) \ldots \log_{99}(100)$

$= \log_{10}(100)$

Thus, $\log_2\left(\frac{2}{1}\right) + \log_2\left(\frac{3}{2}\right) + \log_2\left(\frac{4}{3}\right) + \ldots + \log_2\left(\frac{n+1}{n}\right)$

$= \log_{10}(100)$

$\Rightarrow \quad \log_2\left(\frac{2}{1}\cdot\frac{3}{2}\cdot\frac{4}{3}\ldots\frac{n+1}{n}\right) = 2\log_{10}(10) = 2$

$\Rightarrow \quad \log_2\left(\frac{n+1}{1}\right) = 2$

$\Rightarrow \quad n + 1 = 4$

$\Rightarrow \quad n = 3$

2. Given equation is

$$7^{\log_2 x} = 98 - x^{\log_2 7}$$

$\Rightarrow \quad x^{\log_2 7} = 98 - x^{\log_2 7}$

$\Rightarrow \quad 2x^{\log_2 7} = 98$

$\Rightarrow \quad x^{\log_2 7} = 49$

$\Rightarrow \quad 7^{\log_2 x} = 7^2$

$\Rightarrow \quad \log_2 x = 2$

$\Rightarrow \quad x = 2^2 = 4$

Hence, the solution is $x = 4$.

3. Given equation is

$$4^{\log_3 x} = 32 - 4^{\log_3 x}$$

$\Rightarrow \quad 2\times 4^{\log_3 x} = 32$

$\Rightarrow \quad 4^{\log_3 x} = 16 = 4^2$

$\Rightarrow \quad \log_3 x = 2$

$\Rightarrow \quad x = 3^2 = 9$

Hence, the solution is $x = 3$.

4. Given equation is

$$3\log_x 4 + 2\log_{4x} 4 + 3\log_{16x} 4 = 0$$

$\Rightarrow \quad \frac{3}{\log_4 x} + \frac{2}{\log_4(4x)} + \frac{3}{\log_4(16x)} = 0$

$\Rightarrow \quad \frac{3}{\log_4 x} + \frac{2}{1 + \log_4 x} + \frac{3}{2 + \log_4 x} = 0$

$\Rightarrow \quad \frac{3}{a} + \frac{2}{1+a} + \frac{3}{2+a} = 0$ where $a = \log_4 x$

$\Rightarrow \quad \frac{3}{a} + \frac{3}{2+a} = -\frac{2}{1+a}$

$\Rightarrow \quad \frac{6 + 3a + 3a}{a(2+a)} = -\frac{2}{1+a}$

$\Rightarrow \quad \frac{6(1+a)}{a(2+a)} = -\frac{2}{1+a}$

$\Rightarrow \quad 6(1+a)^2 + 2a(a+2) = 0$

$\Rightarrow \quad 6a^2 + 12a + 6 + 2a^2 + 4a = 0$

$\Rightarrow \quad 8a^2 + 16a + 6 = 0$

$\Rightarrow \quad 4a^2 + 8a + 3 = 0$

$\Rightarrow \quad 4a^2 + 6a + 2a + 3 = 0$

$\Rightarrow \quad 2a(2a+3) + 1(2a+3) = 0$

$\Rightarrow \quad (2a+3)(2a+1) = 0$

$\Rightarrow \quad a = -\frac{3}{2}, -\frac{1}{2}$

$\Rightarrow \quad \log_4 x = -\frac{3}{2}, -\frac{1}{2}$

$\Rightarrow \quad x = 4^{-3/2}, 4^{-1/2}$

$\Rightarrow \quad \alpha = \frac{1}{8}, \beta = \frac{1}{2}$

Hence, the value of

$$\frac{1}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = \frac{1}{2}(8+2) = 5$$

5. Given equation is

$$x + \log_{10}(2x+1) = \log_{10} 6 + x \log_{10} 5$$

$\Rightarrow \quad x + \log_{10}(2x+1) = \log_{10}(6.5^x)$

$\Rightarrow \quad \log_{10}\left(\frac{6.5^x}{2^x+1}\right) = x$

$\Rightarrow \quad \left(\frac{6 \times 5^x}{2^x+1}\right) = 10^x$

$\Rightarrow \quad 10^x(2^x+1) = 6.5^x$

$\Rightarrow \quad 2^x(2^x+1) = 6$

$\Rightarrow \quad (2^x)^2 + (2^x) - 6 = 0$

$\Rightarrow \quad a^2 + a - 6 = 0, a = 2^x$

$\Rightarrow \quad (a+3)(a-2) = 0$

$\Rightarrow \quad a = 2, -3$

$\Rightarrow \quad 2^x = 2, -3$

$\Rightarrow \quad 2^x = 2$

$\Rightarrow \quad x = 1$

Hence, the solution is $x = 1$.

6. We have

$$\log_{(x+1)}(x^2 + x - 6)^2 = 4$$

$\Rightarrow \quad (x^2 + x - 6)^2 = (x+1)^4$

$\Rightarrow \quad x^4 + 2x^3 - 11x^2 - 12x + 36$

$$= x^4 + 4x^3 + 6x^2 + 4x + 1$$

$\Rightarrow \quad 2x^3 + 17x^2 + 16x - 35 = 0$

$\Rightarrow \quad 2x^3 - 2x^2 + 19x^2 - 19x + 35x - 35 = 0$

$\Rightarrow \quad (x-1)(2x^2 + 19x + 35) = 0$

$\Rightarrow \quad x - 1 = 0$

$\Rightarrow \quad x = 1$

7. Given equation is

$$|x-2|^{\log_2(x^3) 3\log_x 4} = (x-2)^3$$

$\Rightarrow \quad \log_2(x^3) - 3\log_x 4 = 3$

$\Rightarrow \quad 3\log_2 x - \frac{3}{\log_4 x} = 3$

$\Rightarrow \quad \log_2 x - \frac{2}{\log_2 x} = 1$

$\Rightarrow \quad a - \frac{2}{a} = 1$ where $a = \log_2 x$

$\Rightarrow \quad a^2 - a - 2 = 0$

$\Rightarrow \quad (a-2)(a+1) = 0$

$\Rightarrow \quad a = 2 - 1$

$\Rightarrow \quad \log_2 x = 2, -1$

$\Rightarrow \quad x = 4, \frac{1}{2}$

Hence, the value of $(\alpha + 2\beta + 3)$ is 8.

8. Given equation is

$$6(\log_x 2 - \log_4 x) + 7 = 0$$

$\Rightarrow \quad 6\left(\log_x 2 - \frac{1}{\log_x 4}\right) + 7 = 0$

$\Rightarrow \quad 6\left(\log_x 2 - \frac{1}{2\log_x 2}\right) + 7 = 0$

$\Rightarrow \quad 6\left(a - \frac{1}{2a}\right) + 7 = 0, a = \log_x 2$

$\Rightarrow \quad 6(2a^2 - 1) + 14a = 0$

$\Rightarrow \quad 3(2a^2 - 1) + 7a = 0$

$\Rightarrow \quad 6a^2 + 7a - 3 = 0$

$\Rightarrow \quad 6a^2 + 9a - 2a - 3 = 0$

$\Rightarrow \quad 3a(2a+3) - (2a+3) = 0$

$\Rightarrow \quad (3a-1)(2a+3) = 0$

$\Rightarrow \quad a = \frac{1}{3}, -\frac{3}{2}$

$\Rightarrow \quad \log_x 2 = \frac{1}{3}, -\frac{3}{2}$

$\Rightarrow \quad \frac{1}{\log_2 x} = \frac{1}{3}, -\frac{3}{2}$

$\Rightarrow \quad \log_2 x = 3, -\frac{2}{3}$

$\Rightarrow \quad x = 2^3, 2^{-2/3}$

Thus the integral solution of x is 8.

Therefore, $\alpha = 8$

Hence, the value of $\left(\frac{2\alpha - 1}{5}\right)$

$$= \left(\frac{16-1}{5}\right) = 3$$

9. We have,

$$N = 6\log_{10} 2 + \log_{10} 31$$

$$= \log_{10} 26 + \log_{10} 31$$

$$= \log_{10}(64 \times 31)$$
$$= \log_{10}(1984)$$
$$< \log_{10}(1000) = 3$$

Also, $N = \log_{10}(1984) > \log_{10}(10000) = 4$

Thus, the sum of successive integers = 3 + 4 = 7.

10. Let $S = \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots = \frac{1/3}{1-(1/3)} = \frac{1}{2}$

We have

$$(0.16)^{\log_{2.5}\left(\frac{1}{2}\right)} = \left(\frac{4}{25}\right)^{\log_{\frac{5}{2}}\left(\frac{1}{2}\right)}$$
$$= \left(\frac{2}{5}\right)^{2\log_{\frac{5}{2}}\left(\frac{1}{2}\right)}$$
$$= \left(\frac{2}{5}\right)^{-2\log_{\frac{5}{2}}\left(\frac{1}{2}\right)}$$
$$= \left(\frac{2}{5}\right)^{\log_{\frac{5}{2}}\left(\frac{1}{2}\right)^{-2}} = \left(\frac{1}{2}\right)^{-2} = 4$$

11. We have,

$$ab = \log_{12} 18 \times \log_{24} 54$$
$$= \frac{\log 18}{\log 12} \times \frac{\log 54}{\log 24}$$
$$= \frac{\log(2.3^2)}{\log(3.2^2)} \times \frac{\log(2.3^3)}{\log(3.2^3)}$$
$$= \frac{\log 2 + 2\log 3}{\log(3) + 2\log 2} \times \frac{\log 2 + 3\log 3}{\log 3 + 3\log 2}$$
$$= \frac{1 + 2\log_2 3}{\log_2(3) + 2} \times \frac{1 + 3\log_2 3}{\log_2 3 + 3}$$
$$= \frac{1+2x}{x+2} \times \frac{1+3x}{x+3},\ x = \log_2 3$$
$$= \frac{1 + 5x + 6x^2}{x^2 + 5x + 6}$$

Also,

$$(a-b) = \log_{12} 18 - \log_{24} 54$$
$$= \frac{\log(2\times 3^2)}{\log(3\times 2^2)} - \frac{\log(2\times 3^3)}{\log(3\times 2^3)}$$
$$= \frac{\log 2 + 2\log 3}{\log(3) + 2\log 2} - \frac{\log 2 + 3\log 3}{\log 3 + 3\log 2}$$
$$= \frac{1 + 2\log_2 3}{\log_2(3) + 2} - \frac{1 + 3\log_2 3}{\log_2 3 + 3}$$
$$= \frac{1+2x}{x+2} - \frac{1+3x}{x+3} \text{ where } x = \log_2 3$$
$$= \frac{(1-x^2)}{(2+x)(3+x)}$$

Hence, the value of

$$5(a-b) + ab = \frac{5(1-x^2)}{(2+x)(3+x)} + \frac{(1+5x+6x^2)}{(2+x)(3+x)}$$
$$= \frac{5(1-x^2) + (1+5x+6x^2)}{(2+x)(3+x)}$$
$$= \frac{x^2+5x+6}{(2+x)(3+x)}$$
$$= \frac{(x+2)(x+3)}{(2+x)(3+x)}$$
$$= 1$$

Previous Years' JEE-Advanced Examinations

1. The given equation is

$$2\log_x a + \log_{ax} a + 3\log_{a^2x} a = 0$$
$$\Rightarrow \frac{2\log a}{\log x} + \frac{\log a}{\log(ax)} + \frac{3\log a}{\log(a^2x)} = 0$$
$$\Rightarrow \frac{2}{\log x} + \frac{1}{\log a + \log x} + \frac{3}{2\log a + \log x} = 0$$
$$\Rightarrow \frac{2}{y} + \frac{1}{b+y} + \frac{3}{2b+y} = 0$$

where $\log a = b$, $\log x = y$

$$\Rightarrow 6y^2 + 11by + 4b^2 = 0$$
$$\Rightarrow 6y^2 + 3by + 8by + 4b^2 = 0$$
$$\Rightarrow (2y+b)(3y+4b) = 0$$
$$\Rightarrow y = -\frac{b}{2}, -\frac{4b}{3}$$

When $y = -\frac{b}{2}$,

$$\log x = -\frac{\log a}{2}$$
$$\Rightarrow \log x^2 = -\log a = \log\left(\frac{1}{a}\right)$$
$$\Rightarrow x^2 = \frac{1}{a}$$
$$\Rightarrow x = a^{-1/2}$$

When $y = -\frac{4b}{3}$,

$$\log x = -\frac{4\log a}{3}$$
$$\Rightarrow 3\log x = 4\log\left(\frac{1}{a}\right)$$
$$\Rightarrow \log x^3 = \log\left(\frac{1}{a}\right)^4$$
$$\Rightarrow x^3 = \left(\frac{1}{a}\right)^4$$
$$\Rightarrow x = a^{-4/3}$$

2. We have $2\log10\, x - \log x\,(0.01),\ x > 1$

$= 2\log_{10} x - \log_x (10)^{-2}$

$= 2(\log_{10} x + \log_x (10))$

$\geq 2 \times 2 = 4$

Thus, the least value is 4.

3. The curve $y = 10^x$ is the reflection of $y = \log_{10} x$ with respect to the line $y = x$.

4. We have $\log_x a + \log_a x$

$$= \left(\log_x a + \frac{1}{\log_x a}\right) \geq 2$$

Thus, the minimum value of the given expression is 2.

5. We have

$\log_{0.3}(x-1) < \log_{0.09}(x-1)$

$\Rightarrow \log_{(0.3)}(x-1) < \log_{(0.3)^2}(x-1)$

$\Rightarrow \log_{(0.3)}(x-1) < \frac{1}{2}\log_{(0.3)}(x-1)$

$\Rightarrow 2\log_{(0.3)}(x-1) < \log_{(0.3)}(x-1)$

$\Rightarrow \log_{(0.3)}(x-1)^2 < \log_{(0.3)}(x-1)$

$\Rightarrow (x-1)^2 > (x-1)$

$\Rightarrow (x-1)^2 - (x-1) > 0$

$\Rightarrow (x-1)(x-1-1) > 0$

$\Rightarrow (x-1)(x-2) > 0$

$\Rightarrow x < 1, x > 2$

Since $x < 1$ does not satisfy the given in-equation,

$\therefore\ x > 2$

Thus, $x \in (2, \infty)$.

6. The given equation is

$\log_7[\log_5\sqrt{x+5} + \sqrt{x}] = 0$

$\Rightarrow \log_5(\sqrt{x+5} + \sqrt{x}) = 7^0 = 1$

$\Rightarrow \sqrt{x+5} + \sqrt{x} = 5^1 = 5$

$\Rightarrow \sqrt{x+5} = 5 - \sqrt{x}$

$\Rightarrow x + 5 = 25 - 10\sqrt{x} + x$

$\Rightarrow 10\sqrt{x} = 20$

$\Rightarrow \sqrt{x} = 2$

$\Rightarrow x = 4$

Hence, the solution is $x = 4$.

7. The given equation is

$\log_{(2x+3)}(6x^2 + 23x + 21) = 4 - \log_{(3x+7)}(4x^2 + 12x + 9)$

$\Rightarrow \log_{(2x+3)}(2x+3)(3x+7) = 4 - \log_{(3x+7)}(2x+3)^2$

$\Rightarrow 1 + \log_{(2x+3)}(3x+7) = 4 - 2\log_{(3x+7)}(2x+3)$

$\Rightarrow \log_{(2x+3)}(3x+7) = 3 - \dfrac{2}{\log_{(2x+3)}(3x+7)}$

$\Rightarrow y = 3 - \dfrac{2}{y}$, where $y = \log_{(2x+3)}(3x+7)$

$\Rightarrow y^2 - 3y + 2 = 0$

$\Rightarrow (y-1)(y-2) = 0$

$\Rightarrow y = 1, 2$

When $y = 1$,

$\log_{(2x+3)}(3x+7) = 1$

$\Rightarrow 3x + 7 = 2x + 3$

$\Rightarrow x = -4$

When $y = 2$,

$\log_{(2x+3)}(3x+7) = 1$

$\Rightarrow 3x + 7 = (2x+3)^2$

$= 4x^2 + 12x + 9$

$\Rightarrow 4x^2 + 9x + 2 = 0$

$\Rightarrow 4x^2 + 8x + x + 2 = 0$

$\Rightarrow 4x(x+2) + 1(x+2) = 0$

$\Rightarrow (x+2)(4x+1) = 0$

$\Rightarrow x = -2, -\dfrac{1}{4}$

As $x > -\dfrac{3}{2}$, so $x = -\dfrac{1}{4}$

Hence, the solution is $x = -\dfrac{1}{4}$.

8. We have,

$x^{(3/4)(\log_2 x)^2 + \log_2 x - \frac{5}{4}} = \sqrt{2}$

$\Rightarrow \left((3/4)(\log_2 x)^2 + \log_2 x - \dfrac{5}{4}\right)\log x = \log(\sqrt{2})$

$\Rightarrow \left((3/4)b^2 + b - \dfrac{5}{4}\right)b = \dfrac{1}{2}$, where $b = \log_2 x$

$\Rightarrow 3b^3 + 4b^2 - 5b - 2 = 0$

$\Rightarrow 3b^3 - 3b^2 + 7b^2 - 7b + 2b - 2 = 0$

$\Rightarrow 3b^2(b-1) + 7b^2 - 7b + 2b - 2 = 0$

$\Rightarrow (b-1)((3b^2 + 7b + 2) = 0$

$\Rightarrow (b-1)(3b^2 + 6b + b + 2) = 0$

$\Rightarrow (b-1)[3b(b+2) + 1(b+2)] = 0$

$\Rightarrow (b-1)(b+2)(3b+1) = 0$

$\Rightarrow b = 1, -2, -\dfrac{1}{3}$

$\Rightarrow \log_2 x = 1, -2, -\dfrac{1}{3}$

$\Rightarrow x = 2, 2^{-2}, 2^{-\frac{1}{3}}$

Thus, the equation has exactly three real solutions of which exactly one is irrational.

9. Given $\log_3 2, \log_3(2^x - 5), \log_3\left(2^x - \dfrac{7}{2}\right) \in$ AP

$\Rightarrow 2\log_3(2^x - 5) = \log_3 2 + \log_3\left(2^x - \dfrac{7}{2}\right)$

$\Rightarrow \log_3(2^x - 5)^2 = \log_3 2 \cdot \left(2^x - \dfrac{7}{2}\right)$

$\Rightarrow (2^x - 5)^2 = 2 \cdot \left(2^x - \dfrac{7}{2}\right)$

$\Rightarrow (2^x)^2 - 10 \times 2^x + 25 = 2 \times 2^x - 7$

$\Rightarrow (2^x)^2 - 12 \times 2^x + 32 = 0$

$\Rightarrow (a)^2 - 12 \times a + 32 = 0$

$\Rightarrow (a-8)(a-4) = 0$

$\Rightarrow a = 8, 4$

$\Rightarrow \quad 2x = 8, 4 = 2^3, 2^2$

$\Rightarrow \quad x = 3, 2$

Since $x = 2$ does not satisfy the logarithmic expression,

$\therefore \quad x = 3$

Hence, the solution is $x = 3$.

10. Let $\log_2 7 = \frac{p}{q}$, where $p, q \in N$ and HCF $(p, q) = 1$

$\Rightarrow \quad 2^{\frac{p}{q}} = 7$

$\Rightarrow \quad 2^p = 7^q$

This is not possible for any $p, q \in N$ and 7 and 2 are prime.

Thus, $\log_2 7$ is an irrational number.

11. The given equation is

$\log_4 (x - 1) = \log_2 (x - 3)$

$\Rightarrow \quad \frac{1}{2}\log_2(x-1) = \log_2(x-3)$

$\Rightarrow \quad \log_2 (x - 1) = \log_2 (x - 3)^2$

$\Rightarrow \quad (x - 3)^2 = x - 1$

$\Rightarrow \quad x^2 - 6x + 9 = x - 1$

$\Rightarrow \quad x^2 - 7x + 10 = 0$

$\Rightarrow \quad (x - 2)(x - 5) = 0$

$\Rightarrow \quad x = 2, 5$

Since $x = 2$ does not satisfy the equation,

$\therefore \quad x = 5$

Thus, the number of solution is one.

12. The given equations are

$(2x)^{\ln 2} = (3y)^{\ln 3}, 3^{\ln x} = 2^{\ln y}$

Now, $3^{\ln x} = 2^{\ln y}$

$\Rightarrow \quad \log(3^{\ln x}) = \log(2^{\ln y})$

$\Rightarrow \quad \log x \log(3) = \log y \log(2)$

$\Rightarrow \quad \frac{\log x}{\log 2} = \frac{\log(y)}{\log(3)} = \lambda(\text{say})$

Also, $(2x)^{\ln 2} = (3y)^{\ln 3}$

$\Rightarrow \quad \log((2x)^{\ln 2}) = \log((3y)^{\ln 3})$

$\Rightarrow \quad \log 2 \log(2x) = \log 3 \log(3y)$

$\Rightarrow \quad \log 2(\log 2 + \log x) = \log 3(\log 3 + \log y)$

$\Rightarrow \quad \log 2(\log 2 + \lambda \log 2) = \log 3(\log 3 + \lambda \log 3)$

$\Rightarrow \quad (\log 2)^2 (1 + \lambda) = (\log 3)^2 (1 + \lambda)$

$\Rightarrow \quad (1 + \lambda)[(\log 2)^2 - (\log 3)^2] = 0$

$\Rightarrow \quad (1 + \lambda) = 0, \qquad [\because (\log 2)^2 - (\log 3)^2 \neq 0]$

$\Rightarrow \quad \lambda = -1$

Thus, $\log x = -\log 2, \log y = -\log 3$

$\Rightarrow \quad \log x = \log\left(\frac{1}{2}\right), \log y = \log\left(\frac{1}{3}\right)$

$\Rightarrow \quad x = \frac{1}{2}, y = \frac{1}{3}$

Hence, $x_0 = \frac{1}{2}$

13. Let

$$x = \sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}} \ldots \text{to } \infty}}$$

$\Rightarrow \quad x = \sqrt{4 - \frac{1}{3\sqrt{2}}x}$

$\Rightarrow \quad x^2 = \left(4 - \frac{1}{3\sqrt{2}}x\right)$

$\Rightarrow \quad 3\sqrt{2}\,x^2 + x - 12\sqrt{2} = 0$

$\Rightarrow \quad 3\sqrt{2}\,x^2 + 9x - 8x - 12\sqrt{2} = 0$

$\Rightarrow \quad 3x(\sqrt{2}x + 3) - 4\sqrt{2}(\sqrt{2}x + 3) = 0$

$\Rightarrow \quad (\sqrt{2}x + 3)(3x - 4\sqrt{2}) = 0$

$\Rightarrow \quad x = -\frac{3}{\sqrt{2}}, \frac{4\sqrt{2}}{3}$

$\Rightarrow \quad x = \frac{4\sqrt{2}}{3}$

Thus,

$$6 + \log_{3/2}\left(\frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}\sqrt{4 - \frac{1}{3\sqrt{2}}} \ldots \text{to } \infty}}\right)$$

$$= 6 + \log_{3/2}\left(\frac{1}{3\sqrt{2}} \times \frac{4\sqrt{2}}{3}\right)$$

$$= 6 + \log_{3/2}\left(\frac{4}{9}\right)$$

$$= 6 + \log_{\left(\frac{2}{3}\right)^{-1}}\left(\frac{2}{3}\right)^2$$

$$= 6 - 2\log_{\left(\frac{2}{3}\right)}\left(\frac{2}{3}\right)$$

$$= 6 - 2 = 4$$

CHAPTER

4

Complex Numbers

4.1 Introduction

There is no real number x which satisfies the polynomial equation $x^2 + 1 = 0$. To permit solutions of this and similar equations, the set of complex number is introduced. We can consider a complex number as having the form $a + ib$, where a and b are real numbers and i, which is called an imaginary unit with the property $i^2 = -1$.

A complex number is generally denoted as z and is defined as $z = a + ib$, where a is called the real part of z, which is denoted as Re(z) and b is called the imaginary part of z, which is denoted as Im(z).

Another way also we can define as, any number is called a complex number

Any complex number $z = a + ib$ is called

(i) a purely real if $b = 0$

(ii) a purely imaginary number, if $a = 0$

(iii) an imaginary number, if $b \neq 0$

Notes

(i) The set R of real numbers is a proper subset of the complex numbers. Thus, the complex number system is

$$N \subset W \subset I \subset Q \subset R \subset C$$

(ii) Zero is the only real number, which is purely real as well as purely imaginary, but not imaginary number.

(iii) i is called an imaginary unit, which is introduced by Swedish mathematician Sir Euler.

Also, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$

(iv) The period of iota is 4.

(v) $i^{4n} = 1$, $i^{4n+1} = i$, $i^{4n+2} = -i$, $i^{4n+3} = -i$, $n \in I$

(vi) The sum of the first four ith consecutive powers is zero. i.e. $i + i^2 + i^3 + i^4 = 0$

(vii) $\sqrt{a}\sqrt{b} = \sqrt{ab}$ holds good only when at least any one of a or b is non negative.

4.2 Algebra of Complex Numbers

Fundamental operations with complex numbers we can proceed as in the algebra of real numbers replacing $i^2 = -1$ when it occurs.

(i) *Addition*

Let $z_1 = a + ib$ and $z_2 = c + id$

Then $z_1 + z_2 = (a + ib) + (c + id)$

$= (a + c) + i(b + d)$

$= A + iB$

$\Rightarrow$ It is a complex number.

(ii) *Subtraction*

Let $z_1 = a + ib$ and $z_2 = c + id$

Then $z_1 - z_2 = (a + ib) - (c + id)$

$= (a - c) + i(b - d)$

$= A + iB$

$\Rightarrow$ It is a complex number.

(iii) *Product*

Let $z_1 = a + b$ and $z_2 = c + b$

Then $z_1 z_2 = (a + ib)(c + id)$

$= (ac - bd) + i(ad + bc)$

$= A + iB$

$\Rightarrow$ It is a complex number.

(iv) *Multiplicative Inverse*

If the product of two non-zero complex numbers is 1, each one is the multiplicative inverse of other one, i.e. if $z_1 z_2 = 1$, then z_1 is the multiplicative inverse of z_2 and conversely.

Example 1: The multiplicative inverse of $3 + 4i$ is

$$\frac{1}{3+4i} = \frac{3-4i}{(3+4i)(3-4i)}$$

$$= \frac{3-4i}{(9+16)} = \frac{3-4i}{25}$$

Example 2: The multiplicative inverse of $2 - 3i$ is

$$\frac{1}{2-3i} = \frac{2+3i}{(2-3i)(2+3i)}$$
$$= \frac{2+3i}{(4+9)} = \frac{2+3i}{13}$$

(v) *Division*

Let $z_1 = a + b$ and $z_2 = c + d$

Then $$\frac{z_1}{z_2} = \frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)}$$
$$= \frac{(ac+bd)+i(bc-ad)}{(c^2+d^2)}$$
$$= \left(\frac{ac+bd}{c^2+d^2}\right) + i\left(\frac{bc-ad}{c^2+d^2}\right)$$

$\Rightarrow$ It is a complex number.

> ***Notes***
> 1. Inequalities of complex numbers are not defined. There is no validity if we say that a complex number is positive or negative.
> 2. $z > 0$ or $4 + 2i > 2 + 4i$ are meaningless.
> 3. In real numbers, if $a^2 + b^2 = 0$, then $a = 0 = b$, however in complex numbers if $z_1^2 + z_2^2 = 0$ does not imply that $z_1 = 0 = z_2$.

4.3 Equality of Complex Numbers

Two complex numbers $z_1 = a + b$ and $z_2 = c + d$ are said to be equal if and only if their real and imaginary parts are separately equal, i.e.

$z_1 = z_2 \Rightarrow a + ib = c + id$

$\Rightarrow$ $a = c$ and $b = d$

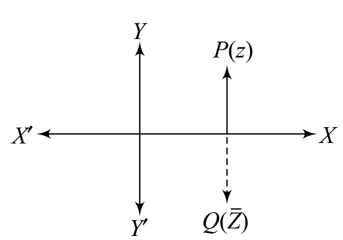
$\Rightarrow$ $\text{Re}(z_1) = \text{Re}(z_2)$ and $\text{Im}(z_1) = \text{Im}(z_2)$

4.4 Conjugate of Complex Numbers

The conjugate of a complex number $z = a + ib$ is denoted as $\bar{z}$ and is defined as $\bar{z} = a - ib$.

In a complex number, if we replace i by $-i$, we get the conjugate of the complex number.

Geometrically, $\bar{z}$ is the mirror image of z with respect to the real axis in the argand plane.



4.4.1 Properties of Conjugates of Complex Numbers

Let z, z_1, z_2 be three complex numbers. Then

(i) $\bar{z}$ is the mirror image of z along the real axis.

(ii) $\overline{(\bar{z})} = z$

(iii) $z = \bar{z} \Rightarrow z$ is purely real.

(iv) $z = -\bar{z} \Rightarrow z$ is purely imaginary.

(v) $\text{Re}(z) = \text{Re}(\bar{z}) = \dfrac{z + \bar{z}}{2}$

(vi) $\text{Im}(z) = \dfrac{z + \bar{z}}{2i}$

(vii) $\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2$

(viii) $\overline{(z_1 - z_2)} = \bar{z}_1 - \bar{z}_2$

(ix) $\overline{(z_1 \cdot z_2)} = \bar{z}_1 \cdot \bar{z}_2$

(x) $\overline{\left(\dfrac{z_1}{z_2}\right)} = \left(\dfrac{\bar{z}_1}{\bar{z}_2}\right)$

(xi) $z_1\bar{z}_2 + z_2\bar{z}_1 = 2Re(z_1\bar{z}_2) = 2Re(\bar{z}_1 z_2)$

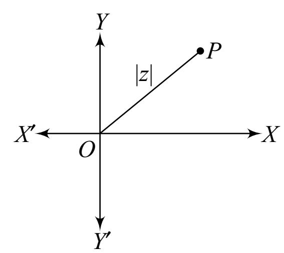
(xii) $\overline{(z^n)} = (\bar{z})^n$

(xiii) If $z = f(z_1)$, then $\bar{z} = f(\bar{z}_1)$

4.5 Modulus of a Complex Number

Let $z = a + ib$ be a complex number. The modulus of a complex number z is denoted as $|z|$ and is defined as $|z| = \sqrt{a^2 + b^2}$

Geometrically, $|z|$ represents the distance between a complex number z and the origin.



4.5.1 Properties of Modulii

(i) $\text{Re}(z) \le |z|$, $\text{Im}(z) \le |z|$

(ii) $\text{Re}(z) \ge -|z|$, $\text{Im}(z) \ge -|z|$

(iii) $|z| = |-z| = |\bar{z}| = |-\bar{z}|$

(iv) $z.\bar{z} = |z|^2$

(v) $|z|^2 = |z^2|$

(vi) $|z|^n = |z^n|$

(vii) $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$

In general, $|z_1 \cdot z_2 \dots z_n| = |z_1| \cdot |z_2| \dots |z_n|$

(viii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

(ix) $|z_1 + z_2|^2 = |z_1|^2 + |z_1|^2 + 2\text{Re}(z_1\, \bar{z}_2)$

Proof $|z_1 + z_2|^2 = (z_1 + z_2)\overline{(z_1 + z_2)}$

$= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$

$= (z_1 \cdot \bar{z}_1 + z_2 \cdot \bar{z}_2 + z_1 \cdot \bar{z}_2 + z_2 \cdot \bar{z}_1)$

$= (|z_1|^2 + |z_2|^2 + (z_1\bar{z}_2 + \overline{z_1 \cdot \bar{z}_2}))$

$= (|z_1|^2 + |z_2|^2 + 2\text{Re}(z_1\bar{z}_2)$

(x) $|z_1 - z_2|^2 = |z_1|^2 + |z_1|^2 - 2\text{Re}(z_1\bar{z}_2)$

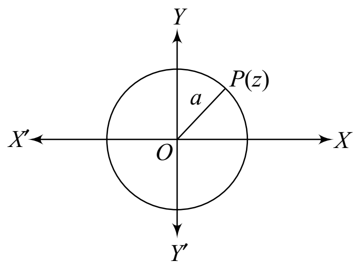
(xi) $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$

(xii) A complex number z is said to be a unimodular complex number if $|z| = 1$.

Thus, $\frac{z}{|z|}$ is always unimodular, if $z \neq 0$. For example,

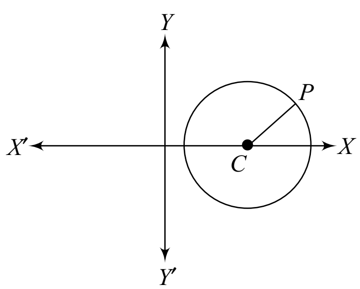
1. Let $z = \cos\theta + i\sin\theta$, then $|z| = 1$
2. Let $z = \frac{\sqrt{3}}{2} + \frac{i}{2}$, then $|z| = 1$
3. Let $z = \frac{1}{3} + i\frac{2\sqrt{2}}{3}$, then $|z| = 1$
4. Let $z = \frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}$, then $|z| = 1$

(xiii) If $|z| = a$, the locus of z is a circle with centre at the origin and the radius a.



Example 1: Let $|z| = 2$. Then its centre is the origin and the radius is 2.

(xiv) If $|z - a| = b$, the locus of z is a circle with centre at a and the radius is b.



Example 1: Let $|z - (3 + 4i)| = 2$, the locus of z is a circle with the centre at (3, 4) and the radius = 2

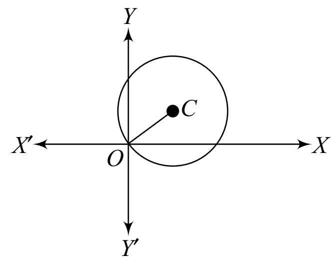
Example 2: Let $|z + (2 + 3i)| = 4$, the locus of z is a circle with the centre at (–2, –3) and the radius = 4

Example 3: Let $|z + (4 - 5i)| = 4$, the locus of z is a circle with the centre at (–4, 5) and the radius = 4

Example 4: Let $|z - 2| = 3$, the locus of z is a circle with the centre at (2, 0) and the radius = 3

Example 5: Let $|z + 3| = 2$, the locus of z is a circle with the centre at (–2, 0) and the radius = 2

Example 6: Let $|z - (3 + 4i)| = 5$, then the locus of z is circle with centre at (3, 4) and radius = 5. Moreover, the circle is passing through the origin.



(xv) If $|z - (\alpha + i\beta)| = a$, the greatest value of $|z|$ is $\sqrt{\alpha^2 + \beta^2} + a$ and the least value of $|z|$ is $\sqrt{\alpha^2 + \beta^2} - a$.

4.6 Argument of a Complex Number

The argument of a complex number $z = a + ib$ is denoted as Arg(z) and is defined as

$$\text{Arg}(z) = \theta = \tan^{-1}\left(\frac{b}{a}\right).$$

The argument of a complex number is the common solution of $\sin\theta$ and $\cos\theta$. Since $\sin\theta$ and $\cos\theta$ are the periodic functions, so the argument of a complex number is not unique.

If the argument of a complex number is θ, it also be $2n\pi + \theta$, $n \in I$.

If the argument of a complex number is unique, it is called the principal argument.

The principal argument of a complex number is denoted as amp(z) i.e. amp(z) = θ, where θ lies in the inequality $-\pi < \theta \leq \pi$.

Note: The argument of a complex number $z = a + ib = r(\cos\theta + \sin\theta)$ is the value of θ satisfying the equations $r\cos\theta = a$ and $r\sin\theta = b$.

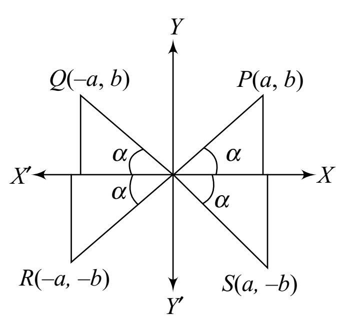
The argument of a complex number zero is not defined.

4.7 Principal Value of Argument of a Complex Number *z*

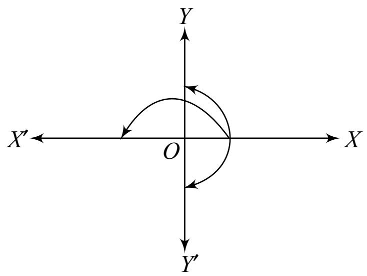
If $z = a + ib$, where $a, b \in R$, the $\arg(z) = \tan^{-1}\left(\frac{b}{a}\right)$ always gives us the principal value. It depends on the quadrant in which the point (a, b) lies.

Consider $\alpha = \tan^{-1}\left|\frac{b}{a}\right|$

1.



(i) If z lies in the first quadrant, then $\arg(z) = \theta = \alpha$
(ii) If z lies in the second quadrant, then $\arg(z) = \theta = (\pi - \alpha)$
(iii) If z lies in the third quadrant, then $\arg(z) = \theta = -(\pi - \alpha)$
(iv) If z lies in the fourth quadrant, then $\arg(z) = \theta = -\alpha$

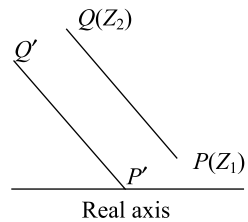
2. 

(i) If z is purely (+ve) real, then $\arg(z) = \theta = 0$
(ii) If z is purely (–ve) real, then $\arg(z) = \theta = \pi$
(iii) If z is purely (+ve) imaginary, then $\arg(z) = \theta = \frac{\pi}{2}$
(iv) If z is purely (–ve) imaginary, then $\arg(z) = \theta = -\frac{\pi}{2}$

Notes
1. The argument of a complex number is a many-valued function. If θ is the argument of a complex number, it also be $2n\pi + \theta$, $n \in I$. Any two arguments of a complex number is differ by $2n\pi$.
2. The unique value of θ such that $-\pi < \theta \le \pi$ is called the principal value of the argument. Unless, otherwise stated, amp(z) implies the principal value of the argument.
3. By specifying the modulus and argument of a complex number is defined completely. For the complex number $0 = 0 + i0$, the argument is not defined and this is the only complex number given by its modulus.

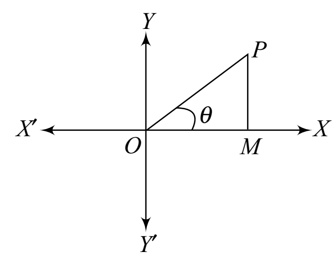
4.7.1 Properties of Arguments of Complex numbers

(i) Argument of a complex number 0 is not defined.
(ii) If z is purely real, then $\arg(z) = 0$ or π
(iii) If z is purely imaginary number, then $\arg(z) = \pm\left(\frac{\pi}{2}\right)$
(iv) $\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2) + 2m\pi$, for some integer m
$\text{Amp}(z_1 z_2) = \text{Amp}(z_1) + \text{Amp}(z_2)$
(v) $\text{Arg}\left(\frac{z_1}{z_2}\right) = \text{Arg}(z_1) - \text{Arg}(z_2) + 2m\pi$, for some integer m.
$$\text{Amp}\left(\frac{z_1}{z_2}\right) = \text{Amp}(z_1) - \text{Amp}(z_2)$$
(vi) $\text{Arg}(z^n) = n\,\text{Arg}(z) + 2m\pi$, for some integer m
$\text{Amp}(z^n) = n\,\text{Amp}(z)$
(vii) If $\text{Arg}\left(\frac{z_2}{z_1}\right) = \theta$, then
$$\text{Arg}\left(\frac{z_1}{z_2}\right) = 2k\pi - \theta, k \in I$$
(viii) $\text{Arg}(z_1 - z_2)$ = angle of the line segment $P'Q' \parallel PQ$, where P lies on real axis, with the real axis.



4.8 Representation of a Complex Number

1. Cartesian form



Every complex number $z = x + iy$ can be represented by a point on the cartesian plane, known as *complex plane* or *z-plane* or *Argand plane* and the diagram is called the *Argand diagram*, by the ordered pair (x, y).

The length OP is called the modulus of the complex number, which is denoted as $|z|$ and θ is called the argument or amplitude of a complex number $z = x + iy$.

Here, $|z| = \sqrt{x^2 + y^2}$

and $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

(where θ is the angle made by OP with positive x-axis)
For example,
(i) $z = a + ib$
(ii) $-z = -(a + ib)$
(iii) $\bar{z} = a - ib$
(iv) If $z_1 = a + ib$ and $z_2 = c + id$, then $z_1 \cdot z_2 = (a + ib)(c + id)$
(v) If $z_1 = a + ib$ and $z_2 = c + id$, then
$$\frac{z_1}{z_2} = \frac{a + ib}{c + id}$$

2. Trigonometric/Polar representation

Let $z = r(\cos\theta + i\sin\theta)$
Then $\text{Arg}(z) = \theta$ and $|z| = r$
Also $z = r(\cos\theta - i\sin\theta)$, then $\text{Arg}(z) = -\theta$ and $|z| = r$
For example,
(i) $z = r(\cos\theta + i\sin\theta)$

(ii) $-z = -r(\cos\theta + i\sin\theta)$
(iii) $\bar{z} = r(\cos\theta - i\sin\theta)$
(iv) If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then
$z_1z_2 = r_1r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$
(v) If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$
and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then $\frac{z_1}{z_2}$
$$= \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$$

> ***Note***
> 1. $\cos\theta + \sin\theta$ is also known as $\cos\theta$ or $e^{i\theta}$.
> Also, $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$ are also known as Euler's formula.

3. Euler's Representation

Let $z = re^{i\theta}$, where $|z| = r$, arg $(z) = \theta$ and $\bar{z} = re^{i\theta}$, where $|z| = r$, $\arg(z) = -\theta$

For example.
(i) $z = re^{i\theta}$
(ii) $-z = -re^{i\theta}$
(iii) $\bar{z} = re^{i\theta}$
(iv) If $z_1 = r_1e^{i\theta_1}, z_2 = r_2e^{i\theta_2}$, then $z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)}$
(v) If $z_1 = r_1e^{i\theta_1}, z_2 = r_2e^{i\theta_2}$, then $\frac{z_1}{z_2} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}$

4. Vector representation

Every complex number can be expressed as the position vector of a point, if the point P represents a complex number z such that $\overrightarrow{OP} = z$ and $|\overrightarrow{OP}| = |z|$.

4.9 Square Root of a Complex Number

Let $z = a + ib$ be a complex number.

Then $\sqrt{z} = \sqrt{a+ib}$
$$= \begin{cases} \pm\left(\sqrt{\frac{|z|+a}{2}} + i\sqrt{\frac{|z|+a}{2}}\right) : b > 0 \\ \pm\left(\sqrt{\frac{|z|+a}{2}} - i\sqrt{\frac{|z|+a}{2}}\right) : b < 0 \end{cases}$$

Proof: Given $z = a + ib$.
Then $\sqrt{a+ib} = x + iy$
$a + ib = (x^2 - y^2) = 2ixy$
Comparing the real and imaginary parts, we get,
$x^2 - y^2 = a, 2xy = b$
Now, $(x^2 + y^2) = (x^2 - y^2) + 4x^2y^2$
$= a^2 + b^2 = |z|^2$
$\Rightarrow \quad (x^2 + y^2) = |z| \quad \ldots(i)$
$\Rightarrow \quad (x^2 - y^2) = a \quad \ldots(ii)$
Adding Eqs (i) and (ii), we get
$2x^2 = |z| + a$
$\Rightarrow \quad x^2 = \frac{|z|+a}{2}$
$\Rightarrow \quad x = \pm\sqrt{\frac{|z|+a}{2}}$
Subtracting Eqs (ii) from (i), we get
$2y^2 = |z| - a$
$\Rightarrow \quad y^2 = \frac{|z|-a}{2}$
$\Rightarrow \quad y = \pm\sqrt{\frac{|z|-a}{2}}$
Thus, $\sqrt{z} = x + iy$
$$= \begin{cases} \pm\left(\sqrt{\frac{|z|+a}{2}} + i\sqrt{\frac{|z|+a}{2}}\right) : b > 0 \\ \pm\left(\sqrt{\frac{|z|+a}{2}} - i\sqrt{\frac{|z|+a}{2}}\right) : b < 0 \end{cases}$$

> ***Note***
> 1. The square roots of i are $\pm\left(\frac{1+i}{\sqrt{2}}\right)$
> 2. The square roots of $-i$ are $\pm\left(\frac{1-i}{\sqrt{2}}\right)$
> 3. The value of $\sqrt{i} + \sqrt{-i} = \pm\sqrt{2}$
> 4. The value of $\sqrt{i} - \sqrt{-i} = \pm i\sqrt{2}$
> 5. The square roots of ω are $\pm\omega^2$
> 6. The square roots of ω^2 are $\pm\omega$

4.10 Cube Roots of Unity

Let $x = \sqrt[3]{1}$.
$\Rightarrow \quad x^3 = 1$
$\Rightarrow \quad x^3 - 1 = 0$
$\Rightarrow \quad (x - 1)(x^2 + x + 1) = 0$
$\Rightarrow \quad (x - 1) = 0, (x^2 + x + 1) = 0$
$\Rightarrow \quad x = 1, x = \frac{-1 \pm i\sqrt{3}}{2}$
$\Rightarrow \quad x = 1, \frac{-1 + i\sqrt{3}}{2}, \frac{-1 - i\sqrt{3}}{2}$
$\Rightarrow \quad x = 1, \omega, \omega^2$
Thus, the cube roots of unity are $1, \omega, \omega^2$.

4.11 Properties of Cube Roots of Unity

(i) 1 is the real cube root of unity whereas ω, ω^2 are the complex cube roots of unity.
(ii) $1 + \omega + \omega^2 = 0$

(iii) $\omega^3 = 1$

(iv) The period of ω is 3, i.e.
$\omega^{3n} = 1,\ \omega^{3n+1} = \omega,\ \omega^{3n+2} = \omega^2,\ n \in I$

(v) $\bar{\omega} = \omega^2,\ \overline{\omega^2} = \omega$

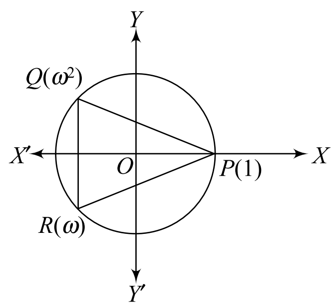
(vi) $\sqrt{\omega} = \pm\omega^2,\ \sqrt{\omega^2} = \pm\omega$

(vii) A complex number $a + ib$ for which
$|a : b| = 1 : \sqrt{3}$ or $\sqrt{3} : 1$
can always be expressed in terms of i, ω, ω^2.
For example,

1. $(1 + i\sqrt{3}) = -2\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = -2\omega^2$

2. $(\sqrt{3} + i) = i(1 - i\sqrt{3}) = -2i\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -2i\omega$

(viii) In the complex plane, the cube roots of unity represent the vertices of an equilateral triangle inscribed in a unit circle, having the centre as origin and one vertex lying on positive real axis.



(ix) If $a + b\omega + c\omega^2 = 0$, $a, b, c \in R$, then $a = b = c$.

(x) The following factors should be remembered.

(a) $x^2 + x + 1 = (x - \omega)(x - \omega^2)$

(b) $x^3 - 1 = (x - 1)(x - \omega)(x - \omega^2)$

(c) $x^2 - x + 1 = (x + \omega)(x + \omega^2)$

(d) $x^3 + 1 = (x + 1)(x + \omega)(x + \omega^2)$

(e) $x^2 + xy + y^2 = (x - y\omega)(x - y\omega^2)$

(f) $x^3 - y^3 = (x - y)(x - y\omega)(x - y\omega)^2$

(g) $x^2 - xy + y^2 = (x + y\omega)(x + y\omega^2)$

(h) $x^3 + y^3 = (x + y)(x + y\omega)(x + y\omega^2)$

(i) $x^2 + y^2 + z^2 - xy - yz - zx$
$= (x + y\omega + z\omega^2)(x + y\omega^2 + z\omega)$

(j) $x^3 + y^3 + z^3 - 3xyz$
$= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$
$= (x + y + z)(x + y\omega + z\omega^2)(x + y\omega^2 + z\omega)$

(k) The cube roots of

(i) $1 = 1, \omega, \omega^2$

(ii) $8 = 2, 2\omega, 2\omega^2$

(iii) $27 = 3, 3\omega, 3\omega^2$

(iv) $5 = \sqrt[3]{5}, \sqrt[3]{5}\omega, \sqrt[3]{5}\omega^2$

(v) $-1 = -1, -\omega, -\omega^2$

(vi) $-8 = -2, -2\omega, -2\omega^2$

(vii) $-27 = -3, -3\omega, -3\omega^2$

(viii) $-4 = -\sqrt[3]{4}, -\sqrt[3]{4}\omega, -\sqrt[3]{4}\,\omega^2$

(l) The roots of

(i) $x^3 - 8 = 0$ are $2, 2\omega, 2\omega^2$

(ii) $(x - 1)^2 - 8 = 0$ are $3, 1 + 2\omega, 1 + 2\omega^2$

(iii) $(x + 1)^3 - 27 = 0$ are $-2, -1 + 3\omega, -1 + 3\omega^2$

(iv) $x^2 - 3x^2 + 3x = 28$ are $4, 1 + 3\omega, 1 + 3\omega^2$.

4.12 DeMoivre's Theorem

Statement

(i) If n be any integer, then
$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

(ii) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \ldots (\cos\theta_n + i\sin\theta_n)$
$= \cos(\theta_1 + \theta_2 + \ldots + q_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)$

(iii) If $p, q \in Z$ and $q \neq 0$, then $(\cos\theta + i\sin\theta)^{p/q}$

$$= \left(\cos\left(\frac{(2k\pi + \theta)p}{q}\right) + i\sin\left(\frac{(2k\pi + \theta)p}{q}\right)\right)$$

Proof

(i) We have,
$(\cos\theta + i\sin\theta)^n = (e^{i\theta})^n$
$= (e^{in\theta})$
$= \cos(n\theta) + i\sin(n\theta)$

(ii) $(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)\ldots(\cos\theta_n + i\sin\theta_n)$
$= e^{i\theta_1}\cdot e^{i\theta_2}\ldots e^{i\theta_n}$
$= e^{i(\theta_1 + \theta_2 + \ldots + \theta_n)}$
$= \cos(\theta_1 + \theta_2 + \ldots + \theta_n) + i\sin(\theta_1 + \theta_2 + \ldots + \theta_n)$

(iii) $(\cos\theta + i\sin\theta)^{p/q}$
$= (\cos(2k\pi + \theta) + i\sin(2k\pi + \theta))^{p/q}$
$= (e^{i(2k\pi + \theta)})^{p/q}$
$= \left(e^{\frac{(2ik\pi + \theta)p}{q}}\right)$
$= \cos\left(\frac{(2k\pi + \theta)p}{q}\right) + i\sin\left(\frac{(2k\pi + \theta)p}{q}\right)$

Hence, the result.
This completes the proof of the theorem.

4.13 *n*th Roots of Unity

Let $x = \sqrt[n]{1}$

$\Rightarrow\quad x^n = 1 = \cos(2k\pi) + i\sin(2k\pi)$

$\Rightarrow\quad x = (\cos(2k\pi) + i\sin(2k\pi))^{1/n}$

$\Rightarrow\quad x = \left(\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)\right)$

where $k = 0, 1, 2, 3, \ldots, n - 1$.

Let $\alpha = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$

Then the nth roots of unity are α^t, where $t = 0, 1, 2, 3, \ldots, n-1$

Thus, the nth roots of unity are

$$1, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^{n-1}.$$

4.13.1 Properties of the *n*th Roots of Unity

(i) The sum of the nth roots of unity is zero.

Proof: We have $1 + \alpha + \alpha^2 + \alpha^3 + \ldots + \alpha^{n-1}$

$$= \frac{1-\alpha^n}{1-\alpha}$$

$$= \frac{1-\left(\cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)\right)^n}{1-\alpha}$$

$$= \frac{1-\left(\cos\left(\frac{2\pi}{n}\times n\right) + i\sin\left(\frac{2\pi}{n}\times n\right)\right)}{1-\alpha}$$

$$= \frac{1-[\cos(2\pi) + i\sin(2\pi)]}{1-\alpha}$$

$$= \frac{1-1}{1-\alpha}$$

$$= 0$$

(ii) The sum of the pth powers of the nth roots of unity is also zero.

Proof: We have $1^p + \alpha^p + \alpha^{2p} + \ldots + \alpha^{(n-1)p}$

$$= 1 + (\alpha^p) + (\alpha^p)^2 + (\alpha^p)^3 + \ldots + (\alpha^p)^{n-1}$$

$$= \frac{1-\alpha^{np}}{1-\alpha^p}$$

$$= \frac{1-1}{1-\alpha^p}\left[\because \alpha^n = \left(\cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)\right)^n\right.$$

$$= \cos\left(\frac{2\pi}{n}\times n\right) + i\sin\left(\frac{2p}{n}\times n\right)$$

$$= \cos(2\pi) + i\sin(2\pi) = 1$$

$$\Rightarrow \alpha^{np} = 1^p = 1]$$

$$= 0$$

Hence, the result.

(iii) The product of the nth roots of unity is $(-1)^{n-1}$.

Proof: Let $P = 1\cdot\alpha\cdot\alpha^2\cdot\alpha^3\ldots\alpha^{n-1}$

$$= \alpha^{1+2+3+\ldots+(n-1)}$$

$$= \alpha^{\frac{n-1}{2}(1+n-1)}$$

$$= \left(\alpha^{\frac{n}{2}}\right)^{n-1}$$

$$= (-1)^{n-1}$$

$$\left[\because \alpha^{\frac{n}{2}} = \left(\cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)\right)^{n/2}\right.$$

$$= \cos\left(\frac{2\pi}{n}\times\frac{n}{2}\right) + i\sin\left(\frac{2p}{n}\times\frac{n}{2}\right)$$

$$= \cos(\pi) + i\sin(\pi) = -1]$$

(iv) The nth roots of unity are in GP with the common ratio α. i.e. $e^{i\left(\frac{2\pi}{n}\right)}$.

(v) In the complex plane, the nth roots of unity are located on the circumference of the unit circle and divide it into n equal arcs.

(vi) If $1, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^{n-1}$ be the nth roots of unity, then $x^n - 1 = (x-1)(x-\alpha)(x-\alpha^2)\ldots(x-\alpha^{n-1})$

(vii) If $1, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^{n-1}$ be the nth roots of unity, then

(a) $$\cos\left(\frac{2\pi}{n}\right) + \cos\left(\frac{4\pi}{n}\right) + \cos\left(\frac{6\pi}{n}\right) + \ldots + \cos\left(\frac{2(n-1)\pi}{n}\right) = 0$$

(b) $$\sin\left(\frac{2\pi}{n}\right) + \sin\left(\frac{4\pi}{n}\right) + \sin\left(\frac{6\pi}{n}\right) + \ldots + \sin\left(\frac{2(n-1)\pi}{n}\right) = 0$$

Proof: Now, $1 + \alpha + \alpha^2 + \ldots + \alpha^{n-1} = 0$

$$\Rightarrow \sum_{k=0}^{n-1}\alpha^k = 0$$

$$\Rightarrow \sum_{k=0}^{n-1}\left(\cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)\right)^k = 0$$

$$\Rightarrow \sum_{k=0}^{n-1}\left(\cos\left(\frac{2k\pi}{n}\right) + i\sin\left(\frac{2k\pi}{n}\right)\right) = 0$$

$$\Rightarrow \left(\cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)\right) + \left(\cos\left(\frac{4\pi}{n}\right) + i\sin\left(\frac{4\pi}{n}\right)\right) + \ldots + \left(\cos\left(\frac{2(n-1)\pi}{n}\right) + i\sin\left(\frac{2(n-1)\pi}{n}\right)\right) = 0$$

$$\Rightarrow \cos\left(\frac{2\pi}{n}\right) + \cos\left(\frac{4\pi}{n}\right) + \cos\left(\frac{6\pi}{n}\right) + \ldots + \cos\left(\frac{2(n-1)\pi}{n}\right) + i\left(\sin\left(\frac{2\pi}{n}\right) + \sin\left(\frac{4\pi}{n}\right) + \ldots + \sin\left(\frac{2(n-1)\pi}{n}\right)\right) = 0$$

Thus,

$$\cos\left(\frac{2\pi}{n}\right) + \cos\left(\frac{4\pi}{n}\right) + \ldots + \cos\left(\frac{2(n-1)\pi}{n}\right)$$

and

$$\sin\left(\frac{2\pi}{n}\right) + \sin\left(\frac{4\pi}{n}\right) + \ldots + \sin\left(\frac{2(n-1)\pi}{n}\right) = 0$$

Hence, the result.

Note The sum of the following series should be remembered.

1. $\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos n\theta$

$$= \frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \times \cos\left(\frac{(n+1)\theta}{2}\right)$$

2. $\sin\theta + \sin 2\theta + \sin 3\theta + \ldots + \sin n\theta$

$$= \frac{\sin\left(\frac{n\theta}{2}\right)}{\sin\left(\frac{\theta}{2}\right)} \times \sin\left(\frac{(n+1)\theta}{2}\right)$$

4.14 Rotation

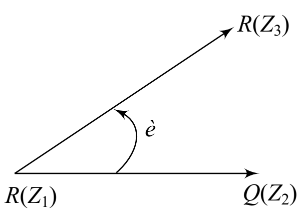
1. Let $z = a + ib$

$= r(\cos\theta + i\sin\theta)$

$= re^{i\theta}$, where $|z| = r$, $\text{Arg}(z) = \theta$

This is known as the rotational form of a complex number.

2. **Concepts of Rotation**



After plotting the points, mark the arrow in anti-clockwise direction.

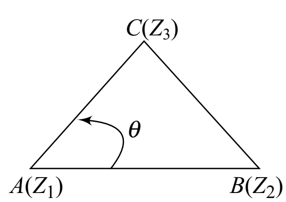
Then $\theta = \left(\frac{z_3 - z_1}{z_2 - z_1}\right)$

Here, $z_3 - z_1 = \overrightarrow{PR}$ where $\overrightarrow{PR}$ is a vector on which arrow goes.

$z_2 - z_1 = \overrightarrow{PQ}$, where $\overrightarrow{PQ}$ is vector from which arrow starts, θ will be +ve or –ve according as it is measured in anti-clockwise or clockwise direction.

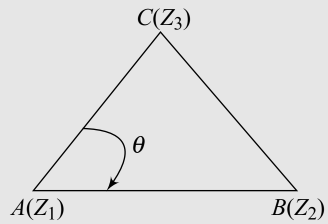
3. **Coni Method**

If z_1, z_2, z_3 be affixes of the vertices of a ΔABC, described in anti-clockwise sense, then $\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \frac{CA}{BA} \times e^{i\theta}$.



Note If θ is measured in anti-clockwise sense, then

$$\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \frac{CA}{BA} \times e^{-i\theta}$$



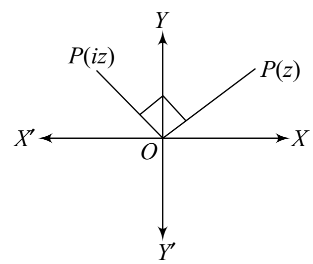
4. **Relation between z and iz**

Let $z = re^{i\theta}$

Now, $i = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = e^{i\frac{\pi}{2}}$

Then $iz = e^{i\frac{\pi}{2}} \cdot e^{i\theta} = e^{i\left(\theta + \frac{\pi}{2}\right)}$

i.e. z represents the point P and iz represents the point Q where $\angle POQ = \frac{\pi}{2}$

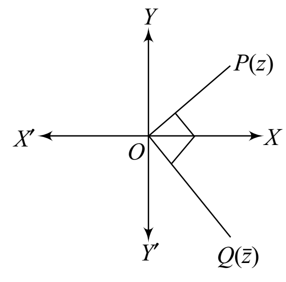


5. **Relation between z and $\bar{z}$**

Let $z = re^{i\theta}$

Then $\bar{z} = re^{-i\theta}$

i.e. z represents the point P and $\bar{z}$ represents the point Q, where $\angle POQ = \frac{\pi}{2}$.



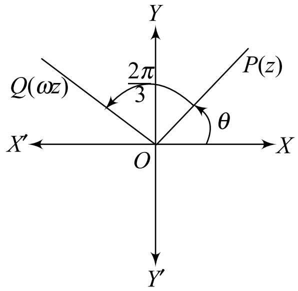
6. **Relation between z and ωz**

Let $z = re^{i\theta}$.

Then $\omega = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = e^{i\left(\frac{2\pi}{3}\right)}$

Now $\omega z = re^{i\theta} \cdot e^{i\left(\frac{2\pi}{3}\right)} = re^{i\left(\theta + \frac{2\pi}{3}\right)}$

i.e. z represents the point P and ωz represents the point Q, where $\angle POQ = \dfrac{2\pi}{3}$.



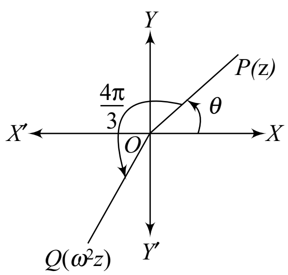
7. **Relation between z and $\omega^2 z$**

Let $z = re^{i\theta}$.

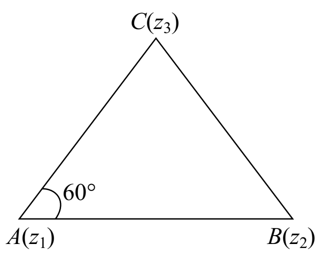
Then $\omega^2 = \cos\left(\dfrac{4\pi}{3}\right) + i\sin\left(\dfrac{4\pi}{3}\right) = e^{i\left(\frac{4\pi}{3}\right)}$

Now, $\omega^2 z = e^{i\left(\frac{4\pi}{3}\right)} \cdot re^{i\theta} = re^{i\left(\theta + \frac{4\pi}{3}\right)}$

i.e. z represents the point P and $\omega^2 z$ represents the point Q, where $\angle POQ = \dfrac{4\pi}{3}$.



8. **Condition of an Equilateral Triangle**



Here, ΔABC is an equilateral triangle.

$$\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \left|\frac{z_3 - z_1}{z_2 - z_1}\right| \times e^{i\frac{\pi}{3}}$$

$$\Rightarrow \left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \frac{|z_3 - z_1|}{|z_2 - z_1|} \times e^{i\frac{\pi}{3}}$$

$$\Rightarrow \left(\frac{z_3 - z_1}{z_2 - z_1}\right) = e^{i\frac{\pi}{3}} = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)$$

$$\Rightarrow \left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$\Rightarrow \left(\frac{z_3 - z_1}{z_2 - z_1} - \frac{1}{2}\right)^2 = \left(i\frac{\sqrt{3}}{2}\right)^2$$

$$\Rightarrow \left(\frac{2z_3 - 2z_1 - z_2 + z_1}{2(z_2 - z_1)}\right)^2 = -\frac{3}{4}$$

$$\Rightarrow (2z_3 - z_1 - z_2)^2 = -3(z_2 - z_1)^2$$

$$\Rightarrow (4z_3^2 + z_1^2 + z_2^2 - 4z_1z_3 - 4z_2z_3 + 2z_1z_2) = -3(z_2^2 + z_1^2 - 2z_1z_2)$$

$$\Rightarrow 4(z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_1z_3) = 0$$

$$\Rightarrow (z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_1z_3) = 0$$

which is the required condition.

Also, $(z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_1z_3) = 0$

$$\Rightarrow (z_1 - z_2)(z_2 - z_3) + (z_2 - z_3)(z_3 - z_1) + (z_3 - z_1)(z_1 - z_2) = 0$$

Dividing both the sides by

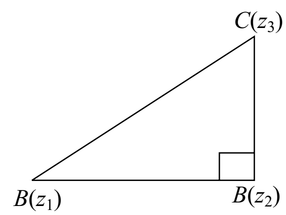
$(z_1 - z_2)(z_2 - z_3)(z_3 - z_1)$, we get,

$$\Rightarrow \frac{1}{z_3 - z_1} + \frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} = 0$$

$$\Rightarrow \frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$$

which is also a condition for an equilateral triangle.

9. **Condition of an Isosceles Right-angled Triangle**



We have $\left(\dfrac{z_3 - z_2}{z_1 - z_2}\right) = \left|\dfrac{z_3 - z_2}{z_1 - z_2}\right| \times e^{-i\frac{\pi}{2}}$

$$\Rightarrow \left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \frac{|z_3 - z_2|}{|z_3 - z_2|} \times e^{-i\frac{\pi}{2}}$$

$$\Rightarrow \left(\frac{z_3 - z_2}{z_1 - z_2}\right) = e^{-i\frac{\pi}{2}} = -i$$

$$\Rightarrow (z_3 - z_2) = -i(z_1 - z_2)$$

$$\Rightarrow (z_3 - z_2)^2 = -(z_1 - 2)^2$$

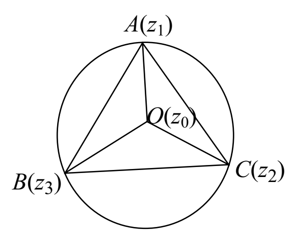
$$\Rightarrow z_3^2 + z_2^2 - 2z_2z_3 = -(z_1^2 + z_2^2 - 2z_1z_2)$$

$$\Rightarrow z_1^2 + z_2^2 - 2z_1z_2 = 2z_1z_3 + 2z_2z_3 - 2z_1z_2 - 2z_3^2$$

$$\Rightarrow (z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$$

which is the required condition.

10. **Condition of Circumcentre w.r.t. an Equilateral Triangle**



In an equilateral triangle, the circumcentre and the centroid are the same point.

Therefore, $z_0 = \left(\dfrac{z_1 + z_2 + z_3}{3}\right)$

$\Rightarrow \quad (z_1 + z_2 + z_3)^2 = 9z_0^2$

$\Rightarrow \quad z_1^2 + z_2^2 + z_3^2 + 2(z_1z_2 + z_2z_3 + z_3z_1) = 9z_0^2$

$\Rightarrow \quad (z_1z_2 + z_2z_3 + z_3z_1)$

$$= \left(\frac{9z_0^2 - z_1^2 - z_2^2 - z_3^2}{2}\right)$$

since z_1, z_2, z_3 are the vertices of an equilateral triangle, then

$\Rightarrow \quad z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

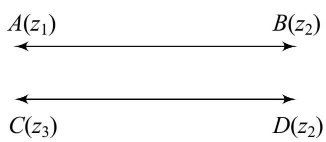
Thus, $z_1^2 + z_2^2 + z_3^2 = \dfrac{9 - z_1^2 - z_2^2 - z_3^2}{2}$

$\Rightarrow \quad 3(z_1^2 + z_2^2 + z_3^2) = 9z_0^2$

$\Rightarrow \quad (z_1^2 + z_2^2 + z_3^2) = 3z_0^2$

which is the required condition.

11. **Condition of Parallelism of Two Lines**



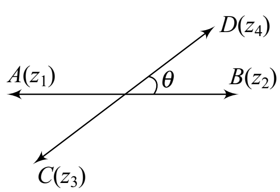
Let $\arg\left(\dfrac{z_1 - z_2}{z_3 - z_4}\right) = \theta$.

If $AB \| CD$, then $\theta = 0$ or $\pm\pi$

$\Rightarrow \quad \arg\left(\dfrac{z_1 - z_2}{z_3 - z_4}\right) = 0 \text{ or } \pm\pi$

$\Rightarrow \quad \left(\dfrac{z_1 - z_2}{z_3 - z_4}\right)$ is real

12. **Condition of Perpendicularity**



If AB ⊥ CD, then $\theta = \pm\dfrac{\pi}{2}$

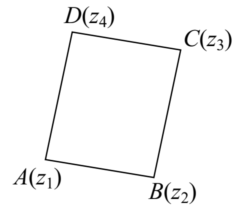
$\arg\left(\dfrac{z_1 - z_2}{z_3 - z_4}\right) = \pm\dfrac{\pi}{2}$

$\left(\dfrac{z_1 - z_2}{z_3 - z_4}\right)$ is purely imaginary number.

$(z_1 - z_2) = \pm k(z_3 - z_4)$,

where k is purely imaginary number.

13. **Condition of a parallelogram**



As we know that the diagonals of a parallelogram bisect each other.

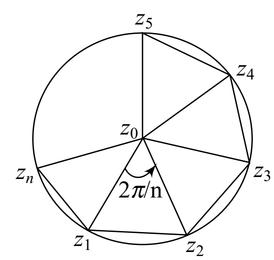
Thus, mid point of AC = Mid-point of BD

$\Rightarrow \quad \dfrac{z_1 + z_3}{2} = \dfrac{z_2 + z_4}{2}$

$\Rightarrow \quad (z_1 + z_3) = (z_2 + z_4)$

14. **Regular polygon of n sides**

(i) Given z_1 and z_0, then find z_2 and z_3 in terms of z_1 and z_0



Centre at Z_0 is at an angle $2\pi/n$.

$$\left(\frac{z_2 - z_0}{z_1 - z_0}\right) = \left|\frac{z_2 - z_0}{z_1 - z_0}\right| \times e^{i\frac{2\pi}{n}}$$

$$\Rightarrow \quad \left(\frac{z_2 - z_0}{z_1 - z_0}\right) = \frac{|z_2 - z_0|}{|z_2 - z_0|} \times e^{i\frac{2\pi}{n}}$$

$$\Rightarrow \quad \left(\frac{z_2 - z_0}{z_1 - z_0}\right) = e^{i\frac{2\pi}{n}}$$

$$\Rightarrow \quad (z_2 - z_0) = (z_1 - z_0) \times e^{i\frac{2\pi}{n}}$$

$$\Rightarrow \quad z_2 = z_0 + (z_1 - z_0) \times e^{i\frac{2\pi}{n}}$$

similarly, we can easily show that,

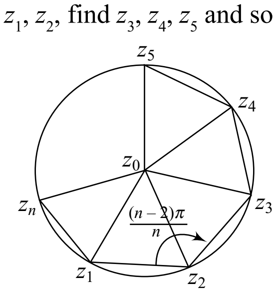
$$z_3 = z_0 + (z_1 - z_0) \times e^{i\frac{4\pi}{n}}$$

$$z_4 = z_0 + (z_1 - z_0) \times e^{i\frac{6\pi}{n}}$$

$\vdots$

$$z_n = z_0 + (z_1 - z_0) \times e^{i\frac{2(n-1)\pi}{n}}$$

(ii) Given z_1, z_2, find z_3, z_4, z_5 and so on.



As we know that, the sum of the angles of a polygon $= (n-2)\pi$

Each interior angle of a polygon,

$$\theta = \left(\frac{(n-2)\pi}{n}\right)$$

We have,

$$(z_3 - z_2) = (z_1 - z_2) \times e^{-i\left(\frac{n-2}{n}\right)\pi}$$

$$z_3 = z_2 + (z_1 - z_2) \times e^{-i\left(\frac{n-2}{n}\right)\pi}$$

Similarly, we can easily show that

$$z_4 = z_2 + (z_1 - z_2) \times e^{-i2\left(\frac{n-2}{n}\right)\pi}$$

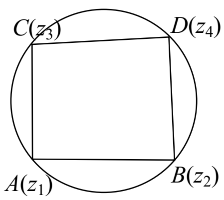
$$z_5 = z_2 + (z_1 - z_2) \times e^{-i4\left(\frac{n-2}{n}\right)\pi}$$

$$z_6 = z_2 + (z_1 - z_2) \times e^{-i6\left(\frac{n-2}{n}\right)\pi}$$

⋮

and so on.

15. **Condition for Points to be Concyclic**



Let z_1, z_2, z_3 and z_4 are taken in order, representing the points A, B, C and D respectively.

If z_1, z_2, z_3 and z_4 are concyclic, then

$$\arg\left(\frac{z_4 - z_2}{z_4 - z_1}\right) = \arg\left(\frac{z_3 - z_2}{z_3 - z_1}\right)$$

$$\Rightarrow \quad \arg\left(\frac{z_4 - z_2}{z_4 - z_1}\right) - \arg\left(\frac{z_3 - z_2}{z_3 - z_1}\right) = 0$$

$$\Rightarrow \quad \arg\left(\left(\frac{z_4 - z_2}{z_4 - z_1}\right) / \left(\frac{z_3 - z_2}{z_3 - z_1}\right)\right) = 0$$

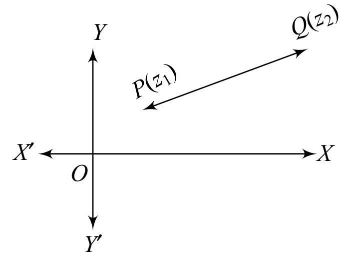
$$\Rightarrow \quad \left(\frac{z_4 - z_2}{z_4 - z_1}\right) \times \left(\frac{z_3 - z_2}{z_3 - z_1}\right) \text{ is real.}$$

Note If z_1, z_2, z_3 and z_4 are not in order, even then this result is also valid.

4.15 Loci in a Complex Plane

1. Distance Formula

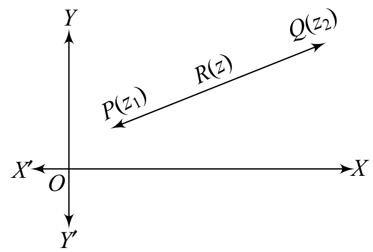
If z_1 and z_2 be the affixes of the two points P and Q respectively, the distance between P and Q is $|z_1 - z_2|$



2. Section Formula

If z_1 and z_2 be the affixes of the two points P and Q respectively and the point R divides the line joining P and Q internally in the ratio $m : n$, the affix z of R is given by

$$z = \frac{mz_2 + nz_1}{m + n}$$

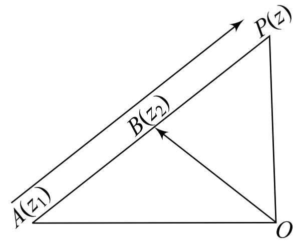


Note If R divides PQ externally in the ratio $m:n$, then $z = \dfrac{mz_2 - nz_1}{m - n}$

1. If a, b, c are three real numbers such that $az_1 + bz_2 + cz_3 = 0$, where $a + b + c = 0$ and a, b, c are not all simultaneously zero, the complex numbers z_1, z_2 and z_3 are collinear.
2. If the vertices A, B, C of a ΔABC represent the complex numbers z_1, z_2 and z_3 respectively and a, b, c are the lengths of its sides, then
   (i) Centroid of ΔABC is $\left(\dfrac{z_1 + z_2 + z_3}{3}\right)$.
   (ii) Orthocentre of the ΔABC is
   $$\frac{(a\sec A)z_1 + (a\sec B)z_2 + (a\sec C)z_3}{a\sec A + b\sec B + c\sec C}.$$
   or
   $$= \frac{(a\tan A)z_1 + (a\tan B)z_2 + (a\tan C)z_3}{a\tan A + b\tan B + c\tan C}$$
   (iii) In-centre of the ΔABC is
   $$= \left(\frac{az_1 + bz_2 + cz_3}{a + b + c}\right).$$
   (iv) Circumcentre of the ΔABC is
   $$= \frac{(\sin 2A)z_1 + (\sin 2B)z_2 + (\sin 2C)z_3}{(\sin 2A) + (\sin 2B) + (\sin 2C)}$$

3. Straight Line

(i) We consider two fixed points $A(z_1)$ and $B(z_2)$ and a variable point P moving on the straight line passing through the points A and B.



By the triangle law of vector addition

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$$

$$\overrightarrow{OP} = \overrightarrow{OA} + t \cdot \overrightarrow{AB}$$

where t is some suitable real number.

Now writing corresponding complex number, we get

$$z = z_1 + t(z_2 - z_1) \qquad \ldots\text{(i)}$$

where t is called the parameter and the Eq. (i) is called the parametric equation of the straight line passing through two fixed points $A(z_1)$ and $B(z_2)$.

(ii) Also Eq. (i) can be written as $\frac{z - z_1}{z_2 - z_1} = t$, where t is a real number.

$$\Rightarrow \quad \text{Im}\left(\frac{z - z_1}{z_2 - z_1}\right) = 0$$

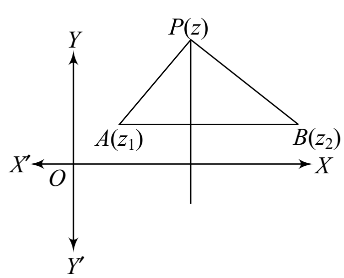
$$\Rightarrow \quad \left(\frac{z - z_1}{z_2 - z_1}\right) = \left(\overline{\frac{z - z_1}{z_2 - z_1}}\right)$$

$$\Rightarrow \quad \left(\frac{z - z_1}{z_2 - z_1}\right) = \left(\frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1}\right)$$

$$\Rightarrow \quad \begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0$$

which represents the non-parametric form of equation of a straight line passing through the points $A(z_1)$ and $B(z_2)$.

(iii) $|z - z_1| + |z - z_2| = |z_1 - z_2|$ is the perpendicular bisector of the line joining z_1 and z_2.

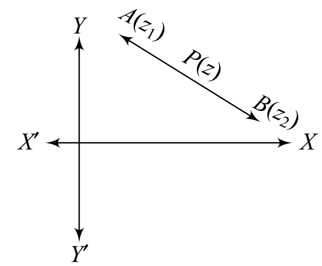


From geometry, it follows that P lies on the perpendicular bisector of the segment AB.

(iv) If $A(z_1)$ and $B(z_2)$ are two fixed points and P is a variable point moving on the line segment AB, then

$$AP + PB = AB$$

i.e. $\quad |z - z_1| + |z - z_2| = |z_1 - z_2|$



(v) From the above figure, we can write

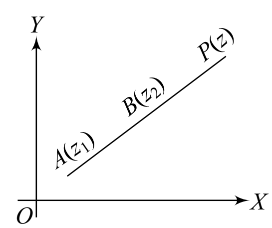
$$\frac{z_1 - z}{z_2 - z} = \frac{PA\, e^{i \arg(z_1 - z)}}{PB e^{i \arg(z_2 - z)}}$$

$$= PA e^{i \arg(z_1 - z) - \arg(z_2 - z)}$$

$$\Rightarrow \quad \frac{z_1 - z}{z_2 - z} = \rho e^{i\pi}$$

This represents the locus of a straight line.

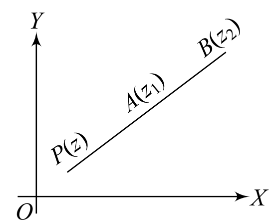
(vi) We consider two fixed points $A(z_1)$, $B(z_2)$ and a variable point $P(z)$ lying on the line passing through A and B but not lying on the line segment AB.



Therefore, $PA - PB = AB$

$$|z - z_1| - |z - z_2| = |z_1 - z_2|$$

which represents a straight line.



Also, $PA - PB = -AB$

$$|z - z_1| - |z - z_2| = -|z_1 - z_2|$$

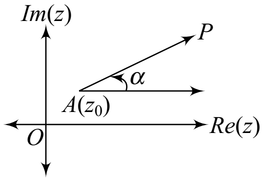
which also represents a straight line.

Also, $\frac{z_1 - z}{z_2 - z} = \frac{AP}{BP} \times e^{i\theta}$

Thus, $\text{Arg}\left(\frac{z_1 - z}{z_2 - z}\right) = 0$

which also represents a straight line.

(vii) We consider a fixed point $A(z_0)$and a variable point P moving in such a way that PA always forms an angle α with positive Re(z) axis.



Thus, $z - z_0 = PA \cdot e^{i\alpha}$

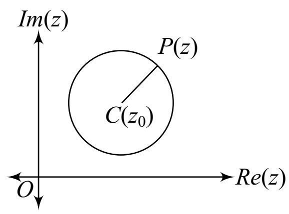
$\Rightarrow \quad \arg(z - z_0) = \alpha$

This equation represents a ray originating from A (but excluding the point A) making an angle with positive Re(z) axis.

> ***Note*** When $z_0 = 10$, the $\arg(z) = \alpha$, which represents a ray originating from origin (but excluding origin) and making an angle α with positive Re(z) axis.

4. Circle

(i) We consider a fixed point $C(z_0)$ and a variable point $P(z)$, which is moving keeping its distance from the point C a constant b.

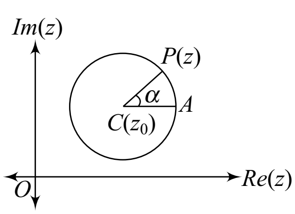


At any point $P(z)$ satisfies the equation

$$|z - z_0| = b \quad \ldots\text{(i)}$$

This equation represents a circle with the centre at (z_0) and the radius is b.

(ii)



For all complex numbers satisfying Eq. (i), then the modulus of $(z - z_0)$ is a constant and its argument is variable.

If we define this variable argument by θ, we can write $(z - z_0) = be^{i\theta}$.

As θ changes $P(z)$ moves on the circle.

Here, θ is called the parameter and the equation is called the parametric equation of the circle whose centre is z_0 and the radius is b.

(iii) From Eq. (i), we have

$$|z - z_0|^2 = b^2$$

$$(z - z_0)\overline{(z - z_0)} = b^2$$

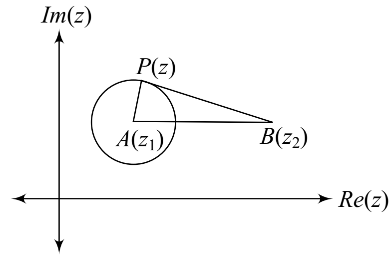
$$(z - z_0)(\bar{z} - \overline{z_0}) = b^2$$

$$z\bar{z} - \bar{z}z_0 - z\overline{z_0} + |z_0|^2 = b^2$$

$|z_0|^2 + \bar{\omega}z + \omega\bar{z} + d = 0$, where $\omega = -z_0$ and $d = |z|^2 - b^2$ which represents a circle, whose centre is $z_0 = -\omega$ and the radius is $b = \sqrt{|z_0|^2 - d}$

(iv) We consider two fixed points $A(z_1)$ and $B(z_2)$ and $P(z)$ is a variable point in such a way that

$\dfrac{PA}{PB}$ = constant = $k(\neq 1)$

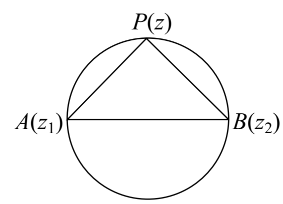


Thus, $\left|\dfrac{z - z_1}{z - z_2}\right| = k$

$\Rightarrow \quad \left|\dfrac{z - z_1}{z - z_2}\right| = k$, where $k \neq 1$

which represents a circle.

(v) If $A(z_1)$ and $B(z_2)$ are two fixed points and $P(z)$ is a variable point, moving on the circle whose diameter is AB.

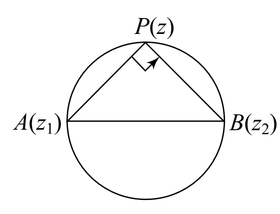


Thus, $PA^2 + PB^2 = AB^2$

$$|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$$

which represents a circle, whose centre is $\left(\dfrac{z_1 + z_2}{2}\right)$ and the radius is $-\dfrac{1}{2}|z_1 - z_2|$

(vi)

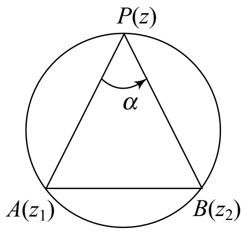


Also, $\left(\dfrac{z_2 - z}{z_1 - z}\right) = \dfrac{PB}{PA} \times e^{i\frac{\pi}{2}}$

$\Rightarrow \quad \arg\left(\dfrac{z_2 - z}{z_1 - z}\right) = \dfrac{\pi}{2}$

which also represents a circle.

(vii) We consider two fixed points $A(z_1)$, $B(z_2)$ and $P(z)$ is a variable point moving in such a way that the angle subtended by segment on the moving point P is a constant (α).

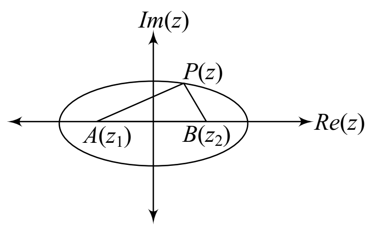


Thus, $\left(\frac{z_2 - z}{z_1 - z}\right) = \rho \times e^{i\alpha}$, where α is a parameter, whose value depends upon the position of P.

Therefore, the equation satisfied by all the complex number z lying on the arc AB of a circle can be written as $\arg\left(\frac{z_2 - z}{z_1 - z}\right) = \alpha$, where $\alpha \neq 0, \pi$

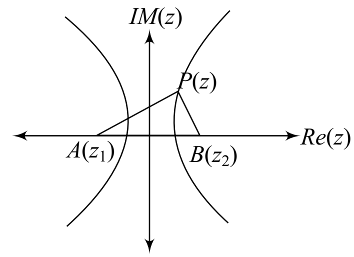
5. Ellipse

If $|z - z_1| + |z - z_2| = 2a$, where $2a > |z_1 - z_2|$, the point z describes an ellipse having foci at z_1 and z_2, respectively and $a \in R^+$.



6. Hyperbola

If $|z - z_1| - |z - z_2| = 2a$, where $2a < |z_1 - z_2|$, the point z describes a hyperbola having foci at z_1 and z_2 respectively and $a \in R^+$.



7. Inverse Point with Respect to a Circle

Two points P and Q are said to be inverse with respect to a circle with the centre O and the radius r, if

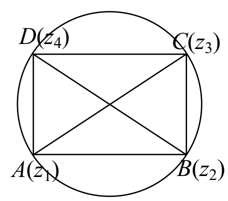
(i) The points O, P, Q are collinear and on the same side of O.

(ii) $OP \cdot OQ = r^2$

> ***Note*** Two points z_1 and z_2 will be the inverse points with respect to the circle $z \cdot \overline{z} + \overline{\alpha} \cdot z + \alpha \cdot \overline{z} + r = 0$ if and only if $z_1 \overline{z_2} + \overline{\alpha} \cdot z_1 + \alpha \cdot \overline{z_2} + r = 0$

26. Ptolemy's Theorem

It states that the product of the lengths of the diagonal of a convex quadrilateral inscribed in a circle is equal to the sum of the lengths of the two pairs of its opposite sides.



i.e $AC.BD = AB.CD + AD.BC$

$\Rightarrow \quad |z_1 - z_3| \cdot |z_2 - z_4|$
$= |z_1 - z_2| \cdot |z_3 - z_4| + |z_1 - z_4| \cdot |z_2 - z_4|$.

Exercises

Level I

(Problems based on Fundamentals)

ABC OF COMPLEX NUMBERS

1. Find the value of $i^n + i^{n+1} + i^{n+2} + i^{n+3}$, $n \in I$.
2. Find the value of $i^{2010} + i^{2011} + i^{2012} + i^{2013}$.
3. Find the smallest integer n for which $\left(\frac{1+i}{1-i}\right)^n = 1$.
4. Find the sum of $\sum_{n=1}^{2013} (i^n + i^{n+1})$.
5. Find the value of $i^P + i^Q + i^R + i^S$ where P, Q, R, S are four consecutive integers.
6. Find the value of $i^{2015} + i^{2016} + i^{2017} + i^{2018}$.
7. If $\sum_{k=0}^{2016} i^k + \sum_{p=0}^{2018} i^p = x + iy$, $i = \sqrt{-1}$, find $x + y + 2$.
8. Find the smallest positive integer n for which $(1 + i)^{2n} = (1 - i)^{2n}$.
9. Find the value of $(1 + i)^5 + (1 + i^3)^5 + (1 + i^5)^7 + (1 + i^7)^7$.
10. Let $z = (n + i)^4$. Find the number of integral values of n for which z is an integer.
11. If $z = 1 + i$, find the multiplicative inverse of z^2.
12. If $z = \frac{1 + 2i}{3 - 4i}$, find the multiplicative inverse of z.

13. If $a + ib > c + id$, find the value of $b + d + 2016$.
14. Find the least positive integer n for which
$$\left(\frac{1+i}{1-i}\right)^n = \frac{2}{\pi}\left(\sin^{-1}x + \sec^{-1}\left(\frac{1}{x}\right)\right).$$
15. Find x and y which satisfy the equation
$$\frac{(1+i)x-2i}{(3+i)} + \frac{(2-3i)y+i}{(3-i)} = i.$$
16. If $x + iy = \frac{2^{1008}}{(1+i)^{2016}} + \frac{(1+i)^{2016}}{2^{1008}}$, find x and y.
17. Let $z = x + iy$. If $z^{1/3} = a + ib$, prove that
$$\frac{x}{a} + \frac{y}{b} = 4(a^2 - b^2).$$
18. If $f(x) = x^4 - 8x^3 + 4x^2 + 4x + 39$ and $f(3 + 2i) = a + ib$, then find $\left(\frac{b}{a} + 10\right)$.
19. Find the least positive integer n for which $z = \left(\frac{2i}{1+i}\right)^n$ is a positive integer.
20 If $x = 3 + 2i$ is a root of a quadratic equation, find its equation.
21. Solve: $z^2 + \bar{z} = 0$.
22. If $(i - i)$ is the root of the equation $z^3 - 2(2 - i)z^2 + (4 - 5i)z + (3i - 1) = 0$, find the other roots.
23. Given that $1 + 2i$ is one root of the equation $x^4 - 3x^3 + 8x^2 - 7x + 5 = 0$, find the other three roots.

MODULUS OF COMPLEX NUMBERS

24. If $|z - (2 + 3i)| = 1$, find the greatest and the least value of $|z|$.
25. If $|z + 3 + 5i| = 2$, find the difference between the greatest and the least value of $|z|$.
26. If $a + ib = (1 + i)(1 + 2i)(1 + 3i) \ldots (1 + ni)$, then find the value of $a^2 + b^2$
27. If $x + iy = \frac{a+ib}{a-ib}$, prove that $x^2 + y^2 = 1$.
28. The complex number z satisfies $z + |z| = 2 + 8i$, find $|z|$.
29. If $z = re^{i\theta}$, then find $|e^{iz}|$.
30. If α, β be different complex numbers, find the maximum value of $\frac{\alpha\bar{\beta} + \beta\bar{\alpha}}{|\alpha\beta|}$.
31. If $|z_1| = |z_2| = |z_3| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1$, find the value of $|z_1 + z_2 + z_3|$.
32. If $z = (3 + 7i)(p + iq)$, $p, q \in I - \{0\}$ is purely imaginary number, find the minimum value of $|z|^2$.
33. If α, β be different complex numbers with $|\beta| = 1$, find the value of $\left|\frac{\beta - \alpha}{1 - \bar{\alpha}\beta}\right|$.
34. Find the complex number z, if $|z + 1| = z + 2(1 + i)$.
35. If $|z - 1|^2 + |z + 1|^2 = 5$, find $|z|^2$.
36. If $|z - 2| = 2|z - 1|$, prove that $|z|^2 = \frac{4}{3}\text{Re}(z)$.
37. If $|z + 6| = |3z + 2|$, prove that $|z| = 2$.
38. Let $z = x + iy$ and $|z + 6| = |2z + 3|$, the locus of z is $x^2 + y^2 = 9$.
39. If $|z| = 1$ and $\omega = \frac{z-1}{z+1}$ (where $z \neq 1$), find $Re(\omega)$.
40. If z be a complex number satisfying the equation $|z + i| + |z - i| = 8$ on the complex plane, find the maximum value of $|z|$.

ARGUMENT OF COMPLEX NUMBERS

41. Find the arguments of

(i) 1+i	(ii) 1 – i
(iii) –1 + i	(iv) –1 – i
(v) 0	(vi) 2013
(vii) –2013	(viii) 2 i
(ix) – 2 i	(x) $\frac{1}{(1+i)}$

42. If $z = \frac{\sqrt{3}}{2} + \frac{i}{2}$, find Arg($-z$).
43. If Arg(z) < 0, find the value of Arg(z) – Arg($-z$).
44. If $z = x + iy$ such that $|z + 1| = |z - 1|$ and $\text{Amp}\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$, find z.
45. If z_1 and z_2 are two non-zero complex numbers such that $|z_1 + z_2| = |z_1 - z_2|$, find $\text{Amp}\left(\frac{z_1}{z_2}\right)$.
46. Let $z = \left(\frac{\cos\theta + i\sin\theta}{\cos\theta - i\sin\theta}\right), \frac{\pi}{4} < \theta < \frac{\pi}{2}$, find Arg($z$).
47. Find the argument of z if
$$z = \sin\left(\frac{\pi}{5}\right) + i\left(1 - \cos\left(\frac{\pi}{5}\right)\right)$$
48. If $\text{Arg}(z) = \frac{\pi}{3}$ and $\text{Arg}(z - 1) = \frac{5\pi}{6}$, find the complex number z.
49. Let z be a complex number having the argument θ, $0 < \theta < \frac{\pi}{2}$ and satisfying the inequality $|z - 3i| = 3$, find $\text{Arg}\left(\cot\theta - \frac{6}{z}\right)$.
50. If $\text{Amp}\left(\frac{z-1}{z+1}\right) = \frac{\pi}{3}$, find the locus of z.
51. Find the angle that the vector representing the complex number $\frac{1}{(\sqrt{3} - i)^{25}}$ makes with the positive direction of the real axis.

MODULUS AND ARGUMENT OF COMPLEX NUMBERS

52. If $z_1, z_2 \in C$, prove that $|z_1 + z_2| \leq |z_1| + |z_2|$.
53. If $z_1, z_2 \in C$, prove that $\big||z_1| - |z_2|\big| \leq |z_1 - z_2|$.
54. If $|z_1 + z_2| = |z_1 - z_2| \Leftrightarrow \text{Arg}(z_1) - \text{Arg}(z_2) = \frac{\pi}{2}$.

55. If $|z_1 + z_2| = |z_1| + |z_2|$, prove that $\text{Arg}(z_1) = \text{Arg}(z_2)$.

56. If $|z_1 + iz_2| = |z_1| + |z_2| \Leftrightarrow \text{Arg}(z_1) - \text{Arg}(z_2) = \frac{\pi}{2}$.

57. If $|z_1 - z_2| = |z_1| + |z_2| \Leftrightarrow z_1 + kz_2 = 0, k \in I^+$

58. If $|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2 \Leftrightarrow \frac{z_1}{z_2}$ is purely imaginary number.

59. If $|z_1| \le 1, |z_2| \le 1$, prove that
$|z_1 - z_2|^2 \le (|z_1| - |z_1|)^2 + (\text{Arg}(z_1) - \text{Arg}(z_2))^2$

60. If $|z_1| \le 1, |z_2| \le 1$, prove that
$|z_1 + z_2|^2 \le (|z_1| - |z_1|)^2 - (\text{Arg}(z_1) - \text{Arg}(z_2))^2$

61. Find the maximum value of $|z + 1|$, where $|z + 4| \le 3$.

62. Find the minimum value of $|z + 2|$, where $|z + 5| \le 4$.

63. Find the minimum values of
(i) $|z| + |z + 2|$
(ii) $|z + 2| + |z - 2|$

64. Find the maximum value of $|z + 2| + |z - 2| + |2z - 7|$.

65. Find the greatest and the least values of $|z_1 + z_2|$, where $z_1 = 6 + 8i$ and $z_2 = 3 + 4i$

66. If $|z - 3i| \le 4$, find the maximum value of $|i(z + 1) + 1|$.

67. If $|z| = 3$, then find the min and maximum values of $\left|z + \frac{1}{z}\right|$.

68. If $|z_1| = 2, |z_2| = 3, |z_3| = 5$ such that $|25z_1z_2 + 9z_1z_3 + 4z_2z_3| = 90$, find the value of $|z_1 + z_2 + z_3|$.

SQUARE ROOT OF A COMPLEX NUMBER

69. Find the square roots of $3 - 4i$.

70. Find the square roots of $5 + 12i$.

71. Find the square roots of $8 - 6i$.

72. Find the square roots of $3i$.

73. Find the square roots of $8 - 15i$.

74. Find the square roots of
$$x^2 + \frac{1}{x^2} + 4i\left(x - \frac{1}{x}\right) - 6.$$

75. If $z^2 + 5 = 12\sqrt{-1}$, find the complex number z.

CUBE ROOTS OF A COMPLEX NUMBER

76. If ω is the complex cube root of unity, find the value of $(2 + 3\omega + 3\omega^2)^{2013}$.

77. If ω is the complex cube root of unity, find the value of $(3 + 4\omega + 5\omega^2)^{10}$.

78. If ω is the non-real cube root of unity, find the sum of $\omega + \omega^{\left(\frac{1}{2}+\frac{3}{8}+\frac{9}{32}+\frac{27}{128}+\dots\right)}$.

79. Find the common roots of $z^3 + 2z^2 + 2z + 1 = 0$ and $z^{2013} + z^{2014} + z^{2015} = 0$.

80. If α, β, γ be the cube roots of (–2013), for any x, y, and z, find the value of $\frac{x\alpha + y\beta + z\gamma}{x\beta + y\gamma + z\alpha}$.

81. Find the value of $\frac{2 + 3\omega + 4\omega^2}{4 + 3\omega^2 + 2\omega}$.

82. Find the value of $\frac{5 + 6\omega + 7\omega^2}{7 + 6\omega^2 + 5\omega} + \frac{5 + 6\omega + 7\omega^2}{6 + 5\omega + 7\omega^2}$

83. If $i = \sqrt{-1}$, find the value of
$$4 + 5\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{334} + 3\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{365}.$$

84. Solve for x: $x^6 - 9x^3 + 8 = 0$.

DE MOIVRES THEOREM

85. If x satisfies the equation $x^2 - 2x\cos\theta + 1 = 0$, find the value of $x^n + \frac{1}{x^n}$.

86. If $z_r = \cos\left(\frac{2r\pi}{5}\right) + i\sin\left(\frac{2r\pi}{5}\right)$, $r = 1, 2, 3, 4, 5$, find the value of $z_1z_2z_3z_4z_5$.

87. If $\sin\alpha + \sin\beta + \sin\gamma = 0 = \cos\alpha + \cos\beta + \cos\gamma$, find the value of
(i) $\sin 3\alpha + \sin 3\beta + \sin 3\gamma$
(ii) $\cos 3\alpha + \cos 3\beta + \cos 3\gamma$

88. If $\sin\alpha + \sin\beta + \sin\gamma = 0 = \cos\alpha + \cos\beta + \cos\gamma$, find the value of
(i) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma$ (ii) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$

89. If $z_r = \cos\left(\frac{\pi}{2^r}\right) + i\sin\left(\frac{\pi}{2^r}\right)$, find the value of $z_1 \cdot z_2 \cdot z_3 \dots$ to ∞.

90. Find the value of $(1 + i)^8 + (1 - i)^8$.

91. Find the value of
$$\sum_{k=1}^{21}\left(\sin\left(\frac{2\pi k}{11}\right) - i\cos\left(\frac{2\pi k}{11}\right)\right).$$

*n*TH ROOTS OF COMPLEX NUMBERS

92. Solve for x: $x^3 - 1 = 0$.

93. Solve for x: $x^5 - 1 = 0$.

94. Solve for x: $x^7 - 1 = 0$.

95. Solve for x: $x^3 + 1 = 0$.

96. Solve for x: $x^5 + 1 = 0$.

97. Solve for x: $x^7 + 1 = 0$.

98. Solve for x: $x^{10} - 1 = 0$.

99. Solve for x: $x^{10} + x^5 + 1 = 0$.

100. Solve for x: $x^{10} - x^9 + x^8 - x^7 + \dots + x^2 - x + 1 = 0$.

101. Solve for z : $z^5 + 1 = 0$ and deduce that
$$4\sin\left(\frac{\pi}{10}\right)\cos\left(\frac{\pi}{5}\right) = 1.$$

102. Solve for z: $z^7 - 1 = 0$ and deduce that
$$\cos\left(\frac{\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right) = \frac{1}{8}.$$

103. Find the integral solutions of $(1 - i)^x = 2^x$.

104. If $z = \left(\frac{\sqrt{3} - i}{2}\right)$, find the value of $(z^{101} + z^{103})^{106}$.

105. Let $z = x + iy$ be a complex number, where x and y are integers. Find the area of the rectangle whose vertices are the roots of $\bar{z}z^3 + z\bar{z}^3 = 350$.

106. Let $z = \cos\theta + i\sin\theta$, find the value of
$$\sum_{m=1}^{15} \text{Im}(z^{2m-1}) \text{ at } \theta = 2°.$$
107. Let a complex number α, $\alpha \neq 1$ be a root of an equation $z^{p+q} - z^p - z^q + 1 = 0$, where p and q are distinct primes. Show that either
$1 + \alpha + \alpha^2 + \ldots + \alpha^{p-1} = 0$ or
$1 + \alpha + \alpha^2 + \ldots + \alpha^{q-1} = 0$ but not both.

ROATATION

108. If a point $P(3, 4)$ is rotated through an angle of 90° in anti-clockwise sense about the origin, find the new position of P.
109. If a point $Q(3, 4)$ is rotated through an angle of 180° in anti-clockwise sense about the origin, find the new position of Q.
110. If a point $P(3, 4)$ is rotated through an angle of 30° in anti-clockwise sense about the point $Q(1, 0)$, find the new position of P.
111. The complex number $\sqrt{3} + i$ becomes $-1 + i\sqrt{3}$ after rotating an angle θ about the origin in anti-clock-wise sense, find the angle θ.
112. The three vertices of a triangle are represented by the complex numbers 0, z_1, z_2. If the triangle is equilateral triangle, prove that $z_1^2 + z_2^2 = z_1 z_2$
113. If the origin and the roots of $z^2 + az + b = 0$ form an equilateral triangle, prove that $a^2 = 3b$.
114. If the area of a triangle on the complex plane formed by the complex numbers z, iz and $z + iz$ is 50 sq.u., find $|z|$.
115. If the area of a triangle on the complex plane formed by the complex numbers z, ωz, $z + \omega z$ is $16\sqrt{3}$ sq.u., find the value of $(|z|^2 + |z| + 2)$.
116. Let z_1 and z_2 be the nth roots of unity which subtend a right angle at the origin, prove that $n = 4k$, where $k \in N$.
117. Suppose z_1, z_2, z_3 be the vertices of an equilateral triangle inscribed in the circle $|z| = 2$. If $z_1 = 1 + i\sqrt{3}$, prove that $z_2 = -2$ and $z_3 = 1 - i\sqrt{3}$.
118. If a and b be real numbers between 0 and 1 such that the points $z_1 = a + i$, $z_2 = 1 + ib$ and $z_3 = 0$ form an equilateral triangle, prove that $a = 2 - \sqrt{3} = b$.
119. The adjacent vertices of a regular polygon of n-sides are the points z and its conjugate $\bar{z}$.
If $\dfrac{\text{Im}(z)}{\text{Re}(z)} = \sqrt{2} - 1$, find the value of n.
120. The vertices A and C of a square are $ABCD$ are $2 + 3i$ and $3 - 2i$ respectively. Find the vertices B and D.
121. A, B, C are the vertices of an equilateral triangle whose centre is i. If A represents the complex number $-i$, find the vertices B and C.
122. A man walks a distance of 3 units from the origin towards the north-east (N45°E) direction. From there, he walks a distance of 4 units towards the north-west (N45°W) direction to reach a point P. Find the position of the point P in the Argand plane.
123. A particle P starts from the point $z_0 = 1 + 2i$, where $i = \sqrt{-1}$. It moves first horizontally away from origin by 5 units and then vertically away from the origin 3 units to reach a point z_1. From z_1 the particle moves $\sqrt{2}$ units in the direction of the vector $\hat{i} + \hat{j}$ and then through an angle $\dfrac{\pi}{2}$ in anti-clockwise direction on a circle with the centre at the origin to reach a point z_2. Find the point z_2.
124. Let z_1 and z_2 be the roots of the equation $z^2 + pz + q = 0$, where the co-efficients p and q may be complex numbers. Let A and B be represent z_1 and z_2 in the complex plane. If $\angle AOB = \alpha$, and $OA = OB$, where O is the origin, prove that $p^2 = 4q\cos^2\left(\dfrac{\alpha}{2}\right)$.

LOCI OF A COMPLEX NUMBER

125. Find the locus of z, if $\text{Arg}\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{4}$.
126. Find the locus of z, if $|z - 1| + |z + 1| \leq 4$.
127. Find the locus of z, if $|z - 2| + |z + 2| \leq 4$.
128. Find the locus of z, if $z = t + 5 + i\sqrt{4 - t^2}$, $t \in R$.
129. If $\left(\dfrac{z^2}{z-1}\right)$ is always real, find the locus of z.
130. If $\text{Re}\left(\dfrac{1}{z}\right) = c, c \neq 0$, find the locus of z.
131. If $|z^2 - 1| = |z|^2 + 1$, find the locus of z.

Level II

(Mixed Problems)

1. If a complex number z satisfying $z + |z| = 1 + 7i$, the value of $|z|^2$ is
(a) 625 (b) 169 (c) 49 (d) 25
2. If $z = (3 + 7i)(p + iq)$, $p, q \in I - \{0\}$ is purely imaginary number, the minimum value of $|z|^2$ is
(a) 0 (b) 58 (c) 3364 (d) 3364/3
3. The number of complex numbers z satisfying $z^3 = \bar{z}$ is
(a) 1 (b) 2 (c) 4 (d) 5
4. The number of real and purely imaginary solution of the equation $z^3 + iz - 1 = 0$ is
(a) 0 (a) 1 (c) 2 (d) 3
5. A point z moves on the curve $|z - 4 - 3i| = 2$ in an argand plane. The maximum and minimum values of $|z|$ are
(a) 2, 1 (b) 6, 5 (c) 4, 3 (d) 7, 3
6. If z be a complex number satisfying the equation $|z + i| + |z - i| = 8$ on the complex plane, the maximum value of $|z|$ is
(a) 2 (b) 4 (c) 6 (d) 8
7. Let z be a complex number satisfying the equation $(z^3 + 3)^2 = -16$, the value of $|z|$ is
(a) $5^{1/2}$ (b) $5^{1/3}$ (c) $5^{2/3}$ (d) 5

8. The area of a triangle whose vertices are the roots of $z^3 + iz^2 + 2i = 0$ is (in sq.u.)
(a) 2 (b) $\frac{3\sqrt{7}}{2}$ (c) $\frac{3\sqrt{7}}{4}$ (d) $\sqrt{7}$

9. The minimum value of $|z - 1 + 2i| + |4i - 3 - z|$ is
(a) $\sqrt{5}$ (b) 5 (c) $2\sqrt{13}$ (d) $\sqrt{15}$

10. The number of complex numbers z such that $|z| = 1$ and $\left|\frac{z}{\overline{z}} + \frac{\overline{z}}{z}\right| = 1$ is
(a) 4 (b) 6 (c) 8 (d) 10

11. The number of ordered pairs (a, b) of real numbers such that $(a + ib)^{2008} = a - ib$ holds good is
(a) 2008 (b) 2009 (c) 2010 (d) 2015

12. The difference between the maximum and minimum values $|z + 1|$, when $|z + 3| \le 3$, is
(a) 6 (b) 5 (c) 4 (d) 3

13. If $z^3 + (3 + 2i)z + (ia - 1) = 0$ has one real root, the value of a $(a \in R)$ lies in
(a) $(-2, -1)$ (b) $(-1, 0)$
(c) $(0, 1)$ (d) $(1, 2)$

14. If $|z| = 1$ and $|\omega - 1| = 1$, where $z, \omega \in C$, the largest set of values of $|2z - 1|^2 + |2\omega - 1|^2$ is
(a) [1, 9] (b) [2, 6]
(c) [2, 12] (d) [2, 18]

15. Let z is a complex number such that $z + \frac{1}{z} = 2\cos(3°)$, the value of $z^{2000} + \frac{1}{z^{2000}} + 1$ is
(a) 0 (b) –1
(c) $(\sqrt{3} + 1)$ (d) $-(\sqrt{3} - 1)$

16. The maximum number of real roots of $x^{2n} - 1 = 0$ is
(a) 2 (b) 3 (c) n (d) $2n$

17. The locus of point z satisfying the condition $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{3}$ is
(a) straight line (b) a circle
(c) a parabola (d) none

18. The region of the z-plane for which $\left|\frac{z-a}{z+a}\right| = 1$, $\text{Re}(a) \ne 0$, is
(a) x-axis (b) y-axis
(c) the straight line $|x| = a$ (d) none

19. If $\text{Re}\left(\frac{z - 8i}{z + 8i}\right) = 0$, then z lies on the curve
(a) $x^2 + y^2 + 6x - 8y = 0$
(b) $4x - 3y + 24 = 0$
(c) $x^2 + y^2 = 8$
(d) none

20 The locus represented by $|z + 1| = |z + i|$ is
(a) a circle of radius 1
(b) an ellipse with foci (1,0) and (0, –1)
(c) a straight line through the origin
(d) a circle on the line joining (1, 0) and (0, 1) as a diameter.

21. If $|\overline{z}| = 25$, the points representing the complex number $(-1 + 75\overline{z})$ will lie on
(a) circle (b) parabola
(c) ellipse (d) hyperbola

22. z_1, z_2, z_3, z_4 are distinct complex numbers representing the vertices $ABCD$ taken in order. If $z_1 - z_4 = z_2 - z_3$ and $\arg\left(\frac{z_4 - z_1}{z_2 - z_1}\right) = \frac{\pi}{2}$, the quadrilateral is a
(a) rectangle (b) rhombus
(c) square (d) trapezium

23. If $\omega = \alpha + i\beta$, $\beta \ne 0$ and $\frac{\omega - \overline{\omega}z}{1 - z}$ is real, then z will satisfy
(a) $z \cdot |z| \ne 1$ (b) $z \cdot |z| = 1$
(c) $z \cdot \overline{z} = 1$ (d) $z \cdot z = \overline{z}$

24. If $u = \frac{2z + 5i}{z - 3}$ and $|u| = 2$, the locus z is
(a) straight line (b) circle
(c) parabola (d) none

25. If $\omega = \frac{z}{z - \frac{i}{3}}$ and $|\omega| = 1$, then z lies on
(a) straight line (b) parabola
(c) circle (d) ellipse

26. Let z_1, z_2, z_3 are the affixes of the vertices of a triangle having its circumcentre at the origin. If z is affix of its orthocentre, then
(a) $z_1 + z_2 + z_3 - z = 0$ (b) $z_1 + z_2 - z_3 + z = 0$
(c) $z_1 - z_2 + z_3 + z = 0$ (d) $-z_1 + z_2 + z_3 + z = 0$

27. Let A, B, C respectively represents the complex numbers z_1, z_2, z_3 on the complex plane. If the circumcentre of the triangle ABC lies at the origin, the orthocentre is represented by the complex number
(a) $z_1 + z_2 - z_3$ (b) $-z_1 + z_2 + z_3$
(c) $z_1 - z_2 + z_3$ (d) $z_1 + z_2 + z_3$

28. The complex number $z = 1 + i$ is rotated through an angle 270° in anti-clockwise direction about the origin and stretch by additional $\sqrt{2}$ units, the new complex number is
(a) $2(1 + i)$ (b) $2(1 - i)$
(c) $(1 - i)$ (d) $-(1 + i)$

29. The vector $z = 4 + 5i$ is turned counter-clockwise through an angle of 180° and stretch 3/2 times. The complex number corresponding to newly obtained vector is
(a) $\left(6 - \frac{15}{2}i\right)$ (b) $\left(-6 + \frac{15}{2}i\right)$
(c) $\left(6 + \frac{15}{2}i\right)$ (d) none

30. The number 15th roots of unity which are also 25th roots of unity is
(a) 3 (b) 5 (c) 10 (d) 13.

31. The complex number z satisfies the equation $\left|z-\frac{25}{z}\right|=24$. The maximum distance from the origin to z is
(a) 25 (b) 30 (c) 32 (d) 40

32. If the area of a triangle formed by the points z, iz and $z + iz$ is 200, then $|z|$ is
(a) 5 (b) 10 (c) 15 (d) 20

33. Let z be a root of $z^5 - 1 = 0$ with $z \neq 1$. The value of $z^{15} + z^{16} + \ldots + z^{50}$ is
(a) 1 (b) –1 (c) 0 (d) 5

34. The set of all real x satisfying the inequality $|4i - 1 - \log_2 x| \geq 5$ is
(a) $\left(0, \frac{1}{16}\right]$ (b) $\left(0, \frac{1}{16}\right] \cup [4, \infty)$
(c) $[4, \infty)$ (d) $\left(\frac{1}{16}, 4\right]$

35. The number of roots of $\left(\frac{x+1}{x-1}\right)^n = 1$, where $n \in R$ and $x \in R$, is
(a) n (b) 1 (c) $n-1$ (d) $n-2$

36. If ω is an imaginary fifth root of 2 and $x = \omega + \omega^2$, the value of $x^5 - 10x^2 - 10x$ is
(a) 4 (b) 6 (c) 20 (d) 12

37. Suppose A is complex number and $n \in N$, such that $A^n = (A + 1)^n = 1$, then the least value of n is
(a) 3 (b) 6 (c) 9 (d) 12

38. If $z^3 - iz^2 - 2iz - 2 = 0$, then z can be
(a) $1 - i$ (b) 1 (c) $1 + i$ (d) $-1 - i$

39. If $(\sqrt{3}+i)^{100} = 2^{99}(a+ib)$, the value of $a^2 + b^2$ is
(a) 1 (b) 2 (c) 3 (d) 4

40. The points of intersection of the two curves $|z - 3| = 2$ and $|z| = 2$ in an argand plane are
(a) $\frac{1}{2}(7 \pm i\sqrt{3})$ (b) $\frac{1}{2}(3 \pm i\sqrt{3})$
(c) $\left(\frac{3}{2} \pm i\sqrt{\frac{7}{2}}\right)$ (d) $\left(\frac{7}{2} \pm i\sqrt{\frac{3}{2}}\right)$

41. Let α and β be the roots of $x^2 + x + 1 = 0$. The equation whose roots are α^{19}, β^7 is
(a) $x^2 - x - 1 = 0$ (b) $x^2 - x + 1 = 0$
(c) $x^2 + x - 1 = 0$ (d) $x^2 + x + 1 = 0$

42. If a, b, c, p, q, r are complex numbers such that $\frac{p}{a}+\frac{q}{b}+\frac{r}{c}=1+i$ and $\frac{a}{p}+\frac{b}{q}+\frac{c}{r}=0$, the value of $\frac{p^2}{a^2}+\frac{q^2}{b^2}+\frac{r^2}{c^2}$ is
(a) 0 (b) –1 (c) $2i$ (d) $-2i$

43. Let $|z_1| = c = |z_2|$, the value of $|z_1 + z_2|^2 + |z_1 - z_2|^2$ is
(a) c^2 (b) $4c^2$ (c) $-c^2$ (d) $\frac{c^2}{2}$

44. The adjacent vertices of a regular polygon of n sides are the points z and its conjugate $\bar{z}$. If $\frac{\text{Im}(z)}{\text{Re}(z)} = \sqrt{2}-1$, the value of n is
(a) 4 (b) 8 (c) 6 (d) 10

45. The vertices A and C of a square $ABCD$ are $2 + 3i$ and $3 - 2i$ respectively. The vertices B and D are
(a) $B = (0, 0), D = (5, 1)$
(b) $B = (0, 0), D = (-5, 1)$
(c) $B = (1, 0), D = (-5, -1)$
(d) $B = (1, 1), D = (-5, -1)$

46. A, B, C are the vertices of an equilateral triangle whose centre is i. If A represents the complex number $-i$, the vertices of B and D are
(a) $B = (2i - \sqrt{3}), C = (2i + \sqrt{3})$
(b) $B = (2i + \sqrt{3}), C = (2i + \sqrt{3})$
(c) $B = (2i - \sqrt{3}), C = (2i - \sqrt{3})$
(d) $B = (2i - \sqrt{3}), C = -(2i + \sqrt{3})$

47. Let A and B represents the complex number $a + i$ and $3 + bi$ and O be the origin. If a triangle OAB forms an isosceles triangle with right angle at B, the value of a and b are
(a) $a = 7, b = 4$ (b) $a = 4, b = 4$
(c) $a = 4, b = 7$ (d) $a = 7, b = 7$

48. The complex number $\sqrt{3}+i$ becomes $-1+i\sqrt{3}$ after rotating by an angle about the origin in anti-clockwise direction. Then the angle θ is
(a) $\pi/2$ (b) $\pi/4$ (c) $-\pi/4$ (d) $\pi/6$

49. $ABCD$ is a rhombus. Its diagonal AC and BD intersect at a point M and satisfy $BD = 2AC$. If the points D and M are $1 + i$ and $2 - i$ respectively, the possible value of A is
(a) $A = 3 - \frac{i}{2}, 1 - \frac{3i}{2}$ (b) $A = 3 - \frac{i}{2}, 3 - \frac{3i}{2}$
(c) $A = 1 - \frac{3i}{2}, 1 - \frac{3i}{2}$ (d) $A = 1 - \frac{i}{2}, 1 - \frac{3i}{2}$

50. If z_1, z_2, z_3 be the vertices of an equilateral triangle in the argand plane such that
$$(z_1^2 + z_2^2 + z_3^2) = k(z_1 z_2 + z_2 z_3 + z_3 z_1),$$
the value of k is
(a) 1 (b) 2 (c) 0 (d) –2

51. For all complex numbers z_1, z_2 satisfying $|z_1| = 12$ and $|z_2 - 3 - 4i| = 5$, the minimum value of $|z_1 - z_2|$ is
(a) 0 (b) 2 (c) 12 (d) 10.

52. The value of $\sum_{k=1}^{10}\left(\sin\left(\frac{2\pi k}{11}\right) - \cos\left(\frac{2\pi k}{11}\right)\right)$ is
(a) –1 (b) 0 (c) –1 (d) i

53. The number of solutions of the system of equations $\begin{cases} |z| = 12 \\ |z-(3+4i)| = 5 \end{cases}$ is
(a) 1 (b) 2 (c) 0 (d) 3

54. If $|z-1| + |z+3| \le 8$, the range of $|z-4|$ is
(a) (0, 7) (b) (1, 8) (c) [1, 9] (d) [2, 5]

55. If z_1, z_2, z_3 be the vertices of an equilateral triangle with centroid z_0, then $z_1^2 + z_2^2 + z_3^2$ is
(a) z_0^2 (b) $2z_0^2$ (c) $3z_0^2$ (d) $9z_0^2$

56. If $z = \frac{i+\sqrt{3}}{2}$, where $i = \sqrt{-1}$, then $(z^{101} + i^{103})^{105}$ is
(a) z (b) z^2 (c) z^3 (d) z^4

57. The locus of z in $|z-5i| + |z+5i| = 12$ represents
(a) a circle (b) an ellipse
(c) a hyperbola (d) a parabola

58. The locus of z satisfying the equation $|iz-1| + |z-i| = 2$ is
(a) a circle (b) an ellipse
(c) a straight line (d) no real curve

59. If $x + \frac{1}{x} = 2\cos\left(\frac{\pi}{18}\right)$, then the value of $x^5 + \frac{1}{x^5}$ is
(a) 10 (b) 32 (c) 0 (d) 2

60. If ω be a complex cube root of $z^3 = 1$, the value of $\omega + \omega^{\left(\frac{1}{2}+\frac{3}{8}+\frac{9}{32}+\frac{27}{128}+\ldots\right)}$ is
(a) 0 (b) 1 (c) –1 (d) –2

61. If α be a complex constant such that $\alpha z^2 + z + \bar{\alpha} = 0$ has a real root, then
(a) $\alpha + \bar{\alpha} = 1$ (b) $\alpha + \bar{\alpha} = 0$
(c) $\alpha + \bar{\alpha} = -1$ (d) $\alpha + \bar{\alpha} = 2$

62. If $z^3 - iz^2 - 2iz - 2 = 0$, then z can be
(a) $1-i$ (b) i
(c) $1+i$ (d) $-(1+i)$

63. If $x = a + b$, $y = a\omega + b\omega^2$, $z = a\omega^2 + b\omega$, where ω is the non real cube root of unity, then
(a) $xyz = a^3 + b^3$ (b) $x^3 + y^3 + z^3 = 3(a^3 + b^3)$
(c) $x^2 + y^2 + z^2 = 6ab$ (d) $x + y + z = 0$

64. If α is the 5th root of unity, then
(a) $|(1 + \alpha + \alpha^2 + \alpha^3 + \alpha^4)| = 0$
(b) $|(1 + \alpha + \alpha^2 + \alpha^3)| = 1$
(c) $|(1 + \alpha + \alpha^2)| = 2\cos\left(\frac{\pi}{5}\right)$
(d) $|(1 + \alpha)| = 2\cos\left(\frac{\pi}{10}\right)$

65. If z_1, z_2, z_3 be the affixes of the vertices of an equilateral triangle and z_0 is the affix of its circumcentre, then
(a) $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_3z_2 + z_1z_3$
(b) $\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$
(c) $z_0 = \frac{1}{3}(z_1 + z_2 + z_3)$
(d) $|z_1 - z_0| = |z_2 - z_0| = |z_3 - z_0|$

66. If ω and ω^2 are non-real cube roots of unity, then
(a) $4 + 5\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{2015} - 5\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{1007} = -1$
(b) $\frac{2+3\omega+5\omega^2}{5+2\omega+3\omega^2} + \frac{2+3\omega+5\omega^2}{3+5\omega+2\omega^2} = -1$
(c) $\frac{1}{1-\alpha} + \frac{1}{1-\beta} + \frac{1}{1-\gamma} = 0$, where α, β, γ are the roots of $x^3 - 3x^2 + 3x + 7 = 0$.
(d) $\frac{1}{1+2\omega} + \frac{1}{2+\omega} - \frac{1}{1+\omega} = 1$

Level III

(Problems for JEE-Advanced)

1. Find the value of $a^6 + a^4 + a^2 + 1$, where $a = \frac{1+i}{\sqrt{2}}$.
2. Find the value of $x^4 + 4x^3 + 6x^2 + 4x + 10$ where $x = -1 + 2i$.
3. If $x = \frac{1+i\sqrt{3}}{2}$, find the value of the expression $y = x^4 - x^2 + 6x - 4$.
4. Find the value of $(i+i)^{6n} + (1-i)^{6n}$, where n is odd integers
5. If $a + ib = \frac{3}{2+\cos\theta + i\sin\theta}$ and $a^2 + b^2 = Ma + N$, find the value of $M + N + 10$.
6. Solve: $z^2 - (5+2i)z + (21+i) = 0$.
7. If c and d be complex numbers such that one root of $x^2 + cx + d = 0$ and the other root is imaginary, prove that $c^2 - \bar{c}^2 = 4d$.
8. Let $z = x + iy$ be a complex number where x and y are integers. Find the area of a rectangle whose vertices are the roots of $z\bar{z}^3 + z^3\bar{z} = 350$.
9. If the complex numbers $\sin x + i\cos 2x$ and $\cos x - i\sin 2x$ are conjugate to each other, then find x.
10. If $z^3 - iz^2 - 2iz - 2 = 0$, then find z.
11. If a, b, c, p, q are complex numbers such that $\frac{p}{a} + \frac{q}{b} + \frac{r}{c} = 1 + i$ and $\frac{a}{p} + \frac{b}{q} + \frac{c}{r} = 0$, find the value of $\frac{p^2}{a^2} + \frac{q^2}{b^2} + \frac{r^2}{c^2}$.
12. If $|z_1| = 1, |z_2| = 2, |z_3| = 3$ and $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 12$, find the value of $|z_1 + z_2 + z_3|$.

13. If $(\sqrt{3}+i)^{100} = 2^{99}(a+ib)$, find $a^2 + b^2$.
14. Find the number of ordered pairs (a, b) of real numbers such that $(a + ib)^{2008} = (a - ib)$.
15. Let $z = x + iy$ and $|z + 6| = |2z + 3|$, the locus of z is $x^2 + y^2 = 9$.
16. If $|z| = 1$ and $\omega = \dfrac{z-1}{z+1}$ (where $z \neq 1$), find Re(ω).
17. If $\dfrac{w - \overline{w}z}{1-z}$ is purely real, where $w = \alpha + i\beta$, $\beta \neq 0$ and $z \neq 1$, find the set of values of z.
18. If $|z| = 1$ and $z \neq 1$, prove that all the values of $u = \dfrac{z}{1-z^2}$ lie on y-axis.
19. If z is represented on the Argand plane by a point on the circle $|z - 1| = 1$, prove that $\dfrac{z-2}{z} = i\tan(\text{Arg } z)$.
20. If $|z| \leq 1$, $|w| \leq 1$, show that $|z - w|^2 \leq (|z| - |w|)^2 + (\text{Arg } z - \text{Arg } w)^2$
21. Let z be a complex number satisfying $|z - 5i| \leq 1$ such that the amplitude is minimum, show that $z = \dfrac{2\sqrt{6}}{5} + i\dfrac{24}{5}$.
22. If $\left|z - \dfrac{25}{z}\right| = 24$, then find the greatest value of $|z|$.
23. For all complex number z_1, z_2 satisfying $|z_1| = 12$ and $|z_2 - 3 - 4i| = 5$, find the minimum value of $|z_1 - z_2|$.
24. Find those complex numbers, which are the points of intersection of two curves $|z - 3| = 2$ and $|z| = 2$ and also find the length of the common chord.
25. If $t^2 + t + 1 = 0$, find the value of $\left(t + \dfrac{1}{t}\right)^2 + \left(t^2 + \dfrac{1}{t^2}\right)^2 + \ldots + \left(t^{2014} + \dfrac{1}{t^{2014}}\right)^2$.
26. If $x + \dfrac{1}{x} = 1$, find the value of $x^{10} + x^{20} + x^{30} + \ldots + x^{100}$.
27. If ω be a complex cube root of unity satisfying the equation $\dfrac{1}{a+\omega} + \dfrac{1}{b+\omega} + \dfrac{1}{c+\omega} = 2\omega^2$ and $\dfrac{1}{a+\omega^2} + \dfrac{1}{b+\omega^2} + \dfrac{1}{c+\omega^2} = 2\omega$, then find the value of $\dfrac{1}{a+1} + \dfrac{1}{b+1} + \dfrac{1}{c+1}$.
28. If 1, ω, ω^2 be the cube roots of unity, prove that $\dfrac{1}{x-1} + \dfrac{1}{x\omega - 1} + \dfrac{1}{x\omega^2 - 1} = \dfrac{3}{x^3 - 1}$.
29. If ω is the non-real cube root of unity and $(1 + \omega^2)^n = (1 + \omega^4)^n$, find the least value of n.
30. Let ω be the complex number $\cos\left(\dfrac{2\pi}{3}\right) + i\sin\left(\dfrac{2\pi}{3}\right)$, find the number of distinct complex number z satisfying
$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0.$$
31. If a, b, c be integers, not all simultaneously equal and ω be the non-real cube root of unity, find the minimum value of $|a + b\omega + c\omega^2|$.
32 If ω be a cube root of unity and $(1 + \omega)^7 = A + B\omega$, find the value $A + B + 10$.
33. If $x + \dfrac{1}{x} = 2\cos\alpha$, $y + \dfrac{1}{y} = 2\cos\beta$, prove that $\dfrac{x}{y} + \dfrac{y}{x} = 2\cos(\alpha \pm \beta)$.
34. Solve for x: $x^4 + x^3 + x^2 + x + 1 = 0$.
35. Solve for x:
$$x^{14} - x^{13} + x^{11} - x^{10} + \ldots + -x + 1 = 0.$$
36. Solve for x:
$$x^{12} - x^{11} + x^{10} - x^9 + \ldots -x + 1 = 0.$$
37. Find the value of
$$\sum_{k=1}^{10}\left(\sin\left(\frac{2\pi k}{11}\right) - i\cos\left(\frac{2\pi k}{11}\right)\right).$$
38. Solve for z: $z^8 + 1 = 0$.
39. Solve for z: $z^7 - 1 = 0$ and deduce that $\cos\left(\dfrac{\pi}{7}\right)\cdot\cos\left(\dfrac{2\pi}{7}\right)\cdot\cos\left(\dfrac{4\pi}{7}\right) = \dfrac{1}{8}$.
40 Solve for z: $z^5 + 1 = 0$ and deduce that $4\sin\left(\dfrac{\pi}{10}\right)\cdot\cos\left(\dfrac{\pi}{5}\right) = 1$.
41. Solve for x: $x^{10} + x^5 + 1 = 0$.
42. If ω is an imaginary 5th root of 2 and $x = \omega + \omega^2$, find the value of $x^5 - 10x^3 - 10x^2$.
43. Show that the roots of $8x^3 - 4x^2 - 4x + 1 = 0$ are $\cos\left(\dfrac{\pi}{7}\right)$, $\cos\left(\dfrac{3\pi}{7}\right)$, $\cos\left(\dfrac{5\pi}{7}\right)$.
44. Given $z = \cos\left(\dfrac{2\pi}{2n+1}\right) + i\sin\left(\dfrac{2\pi}{2n+1}\right)$, $n \in I^+$, find an equation whose roots are $\alpha = z + z^3 + z^5 + \ldots + z^{2n-1}$ and $\beta = z^2 + z^4 + z^6 + \ldots + z^{2n}$.
45. Let z_1 and z_2 be the nth roots of unity which subtend a right angle at the origin, prove that $n = 4k$, where $k \in N$.
46. If ω be the fifth root of unity, find the value of $\log_2|1 + \omega + \omega^2 + \omega^3 - \omega^{-1}|$.
47. If $\beta \neq 1$ be any nth root of unity, prove that $1 + 3\beta + 5\beta^2 + \ldots$ to n terms $= \dfrac{2n}{1-\beta}$.
48. If 5th root of ω is 3 such that $x = \omega + \omega^2$, find the value of $x^5 - 15x^2 - 15x + 18$.

49. Find the real values of x and y if
$$(x^4 + 2xi) - (3x^2 + iy) = (3 - 5i) + (1 + i2y)$$
[Roorkee, 1984]

Note: No questions asked in 1985, 1986, 1987, 1988.

50. Let A and B be two complex numbers such that $\frac{A}{B} + \frac{B}{A} = 1$. Prove that the origin and the two points represented by A and B form the vertices of an equilateral triangle. **[Roorkee, 1989]**
51. For every real value of $a > 0$, determine the complex numbers which will satisfy the equation
$|z|^2 - 2iz + 2a(1 + i) = 0$ **[Roorkee, 1990]**
52. Find the range of the real number a for which the equation $z + a|z - 1| + 2i = 0$ has a solution and also find the solution. **[Roorkee, 1991]**
53. Find the equation in complex variable of all circles which are orthogonal to $|z| = 1$ and $|z - 1| = 4$. **[Roorkee, 1992]**
54. Find the complex number z which simulatenously satisfies the equations
$$\left|\frac{z-12}{z-8i}\right| = \frac{5}{3} \text{ and } \left|\frac{z-4}{z-8}\right| = 1.$$
[Roorkee, 1993]
55. Use De Moivre's theorem, solve the equation
$2\sqrt{2}x^4 = (\sqrt{3} - 1) + i(\sqrt{3} + 1).$ **[Roorkee, 1994]**
56. Find all complex number z for which
$$\arg\left(\frac{3z - 6 - 3i}{2z - 8 - 6i}\right) = \frac{\pi}{4} \text{ and } |z - 3 + i| = 3.$$
[Roorkee, 1995]
57. Find all complex numbers satisfying the equation $2|z|^2 + z^2 - 5 + i\sqrt{3} = 0.$ **[Roorkee, 1996]**
58. Evaluate:
$$\sum_{p=1}^{32}(3p+2)\left(\sum_{q=1}^{10}\left(\sin\left(\frac{2q\pi}{11}\right) - i\cos\left(\frac{2q\pi}{11}\right)\right)\right)^p.$$
[Roorkee, 1997]
59. Find all roots of the equation
$(3z - 1)^4 + (z - 2)^4 = 0$ in the simplified form of $a + ib$. **[Roorkee, 1998]**
60. If $\alpha = e^{i\frac{2\pi}{7}}$ and $f(x) = A_0 + \sum_{k=1}^{20} A_k x^k$,
find the value of $f(x) + f(\alpha x) + f(\alpha^2 x) + \ldots + f(\alpha^6 x)$ which is independent of α. **[Roorkee, 1999]**
61. Given $z = \cos\left(\frac{2\pi}{2n+1}\right) + i\sin\left(\frac{2\pi}{2n+1}\right)$, n being a positive integer, find the equation whose roots are
$\alpha = z + z^3 + z^5 + \ldots + z^{2n-1}$ and $\beta = z^2 + z^4 + z^6 + \ldots + z^{2n}$. **[Roorkee, 2000]**
62. Find all those roots of the equation $z^{12} - 56z^6 - 512 = 0$, whose imaginary part is positive. **[Roorkee, 2001]**
63. If z be a complex number satisfying the equations $3|z - 12| = 5|z - 8i|$ and $|z - 4| = |z - 8|$ find Im(z).
64. Let z be a complex number such that $z + \frac{1}{z} = 2\cos(2°)$, find the value of $z^{2010} + \frac{1}{z^{2010}} + 3$.
65. Let z be a complex number satisfying the equation $(z^5 + 3)^2 = -16$, find $|z|$.
66. Find the equation of the radical axis of two circles represented by the equations
$$|z - 2| = 3 \text{ and } |z - 2 - 3i| = 4.$$
on the complex plane.
67. If $x = a + ib$ be a complex number such that $x^2 = 3 + 4i$, $x^3 = 2 + 11i$, where $i = \sqrt{-1}$, find the value of $(1 + b + 2)$.
68. Let z be a complex number such that
$2016\ \text{Im}\left(\frac{1}{z}\right) - 1 = 0$, find the locus of z.
69. Solve: $z^{n-1} = \bar{z}, n \in N$.
70. Prove that $\sum_{k=1}^{n-1}(n-k)\cos\left(\frac{2k\pi}{n}\right) = -\frac{n}{2}$.
71. Given $z_1 + z_2 + z_3 = A$, $z_1 + z_2\omega + z_3\omega^2 = B$ and $z_1 + z_2\omega^2 + z_3\omega = C$, find z_1, z_2, z_3 in terms of A, B, C.
72. Let z be a non-zero complex numbers lying on the circle $|z| = 1$, then prove that
$$z = \frac{1 + i\tan\left(\frac{\text{Arg } z}{2}\right)}{1 - i\tan\left(\frac{\text{Arg } z}{2}\right)}.$$
73. If $\sin\theta + 2\sin\varphi + 3\sin\psi = 0$
and $\cos\theta + 2\cos\varphi + 3\cos\psi = 0$,
find
(i) $\cos 3\theta + 8\cos 3\varphi + 27\cos 3\psi$
(ii) $\sin 3\theta + 8\sin 3\varphi + 27\sin 3\psi$
(ii) $\sin(\varphi + \psi) + 2\sin(\psi + \theta) + 3\sin(\theta + \varphi)$
74. If $1, \alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the roots of $x^5 - 1 = 0$, find the value of
$$\frac{\omega - \alpha_1}{\omega^2 - \alpha_1} \cdot \frac{\omega - \alpha_2}{\omega^2 - \alpha_2} \cdot \frac{\omega - \alpha_3}{\omega^2 - \alpha_3} \cdot \frac{\omega - \alpha_4}{\omega^2 - \alpha_4}.$$
75. Assume that $A_i (i = 1, 2, \ldots, n)$ be the vertices of a regular polygon inscribed in a unit circle. Find the value of $|A_1A_2|^2 + |A_1A_3|^2 + \ldots + |A_1A_n|^2$.

Level IV

(Tougher Problems for JEE-Advanced Examination only)

1. If z_1, z_2, z_3 be three non-collinear unimodular complex numbers, prove that
$$E = |z_1 - z_2|^2 + |z_2 - z_3|^2 + |z_3 - z_1|^2$$
does not exceed 9.

2. Find the area of a triangle on the Argand plane formed by the complex numbers $-z$, iz, $z - iz$.
3. If a complex number z lies on a circle of radius 1/2, prove that the complex number $(-1 + 4z)$ lies on a circle of radius 2.
4. If $|z_1| = 1$, $|z_2| = 2$, $|z_3| = 3$ and $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 12$, find the value of $|z_1 + z_2 + z_3|$.
5. If z_1, z_2, z_3 be the complex numbers representing the points A, B, C such that $\frac{2}{z_1} = \frac{1}{z_2} + \frac{1}{z_3}$, prove that a circle through A, B, C passes through the origin.
6. Let a, b, c be three distinct complex numbers such that $\frac{a}{1-b} = \frac{b}{1-c} = \frac{c}{1-a} = k$, find the value of k.
7. If a and b are positive integers such that $N = (a + ib)^3 - 107i$ is a positive integer, find N.
8. Find the set of points on the Argand plane for which the real part of the complex number $(1 + i)z^2$ is positive, where $z = x + iy$, $x, y \in R$ and $i = \sqrt{-1}$.
9. C is the complex number and $f : C \to R$ is defined by $f(z) = |z^3 - z + 2|$. What is the maximum value of f on the unit circle $|z| = 1$.
10. Let $z = x + iy$ be a complex number, where x and y are real numbers. Let A and B be two sets defined as $A = \{z : |z| \le 2\}$ and $B = \{z : (1-i)z + (1+i)\bar{z} \le 4\}$. Find the area of the region $A \cap B$.
11. Show that
$$\left[1+\left(\frac{1+i}{2}\right)\right]\left[1+\left(\frac{1+i}{2}\right)^2\right]\left[1+\left(\frac{1+i}{2}\right)^{2^2}\right]\left[1+\left(\frac{1+i}{2}\right)^{2^n}\right] = \left(1 - \frac{1}{2^{2^n}}\right)(1+i),\ n \ge 2$$
12. Show that the locus formed by z in the equation $z^3 + iz = 1$ never crosses the co-ordinate axes in the Argand plane. Further show that
$$|z| = \sqrt{\frac{-\operatorname{Im}(z)}{2\operatorname{Re}(z)\operatorname{Im}(z)+1}}$$
13. For all real numbers x, let the function $f(x) = \frac{1}{x-1}$, where $i = \sqrt{-1}$. If there exist real number a, b, c and d for which $f(a)$, $f(b)$, $f(c)$ and $f(d)$ form a square on the complex plane, find the area of the square.
14. If $\alpha = e^{i\frac{2\pi}{n}}$ and $f(x) = A_0 + \sum_{k=1}^{20} A_k x^k$, find the value of $f(x) + f(\alpha x) + f(\alpha^2 x) + \ldots + f(\alpha^k x)$, which is independent of α
15. Solve the equation $z^7 - 1 = 0$ and deduce that
$$\cos\left(\frac{2\pi}{7}\right) + \cos\left(\frac{4\pi}{7}\right) + \cos\left(\frac{6\pi}{7}\right) = -\frac{1}{2}.$$
16. Solve the equation $z^7 + 1 = 0$ and deduce that
$$\cos\left(\frac{\pi}{7}\right)\cdot\cos\left(\frac{3\pi}{7}\right)\cdot\cos\left(\frac{5\pi}{7}\right) = -\frac{1}{8}.$$
17. If ω be the nth root of unity and z_1, z_2 be any two complex numbers, prove that
$$\sum_{k=0}^{n-1} |z_1 + \omega^k z_2|^2 = n(|z_1|^2 + |z_2|^2).$$
18. Given that $|z - 1| = 1$, where z is a point on the complex plane, show that
$$\frac{z-2}{z} = i\tan(\operatorname{Arg}(z))$$
19. Given $z = \cos\left(\frac{2\pi}{2n+1}\right) + i\sin\left(\frac{2\pi}{2n+1}\right)$, where n is a positive integer, find the equation whose roots are $\alpha = z + z^3 + \ldots + z^{2n-1}$ and $\beta = z^2 + z^4 + \ldots + z^{2n}$.
20. If α and β be the roots of $z + \frac{1}{z} = 2(\cos\theta + i\sin\theta)$, where $0 < \theta < \pi$, prove that $|\alpha - i| = |\beta - i|$.
21. If a complex number z lies on the curve $|z - (-1 + i)| = 1$, then find the locus of
$$\omega = \frac{z+i}{z-i},\ i = \sqrt{-1}.$$
22. Consider a triangle formed by the points $A\left(\frac{2}{\sqrt{3}}e^{\frac{i\pi}{2}}\right)$, $B\left(\frac{2}{\sqrt{3}}e^{-\frac{i\pi}{6}}\right)$, $C\left(\frac{2}{\sqrt{3}}e^{-\frac{i5\pi}{6}}\right)$. Let $P(z)$ be any point on its circle, prove that $AP^2 + BP^2 + CP^2 = 5$.
23. Find the common tangents of the curves $\operatorname{Re}(z) = |z - 2a|$ and $|z - 4a| = 3a$.
24. Show that the complex numbers whose real and imaginary parts are integers and satisfy the equation $z\bar{z}^3 + \bar{z}z^3 = 350$ form a rectangle in the Argand plane with the length of the diagonal having an integral number of units.
25. If $\beta \ne 1$ be any nth root of unity, prove that the value of $1 + 3\beta + 5\beta^2 + \ldots$ to n terms is $\frac{2n}{\beta - 1}$.
26. The equation $x^3 = 9 + 46i$, where $i = \sqrt{-1}$ has a solution of the form $a + ib$, where a and b are integers. Find the value of $(a^3 + b^3)$.
27. If ω be the ffth root of 2 and $x = \omega + \omega^2$, prove that $x^5 = 10x^2 + 10x + 6$.
28. Let Z is a complex number satisfying the equation $z^2 - (3 + i)z + m + 2i = 0$, where $m \in R$. Suppose the equation has a real root, find the value of m.
29. a, b, c are real numbers in the polynomial
$P(Z) = 2Z^4 + aZ^3 + bZ^2 + cZ + 3$.
If two roots of $P(Z) = 0$ are 2 and i (iota), find the value of a.

30. Resolve $z^5 + 1$ into linear and quadratic factors with real co-efficients. Deduce that

$$4\sin\left(\frac{\pi}{10}\right)\cos\left(\frac{\pi}{5}\right) = 1.$$

31. If the equation $(z + 1)^7 + z^7 = 0$ has roots $z_1, z_2, \ldots, z_7$, find the value of

(i) $\sum_{r=1}^{7} \text{Re}(Z_r)$ (ii) $\sum_{r=1}^{7} \text{Im}(Z_r)$

32. Find all real values of the parameter a for which the equation $(a - 1)z^4 - 4z^2 - a + 2 = 0$
has only pure imaginary roots.

33. If the biquadratic $x^4 + ax^3 + bx^2 + cx + d = 0$ has 4 non-real roots, two with sum $3 + 4i$ and the other two with product $13 + 11i$, find the value of b.

34. If Z_r, $r = 1, 2, 3, \ldots, 2m$, $m \in N$ are the roots of the equation $Z^{2m} + Z^{2m-1} + Z^{2m-2} + \ldots + Z + 1 = 0$, prove that $\sum_{r=1}^{2m}\left(\frac{1}{Z_r - 1}\right) = -m.$

Integer Type Questions

Note The answer to each question is a SINGLE DIGIT INTEGER ranging from 0 to 9, both inclusive.

1. If ω and ω^2 be the cube roots of unity

$$\frac{1}{a+\omega} + \frac{1}{b+\omega} + \frac{1}{c+\omega} = 2\omega^2$$

and $\frac{1}{a+\omega^2} + \frac{1}{b+\omega^2} + \frac{1}{c+\omega^2} = 2\omega,$

the value of $\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}$ is......

2. If ω be a complex cube root of unity, the value of

$$1 + \cos\left[\{(1-\omega)(1-\omega^2) + \ldots + (10-\omega)(10-\omega^2)\}\frac{\pi}{900}\right]$$

is......

3. Let z_1 and z_2 be two non-real complex cube roots of unity and $|z - z_1|^2 + |z - z_1|^2 = \lambda$ be the equation of a circle with z_1, z_2 as ends of a diameter, the value of $(\lambda + 4)$ is......

4. Let z_k where $k = 0, 1, 2, \ldots, 6$ be the roots of $z^7 + (z + 1)^7 = 0$ such that $2\sum_{k=0}^{6} \text{Re}(z_k) = \lambda$, the value of $(\lambda + 10)$ is......

5. If $x + \frac{1}{x} = 1$, the value of

$1 + x^{20} + x^{30} + x^{40})(2 + x^{50} + x^{60} + x^{70}) \times (3 + x^{80} + x^{90} + x^{100})$ is......

6. If z_1, z_2 represent adjacent vertices of a regular polygon of n sides and if

$\frac{\text{Im}(z_1)}{\text{Re}(z_1)} = (\sqrt{2} - 1)$, then n is......

7. If z be a non-real root of $(-1)^{1/7}$, the value of $(z^{86} + z^{175} + z^{289} + 3)$ is......

8. The value of $\left(\frac{i+\sqrt{3}}{2}\right)^{2016} + \left(\frac{i-\sqrt{3}}{2}\right)^{2016} + 3$ is......

9. The number of common roots of the system of equations $\begin{cases} x^4 - 1 = 0 \\ x^5 - x^3 + x^2 - 1 = 0 \end{cases}$ is......

10. If m be the number of integral solutions of $(1 - i)^x = 2^x$ and n be the number of common roots of $\begin{cases} 1 + z^{100} + z^{1985} = 0 \\ 1 + 2z + 2z^2 + z^3 = 0 \end{cases}$, the value of $(m + n + 3)$ is......

11. If z_1, z_2, z_3 be three points lying on the circle $|z| = 1$, the maximum value of

$|z_1 - z_2|^2 + |z_2 - z_3|^2 + |z_3 - z_1|^2$ is.....

12. If $|z_1| = 1$, $|z_2| = 2$, $|z_3| = 3$ and

$|9z_1z_2 + 4z_1z_3 + z_2z_3| = 12$, the value of $|z_1 + z_2 + z_3|$ is.....

13. The sum if

$$\left(1 + \sum_{k=0}^{13}\left\{\cos\left(\frac{2k+1}{13}\right) + i\sin\left(\frac{2k+1}{13}\right)\right\}\right)$$

is......

14. Consider a triangle formed by the points

$$A\left(\frac{2}{\sqrt{3}}e^{\frac{i\pi}{2}}\right), B\left(\frac{2}{\sqrt{3}}e^{-\frac{i\pi}{6}}\right), C\left(\frac{2}{\sqrt{3}}e^{-\frac{i5\pi}{6}}\right).$$

Let $P(z)$ be any point on its circle, the value of $(AP^2 + BP^2 + CP^2)$ is.....

15. If $x^2 + x + 1 = 0$, then the value of

$$\frac{1}{9}\times\left[\left(x + \frac{1}{x}\right)^2 + \left(x^2 + \frac{1}{x^2}\right)^2 + \ldots + \left(x^{27} + \frac{1}{x^{27}}\right)^2\right]$$

is......

Comprehensive Link Passages

Passage I

The polynomial equation $x^2 + 1 = 0$ gives us solution $x = \pm\sqrt{-1} = \pm i$,

where i (iota) is known as imaginary unit.

Thus, $i^2 = -1$, $i^3 = -i$, $i^4 = 1$

1. The sum of $i^n + i^{n+1} + i^{n+2} + i^{n+3}$, $n \in N$ is
(a) –1 (b) 1 (c) 0 (d) 2

2. The sum of $\sum_{n=1}^{13}(i^n + i^{n+1})$ is
   (a) $i - 1$ (b) i (c) 0 (d) i + 1
3. The smallest integer n for which $\left(\frac{1+i}{1-i}\right)^n = 1$ is
   (a) 4 (b) 0 (c) 8 (d) none
4. The smallest positive integer n for which $\left(\frac{1+i}{1-\text{i}}\right)^n = 1$ is
   (a) 1 (b) 4 (c) 2 (d) 6
5. The smallest positive integer n for which $\left(\frac{1+i}{1-i}\right)^n$ is real, is
   (a) 1 (b) 2 (c) 3 (d) 4.

Passage II

The roots of the polynomial equation $x^3 - 1 = 0$ are 1, ω, ω^2 such that $\omega^n + \omega^{n+1} + \omega^{n+2} = 0$, $n \in I$. Also, $x^3 - 1 = (x - 1)(x - \omega)(x - \omega^2)$.

1. The value of
   $$4+5\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{2015}-5\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)^{1007} \text{ is}$$
   (a) 4 (b) 5 (c) –1 (d) 2
2. If $\left(\frac{3}{2}+i\frac{\sqrt{3}}{2}\right)^{50} = 3^{25}(x+iy), x, y \in R$, the value of xy is
   (a) $\frac{1}{4}$ (b) $\frac{\sqrt{3}}{4}$ (c) $-\frac{\sqrt{3}}{4}$ (d) 1
3. If α, β, γ be the roots of $x^3 - 3x^2 + 3x + 7 = 0$, the value of $\sum\left(\frac{\alpha-1}{\beta-1}\right)$ is
   (a) 0 (b) 3ω (c) $\frac{3}{\omega}$ (d) 4

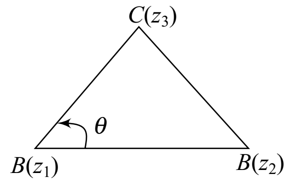
Passage III

A complex number z is unimodular if $|z| = 1$. Also, $|z|^2 = z \cdot \bar{z}$.

1. If z_1, z_2, z_3 be complex numbers such that $|z_1| = |z_2| = |z_3| = 1 = |z_1 + z_2 + z_3|$, then the value of $\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right|$ is
   (a) 1 (b) 3 (c) < 3 (d) none
2. If $z_1, z_2, z_3, \ldots, z_n$ be complex numbers such that $|z_1| = |z_2| = |z_3| = \ldots = |z_n| = 1$. The value of $|z_1 + z_2 + z_3 + \ldots + z_n|$ is
   (a) 1
   (b) $|z_1| + |z_2| + |z_3| + \ldots + |z_n|$
   (c) n
   (d) $\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}+\ldots+\frac{1}{z_n}\right|$
3. If $|z_1| = 1$, $|z_2| = 2$, $|z_3| = 3$ and $|z_1 + z_2 + z_3| = 1$, the value of $|9z_1z_2 + 4z_1z_3 + z_2z_3|$ is
   (a) 6 (b) 16 (c) 216 (d) 1296
4. If all the roots of $z^3 + az^2 + bz + c = 0$ be the unit modulus, then
   (a) $|a| \le 3$ (b) $|c| \le 3$ (c) $|b| > 3$ (d) none

Passage IV

If z_1, z_2, z_3 be affixes of the vertices of a ΔABC, described in anti-clockwise sense, then $\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \frac{CA}{BA} \times e^{i\theta}$



1. If the area of a triangle on the complex plane z, iz and $z + iz$ is 50 sq units, then $|z|$ is
   (a) 5 (b) 10 (c) 20 (d) 15
2. Let z_1, z_2, z_3 the vertices of an equilateral triangle inscribed in the circle $|z| = 2$. If $z_1 = 1 + i\sqrt{3}$, then $z_3 - z_2$ is
   (a) $3 - i\sqrt{3}$ (b) $1 - i\sqrt{3}$
   (c) –2 (d) $2 - i\sqrt{3}$
3. The centre of the arc represented by
   $\arg\left(\frac{z-3i}{z-2i+4}\right) = \frac{\pi}{4}$ is
   (a) $\frac{1}{2}(5i+5)$ (b) $\frac{1}{2}(5i-5)$
   (c) $\frac{1}{2}(9i+5)$ (d) $\frac{1}{2}(9i-5)$

Passage V

If $z_1, z_2 \in C$, then $||z_1| - |z_2|| \le |z_1 + z_2| \le |z_1| + |z_2|$.

1. If $|z| \le 4$, the maximum value of $|iz - 3 - 4i|$ is
   (a) 2 (b) 3 (c) 4 (d) 9
2. If the greatest value of $|z|$ in $|z - 3 - 4i| \le a$ is equal to the least value of $\left(x^4 + x + \frac{5}{x}\right), x > 0,$ the value of a is
   (a) 1 (b) 4 (c) 5 (d) 2
3. The minimum value of $|z - 2| + |z + 2|$ is
   (a) 0 (b) 2 (c) 1 (d) 4
4. For any complex number z, the maximum value of $|z| - |z - 2|$ is
   (a) 1 (b) 2 (c) 0 (d) 3.
5. If $|z - 3 + 2i| \le 4$, the difference between the greatest and the least value of $|z|$ is
   (a) 4 (b) 6 (c) 8 (d) 2

Passage VI

The roots of $x^n - 1 = 0$ are 1, α, α^2, α^3, …, α^{n-1} where $\alpha = \cos\left(\frac{2\pi}{n}\right) + i\sin\left(\frac{2\pi}{n}\right)$, which is also known as nth roots of unity.

1. The value of $(1-\alpha_1)(1-\alpha_2)(1-\alpha_3)\ldots(1-\alpha_{n-1})$ is
   (a) n (b) 0 (c) 1 (d) none
2. The value of $1+\alpha+\alpha^2+\ldots+\alpha^{n-1}$ is
   (a) 1 (b) n (c) 0 (d) none
3. The value of $\frac{1}{(2-\alpha)}+\frac{1}{(2-\alpha^2)}+\ldots+\frac{1}{(2-\alpha^{n-1})}$ is
   (a) $\left(\frac{(n-2)2^{n-1}+1}{2^n-1}\right)$ (b) $\left(\frac{(n-2)2^{n-1}-1}{2^n-1}\right)$
   (c) $\left(\frac{(n-2)2^{n}-1}{2^n-1}\right)$ (d) $\left(\frac{(n-2)2^{n-1}-2}{2^n-1}\right)$
4. The value of
   $\cos\left(\frac{2\pi}{11}\right)+\cos\left(\frac{4\pi}{11}\right)+\cos\left(\frac{6\pi}{11}\right)+\cos\left(\frac{8\pi}{11}\right)+\cos\left(\frac{10\pi}{11}\right)$,
   where $x^{11}-1=0$, is
   (a) 1/2 (b) 0 (c) –1/2 (d) 1/4
5. The value of $\cos\left(\frac{\pi}{7}\right)\cos\left(\frac{3\pi}{7}\right)\cos\left(\frac{5\pi}{7}\right)$, where $x^7+1=0$, is
   (a) –1/2 (b) –1/4 (c) –1/8 (d) 1/16
6. The value of $1+2\alpha+3\alpha^2+\ldots+n\alpha^{n-1}$ is
   (a) $\frac{1-\alpha^n}{(1-\alpha)^2}-\frac{n\alpha^n}{(1-\alpha)}$ (b) $\frac{1-\alpha^n}{(1-\alpha)^2}+\frac{n\alpha^n}{(1-\alpha)}$
   (c) $\frac{1-\alpha^n}{(1-\alpha)^2}+\frac{(n-1)\alpha^n}{(1-\alpha)}$ (d) $\frac{1-\alpha^n}{(1-\alpha)^2}-\frac{(n-1)\alpha^n}{(1-\alpha)}$

Matrix Match

1. Match the following columns.

Column I		Column II	
(A)	If three complex numbers are in AP, then they lie on	(P)	a circle
(B)	If $\left(\frac{z^2}{z-1}\right)$ is always real then z lies on	(Q)	a straight line
(C)	If $\lvert z^2-1\rvert=\lvert z\rvert^2+1$, then z lies on	(R)	an ellipse
(D)	If $\text{Im}\left(\frac{2z+1}{1+iz}\right)=-2$, then z lies on	(S)	a parabola

2. Match the following columns.

Column I		Column II	
(A)	The simplified form of $\left(\frac{1-i}{1+i}\right)^{4n+1}, n\in I^+$ is	(P)	1
(B)	If m, n, p, q be four consecutive integers, then the value of $i^m+i^n+i^p+i^q$ is	(Q)	3
(C)	The number of values of i^n+i^{-n} for different $n\in I$ is	(R)	0
(D)	The value of $\sum_{k=0}^{200} i^k+\sum_{p=0}^{50} i^p$ is	(S)	$-i$

3. Match the following columns.

Column I		Column II	
(A)	For any complex number z, the minimum value of $\lvert z-2\rvert+\lvert z+3\rvert$ is	(P)	1
(B)	For any complex number z, the minimum value of $\lvert z-1007\rvert+\lvert z-1008\rvert+\lvert 2z-2016\rvert$ is	(Q)	2
(C)	If α, β be the complex numbers, the maximum value of $\frac{\alpha\bar{\beta}+\beta\bar{\alpha}}{\lvert\alpha\beta\rvert}$ is	(R)	3
(D)	For a complex number z, the minimum value of $\lvert z\rvert+\lvert z-\cos\alpha-i\sin\alpha\rvert$ is	(S)	5

Matching List Type (Only One Option is Correct)

This section contains four questions, each having two matching list. Choices for the correct combination of elements from List I and List II are given as options (A), (B), (C) and (D), out of which ONLY ONE is correct.

4. Match the following lists

List I		List II	
If $1, z_1, z_2, \ldots, z_{10}$ are the 11th roots of unity, then			
(P)	the value of $(1-z_1)(1-z_2)\ldots(1-z_{10})$ is	(1)	1
(Q)	the value of $z_1^{100}+z_2^{100}+\ldots+z_{10}^{100}$ is	(2)	–1
(R)	the value of $(1+z_1)(1+z_2)\ldots(1+z_{10})$ is	(3)	0
(S)	the value of $1-z_1\cdot z_2\ldots z_{10}$ is	(4)	11

Codes:

	P	Q	R	S
(A)	4	2	1	3
(B)	2	4	3	1
(C)	4	3	1	2
(D)	2	4	1	3

5. Match the following lists.

	List I		List II
(P)	Let z be a root of $x^5 - 1 = 0$, the value of $z^{15} + z^{16} + z^{17} + \ldots + z^{50}$ is	(1)	4
(Q)	If $z = x + iy$, $z^{1/3} = a - ib$ and $\frac{x}{a} - \frac{y}{b} = k(a^2 - b^2)$, the value of k is	(2)	2
(R)	The number of common roots of the equations $z^3 + 2z^2 + 2z + 1 = 0$ and $z^{2015} + z^{2014} + 1 = 0$ is	(3)	1
(S)	If $z_r = \cos\left(\frac{2r\pi}{5}\right) + i\sin\left(\frac{2r\pi}{5}\right)$, where $r = 1, 2, 3, 4, 5$, then the value of $(z_1 z_2 z_3 z_4 z_5 + 2)$ is	(4)	3

Codes:

	P	Q	R	S
(A)	3	4	2	1
(B)	3	1	2	4
(C)	1	4	3	1
(D)	3	1	4	2

6. Match the following lists.

	List I		List II
(P)	The number of solution of the equation $2z = \|z\| + 2i$ is	(1)	4
(Q)	The sum of the non-real complex values of x in $(x - 1)^4 - 16 = 0$ is	(2)	1
(R)	If $\left\|\frac{z_1 - 3z_2}{3 - z_1\bar{z}_2}\right\| = 1$ and $\|z_2\| \neq 1$, the value of $\|z_1\|$ is	(3)	2
(S)	If $x = a + b$, $y = a\omega + b\omega^2$ and $z = a\omega^2 + b\omega$ such that $x^3 + y^3 + z^3 = \lambda(a^3 + b^3)$, the value of $(\lambda + 1)$ is	(4)	3

Codes

	P	Q	R	S
(A)	2	3	1	4
(B)	3	2	1	4
(C)	2	3	4	1
(D)	3	2	4	2

Questions asked in Previous Years' JEE-Advanced Examinations

1. If $x = a + b$, $y = a\alpha + b\beta$ and $z = a\beta + b\alpha$ where α, β are complex cube roots of unity, prove that $xyz = a^3 + b^3$. **[IIT-JEE, 1978]**

2. If the cube roots of unity are 1, ω, ω^2, the roots of $(x - 1)^3 + 8 = 0$ are
   (a) $-1, 1 + 2\omega, 1 + 2\omega^2$ (b) $-1, 1 - 2\omega, 1 - 2\omega^2$
   (c) $-1, -1, -1$ (d) none
   [IIT-JEE, 1979]

3. The smallest positive integer n for which $\left(\frac{1+i}{1-i}\right)^n = 1$ is
   (a) $n = 8$ (b) $n = 16$ (c) $n = 12$ (d) none
   [IIT-JEE, 1980]

4. The complex number $z = x + iy$ which satisfies the equation $\left|\frac{z - 5i}{z + 5i}\right| = 1$ lie on
   (a) the x-axis
   (b) the straight line $y = 5$
   (c) a circle passing through the origin
   (d) none **[IIT-JEE, 1981]**

5. The inequality $|z + 2| < |z - 2|$ represents the region given by
   (a) $\text{Re}(z) \geq 0$ (b) $\text{Re}(z) > 0$
   (c) $\text{Re}(z) < 0$ (d) none
   [IIT-JEE, 1982]

6. If $z = \left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right)^5$, then
   (a) $\text{Re}(z) = 0$
   (b) $\text{Im}(z) = 0$
   (c) $\text{Re}(z) > 0$, $\text{Im}(z) > 0$
   (d) $\text{Re}(z) > 0$, $\text{Im}(z) < 0$ **[IIT-JEE, 1982]**

7. If $z = x + iy$ and $\omega = \left(\frac{1 - iz}{z - i}\right)$, then $|\omega| = 1$ implies that in the complex plane
   (a) z lies on the imaginary axis
   (b) z lies on the real axis
   (c) z lies on the unit circle
   (d) none
   [IIT-JEE, 1983]

8. The points z_1, z_2, z_3, z_4 in the complex plane are the vertices of a parallelogram taken in order if and only if
   (a) $z_1 + z_4 = z_2 + z_3$ (b) $z_1 + z_3 = z_2 + z_4$
   (c) $z_1 + z_2 = z_3 + z_4$ (d) none
   [IIT-JEE, 1983]

9. If the complex numbers z_1, z_2, z_3 represent the vertices of an equilateral triangle such that $|z_1| = |z_2| = |z_3|$, then $z_1 + z_2 + z_3 = 0$. Is it true or false? **[IIT-JEE, 1984]**

10. If three complex numbers are in AP, they lie on a circle in the complex plane. Is it true or false?
    [IIT-JEE, 1985]

11. If a, b, c and u, v, w be complex numbers representing the vertices of two triangles such that $c = (1 - r)a + rb$ and $w = (1 - r)u + rv$, where r is a complex number, the two triangles
    (a) have the same area (b) are similar
    (c) are congruent (d) none
    [IIT-JEE, 1985]

12. Show that the area of a triangle on the Argand diagram formed by the complex numbers z, iz, $z + iz$ is $\frac{1}{2}|z|^2$.
[IIT-JEE, 1986]

13. Complex numbers z_1, z_2, z_3 are the vertices A, B and C respectively of an isosceles right-angled triangle with right angle at C. Show that
$$(z_1 - z_2)^2 = 2(z_1 - z_3)(z_3 - z_2)$$
[IIT-JEE, 1986]

14. If z_1 and z_2 be two non-zero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then $\text{Arg}(z_1) - \text{Arg}(z_2)$ is equal to
(a) $-\pi$ (b) $-\frac{\pi}{2}$ (c) 0 (d) $\frac{\pi}{2}$
[IIT-JEE, 1987]

15. The value of $\sum_{k=1}^{6}\left(\sin\left(\frac{2\pi k}{7}\right) - i\cos\left(\frac{2\pi k}{7}\right)\right)$ is
(a) -1 (b) 0 (c) $-i$ (d) i
[IIT-JEE, 1987]

16. The complex numbers
$\sin x + i\cos 2x$ and $\cos x - i\sin 2x$
are conjugate to each other for
(a) $x = n\pi$ (b) $x = 0$
(c) $\left(n + \frac{1}{2}\right)\pi$ (d) no values of x
[IIT-JEE, 1988]

No questions asked in 1989.

17. Let $z_1 = 10 + 6i$ and $z_2 = 4 + 6i$. If z be any complex number such that the argument of $\left(\frac{z - z_1}{z - z_2}\right)$ is $\frac{\pi}{4}$, then prove that $|z - 7 - 9i| = 3\sqrt{2}$
[IIT-JEE, 1990]

18. The equation not representing a circle is given by
(a) $\text{Re}\left(\frac{1+z}{1-z}\right) = 0$ (b) $z\bar{z} + iz - i\bar{z} + 1 = 0$
(c) $\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}$ (d) $\left|\frac{z-1}{z+1}\right| = 1$
[IIT-JEE, 1991]

19. If $z = -1$, the principal value of the $\text{Arg}(z^{2/3})$ is equal to
(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$ or 0
(c) $\frac{10\pi}{3}$ (d) π
[IIT-JEE, 1991]

20. If z be a complex number such that $z \neq 0$ and $\text{Re}(z) = 0$, then
(a) $\text{Re}(z^2) = 0$ (b) $\text{Im}(z^2) = 0$
(c) $\text{Re}(z^2) = \text{Im}(z^2)$ (d) none
[IIT-JEE, 1992]

21. The complex number $\left(\frac{1+2i}{1-i}\right)$ lies in the
(a) 1st quadrant (b) IInd quadrant
(c) IIIrd quadrant (d) IVth quadrant.
[IIT-JEE, 1992]

22. If α and β be different complex numbers with $|\beta| = 1$, then $\left|\frac{\beta - \alpha}{1 - \bar{\alpha}\beta}\right|$ is equal to
(a) 0 (b) 1/2 (c) 1 (d) 2
[IIT-JEE, 1992]

23. 1, ω, ω^2 be the cube roots of unity, the value of $(1 + \omega)^3 - (1 + \omega^2)^3$ is
(a) 2ω (b) 2 (c) -2 (d) 0
[IIT-JEE, 1993]

24. If α and β be two non-zero complex numbers and z be a variable complex number. If the lines $\alpha\bar{z} + \bar{\alpha}z + 1 = 0$ and $\beta\bar{z} + \bar{\beta}z - 1 = 0$ are mutually perpendicular, then
(a) $\alpha\beta + \bar{\alpha}\bar{\beta} = 0$ (b) $\alpha\beta - \bar{\alpha}\bar{\beta} = 0$
(c) $\bar{\alpha}\beta - \alpha\bar{\beta} = 0$ (d) $\alpha\bar{\beta} + \bar{\alpha}\beta = 0$
[IIT-JEE, 1993]

25. If z_1, z_2, z_3, be the vertices of an equilateral triangle inscribed in the circle $|z| = 2$ and if $z_1 = 1 + i\sqrt{3}$, then
(a) $z_2 = -2, z_3 = 1 - i\sqrt{3}$
(b) $z_2 = 2, z_3 = 1 - i\sqrt{3}$
(c) $z_2 = -2, z_3 = -1 - i\sqrt{3}$
(d) $z_2 = 1 - i\sqrt{3}, z_3 = -1 - i\sqrt{3}$
[IIT-JEE, 1994]

26. If $\omega(\neq 1)$ be a cube root of unity and $(1 + \omega)^7 = A + B\omega$, then A and B are respectively the numbers are
(a) 0, 1 (b) 1, 1 (c) 1, 0 (d) -1, 1
[IIT-JEE, 1995]

27. Let z and ω be two non-zero complex numbers such that $|z| = |\omega|$ and $\text{Arg}(z) + \text{Arg}(\omega) = \pi$, then z is equal to
(a) ω (b) $-\omega$ (c) $\bar{\omega}$ (d) $-\bar{\omega}$
[IIT-JEE, 1995]

28. Let z and ω be two complex numbers such that $|z| \leq 1$, $|\omega| \leq 1$ and $|z + i\omega| = |z - i\bar{\omega}| = 2$, then z is equal to
(a) 1 or i (b) i or $-i$
(c) 1 or -1 (d) i or -1
[IIT-JEE, 1995]

29. If $iz^3 + z^2 - z + i = 0$, show that $|z| = 1$
[IIT-JEE, 1995]

30. If $|z| \leq 1$, $|\omega| \leq 1$, show that
$$|z - \omega|^2 \leq (|z| - |\omega|)^2 + (\text{Arg}(z) - \text{Arg}(\omega))^2.$$
[IIT-JEE, 1995]

31. For positive integers n_1, n_2, the value of the expression
$$(1 + i)^{n1} + (1 + i^3)^{n1} + (1 + i^5)^{n2} + (1 + i^7)^{n2}$$
is a real number if and only if
(a) $n_1 = n_2 + 1$ (b) $n_1 = n_2 - 1$
(c) $n_1 = n_2$ (d) $n_1 > 0, n_2 > 0$
[IIT-JEE, 1996]

32. Find all complex numbers z satisfying $\bar{z} = iz^2$.
[IIT-JEE, 1996]

33. Let $\bar{b}z + b\bar{z} = c$, $b \neq 0$ be a line in the complex plane, where $\bar{b}$ is the complex conjugate of b. If a point z_1 is the reflection of a point z_2 through the line, show that $\bar{z}_1 b + z_2\bar{b} = c$. **[IIT-JEE, 1997]**

34. Let z_1 and z_2 be the roots of $z^2 + pz + q = 0$, where the co-efficients p and q may be complex numbers. Let A and B represent z_1 and z_2 in the complex plane. If $\angle AOB = \alpha$ and $OA = OB$, where O is the origin, prove that $p^2 = 4q\cos^2\left(\frac{\alpha}{2}\right)$. **[IIT-JEE, 1997]**

35. Prove that $\sum_{k=1}^{n-1}(n-k)\cos\left(\frac{2k\pi}{n}\right) = -\frac{n}{2}$ where $n \geq 3$ is an integer. **[IIT-JEE, 1997]**

36. If ω be an imaginary cube root of unity, then $(1 + \omega - \omega^2)^7$ equals
(a) $128\,\omega$ (b) $-128\,\omega$
(c) $128\,\omega^2$ (d) $-128\,\omega^2$
[IIT-JEE, 1998]

37. The sum of $\sum_{n=1}^{13} i^n + i^{n+1}$ is
(a) i (b) $i - 1$ (c) $-i$ (d) 0
[IIT-JEE, 1998]

38. If $i = \sqrt{-1}$, the value of
$$4 + 5\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{334} + 3\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{365}$$ is
(a) $1 - i\sqrt{3}$ (b) $-1 + i\sqrt{3}$
(c) $i\sqrt{3}$ (d) $-i\sqrt{3}$
[IIT-JEE, 1999]

39. For complex numbers z and ω, prove that $|z^2|\omega - |\omega^2|z = z - \omega$ if $z = \omega$ or $z\bar{\omega} = 1$. **[IIT-JEE, 1999]**

40. Find all the roots of $(3z - 1)^4 + (z - 2)^4 = 0$. **[IIT-JEE, 1999]**

41. If arg $(z) < 0$, then Arg$(-z)$ – Arg(z) =
(a) π (b) $-\pi$ (c) $-\frac{\pi}{2}$ (d) $\frac{\pi}{2}$
[IIT-JEE, 2000]

42. If z_1, z_2, z_3 be three complex numbers such that $|z_1| = |z_2| = |z_3| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right| = 1$, then $|z_1 + z_2 + z_3|$ is
(a) 1 (b) <1 (c) > 3 (d) 3
[IIT-JEE, 2000]

43. Let z_1 and z_2 be the nth roots of unity which subtend a right-angle at the origin, then n must be of the form
(a) $4k + 1$ (b) $4k + 2$
(c) $4k + 3$ (d) $4k$ **[IIT-JEE, 2001]**

44. The complex numbers z_1, z_2 and z_3 satisfying $\left(\frac{z_1 - z_2}{z_2 - z_3}\right) = \frac{1 - i\sqrt{3}}{2}$ are the vertices of the triangle which is
(a) of area zero (b) rt angled
(c) equilateral (d) obtuse angled
[IIT-JEE, 2001]

45. Let $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. Then the value of
$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix}$$ is
(a) 3ω (b) $3\omega(\omega - 1)$
(c) $3\omega^2$ (d) $-3\omega(\omega - 1)$
[IIT-JEE, 2002]

46. For all complex numbers z_1, z_2 satisfying $|z_1| = 12$ and $|z_2 - 3 - 4i| = 5$, find the minimum value of $|z_1 - z_2|$ is
(a) 0 (b) 2 (c) 7 (d) 17
[IIT-JEE, 2002]

47. Let a complex number α, $\alpha \neq 1$ be a root of
$z^{p+q} - z^p - z^q + 1 = 0$,
where p and q are distinct primes. Show that either
$1 + \alpha + \alpha^2 + \ldots + \alpha^{p-1} = 0$
or $1 + \alpha + \alpha^2 + \ldots + \alpha^{q-1} = 0$
but not both together. **[IIT-JEE, 2002]**

48. If $|z| = 1$ and $\omega = \frac{z-1}{z+1}$, where $z \neq -1$, then Re(ω) is
(a) 0 (b) $-\frac{1}{|z+1|^2}$
(c) $\frac{1}{|z+1|^2}$ (d) $\frac{\sqrt{2}}{|z+1|^2}$
[IIT-JEE, 2003]

49. If z_1 and z_2 be two complex numbers such that $|z_1| < 1$, $|z_2| > 1$, prove that $\left|\frac{1 - z_1\overline{z_2}}{z_1 - z_2}\right| < 1$
[IIT-JEE, 2003]

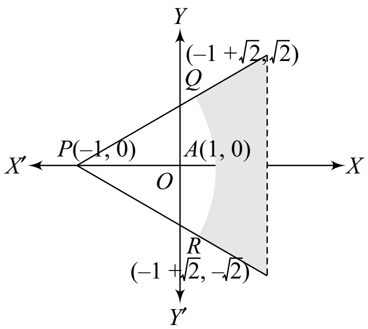
50. Prove that there exist no complex number z such that $|z| < \frac{1}{3}$ and $\sum_{r=1}^{n}(a_r z^r) = 1$, where $|a_r| < 2$ **[IIT-JEE-2003]**

51. If $\omega(\neq 1)$ be a cube root of unity and $(1 + \omega^2)^n = (1 + \omega^4)^n$, the least value of n is
(a) 2 (b) 3 (c) 5 (d) 6
[IIT-JEE, 2004]

52. Find the centre and the radius of the circle given by $\left|\frac{z - \alpha}{z - \beta}\right| = k$, where $k \neq 1$
$z = x + iy$, $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$
[IIT-JEE, 2004]

53. Let $|z - 1| = \sqrt{2}$ is a circle inscribed in a square whose one vertex is $2 + i\sqrt{3}$. Find the remaining vertices. **[IIT-JEE, 2005]**

54. PQ and PR are two infinite rays and QAR is an arc. Point lying in the shaded region excluding the boundary satisfies



(a) $|z+1|>2; |\text{Arg}(z+1)|<\frac{\pi}{4}$

(b) $|z+1|>2; |\text{Arg}(z+1)|<\frac{\pi}{2}$

(c) $|z-1|>2; |\text{Arg}(z-1)|<\frac{\pi}{4}$

(d) $|z-1|>2; |\text{Arg}(z-1)|<\frac{\pi}{2}$ **[IIT-JEE, 2005]**

55. If a, b, c be integers, not all simultaneously equal and ω a cube root of unity ($\omega \neq 1$), the minimum value of $|a + b\omega + c\omega^2|$ is

(a) 0 (b) 1 (c) $\frac{\sqrt{3}}{2}$ (d) $\frac{1}{2}$

[IIT-JEE, 2005]

56. If $\left(\frac{w-\bar{w}z}{1-z}\right)$ is purely real, where $w = \alpha + i\beta$, $\beta \neq 0$ and $z \neq 1$, the set of values of z is

(a) $\{z: |z| = 1\}$ (b) $\{z: z = \bar{z}\}$
(c) $\{z: z \neq 1\}$ (d) $\{z: |z| = 1, z \geq 1\}$

[IIT-JEE, 2006]

57. A man walks a distance of 3 units from the origin towards the north-east (N45°W) direction. From there, he walks a distance of 4 units towards the north-west (N45°W) direction to reach a point P. Then the position of P in the Argand plane is

(a) $3e^{\frac{i\pi}{4}} + 4i$ (b) $(3-4i)e^{\frac{i\pi}{4}}$
(c) $(4+3i)e^{\frac{i\pi}{4}}$ (d) $(3+4i)e^{\frac{i\pi}{4}}$

[IIT-JEE, 2007]

58. If $|z| = 1$ and $|z| = \pm 1$, all the values of $\left(\frac{z}{1-z^2}\right)$ lie on

(a) a line not passing through the origin
(b) $|z| = \sqrt{2}$
(c) the x-axis
(d) the y-axis **[IIT-JEE, 2007]**

59. A particle P starts from the point $z_0 = 1 + 2i$, where $i = \sqrt{-1}$. It moves first horizontally away from the origin by 5 units and then vertically away from origin 3 units to reach a point z_1. From z_1 the particle moves $\sqrt{2}$ units in the direction of the vector $\hat{i} + \hat{j}$ and then it moves through an angle $\frac{\pi}{2}$ in anti-clockwise direction on a circle with centre at origin to reach a point z_2. The point z_2 is given by

(a) $6 + 7i$ (b) $-7 + 6i$
(c) $7 + 6i$ (d) $-6 + 7i$

[IIT-JEE, 2008]

60. Let $z = x + iy$ be a complex number, where x and y are integers. Then the area of the rectangle whose vertices are the roots of $\bar{z}z^3 + z\bar{z}^3 = 350$ is

(a) 48 (b) 32 (c) 40 (d) 80

[IIT-JEE, 2009]

61. Let $z = \cos\theta + i\sin\theta$. Then the value of $\sum_{m=1}^{15} \text{Im}(z^{2m-1})$ at $\theta = 2°$, is

(a) $\frac{1}{\sin(2°)}$ (b) $\frac{1}{3\sin(2°)}$
(c) $\frac{1}{2\sin(2°)}$ (d) $\frac{1}{4\sin(2°)}$

[IIT-JEE, 2009]

62. Match the following columns:

	Column I		Column II
(A)	The set of points z satisfying $\|z - i\|z\|\| = \|z + i\|z\|\|$ is contained in or equal to	(P)	an ellipse with eccentricity $\frac{4}{5}$
(B)	The set of points z satisfying $\|z+4\| + \|z-4\| = 10$ is contained in or equal to	(Q)	the set of points z satisfying Im $z = 0$
(C)	If $\|w\| = 2$, the set of points $z = w - \frac{1}{w}$ is contained in or equal to	(R)	the set of points z satisfying $\|\text{Im}(z)\| \leq 1$
(D)	If $\|w\| = 1$, the set of points $z = w + \frac{1}{w}$ is contained in or equal to	(S)	the set of points z satisfying $\|\text{Re}(z)\| \leq 2$
		(T)	the set of points z satisfying $\|z\| \leq 3$

[IIT-JEE, 2010]

63. Let z_1 and z_2 be two distinct complex numbers and let $z = (1-t)z_1 + tz_2$ for some real number 1 with $0 < t < 1$. If Arg(W) denotes the principal argument of a non-zero complex number w, then

(a) $|z - z_1| + |z - z_2| = |z_1 - z_2|$
(b) $\text{Arg}(z - z_1) = \text{Arg}(z - z_2)$
(c) $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$
(d) $\text{Arg}(z - z_1) = \text{Arg}(z_2 - z_1)$ **[IIT-JEE, 2010]**

64. Let ω be the complex number $\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$. Then the number of distinct complex number z satisfying

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0 \text{ is equal to } \ldots$$

[IIT-JEE, 2010]

65. If z be any complex number satisfying $|z - 3 - 2i| \le 2$, then the minimum value of $|2z - 6 + 5i|$ is......

[IIT-JEE, 2011]

66. **Comprehension**

Let a, b and c be three real numbers satisfying

$$[abc]\begin{bmatrix} 1 & 9 & 7 \\ 8 & 2 & 7 \\ 7 & 3 & 7 \end{bmatrix} = [000] \qquad \ldots(E)$$

(i) If the point $P(a, b, c)$ with reference to (E) lies on the plane $2x + y + z = 1$, the value of $7a + b + c$ is

(a) 0 (b) 12 (c) 7 (d) 6

(ii) Let ω be a solution of $x^3 - 1 = 0$ with $\text{Im}(\omega) > 0$. If $a = 2$ with b and c satisfying (E), the value of $\dfrac{3}{\omega^a} + \dfrac{1}{\omega^b} + \dfrac{3}{\omega^c}$ is....

(a) –2 (b) 2 (c) 3 (d) –3

[IIT-JEE, 2011]

67. Let z be a complex number such that the imaginary part of z is non-zero and $a = z^2 + z + 1$ is real. Then a cannot take the value

(a) –1 (b) $\dfrac{1}{3}$ (c) $\dfrac{1}{2}$ (d) $\dfrac{3}{4}$

[IIT-JEE, 2012]

68. Let the complex numbers α and $\dfrac{1}{\alpha}$ lie on circles

$(x - x_0)^2 + (y - y_0)^2 = r^2$ and $(x - x_0)^2 + (y - y_0)^2 = 4r^2$

respectively. If $z_0 = x_0 + iy_0$ satisfies the equation $2|z_0|^2 = r^2 + 2$,

then $|\alpha|$ is

(a) $\dfrac{1}{\sqrt{2}}$ (b) $\dfrac{1}{2}$ (c) $\dfrac{1}{\sqrt{7}}$ (d) $\dfrac{1}{3}$

[IIT-JEE, 2013]

69. Let $w = \dfrac{\sqrt{3}+i}{2}$ and $P = \{w^n : n = 1, 2, 3, \ldots\}$

Further $H_1 = \left\{ z \in C : \text{Re}(z) > \dfrac{1}{2} \right\}$

and $H_2 = \left\{ z \in C : \text{Re}(z) < -\dfrac{1}{2} \right\}$,

where c is the set of all complex numbers

$z_1 \in P \cap H_1$, $z_2 \in P \cap H_2$ and O represent the origin, then $\angle z_1 O z_2$ is

(a) $\dfrac{\pi}{2}$ (b) $\dfrac{\pi}{6}$ (c) $\dfrac{2\pi}{3}$ (d) $\dfrac{5\pi}{6}$

[IIT-JEE, 2013]

70. Let $S = S_1 \cap S_2 \cap S_3$, where

$S_1 = \{z \in c : |z| < 4\}$,

$$S_2 = \left\{ z \in c : \text{Im}\left(\frac{z - 1 + i\sqrt{3}}{1 - i\sqrt{3}} \right) > 0 \right\}$$

and $S_3 = \{z \in c : \text{Re}(z) > 0\}$

(i) Area of S

(a) $\dfrac{10\pi}{3}$ (b) $\dfrac{20\pi}{3}$ (c) $\dfrac{16\pi}{3}$ (d) $\dfrac{32\pi}{3}$

(ii) $\min\limits_{z \in S} |1 - 3i - z|$ is

(a) $\dfrac{2-\sqrt{3}}{2}$ (b) $\dfrac{2+\sqrt{3}}{2}$

(c) $\dfrac{3-\sqrt{3}}{2}$ (d) $\dfrac{3+\sqrt{3}}{2}$

[IIT-JEE, 2013]

71. Let $z_k = \cos\left(\dfrac{2k\pi}{10}\right) + i\sin\left(\dfrac{2k\pi}{10}\right)$,

where $k = 1, 2, 3, \ldots, 9$.

List I		List II	
(P)	For each z_k, there exist z_j such that $z_k \cdot z_j = 1$.	(1)	True
(Q)	There exist a $k \in \{1, 2, \ldots, 9\}$ such that $z_k \cdot z_j = 1$. has no solution z in the set of complex numbers	(2)	False
(R)	$\dfrac{\lvert 1-z_1\rvert \lvert 1-z_2\rvert \ldots \lvert 1-z_9\rvert}{10}$ equals	(3)	1
(S)	$1 - \sum_{k=1}^{9} \cos\left(\dfrac{2k\pi}{10}\right)$ equals	(4)	2

[IIT-JEE, 2014]

Answers

Level II

1. (a) 2. (c) 3. (d) 4. (a) 5. (d)
6. (b) 7. (b) 8. (a) 9. (c) 10. (c)
11. (c) 12. (a) 13. (b) 14. (d) 15. (a)
16. (a) 17. (b) 18. (b) 19. (a) 20. (c)
21. (d) 22. (a) 23. (b) 24. (a) 25. (a)
26. (a) 27. (d) 28. (b) 29. (a) 30. (b)
31. (a) 32. (d) 33. (a) 34. (b) 35. (c)
36. (b) 37. (d) 38. (c) 39. (d) 40. (b)

41. (d) 42. (c) 43. (b) 44. (b) 45. (a)
46. (a) 47. (a) 48. (a) 49. (a) 50. (a)
51. (d) 52. (b) 53. (c) 54. (c) 55. (c)
56. (c) 57. (a) 58. (a) 59. (c) 60. (c)
61. (a, c) 62. (b,c) 63. (a,b,c,d)
64. (a, b, c) 65. (a, b, c, d)
66. (a, b, c)

Level III

1. 0
2. 25
3. $-1+i2\sqrt{3}$
4. 0
5. 11
6. $3 + 5i, 2 - 3i$
7. $4d$
8. 48
9. $x = \varphi$
10. $z = i, \pm 2\sqrt{i}$
11. $2i$
12. $(a^2 + b^2) = 4$
13. $(a^2 + b^2) = 4$
14. 2010
15. $x^2 + y^2 = 9$
16. $\{z: |z| = 1, z \neq 1\}$.
17. $\{z: |z| = 1, z \neq 1\}$.
18. u lies on y-axis
19. $i \tan(\text{Arg } z)$
20. $(|z| - |w|)^2 + (\text{Arg}(z) - \text{Arg}(w))^2$
21. $\frac{2\sqrt{6}}{5} + i\frac{24}{5}$
22. 25
23. 2
24. $\sqrt{7}$
25. 4027
26. ω
27. 2
28. $\frac{3}{(x^3-1)}$
29. 3
30. 1
31. $a = 2, b = 1, c = 1$
32. 12
33. $\frac{x}{y} + \frac{y}{x} = 2\cos(\alpha + \beta)$
34. $1, \left(\cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)\right), \left(\cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right)\right)$
$\left(\cos\left(\frac{6\pi}{5}\right) + i\sin\left(\frac{6\pi}{5}\right)\right), \left(\cos\left(\frac{8\pi}{5}\right) + i\sin\left(\frac{8\pi}{5}\right)\right)$
35. $-1, \cos\left(\frac{(2r+1)\pi}{15}\right) \pm i\sin\left(\frac{(2r+1)\pi}{15}\right)$,
where $r = 1, 2, 3, 4, 5, 6, 7$.
36. $-1, \cos\left(\frac{(2r+1)\pi}{13}\right) \pm i\sin\left(\frac{(2r+1)\pi}{13}\right)$,
where $r = 1, 2, 3, 4, 5, 6$.
37. 1
38. $z = \cos\left(\frac{\pi}{8}\right) \pm i\sin\left(\frac{\pi}{8}\right), \cos\left(\frac{3\pi}{8}\right) \pm i\sin\left(\frac{3p}{8}\right)$,
$\cos\left(\frac{5\pi}{8}\right) \pm i\sin\left(\frac{5\pi}{8}\right), \cos\left(\frac{7\pi}{8}\right) \pm i\sin\left(\frac{7p}{8}\right)$
39. $\cos\left(\frac{\pi}{7}\right) \cdot \cos\left(\frac{2\pi}{7}\right) \cdot \cos\left(\frac{4\pi}{7}\right) = \frac{1}{8}$
40. $4\cos\left(\frac{\pi}{5}\right)\cos\left(\frac{2\pi}{5}\right) = 1$
41. $x = \cos\left(\frac{2\pi}{15}\right) + i\sin\left(\frac{2\pi}{15}\right), \cos\left(\frac{8\pi}{15}\right) + i\sin\left(\frac{8\pi}{15}\right)$,
$\cos\left(\frac{14\pi}{15}\right) + i\sin\left(\frac{14\pi}{15}\right), \cos\left(\frac{20\pi}{15}\right) + i\sin\left(\frac{20\pi}{15}\right)$
$\cos\left(\frac{26\pi}{15}\right) + i\sin\left(\frac{26\pi}{15}\right)$
similarly, we can easily find the other roots.
42. 0
44. $x^2 + x + \frac{1}{4\cos^2\left(\frac{\pi}{2n+1}\right)} = 0$
45. $n = 4k$
46. 1
47. $S_n = \frac{2n}{\beta - 1}$
48. 30
49. $x = \pm\sqrt{\frac{3}{5}}, y = \pm\sqrt{\frac{3}{5}} + \frac{5}{2}$
51. $a + (-1 \pm \sqrt{1 - 2a - a^2})$
52. $a \in \left[-\frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2}\right] - \{-1, 1\}$
53. $|z - (-7 + ib)| = \sqrt{48 + b^2}$
54. $6 + 8i, 6 + 18i$
55. $x = \cos\left(\frac{\pi k}{2} + \frac{\pi}{48}\right) + i\sin\left(\frac{\pi k}{2} + \frac{\pi}{48}\right)$,
where $k = 0, 1, 2, 3$.
56. $\left(4 - \frac{4}{\sqrt{5}}\right) + i\left(1 + \frac{2}{\sqrt{5}}\right)$,
$\left(4 + \frac{4}{\sqrt{5}}\right) + i\left(1 - \frac{2}{\sqrt{5}}\right)$

57. $\left(\pm\frac{1}{\sqrt{6}}, \mp\frac{3}{\sqrt{2}}\right), \left(\pm\sqrt{\frac{3}{2}}, \mp\frac{1}{\sqrt{2}}\right)$

58. 48 $(1 - i)$, $S_1 = (-16 + 16i)$

59. $z = \frac{10 - 7\sqrt{2}}{20 - 6\sqrt{2}} + i\frac{(-5\sqrt{2})}{20 - 6\sqrt{2}} = a + ib$

similarly for n = 1, 2, 3, we get three other roots of the given equation.

60. $7A_0$

61. $x^2 + x + \frac{1}{4\cos^2\left(\frac{\pi}{2n+1}\right)} = 0$

62. $z = \pm 2, \pm i\sqrt{2}$

63. Im $(z) = y = 8, 17$

64. 3

65. $|z| = 5^{1/5}$

66. $3y - 1 = 0$

67. 5

68. Locus of z is a circle

69. $z = \begin{cases} 0 \\ e^{i\frac{2k\pi}{n}} : & k \in I, n \neq 2 \\ re^{i(k\pi)} : & k \in I, k \in R - \{1\}, n = 2 \end{cases}$

70. $\sum_{k=1}^{n-1} (n-k)\cos(k\theta) = -\frac{n}{2}$

71. $z_3 = \frac{1}{3}(A + B\omega + C\omega^2)$

72. $z = \frac{1 + i\tan\left(\frac{\arg z}{2}\right)}{1 - i\tan\left(\frac{\arg z}{2}\right)}$

74. ω

75. $2n$

Level IV

2. $\frac{3}{2}|z|^2$

4. 2

6. $k = \omega$ or $-\omega^2$

7. 198

8. Required set is constituted by the angles without their boundaries, whose sides are straight lines $y = (\sqrt{2} - 1)x$, $y = -(\sqrt{2} + 1)x$ containing the x-axis.

9. $|f(z)|$ is maximum when $z = w$, where w is the cube root of unity and $|f(z)| = \sqrt{13}$.

10. $(\pi - 2)$

13. 51

14. 14

19. $4x^2 + 4x + \sec^2\left(\frac{\pi}{2n+1}\right) = 0$

21. The locus of ω is a circle, whose centre is $\left(-\frac{3}{2}, \frac{1}{2}\right)$ and the radius is $\frac{1}{\sqrt{2}}$.

23. $x = a$, $y = \frac{x}{\sqrt{3}} + \frac{2a}{\sqrt{3}}$ and $y = \frac{-x}{\sqrt{3}} - \frac{2a}{\sqrt{3}}$

26. 35

28. $m = 2$

29. –11/2

31. (i) –7/2 (ii) 0

32. [–3, –2]

33. 31

INTEGER TYPE QUESTIONS

1. 2	2. 1	3. 7	4. 3	5. 6
6. 8	7. 2	8. 4	9. 2	10. 6
11. 9	12. 2	13. 1	14. 5	15. 6

COMPREHENSIVE LINK PASSAGES

Passage I:	1. (c)	2. (a)	3. (d)		
Passage II:	1. (c)	2. (b)	3. (c)		
Passage III:	1. (a)	2. (d)	3. (a)	4. (a)	
Passage IV:	1. (b)	2. (a)	3. (d)		
Passage V:	1. (d)	2. (c)	3. (d)	4. (b)	5. (c)
Passage VI:	1. (a)	2. (c)	3. (a)	4. (b)	5. (c)
	6. (a)				

MATRIX MATCH

1. (A) → Q; (B) → P; (C) → Q; (D) → S
2. (A) → S; (B) → R; (C) → Q; (D) → P
3. (A) → S; (B) → P; (C) → Q; (D) → P
4. (A)
5. (B)
6. (C)

HINTS AND SOLUTIONS

LEVEL I

1. We have,

$i^n + i^{n+1} + i^{n+2} + i^{n+3}$
$= i^n(1 + i + i^2 + i^3)$
$= 0$

2. We have,

$i^{2010} + i^{2011} + i^{2012} + i^{2013}$
$= i^{2010}(1 + i + i^2 + i^3)$
$= 0$

3. We have,

$$\left(\frac{1+i}{1-\mathrm{i}}\right)^n = 1$$

$$\Rightarrow \left(\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right)^n = 1$$

$$\Rightarrow \left(\frac{(1+i)^2}{1^2-(-i)^2}\right)^n = 1$$

$$\Rightarrow \left(\frac{1+i^2+2i}{1+1}\right)^n = 1$$

$\Rightarrow (i)^n = 1$

$\Rightarrow n = \ldots, -8, -4, 0, 4, 8, 12, \ldots$

Clearly, n is not defined

> ***Notes***
> 1. The smallest positive integer n for which $\left(\frac{1+i}{1-i}\right)^n = 1$, then $n = 4$
> 2. The smallest non negative integer n for which $\left(\frac{1+i}{1-i}\right)^n = 1$, then $n = 0$
> 3. The smallest positive integer n for which $\left(\frac{1+i}{1-\mathrm{i}}\right)^n$ is real, then $n = 2$.

4. We have,

$$\sum_{n=1}^{2013} (i^n + i^{n+1})$$

$$= \left(\underbrace{i + i^2 + i^3 + i^4 + \ldots + i^{2012}}_{=0} + i^{2013}\right)$$

$$+ \left(\underbrace{i^2 + i^3 + i^4 + i^5 + \ldots + i^{2013}}_{=0} + \mathrm{i}^{2014}\right)$$

$= i^{2013} + i^{2014}$
$= i + i^2$
$= i - 1$

5. We have,

$i^P + i^Q + i^R + i^S$
$= i^P(1 + i + i^2 + i^3)$
$= i^P \times 0$
$= 0$

6. We have,

$i^{2015} + i^{2016} + i^{2017} + i^{2018}$
$= i^{2015}(1 + i + i^2 + i^3)$
$= i^{2015} \times 0$
$= 0$

7. We have,

$$\sum_{k=0}^{2016} i^k + \sum_{p=0}^{2018} i^p = x + iy$$

$$\Rightarrow \quad x + iy = \sum_{k=0}^{2016} i^k + \sum_{p=0}^{2018} i^p$$

$\Rightarrow \quad x + iy = i^{2016} + i^{2016} + i^{2017} + i^{2018}$

$\Rightarrow \quad x + iy = 1 + 1 + i + i^2 = 1 + i$

Thus, $x = 1, y = 1$

Hence, the value of

$x + y + 2$
$= 1 + 1 + 2$
$= 4$

8. We have,

$(1 + i)^{2n} = (1 - i)^{2n}$

$$\Rightarrow \frac{(1+i)^{2n}}{(1-i)^{2n}} = 1$$

$$\Rightarrow \left(\frac{1+i}{1-i}\right)^{2n} = 1$$

$$\Rightarrow \left(\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right)^{2n} = 1$$

$$\Rightarrow \left(\frac{(1+\mathrm{i})^2}{(1)^2-(i)^2}\right)^{2n} = 1$$

$$\Rightarrow \left(\frac{1+i^2+2i}{1+1}\right)^{2n} = 1$$

$\Rightarrow (i)^{2n} = 1$

$\Rightarrow n = 2$

Hence, the positive integer n is 2.

9. We have

$(1 + i)^5 + (1 + i^3)^5 + (1 + i^5)^7 + (1 + i^7)^7$
$= (1 + i)^5 + (1 - i)^5\} + \{(1 + i)^7 + (1 - i)^7$
$= 21 \pm {}^5C_1 i + {}^5C_1 i^2 \pm {}^5C_1 i^3 + {}^5C_1 i^4 \pm {}^5C_1 i^5\}$
$2(1 \pm {}^7C_1 i + {}^7C_2 i^2 \pm {}^7C_3 i^3 + {}^7C_4 i^4 \pm {}^7C_5 i^5 + {}^7C_6 i^6 \pm {}^7C_7 i^7)$
$= (2 - 20 + 10) + (1 - 21 + 35 - 7)$
$= -8 + 8$
$= 0$

10. We have $z = (n + i)^4$

For $n = 0$, $z = i^4 = 1$, an integer

For $n = 1$, $z = (1 + i)^4 = 1 - 6 + 1 = -4$, an integer

For $n = -1$, $z = (1 + i)^4 = 1 - 6 + 1 = -4$, an integer

For $n = R - \{-1, 0, 1\}$, z, an imaginary number.

Thus, the number of integral values of n is 3.

11. We have,

$$z^2 = (1 + i)^2 = 1 + i^2 + 2i = 2i$$

Hence, the multiplicative inverse of z^2 is $= \frac{1}{2i} = -\frac{i}{2}$.

12 We have $z = \frac{1+2i}{3-4i} = \frac{(1+2i)(3+4i)}{(3-4i)(3+4i)}$

$$z = \frac{3+4i+6i-8}{25} = \frac{-5+10i}{25}$$

Hence, the multiplicative inverse of z is

$$\frac{25}{-5+10i}$$

$$= \frac{25(-5-10i)}{(-5+10i)(-5-10i)}$$

$$= \frac{25(-5-10i)}{125}$$

$$= \frac{(-5-10i)}{5} = -1-2i$$

13. The given relation $a + ib > c + id$ holds good only when if $b = 0, d = 0$.

Thus, $b + d + 2016 = 0 + 0 + 2016 = 2016$.

14. We have,

$$\left(\frac{1+i}{1-i}\right)^n = \frac{2}{\pi}\left(\sin^{-1}x + \sec^{-1}\left(\frac{1}{x}\right)\right)$$

$$\Rightarrow \left(\frac{1+i}{1-i}\right)^n = \frac{2}{\pi}(\sin^{-1}x + \cos^{-1}x)$$

$$\Rightarrow \left(\frac{1+i}{1-i}\right)^n = \frac{2}{\pi} \times \frac{\pi}{2} = 1$$

$$\Rightarrow \left(\frac{1+i}{1-i}\right)^n = 1$$

$\Rightarrow$ $i^n = 1$

$\Rightarrow$ $i^n = i^4$

$\Rightarrow$ $n = 4$

Thus, the positive integer is 4.

15. We have

$$\frac{(1+i)x-2i}{(3+i)} + \frac{(2-3i)y+\mathrm{i}}{(3-i)} = i$$

$$\Rightarrow \frac{1+i(x-2)}{(3+i)} + \frac{2+\mathrm{i}(1-3y)}{(3-i)} = \mathrm{i}$$

$\Rightarrow$ $(4 + 2i)x + (9 - 7i)y - 3i - 3 = 10i$

Equating the real and imaginary parts, we get,

$2x - 7y = 13$ and $4x + 9y = 3$.

Hence, $x = 3, y = -1$.

16. We have

$$x + iy = \frac{2^{1008}}{(1+i)^{2016}} + \frac{(1+i)^{2016}}{2^{1008}}$$

$$= \left(\frac{2}{(1+i)^2}\right)^{1008} + \left(\frac{(1+\mathrm{i})^2}{2}\right)^{1008}$$

$$= \left(\frac{2}{2i}\right)^{1008} + \left(\frac{2i}{2}\right)^{1008}$$

$$= \left(\frac{1}{i}\right)^{1008} + (i)^{1008}$$

$$= (-i)^{1008} + (i)^{1008} = 1 + 1 = 2$$

Thus, $x = 2$ and $y = 0$.

17. We have

$z^{1/3} = a + b$

$\Rightarrow$ $(x + i\,y)^{1/3} = a + i\,b$

$\Rightarrow$ $(x + i\,y) = (a + i\,b)^3$

$= \mathrm{a}^3 + 3a^2(i\,b) + 3a(i\,b)^2 + (i\,b)^3$

$\Rightarrow$ $(x + iy) = a^3 + i^3a^2b - 3ab^2 - ib^3$

$\Rightarrow$ $(x + iy) = (a^3 - 3ab^2) + i(3a^2b - b^3)$

$\Rightarrow$ $x = (a^3 - 3ab^2)$ and $y = (3a^2b - b^3)$

$\Rightarrow$ $\frac{x}{a} = a^2 - 3b^2$ and $\frac{y}{b} = 3a^2 - b^2$

$\Rightarrow$ $\frac{x}{a} + \frac{y}{b} = (a^2 - 3b^2) + (3a^2 - b^2)$

Adding, we get

$\Rightarrow$ $\frac{x}{a} + \frac{y}{b} = (4a^2 - 4b^2)$

$\Rightarrow$ $\frac{x}{a} + \frac{y}{b} = 4(a^2 - b^2)$

18. Given

$x = 3 + 2i$

$\Rightarrow$ $(x - 3)^2 = -4$

$\Rightarrow$ $x^2 - 6x + 9 = -4$

$\Rightarrow$ $x^2 - 6x + 13 = 0$

We have $x^4 - 8x^3 + 4x^2 + 4x + 39$

$= x^2(x^2 - 6x + 13) - 2x^3 - 9x^2 + 4x + 39$

$= -2x^3 - 9x^2 + 4x + 39$

$= -2x(x^2 - 6x + 13) - 21x^2 + 30x + 39$

$= -21x^2 + 30x + 39$

$= -21(x^2 - 6x + 13) - 96x + 312$

$= -96x + 312$

$= -96(3 + 2i) + 312$

$= -288 - 192i + 312$

$= 24 - 192i$

Thus, $a = 24$ and $b = -192$.

Hence, the value of

$$\left(\frac{b}{a}+10\right)$$

$$=\left(-\frac{192}{24}+10\right)$$

$= -8 + 10$

$= 2$

19. We have,

$$z=\left(\frac{2i}{1+i}\right)^n$$

$$=\left(\frac{2i}{1+i}\times\frac{(1-i)}{(1-i)}\right)^n$$

$$=\left(\frac{2i(1-i)}{2}\right)^n$$

$$= (1+i)^n$$

$$=\left(\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)\right)^n$$

$$=\left(\sqrt{2}\left(\cos\left(\frac{\pi}{4}\right)+i\sin\left(\frac{\pi}{4}\right)\right)\right)^n$$

$$=(2)^{\frac{n}{2}}\left(\cos\left(\frac{n\pi}{4}\right)+i\sin\left(\frac{n\pi}{4}\right)\right)$$

It will be a real number if

$$\sin\left(\frac{n\pi}{4}\right)=0$$

$$\Rightarrow \quad \left(\frac{n\pi}{4}\right)=k\pi,\ k=1,2,3\ldots$$

$\Rightarrow \quad n = 4k,\ k = 1, 2, 3, \ldots$

For $n = 4$, $z=4\cos\left(\frac{4\pi}{4}\right)=-4$

and for $n = 8$, $z=16\cos\left(\frac{8\pi}{4}\right)=16$

Hence, the minimum value of n is 8.

20. As we know that, if one root of a quadratic equation is imaginary, its other one will be its conjugate.

Thus, the another root is $3 - 2i$.

Now, the sum of the roots $= 3 + 2i + 3 - 2i$

$= 6$

and product of the roots

$= (3 + 2i)(3 - 2i)$

$= (3)^2 - (2i)^2 = 9 + 4 = 13$

Hence, the required equation is

$x^2 - Sx + P = 0$

$\Rightarrow \quad x^2 - 6x + 13 = 0$

21. We have $z^2 + \bar{z} = 0$.

Let $z = x + iy$

Then $(x + iy)^2 + (x - iy) = 0$

$\Rightarrow \quad (x^2 - y^2 + i2xy) + (x - iy) = 0$

$\Rightarrow \quad (x^2 - y^2 + x) + i(2xy - y) = 0$

$\Rightarrow \quad (x^2 - y^2 + x) = 0,\ (2xy - y) = 0$

Now, $2xy - y = 0$ gives

$\Rightarrow \quad y(2x - 1) = 0$

$\Rightarrow \quad y = 0,\ x = 1/2$

When $y = 0$, then

$x^2 + x = 0$

$\Rightarrow \quad x(x + 1) = 0 \quad \Rightarrow \quad x = 0, -1$

So the solutions are $(0, 0)$, $(-1, 0)$.

When $x = 1/2$, then

$$\frac{1}{4}-y^2+\frac{1}{2}=0$$

$$\Rightarrow \quad y^2=\frac{3}{4}$$

$$\Rightarrow \quad y=\pm\frac{\sqrt{3}}{2}$$

So, the solutions are $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right),\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$.

Hence, the solutions of the given equations are

$(0, 0)$, $(-1, 0)$, $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right),\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$

i.e. $0, -1,\ \frac{1}{2}+i\frac{\sqrt{3}}{2},\frac{1}{2}-i\frac{\sqrt{3}}{2}$

22. Let z_1, z_2, z_3 are the roots of the given equation.

Thus, $z_1 + z_2 + z_3 = 2(2 - i)$

$\Rightarrow \quad z_2 + z_3 = 2(2 - i) - z_1$

$= 2(2 - i) - (1 - i)$

$= 3 - i$ …(i)

Also, $z_1.z_2.z_3 = (1 - 3i)$

$$z_2\cdot z_3=\frac{1-3i}{1-i}=\frac{(1-3i)(1+i)}{(1-i)(1+i)}$$

$$\Rightarrow \quad =\frac{1+i-3i+3}{2}$$

$$=\frac{4-2i}{2}=2-i \quad \ldots\text{(ii)}$$

Solving Eqs (i) and (ii), we get

$z_1 = 1,\ z_2 = 2 - i$

23. If one of the given equation is $1 + 2i$, then its other root will be its conjugate. Thus $1 - 2i$ is the other root. Now,

$S = 1 + 2i + 1 - 2i$

$= 2$

and $\quad P = (1 + 2i)(1 - 2i)$

$= 5$

Therefore, $(x^2 - 2x + 5)$ is a factor of the given equation.

The given equation can also be written as

$x^2(x^2 - 2x + 5) - x(x^2 - 2x + 5) + (x^2 - 2x + 5) = 0$

$\Rightarrow \quad (x^2 - 2x + 5)(x^2 - x + 1) = 0$

$\Rightarrow \quad (x^2 - 2x + 5) = 0,\ (x^2 - x + 1) = 0$

$$\Rightarrow \quad x=1\pm 2i,\ x=\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right),\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$$

Hence, the solutions are

$$\left\{1 \pm 2i, \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right), \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\right\}$$

24. Thus, the greatest and the least value of $|z|$ are $\sqrt{13}+1, \sqrt{13}-1$ respectively.

25. Thus, the greatest and the least value of $|z|$ are $\sqrt{34}+2, \sqrt{34}-2$.

Hence, the required difference

$$= (\sqrt{34}+2) - (\sqrt{34}-2)$$
$$= 4$$

26. Given $a + ib = (1 + i)(1 + 2i)(1 + 3i) \dots (1 + ni)$

$\Rightarrow \quad |a + ib| = |(1 + i)(1 + 2i)(1 + 3i) \dots (1 + ni)|$

$\qquad = |(1 + i)||(1 + 2i)||(1 + 3i)| \dots |(1 + ni)|$

$\Rightarrow \quad \sqrt{a^2+b^2} = \sqrt{1^2+1^2}\sqrt{1^2+2^2}\sqrt{1^2+3^2}\dots\sqrt{1^2+n^2}$

$\Rightarrow \quad (a^2 + b^2) = 2.5.10 \dots (n^2 + 1)$

Hence, the result.

27. We have,

$$x + iy = \frac{a+ib}{a-ib}$$

$\Rightarrow \quad |x+iy| = \left|\frac{a+ib}{a-ib}\right|$

$\Rightarrow \quad = \frac{|a+ib|}{|a-ib|}$

$\Rightarrow \quad |x+iy|^2 = \frac{|a+ib|^2}{|a-ib|^2}$

$\Rightarrow \quad x^2+y^2 = \frac{(a^2+b^2)}{(a^2+b^2)} = 1$

Hence, the result.

28. We have,

$$z + |z| = 2 + 8i$$

$\Rightarrow \quad (x+iy) + \sqrt{x^2+y^2} = 2 + 8i$

$\Rightarrow \quad (x + \sqrt{x^2+y^2}) + iy = 2 + 8i$

Thus, $x + \sqrt{x^2+y^2} = 2, y = 8$

$\Rightarrow \quad (x + \sqrt{x^2+64}) = 2$

$\Rightarrow \quad \sqrt{x^2+64} = (2-x)$

$\Rightarrow \quad (x^2 + 64) = (2 - x)^2$

$\Rightarrow \quad = 4 - 4x + x^2$

$\Rightarrow \quad 4x = -60$

$\Rightarrow \quad x = -15$

Hence, $|z| = \sqrt{x^2+y^2}$

$$= \sqrt{225+64} = \sqrt{289} = 17$$

29. We have,

$$iz = ire^{i\theta} = e^{i\frac{\pi}{2}}re^{i\theta} = re^{i\left(\frac{\pi}{2}+\theta\right)}$$
$$= r\left(\cos\left(\frac{\pi}{2}+\theta\right) + i\sin\left(\frac{\pi}{2}+\theta\right)\right)$$
$$= r(-\sin\theta + i\cos\theta)$$

Now, $e^{iz} = e^{r(-\sin\theta + i\cos\theta)} = e^{-r\sin\theta + i(r\cos\theta)}$

$\qquad = e^{-r\sin\theta}e^{-i(r\cos\theta)}$

$\Rightarrow \quad |e^{iz}| = |e^{-r\sin\theta}e^{-i(r\cos\theta)}|$

$\qquad = e^{-r\sin\theta}|e^{-i(r\cos\theta)}|$

$\qquad = e^{-r\sin\theta} \times 1$

$\qquad = e^{-r\sin\theta}$

30. We have,

$$\left|\frac{\alpha\bar{\beta}+\beta\bar{\alpha}}{|\alpha\beta|}\right|$$
$$= \left|\frac{|\alpha\bar{\beta}+\beta\bar{\alpha}|}{|\alpha\beta|}\right|$$
$$\leq \frac{|\alpha\bar{\beta}|+|\beta\bar{\alpha}|}{|\alpha\beta|}$$
$$= \frac{|\alpha||\bar{\beta}|+|\beta||\bar{\alpha}|}{|\alpha||\beta|}$$
$$= \frac{|\alpha||\bar{\beta}|}{|\alpha||\beta|} + \frac{|\beta||\bar{\alpha}|}{|\alpha||\beta|}$$
$$= 2 \qquad (\because |\alpha| = |\bar{\alpha}|, |\beta| = |\bar{\beta}|)$$

Thus, the maximum value is 2.

31. Given $|z_1| = 1$

$\Rightarrow \quad |z_1|^2 = 1 \qquad \Rightarrow z_1 \cdot \bar{z}_1 = 1$

$\Rightarrow \quad \bar{z}_1 = \frac{1}{z_1}$

Similarly, $\bar{z}_2 = \frac{1}{z_2}, \bar{z}_3 = \frac{1}{z_3}$

Now, $\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right| = 1$

$\Rightarrow \quad |\bar{z}_1+\bar{z}_2+\bar{z}_3| = 1$

$\Rightarrow \quad |\overline{z_1+z_2+z_3}| = 1$

$\Rightarrow \quad |z_1+z_2+z_3| = 1$

32. We have $z = (3 + 7i)(p + iq), p, q \in I - \{0\}$

$\Rightarrow \quad z = (3p - 7q) + i(7p + 3q)$

$\Rightarrow \quad |z| = \sqrt{(3p-7q)^2 + (7p+3q)^2}$

$\Rightarrow \quad |z|^2 = (3p - 7q)^2 + (7p + 3q)^2$

$\qquad = 9p^2 + 49q^2 + 49p^2 + 9q^2$

$\qquad = 58(p^2 + q^2) \qquad \dots$(i)

Since z is purely imaginary number, so

$$(3p - 7q) = 0$$

$\Rightarrow \quad \frac{p}{7} = \frac{q}{3} = \lambda(\text{say}), \lambda \in I - \{0\}$

Thus, $|z|^2$ will be minimum only when $\lambda = 1$.

Therefore, $p = 7$ and $q = 3$

Hence, the minimum value of $|z|^2 = 58(49 + 9)$
$= 58 \times 58$
$= 3364$

33. Given,

$|\beta| = 1$

$\Rightarrow |\beta|^2 = 1$

$\Rightarrow \beta \cdot \bar{\beta} = 1$

$\Rightarrow \beta = \frac{1}{\bar{\beta}}$

Now, $\left|\frac{\beta - \alpha}{1 - \bar{\alpha}\beta}\right| = \left|\frac{\beta - \alpha}{1 - \bar{\alpha} \cdot \frac{1}{\bar{\beta}}}\right|$

$= \left|\frac{\bar{\beta}(\beta - \alpha)}{\bar{\beta} - \bar{\alpha}}\right|$

$= \frac{|\bar{\beta}||\beta - \alpha|}{|\bar{\beta} - \bar{\alpha}|}$

$= \frac{|\bar{\beta}||\beta - \alpha|}{|\overline{(\beta - \alpha)}|} = 1$

34. We have $|z + 1| = z + 2(1 + i)$.

Let $z = x + iy$.

Then $|x + iy + 1| = (x + iy) + 2(1 + i)$

$\Rightarrow |(x + 1) + iy| = (x + 2) + i(y + 2)$

$\Rightarrow \sqrt{(x+1)^2 + y^2} = (x + 2) + i(y + 2)$

Comparing the real and imaginary parts, we get

$\sqrt{(x+1)^2 + y^2} = (x + 2), (y + 2) = 0$

when $y = -2, x = 1/2$

Hence, the complex number is

$z = x + iy = \frac{1}{2} - 2i$

35. We have

$|z - 1|^2 + |z + 1|^2 = 5$

$\Rightarrow |(x + iy) - 1|^2 + |(x + iy) + 1|^2 = 5$

$\Rightarrow |(x - 1) - iy|^2 + |(x + 1) + iy|^2 = 5$

$\Rightarrow (x - 1)^2 + y^2 + (x + 1)^2 + y^2 = 8$

$\Rightarrow 2(x^2 + y^2) + 2 = 8$

$\Rightarrow 2(x^2 + y^2) = 6$

$\Rightarrow (x^2 + y^2) = 3$

$\Rightarrow |z|^2 = 3$

36. Given,

$|z - 2| = 2|z - 1|$

$\Rightarrow |z - 2|^2 = 4|z - 1|^2$

$\Rightarrow (z - 2)(\bar{z}_1 - 2) = 4(z - 1)(\bar{z}_1 - 1)$

$\Rightarrow (z \cdot \bar{z} - 2z - 2\bar{z} + 4) = 4(z \cdot \bar{z} - z - \bar{z} + 1$

$\Rightarrow (|z|^2 - 2z - 2\bar{z} + 4) = 4(|z|^2 - z - \bar{z} + 1)$

$\Rightarrow 3|z|^2 - 2z - 2\bar{z} = 0$

$\Rightarrow 3|z|^2 - 2(z - 2\bar{z}) = 0$

$\Rightarrow 3|z|^2 - 2(z - 2\bar{z}) = 2.2\text{Re}(z) = 4\text{Re}(z)$

$\Rightarrow |z|^2 = \frac{4}{3}\text{Re}(z)$

Hence, the result.

37. Given,

$|z + 6| = |3z + 2|$

$\Rightarrow |z + 6|^2 = |3z + 2|^2$

$\Rightarrow (z + 6)(\bar{z} + 6) = (3z + 2)(3\bar{z} + 2)$

$\Rightarrow (z.\bar{z} + 6(z + \bar{z}) + 36) = (9z.\bar{z} + 6(z + \bar{z}) + 4)$

$\Rightarrow (z.\bar{z} + 36) = (9z.\bar{z} + 4)$

$\Rightarrow (|z|^2 + 36) = (9|z|^2 + 4)$

$\Rightarrow 8|z|^2 = 32$

$\Rightarrow |z|^2 = 4$

$\Rightarrow |z| = 2$

Hence, the value of $|z|$ is 2.

38. Given,

$|z + 6| = |2z + 3|$

$\Rightarrow |x + iy + 6| = |2(x + iy) + 3|$

$\Rightarrow |(x + 6) + iy| = |(2x + 3) + i.2y|$

$\Rightarrow |(x + 6) + iy|^2 = |(2x + 3) + i.2y|^2$

$\Rightarrow (\text{x} + 6)^2 + y^2 = (2x + 3)^2 + 4y^2$

$\Rightarrow x^2 + 12x + 36 + y^2 = 4x^2 + 12x + 9 + 4y^2$

$\Rightarrow 3x^2 + 3y^2 = 27$

$\Rightarrow x^2 + y^2 = 9$

Hence, the locus of z is $x^2 + y^2 = 9$.

39. Given,

$|z| = 1$

$\Rightarrow |z|^2 = 1$

$\Rightarrow z.\bar{z} = 1$

$\Rightarrow \bar{z} = \frac{1}{z}$

Now, $2\text{Re}(\omega) = (\omega + \bar{\omega})$

$= \left(\frac{z-1}{z+1}\right) + \overline{\left(\frac{z-1}{z+1}\right)}$

$= \left(\frac{z-1}{z+1}\right) + \left(\frac{\bar{z}-1}{\bar{z}+1}\right)$

$= \left(\frac{z-1}{z+1}\right) + \left(\frac{(1/z)-1}{(1/z)+1}\right)$

$= \left(\frac{z-1}{z+1}\right) + \left(\frac{1-z}{1+z}\right)$

$= \left(\frac{z-1+1-z}{z+1}\right)$

$= 0$

$\Rightarrow \text{Re}(\omega) = 0$

40. We have,

$|z + i| + |z - i| = 8$

$\Rightarrow \quad 8 = |z + i| + |z - i|$
$\Rightarrow \quad 8 = |z + i| + |z - i| \geq |z + i + z - i|$
$\Rightarrow \quad |z + i + z - i| \leq 8$
$\Rightarrow \quad |2z| \leq 8$
$\Rightarrow \quad |z| \leq 4$

Hence, the maximum value of $|z|$ is 4.

41. (i) Let $z = 1 + i = (1, 1)$.

We have $\alpha = \tan^{-1}\left|\frac{1}{1}\right| = \tan^{-1}(1) = \frac{\pi}{4}$

Since, the given complex number lies in the first first quadrant, so $\text{Arg}(z) = \theta = \alpha = \frac{\pi}{4}$

(ii) Let $z = 1 - i = (1, -1)$.

We have $\alpha = \tan^{-1}\left|\frac{-1}{1}\right| = \tan^{-1}(1) = \frac{\pi}{4}$

Since the complex number z lies in the fourth quadrant, so

$\text{Arg}(z) = \theta = -\alpha = -\frac{\pi}{4}$

(iii) Let $z = -1 + i = (-1, 1)$

We have $\alpha = \tan^{-1}\left|\frac{1}{-1}\right| = \tan^{-1}(1) = \frac{\pi}{4}$

Since the complex number lies in the second quadrant, so

$$\text{Arg}(z) = \theta = \pi - \alpha = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

(iv) Let $z = -1 - i = (-1, -1)$

We have $\alpha = \tan^{-1}\left|\frac{-1}{-1}\right| = \tan^{-1}(1) = \frac{\pi}{4}$

Since the complex number z lies in the third quadrant, so

$$\text{Arg}(z) = \theta = -(\pi - \alpha)$$

42. Now $-z = -\frac{\sqrt{3}}{2} - \frac{i}{2} = \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

We have $\alpha = \tan^{-1}\left|\frac{-1/2}{-\sqrt{3/2}}\right| = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$

Since the complex number $-z$ lies in the third quadrant, so

$\text{Arg}(-z) = \theta = -(\pi - \alpha)$

$$= -\left(\pi - \frac{\pi}{6}\right) = -\frac{5\pi}{6}$$

$$= -\left(\pi - \frac{\pi}{4}\right) = -\frac{3\pi}{4}$$

Similarly, you can solve the other parts easily.

43. Since $\text{Arg}(z) < 0$, so z lies in the third quadrant and $(-z)$ lies in the first quadrant

Thus,

$$\text{Arg}(z) - \text{Arg}(-z) = -(\pi - \alpha) - \theta = -\pi$$

44. Given $|z + 1| = |z - 1|$

$\Rightarrow \quad |x + iy + 1| = |x + iy - 1|$
$\Rightarrow \quad |(x + 1) + iy| = |(x - 1) + iy|$
$\Rightarrow \quad \sqrt{(x+1)^2 + y^2} = \sqrt{(x-1)^2 + y^2}$
$\Rightarrow \quad (x + 1)^2 + y^2 = (x - 1)^2 + y^2$
$\Rightarrow \quad 4x = 0 \Rightarrow x = 0$

Also, $\text{Amp}\left(\frac{z-1}{z+1}\right) = \frac{\pi}{4}$

$\Rightarrow \quad \text{Amp}(z - 1) - \text{Amp}(z + 1) = \frac{\pi}{4}$

$\Rightarrow \quad \tan^{-1}\left(\frac{y}{x-1}\right) - \tan^{-1}\left(\frac{y}{x+1}\right) = \frac{\pi}{4}$

$\Rightarrow \quad \tan^{-1}\left(\frac{\frac{y}{x-1} - \frac{y}{x+1}}{1 + \frac{y}{x-1} \cdot \frac{y}{x+1}}\right) = \frac{\pi}{4}$

$\Rightarrow \quad \tan^{-1}\left(\frac{xy + y - xy + y}{x^2 + y^2 - 1}\right) = \frac{\pi}{4}$

$\Rightarrow \quad \tan^{-1}\left(\frac{2y}{x^2 + y^2 - 1}\right) = \frac{\pi}{4}$

$\Rightarrow \quad \frac{2y}{x^2 + y^2 - 1} = 1$

$\Rightarrow \quad x^2 + y^2 - 1 = 2y$
$\Rightarrow \quad x^2 + y^2 - 2y - 1 = 0$
$\Rightarrow \quad y^2 - 2y - 1 = 0$, since $x = 0$

$\Rightarrow \quad y = \frac{2 \pm \sqrt{8}}{2} = (1 \pm \sqrt{2})$

Thus, the complex number,

$$z = x + iy = i(1 \pm \sqrt{2})$$

45. Given,

$|z_1 + z_2| = |z_1 - z_2|$
$\Rightarrow \quad |z_1 + z_2|^2 = |z_1 - z_2|^2$
$\Rightarrow \quad |z_1|^2 + |z_1|^2 + 2\text{Re}(z_1\bar{z}_2) = |z_1|^2 + |z_1|^2 + 2\text{Re}(z_1\bar{z}_2)$
$\Rightarrow \quad 4\text{Re}(z_1\bar{z}_2) = 0$
$\Rightarrow \quad \text{Re}(e^{i(\theta_1 + \theta_2)}) = 0$
$\Rightarrow \quad \text{Re}(\cos(\theta_1 - \theta_2) - i(\theta_1 - \theta_2)) = 0$
$\Rightarrow \quad \cos(\theta_1 - \theta_2) = 0$
$\Rightarrow \quad \theta_1 - \theta_2 = \frac{\pi}{2}$

Thus, $\text{Amp}\left(\frac{z_1}{z_2}\right)(\theta_1 - \theta_2) = \frac{\pi}{2}$

46. We have,

$$z = \left(\frac{\cos\theta + i\sin\theta}{\cos\theta - i\sin\theta}\right)$$

$$= (\cos 2\theta + i\sin 2\theta)$$

Also, it is given that $\frac{\pi}{4} < \theta < \frac{\pi}{2}$

$\Rightarrow \quad \frac{\pi}{2} < 2\theta < \pi$

Thus, Arg $(z) = \pi - 2\theta$

47. We have,

$$z = \sin\left(\frac{\pi}{5}\right) + i\left(1 - \cos\left(\frac{\pi}{5}\right)\right)$$

$$= \sin\left(\frac{\pi}{5}\right) + \text{i}\left(1 - \cos\left(\frac{\pi}{5}\right)\right)$$

$$= \sin\left(\frac{\pi}{5}\right) + 2 \cdot i \sin^2\left(\frac{\pi}{10}\right)$$

Thus, $\text{Arg}(z) = \tan^{-1}\left(\frac{2\sin^2(\pi/10)}{\sin(\pi/5)}\right)$

$$= \tan^{-1}\left(\frac{2\sin^2(\pi/10)}{2\sin(\pi/10)\cos(\pi/10)}\right)$$

$$= \tan^{-1}\left(\tan\left(\frac{\pi}{10}\right)\right)$$

$$= \frac{\pi}{10}$$

48. Let $z = x + iy$.
Given,

$$\text{Arg}(z) = \frac{\pi}{3}$$

$\Rightarrow \quad \tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{3}$

$\Rightarrow \quad \frac{y}{x} = \tan\left(\frac{\pi}{3}\right) = \sqrt{3}$

$\Rightarrow \quad y = x\sqrt{3}$

Now, $z - 1 = (x + iy) - 1 = (x - 1) + iy$

Also, $\text{Arg}(z - 1) = \frac{5\pi}{6}$

$\Rightarrow \quad \tan^{-1}\left(\frac{y}{x-1}\right) = \frac{5\pi}{6}$

$\Rightarrow \quad \frac{y}{x-1} = \tan\left(\frac{5\pi}{6}\right) = -\sqrt{3}$

$\Rightarrow \quad x\sqrt{3} = -\sqrt{3}(x-1)$

$\Rightarrow \quad x = -x + 1$

$\Rightarrow \quad 2x = 1$

$\Rightarrow \quad x = \frac{1}{2}$ and $y = \frac{\sqrt{3}}{2}$

Thus, the complex number is $\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$.

49. Let $z = e^{i\theta} = \cos\theta + i\sin\theta$

We have $|z - 3i| = 3$

$\Rightarrow \quad |\cos\theta + i(\sin\theta - 3)| = 3$

$\Rightarrow \quad \sqrt{\cos^2\theta + (\sin\theta - 3)^2} = 3$

$\Rightarrow \quad \cos^2\theta + (\sin\theta - 3)^2 = 9$

$\Rightarrow \quad \cos^2\theta + \sin^2\theta - 6\sin\theta + 9 = 9$

$\Rightarrow \quad 1 - 6\sin\theta = 0$

$\Rightarrow \quad \sin\theta = \frac{1}{6}$ (i)

Now, $\cot\theta - \frac{6}{z} = \cot\theta - \frac{6}{(\cos\theta + i\sin\theta)}$

$$= \cot\theta - 6(\cos\theta - i\sin\theta)$$

$$= \cot\theta - \frac{1}{\sin\theta}(\cos\theta - i\sin\theta)$$

$$= \cot\theta - \cot\theta + i$$

$$= i$$

Therefore,

$$\text{Arg}\left(\cot\theta - \frac{6}{z}\right) = \text{Arg}(i) = \frac{\pi}{2}$$

50. We have,

$$\text{Amp}\left(\frac{z-1}{z+1}\right) = \frac{\pi}{3}$$

$\Rightarrow \quad \text{Amp}\left(\frac{x+iy-1}{x+iy+1}\right) = \frac{\pi}{3}$

$\Rightarrow \quad \text{Amp}\left(\frac{(x-1)+iy}{(x+1)+iy}\right) = \frac{\pi}{3}$

$\Rightarrow \quad \text{Amp}\left(\frac{(x-1)+iy}{(x+1)+iy} \times \frac{(x+1)-iy}{(x+1)-iy}\right) = \frac{\pi}{3}$

$\Rightarrow \quad \text{Amp}\left(\frac{(x^2-1)+y^2+i(xy+y-xy+y)}{(x+1)^2+\text{y}^2}\right) = \frac{\pi}{3}$

$\Rightarrow \quad \text{Amp}\left(\frac{(x^2+y^2-1)}{(x+1)^2+y^2} + i\frac{(2y)}{(x+1)^2+y^2}\right) = \frac{\pi}{3}$

$\Rightarrow \quad \tan^{-1}\left(\frac{2y}{(x^2+y^2-1)}\right) = \frac{\pi}{3}$

$\Rightarrow \quad \frac{2y}{(x^2+y^2-1)} = \tan\left(\frac{\pi}{3}\right)$

$\Rightarrow \quad \frac{2y}{(x^2+y^2-1)} = \sqrt{3}$

$\Rightarrow \quad x^2 + y^2 - \frac{2}{\sqrt{3}}y - 1 = 0$

Hence, the locus of z is a circle.

51. We have $\frac{1}{(\sqrt{3}-i)^{25}}$.

Let $z = \left(\frac{\sqrt{3}+i}{4}\right)$

Then $\text{Arg}(z) = \tan^{-1}\left(\frac{1/4}{\sqrt{3}/4}\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$

$\Rightarrow \quad \text{Arg}(z) = \dfrac{\pi}{6}$

Hence, the angle is $\dfrac{\pi}{6}$.

52. We have, $|z^1 + z_2|^2$

$= r_1(\cos\theta_1 + i\sin\theta_2) + r_2(\cos\theta_2 + i\sin\theta_2)|^2$

$= r_1|(\cos\theta_1 + r_2\cos\theta_2) + i(r_2\sin\theta_1 + r_2\sin\theta_2)|^2$

$= r_1(\cos\theta_1 + r_2\cos\theta_2)^2 + (r_1\sin\theta_1 + r_2\sin\theta_2)^2$

$= r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)$

$\le r_1^2 + r_2^2 + 2r_1r_2$

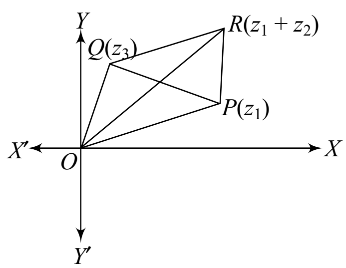
$= (r_1 + r_2)^2$

$= (|z_1| + |z_2|)^2$

$\Rightarrow \quad |z_1 + z_2| \le |z_1| + |z_2|$

Hence, the result.

Alternate method:



In ΔOPR, $OP + PR \ge OR$

$\Rightarrow \quad |z_1| + |z_2| \ge |z_1 + z_2|$

$\Rightarrow \quad |z_1| + |z_2| \le |z_1| + |z_2|$

53. In ΔOPQ, $|OP - OQ| \le PQ$

$\Rightarrow \quad ||z_1| - |z_2|| \le |z_1 - z_2|$

Hence, the result.

54. We have,

$|z_1 + z_2| = |z_1 - z_2|$

$\Rightarrow \quad |z_1 + z_2|^2 = |z_1 - z_2|^2$

$\Rightarrow \quad |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2)$
$= |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

$\Rightarrow \quad 4|z_1||z_2|\cos(\theta_1 - \theta_2) = 0$

$\Rightarrow \quad \cos(\theta_1 - \theta_2) = 0$

$\Rightarrow \quad (\theta_1 - \theta_2) = \dfrac{\pi}{2}$

$\Rightarrow \quad \text{Arg}(z_1) - \text{Arg}(z_2) = \dfrac{\pi}{2}$

55. We have,

$|z_1 + z_2| = |z_1| + |z_2|$

$\Rightarrow \quad |z_1 + z_2|^2 = (|z_1| + |z_2|)^2$

$\Rightarrow \quad |z_1|^2 + |z_2|^2 + 2|z_1||\,z_2|\cos(\theta_1 - \theta_2)$
$= (|z_1|^2 + |z_2|^2 + 2|z_1||\,z_2|)$

$\Rightarrow \quad 2|z_1||z_2|\cos(\theta_1 - \theta_2) = 2|z_1||z_2|$

$\Rightarrow \quad \cos(\theta_1 - \theta_2) = 1 = \cos(0)$

$\Rightarrow \quad \theta_1 = \theta_2$

$\Rightarrow \quad \text{Arg}(z_1) = \text{Arg}(z_2)$

Hence, the result.

56. This is possible only when $0, z_1, iz_2$ are on the same side of a straight line.

Thus, $\text{Arg}(z_1) = \text{Arg}(iz_2)$

$\Rightarrow \quad \text{Arg}(z_1) = \text{Arg}(i) + \text{Arg}(iz_2)$

$\Rightarrow \quad \text{Arg}(z_1) = \dfrac{\pi}{2} + \text{Arg}(z_2)$

$\Rightarrow \quad \text{Arg}(z_1) - \text{Arg}(z_2) = \dfrac{\pi}{2}$

Hence, the result.

57. Given,

$|z_1 - z_2| = |z_1| + |z_2|$

$\Rightarrow \quad |z_1 + z_2|^2 = (|z_1| + |z_2|)^2$

$\Rightarrow \quad |z_1|^2 + |z_2|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2)$
$= |z_1|^2 + |z_2|^2 - 2|z_1||z_2|$

$\Rightarrow \quad \cos(\theta_1 - \theta_2) = -1 = \cos(\pi)$

$\Rightarrow \quad \theta_1 - \theta_2 = \pi$

$\Rightarrow \quad \text{Arg}(z_1) - \text{Arg}(z_2) = \pi$

$\Rightarrow \quad \text{Arg}\left(\dfrac{z_1}{z_2}\right) = \pi$

$\Rightarrow \quad \left(\dfrac{z_1}{z_2}\right) = -k, k \in I^+$

$\Rightarrow \quad z_1 = -kz_2$

$\Rightarrow \quad z_1 + kz_2 = 0$

Hence, the result.

58. Given,

$|z_1 + z_2|^2 = |z_1|^2 + |z_2|^2$

$\Rightarrow \quad |z_1|^2 + |z_2|^2 + 2|z_1||z_2|\cos(\theta_1 - \theta_2) = |z_1|^2 + |z_2|^2$

$\Rightarrow \quad \cos(\theta_1 - \theta_2) = 0$

$\Rightarrow \quad \theta_1 - \theta_2 = \dfrac{\pi}{2}$

Now $\dfrac{z_1}{z_2} = \dfrac{|z_1|}{|z_2|}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))$

$= i\dfrac{|z_1|}{|z_2|}$

$\Rightarrow \quad \dfrac{z_1}{z_2}$ is purely an imaginary number.

59. We have $|z_1 - z_2|^2$

$= |z_1|^2 + |z_1|^2 - 2|z_1||z_2|\cos(\theta_1 - \theta_2)$

$= r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)$

$= r_1^2 + r_2^2 - 2r_1r_2 + 2r_1r_2 - 2r_1r_2\cos(\theta_1 - \theta_2)$

$= (r_1 - r_2)^2 + 2r_1r_2(1 - \cos(\theta_1 - \theta_2))$

$$= (r_1 - r_2)^2 + 2r_1r_2 \cdot 2\sin^2\left(\frac{\theta_1 - \theta_2}{2}\right)$$
$$\le (r_1 - r_2)^2 + 2r_1r_2 \cdot 2 \cdot \left(\frac{\theta_1 - \theta_2}{2}\right)^2$$
$$= (r_1 - r_2)^2 + r_1r_2(\theta_1 - \theta_2)^2$$
$$\le (r_1 - r_2)^2 + (\theta_1 - \theta_2)^2$$
$$\Rightarrow \quad |z_1 - z_2|^2 \le (|z_1|^2 - |z_2|^2) + (\text{Arg}(z_1) - \text{Arg}(z_2))^2$$

Hence, the result.

60. Do yourself.

61. We have,
$$|z+1| = |z+4-3|$$
$$= |(z+4)+(-3)|$$
$$\le |z+4| + |-3|$$
$$\le 3+3 = 6$$
Hence, the maximum value of $|z+1|$ is 6.

62. We have,
$$|z+2| = |(z+5)-3|$$
$$= |(z+5)+(-3)|$$
$$\ge |(z+5)+|-3| = 4-3 = 1$$
Hence, the minimum value of $|z+2|$ is 1.

63. (i) We have,
$$|z| + |z+2| \ge |z-(z+2)| = 2$$
Hence, the minimum value is 2.

(ii) We have,
$$|z+2| + |z-2| \ge |(z+2)-(z-2)| = 4$$
Hence, the minimum value is 4

64. We have,
$$|z+2| + |z-2| + |2z-7|$$
$$= |z+2| + |z-2| + |7-2z|$$
$$\le |z+2+z-2+2z-7|$$
$$= 7$$
Hence, the maximum value is 7.

65. We have,
$$|z_1+z_2| \le |z_1| + |z_2| = 10+5 = 15$$
Also,
$$|z_1+z_2| \ge |z_1| - |z_2| = 10-5 = 5$$
Thus, the greatest value is 15 and the least value is 5.

66. We have,
$$|i(z+1)+1| = |i(z-3i)+i-2|$$
$$= |i(z-3i)+(i-2)|$$
$$\le |i(z-3i)| + |(i-2)|$$
$$\le 4+\sqrt{5}$$
Hence, the maximum value is $4+\sqrt{5}$.

67. We have,
$$\left|z+\frac{1}{z}\right| \ge |z| - \frac{1}{|z|} = 3 - \frac{1}{3} = \frac{8}{3}$$
Hence, the minimum value is $\frac{8}{3}$.

Also,
$$\left|z+\frac{1}{z}\right| \le |z| + \frac{1}{|z|} = 3 + \frac{1}{3} = \frac{10}{3}$$
Hence, the maximum value is $\frac{10}{3}$.

68. Given $|z_1| = 2, |z_2| = 3, |z_3| = 5$
$$\Rightarrow \quad |z_1|^2 = 4, |z_2|^2 = 9, |z_3|^2 = 25$$
$$\Rightarrow \quad z_1\overline{z_1} = 4, z_2\overline{z_2} = 9, z_3\overline{z_3} = 25$$
$$\Rightarrow \quad \overline{z_1} = \frac{4}{z_1}, \overline{z_2} = \frac{9}{z_2}, \overline{z_3} = \frac{25}{z_3}$$
Also, $|25z_1z_2 + 9z_1z_3 + 4z_3z_2| = 90$
$$\Rightarrow \quad |25z_1z_2 + 9z_1z_3 + 4z_3z_2| = 90$$
$$\Rightarrow \quad \left|z_1z_2z_3\left(\frac{25}{z_3} + \frac{9}{z_2} + \frac{4}{z_1}\right)\right| = 90$$
$$\Rightarrow \quad |z_1z_2z_3|\left|\left(\frac{25}{z_3} + \frac{9}{z_2} + \frac{4}{z_1}\right)\right| = 90$$
$$\Rightarrow \quad |z_1||z_2||z_3||\overline{z_3} + \overline{z_2} + \overline{z_1}| = 90$$
$$\Rightarrow \quad |z_1||z_2||z_3||\overline{z_1 + z_2 + z_3}| = 90$$
$$\Rightarrow \quad |z_1||z_2||z_3||z_1 + z_2 + z_3| = 90$$
$$\Rightarrow \quad 30 \times |z_1 + z_2 + z_3| = 90$$
$$\Rightarrow \quad |z_1 + z_2 + z_3| = 3$$

69. We have,
$$3-4i = 3-2.\,2.i$$
$$= (2)2 + i^2 - 2.2.i$$
$$= (2-i)^2$$
Thus, $\sqrt{3-4i} = \pm(2-i)$

70. We have,
$$5+12i = 5+2.\,3.\,2i$$
$$= (3)^2 + (2i)^2 + 2.3.2i$$
$$= (3+2i)^2$$
Thus, $\sqrt{(5+12i)} = \pm(3+2i)$

71. We have,
$$8-6i = 8-2.\,3.i$$
$$= (3)^2 + (2i)^2 - 2.3.2i$$
$$= (3-i)^2$$
Thus, $\sqrt{8-6i} = \pm(3-i)$

72. We have,
$$3i = \frac{3}{2}(2i)$$
$$= \frac{3}{2}(2\cdot 1\cdot i)$$
$$= \frac{3}{2}(1^2 + i^2 + 2\cdot 1\cdot i)$$
$$= \frac{3}{2}(1+i)^2$$

Thus, $\sqrt{3i} = \pm\left(\frac{3}{2}(1+i)\right)$

73. We have,

$$8 - 15i = \frac{1}{2}(16 - 30i)$$
$$= \frac{1}{2}(16 - 2\cdot 5\cdot 3i)$$
$$= \frac{1}{2}[5^2 + (3i)^2 - 2\cdot 5\cdot 3i]$$
$$= \frac{1}{2}(5 - 3i)^2$$

Thus, $\sqrt{8-15i} = \pm\frac{1}{\sqrt{2}}(5-3i)$

74. We have,

$$x^2 + \frac{1}{x^2} + 4i\left(x - \frac{1}{x}\right) - 6$$
$$= \left(x - \frac{1}{x}\right)^2 + 2\cdot x\cdot\frac{1}{x} + 4i\left(x - \frac{1}{x}\right) - 6$$
$$= \left(x - \frac{1}{x}\right)^2 + 4i\left(x - \frac{1}{x}\right) - 4$$
$$= \left(x - \frac{1}{x}\right)^2 + 2\cdot\left(x - \frac{1}{x}\right)\cdot 2i - 4$$
$$= \left(x - \frac{1}{x}\right)^2 + 2\cdot\left(x - \frac{1}{x}\right)\cdot 2i + (2i)^2$$
$$= \left(x - \frac{1}{x} + 2i\right)^2$$

Thus, $\sqrt{x^2 + \frac{1}{x^2} + 4i\left(x - \frac{1}{x}\right) - 6}$

$$= \pm\left(x - \frac{1}{x} + 2i\right)$$

75. We have,

$z^2 + 5 = 12\sqrt{-1} = 12i$

$\Rightarrow$ $z^2 = -5 + 12i$

$\Rightarrow$ $z^2 = -5 + 2.2.3i$

$\Rightarrow$ $z^2 = (2 + 3i)^2$

$\Rightarrow$ $z = \pm(2 + 3i)$

Thus, the complex number z can be $(2 + 3i)$ or $(-2 - 3i)$.

76. We have,

$$(2 + 3\omega + 3\omega^2)^{2013} = (2 + 3(\omega + \omega^2))^{2013}$$
$$= (2 + 3(-1))^{2013}$$
$$= (-1)^{2013}$$
$$= -1$$

77. We have,

$$(3 + 4\omega + 5\omega^2)^{10} = (3 + 4(\omega + \omega^2) + \omega^2)^{10}$$
$$= (3 - 4 + \omega^2)^{10}$$
$$= (-1 + \omega^2)^{10}$$
$$= \left(-1 - \frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^{10}$$
$$= \left(\frac{3}{2} + i\frac{\sqrt{3}}{2}\right)^{10}$$
$$= (i\sqrt{3})^{10}\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^{10}$$
$$= -3^5\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{10}$$
$$= -3^5\omega^{10}$$

78. Let $S = \frac{1}{2} + \frac{3}{8} + \frac{9}{32} + \frac{27}{128} + \ldots$

$$= \frac{1}{2}\left(1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \ldots\right)$$
$$= \frac{1}{2}\times\frac{1}{1 - \frac{3}{4}}$$
$$= \frac{1}{2}\times\frac{4}{4-3} = 2$$

Thus, $\omega + \omega^s = \omega + \omega^2 = -1$

79. We have,

$z^2 + 2z^2 + 2z + 1 = 0$

$\Rightarrow$ $(z^3 + 1) + 2z(z + 1) = 0$

$\Rightarrow$ $(z + 1)(z^2 - z + 1) + 2z(z + 1) = 0$

$\Rightarrow$ $(z + 1)(z^2 - z + 1 + 2z) = 0$

$\Rightarrow$ $(z + 1)(z^2 + z + 1) = 0$

$\Rightarrow$ $z = -1, -\omega, -\omega^2$

When $z = -1$, then

$z^{2013} + z^{2014} + z^{2015}$

$= -1 + 1 - 1 = -1 \neq 0$

When $z = \omega$, then

$z^{2013} + z^{2014} + z^{2015}$

$= \omega^{2013} + \omega^{2014} + \omega^{2015}$

$= 1 + \omega + \omega^2 = 0$

When $z = \omega^2$, then

$z^{2013} + z^{2014} + z^{2015}$

$= 1 + \omega^2 + \omega = 0$

Hence, the common roots are ω, ω^2.

80. Let $p = -2013$.

Then $\sqrt[3]{p} = \sqrt[3]{p}, \sqrt[3]{p}\,\omega, \sqrt[3]{\pi}\,\omega^2$

So, $\alpha = \sqrt[3]{p}, \beta = \sqrt[3]{p}\,\omega, \gamma = \sqrt[3]{p}\,\omega^2$

Now, $\frac{x\alpha + y\beta + z\gamma}{x\beta + y\gamma + z\alpha}$

$$= \frac{x \cdot \sqrt[3]{p} + y \cdot \sqrt[3]{p} \cdot \omega + z \cdot \sqrt[3]{p} \cdot \omega^2}{x \cdot \sqrt[3]{p} \cdot \omega + y \cdot \sqrt[3]{p} \cdot \omega^2 + z \cdot \sqrt[3]{p}}$$

$$= \frac{x + y\omega + z\omega^2}{x\omega + y\omega^2 + z}$$

$$= \frac{x + y\omega + z\omega^2}{\omega(x + y\omega + z\omega^2)}$$

$$= \frac{1}{\omega}$$

$$= \omega^2$$

81. We have,

$$\frac{2+3\omega+4\omega^2}{4+3\omega^2+2\omega} = \frac{2+3\omega+4\omega^2}{\omega(2+3\omega+4\omega^2)}$$

$$= \frac{1}{\omega}$$

$$= \omega^2$$

82. We have,

$$\frac{5+6\omega+7\omega^2}{7+6\omega^2+5\omega} + \frac{5+6\omega+7\omega^2}{6+5\omega+7\omega^2}$$

$$= \frac{5+6\omega+7\omega^2}{\omega(5+6\omega+7\omega^2)} + \frac{5+6\omega+7\omega^2}{\omega^2(5+6\omega+7\omega^2)}$$

$$= \frac{1}{\omega} + \frac{1}{\omega^2} = \omega^2 + \omega$$

$$= -1$$

83. We have,

$$4 + 5\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{334} + 3\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{365}$$

$$= 4 + 5\omega^{334} + 3\omega^{365}$$

$$= 4 + 5\omega + 3\omega^2$$

$$= 3(1 + \omega + \omega^2) + 1 + 2\omega$$

$$= 0 + 1 + 2\omega$$

$$= 1 + 2\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$$

$$= 1 - 1 + i\sqrt{3}$$

$$= i\sqrt{3}$$

84. We have

$x^6 - 9x^3 + 8 = 0$

$\Rightarrow \quad (x^3 - 1)(x^3 - 8) = 0$

$\Rightarrow \quad (x^3 - 1) = 0, (x^3 - 8) = 0$

$\Rightarrow \quad x^3 = 1$ and $x^3 = 8$

$\Rightarrow \quad x = 1, \omega, \omega^2$ and $x = 2, 2\omega, 2\omega^2$

$\Rightarrow \quad x = 1, 2, \omega, 2\omega, \omega^2, 2\omega^2$

85. We have,

$x^2 - 2x\cos\theta + 1 = 0$

$$\Rightarrow \quad x = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2}$$

$$\Rightarrow \quad x = \frac{2\cos\theta \pm 2i\sin\theta}{2} = (\cos\theta \pm i\sin\theta)$$

When $x = \cos\theta + i\sin\theta$, then

$\Rightarrow \quad x^n = (\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$

$$\Rightarrow \quad \frac{1}{x} = \cos\theta - i\sin\theta$$

$$\Rightarrow \quad \frac{1}{x^n} = (\cos\theta - i\sin\theta)^n = \cos(n\theta) - i\sin(n\theta)$$

Thus, $x^n + \dfrac{1}{x^n} = 2\cos(n\theta)$

Similarly, we can easily prove that,

when $x = \cos\theta - i\sin\theta$, then

$$x^n + \frac{1}{x^n} = 2\cos(n\theta)$$

86. We have,

$$z_r = \cos\left(\frac{2r\pi}{5}\right) + i\sin\left(\frac{2r\pi}{5}\right)$$

$$= e^{i\left(\frac{2r\pi}{5}\right)}$$

Now,

$$z_1 z_2 z_3 z_4 z_5 = e^{i\frac{2\pi}{5}(1+2+3+4+5)}$$

$$= e^{i\frac{2\pi}{5}\times 15}$$

$$= e^{16\pi}$$

$$= \cos(6\pi) + i\sin(6\pi)$$

$$= 1 + i.0$$

$$= 1$$

87. Let $x = \cos\alpha + i\sin\alpha$,

$y = \cos\alpha + i\sin\beta$

and $z = \cos\gamma + i\sin\gamma$

Now, $x + y + z$

$= (\cos\alpha + i\sin\alpha) + (\cos\beta + i\sin\beta) + (\cos\gamma + i\sin\gamma)$

$= (\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma)$

$= 0 + i.0$

$= 0$

$\Rightarrow \quad x^3 + y^3 + z^3 = 3xyz$

$\Rightarrow \quad (\cos\alpha + i\sin\alpha)^3 + (\cos\beta + i\sin\beta)^3 + (\cos\gamma + i\sin\gamma)^3$

$= 3(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$

$\Rightarrow \quad (\cos 3\alpha + i\sin 3\alpha) + (\cos 3\beta + i\sin 3\beta) + (\cos 3\gamma + i\sin 3\gamma)$

$= 3(\cos(\alpha + \beta + \gamma) + i\sin(\alpha + \beta + \gamma))$

$\Rightarrow \quad (\cos 3\alpha + \cos 3\beta + \cos 3\gamma) + i(\sin 3\alpha + \sin 3\beta + \sin 3\gamma)$

$= 3\cos(\alpha + \beta + \gamma) + i\,3\sin(\alpha + \beta + \gamma)$

Comparing the real and imaginary parts, we get

$\cos 3\alpha + \cos 3\beta + \cos 3\gamma = 3\cos(\alpha + \beta + \gamma)$

and $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = 3\sin(\alpha + \beta + \gamma)$

Hence, the result.

88. Let $x = \cos\alpha + i\sin\alpha$,

$y = \cos\beta + i\sin\beta$

and $z = \cos\gamma + i\sin\gamma$

Now, $\frac{1}{x} + \frac{1}{y} + \frac{1}{z}$

$= \cos\alpha + i\sin\alpha + \cos\beta + i\sin\beta + \cos\gamma + i\sin\gamma$

$= (\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma)$

$= 0 - i.0$

$= 0$

$\Rightarrow \quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0$

$\Rightarrow \quad \frac{yz + xz + xy}{xyz} = 0$

$\Rightarrow \quad xy + yz + zx = 0$

Also, $(x + y + z)^2 = (x^2 + y^2 + z^2) + 2(xy + yz + zx)$

$= x^2 + y^2 + z^2$

$\Rightarrow \quad x^2 + y^2 + z^2 = 0 \quad (\because x + y + z = 0)$

$\Rightarrow \quad (\cos\alpha + i\sin\alpha)^2 + (\cos\beta + i\sin\beta)^2 + (\cos\gamma + i\sin\gamma)^2 = 0$

$\Rightarrow \quad (\cos 2\alpha + i\sin 2\alpha) + (\cos 2\beta + i\sin 2\beta) + (\cos 2\gamma + i\sin 2\gamma) = 0$

$\Rightarrow \quad (\cos 2\alpha + \cos 2\beta + \cos 2\gamma) + (\sin 2\alpha + \sin 2\beta + \sin 2\gamma) = 0 + i.0$

Comparing the real and imaginary parts, we get

$(\cos 2\alpha + \cos 2\beta + \cos 2\gamma) = 0$

and $(\sin 2\alpha + \sin 2\beta + \sin 2\gamma) = 0$

Thus, $(\cos 2\alpha + \cos 2\beta + \cos 2\gamma) = 0$

$\Rightarrow \quad (2\cos^2\alpha - 1 + 2\cos^2\beta - 1 + 2\cos^2\gamma - 1) = 0$

$\Rightarrow \quad (2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) - 3) = 0$

$\Rightarrow \quad (\cos^2\alpha + \cos^2\beta + \cos^2\gamma) = \frac{3}{2}$

and

$(1 - \sin^2\alpha) + (1 - \sin^2\beta) + (1 - \sin^2\gamma) = \frac{3}{2}$

$\Rightarrow \quad \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 3 - \frac{3}{2} = \frac{3}{2}$

Hence, the result.

89. Given $z_r = \cos\left(\frac{\pi}{2^r}\right) + i\sin\left(\frac{\pi}{2^r}\right) = e^{i\left(\frac{\pi}{2^r}\right)}$

Now, $z_1.z_2.z_3 \ldots$ to ∞

$= e^{i\left(\frac{\pi}{2}\right)} \cdot e^{i\left(\frac{\pi}{2^2}\right)} \cdot e^{i\left(\frac{\pi}{2^3}\right)} \ldots$ to ∞

$= e^{i\left(\frac{\pi}{2}\right)\left(1 + \frac{1}{2} + \frac{1}{2^2} + \ldots \text{to} \infty\right)}$

$= e^{i\left(\frac{\pi}{2}\right)\left(\frac{1}{1 - \frac{1}{2}}\right)}$

$= e^{i\left(\frac{\pi}{2}\right) \times 2i}$

$= e^{i\pi}$

$= \cos(\pi) + i\sin(\pi)$

$= -1$

90. We have,

$(1 + i) = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$

$= \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)$

$\Rightarrow \quad (1 + i)^8 = (\sqrt{2})^8\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right)^8$

$= 2^4\left(\cos\left(\frac{\pi}{4} \times 8\right) + i\sin\left(\frac{\pi}{4} \times 8\right)\right)$

$= 24(\cos(2\pi) + i\sin(2\pi))$

Similarly,

$(1 - i)^8 = 2^4(\cos(2\pi) - i\sin(2\pi))$

Therefore, $(1 + i)^8 + (1 - i)^8$

$= 2^4(\cos(2\pi) + i\sin(2\pi))$

$+ 2^4[\cos(2\pi) - i\sin(2\pi)]$

$= 2^4[2 \cdot \cos(2\pi)]$

$= 2^5 \cdot 1$

$= 32$

91. We have,

$\sum_{k=1}^{21}\left(\sin\left(\frac{2\pi k}{11}\right) - i\cos\left(\frac{2\pi k}{11}\right)\right)$

$= \sum_{k=1}^{21}\left(-i^2\sin\left(\frac{2\pi k}{11}\right) - i\cos\left(\frac{2\pi k}{11}\right)\right)$

$= \sum_{k=1}^{21} -i\left(\cos\left(\frac{2\pi k}{11}\right) + i\sin\left(\frac{2\pi k}{11}\right)\right)$

$= \sum_{k=1}^{21} -ie^{i\left(\frac{2\pi k}{11}\right)}$

$= -i\left(\sum_{k=1}^{21} e^{i\left(\frac{2\pi k}{11}\right)}\right)$

$= -i\left(e^{i\left(\frac{2\pi}{11}\right)} + e^{i\left(\frac{4\pi}{11}\right)} + e^{i\left(\frac{6\pi}{11}\right)} + \ldots + e^{e^{i\left(\frac{42\pi}{11}\right)}}\right)$

$= -ie^{i\left(\frac{2\pi}{11}\right)}\left(1 + e^{i\left(\frac{2\pi}{11}\right)} + e^{i\left(\frac{4\pi}{11}\right)} + \ldots + e^{i\left(\frac{40\pi}{11}\right)}\right)$

$= -ie^{i\left(\frac{2\pi}{11}\right)}\left(\frac{1 - e^{i\left(\frac{42\pi}{11}\right)}}{1 - e^{i\left(\frac{2\pi}{11}\right)}}\right)$

$= -i\left(\frac{e^{i\left(\frac{2\pi}{11}\right)} - e^{i\left(\frac{44\pi}{11}\right)}}{1 - e^{i\left(\frac{2\pi}{11}\right)}}\right)$

$= -i\left(\frac{e^{i\left(\frac{2\pi}{11}\right)} -}{1 - e^{i\left(\frac{2\pi}{11}\right)}}\right)$

$= -i \times -1$

$= i$

92. We have

$x^3 - 1 = 0$

$\Rightarrow \quad x^3 = 1 = \cos(2r\pi) + i\sin(2r\pi)$

$\Rightarrow \quad x = \cos\left(\frac{2r\pi}{3}\right) + i\sin\left(\frac{2r\pi}{3}\right), r = 0, 1, 2$

When $r = 0$, $x = 1$

When $r = 1$, $x = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right) = e^{i\frac{2\pi}{3}}$

When $r = 2$, $x = \cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right) = e^{i\frac{4\pi}{3}}$

Hence, the solutions are $\left\{1, e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}\right\}$.

93. We have $x^5 - 1 = 0$

$\Rightarrow \quad x^5 = 1 = \cos(2r\pi) + i\sin(2r\pi)$

$\Rightarrow \quad x = \cos\left(\frac{2r\pi}{5}\right) + i\sin\left(\frac{2r\pi}{5}\right)$

Where $r = 0, 1, 2, 3, 4$

When $r = 0$, then $x = 1$

When $r = 1$, $x = \cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right) = e^{i\frac{2\pi}{5}}$

When $r = 2$, $x = \cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right) = e^{i\frac{4\pi}{5}}$

When $r = 3$, $x = \cos\left(\frac{6\pi}{5}\right) + i\sin\left(\frac{6\pi}{5}\right)$

$= \cos\left(2\pi - \frac{4\pi}{5}\right) + i\sin\left(2\pi - \frac{4\pi}{5}\right)$

$= \cos\left(\frac{4\pi}{5}\right) - i\sin\left(\frac{4\pi}{5}\right) = e^{-i\frac{4\pi}{5}}$

When $r = 4$, $x = \cos\left(\frac{8\pi}{5}\right) + i\sin\left(\frac{8\pi}{5}\right)$

$= \cos\left(2\pi - \frac{2\pi}{5}\right) + i\sin\left(2\pi - \frac{2\pi}{5}\right)$

$= \cos\left(\frac{2\pi}{5}\right) - i\sin\left(\frac{2\pi}{5}\right) = e^{-i\frac{2\pi}{5}}$

Hence, the solutions are $\left\{1, e^{\pm i\frac{2\pi}{5}}, e^{\pm i\frac{4\pi}{5}}\right\}$,

i.e. $\left\{e^{\pm i\frac{2r\pi}{5}}\right\}$, where $r = 0, 1, 2$.

94. We have,

$x^7 - 1 = 0$

$\Rightarrow \quad x = \cos\left(\frac{2r\pi}{7}\right) + i\sin\left(\frac{2r\pi}{7}\right)$

where $r = 0, 1, 2, 3, 4, 5, 6$

$\Rightarrow \quad x = e^{i\frac{2r\pi}{7}}, r = 0, 1, 2, 3, 4, 5, 6$

Hence, the solution set is

$\left\{e^{\pm i\frac{2r\pi}{7}}\right\}$, where $r = 0, 1, 2, 3$

95. We have,

$x^3 + 1 = 0$

$\Rightarrow \quad x^3 = -1 = \cos(2r + 1)\pi + i\sin(2r + 1)\pi$

where $r = 0, 1, 2$

$\Rightarrow \quad x = \cos\left(\frac{2r+1}{3}\right)\pi + i\sin\left(\frac{2r+1}{3}\right)\pi$

When $r = 0$, $x = \cos\left(\frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{3}\right)$

$p = -\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = -\omega^2$

When $r = 1$, $x = -1$

When $r = 2$, $x = \cos\left(\frac{5\pi}{3}\right) + i\sin\left(\frac{5\pi}{3}\right)$

$= \cos\left(2\pi - \frac{\pi}{3}\right) + i\sin\left(2\pi - \frac{\pi}{3}\right)$

$= \cos\left(\frac{\pi}{3}\right) - i\sin\left(\frac{\pi}{3}\right)$

$= \frac{1}{2} - i\frac{\sqrt{3}}{2} = -\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$

$= -\omega$

Hence, the solutions are $\{-1, -\omega, -\omega^2\}$.

96. We have,

$x^5 + 1 = 0$

$\Rightarrow \quad x = \cos\left(\frac{2r+1}{5}\right)\pi + i\sin\left(\frac{2r+1}{5}\right)\pi$

Where $r = 0, 1, 2, 3, 4$.

When $r = 0$, $x = \cos\left(\frac{\pi}{5}\right) + i\sin\left(\frac{\pi}{5}\right) = e^{i\frac{\pi}{5}}$

When $r = 1$, $x = \cos\left(\frac{3\pi}{5}\right) + i\sin\left(\frac{3\pi}{5}\right) = e^{i\frac{3\pi}{4}}$

When $r = 2$, $x = -1$

When $r = 3$, $x = \cos\left(\frac{7\pi}{5}\right) + i\sin\left(\frac{7\pi}{5}\right)$

$= \cos\left(2\pi - \frac{3\pi}{5}\right) + i\sin\left(2\pi - \frac{3\pi}{5}\right)$

$= \cos\left(\frac{3\pi}{5}\right) - i\sin\left(\frac{3\pi}{5}\right) = e^{-i\frac{3\pi}{5}}$

When $r = 4$, $x = \cos\left(\frac{9\pi}{5}\right) + i\sin\left(\frac{9\pi}{5}\right)$

$x = \cos\left(2\pi - \frac{\pi}{5}\right) + i\sin\left(2\pi - \frac{\pi}{5}\right)$

$= \cos\left(\frac{\pi}{5}\right) - i\sin\left(\frac{\pi}{5}\right) = e^{-i\frac{\pi}{5}}$

Hence, the solutions are

$\left\{-1, e^{\pm i\frac{\pi}{5}}, e^{\pm i\frac{3\pi}{5}}\right\}$

i.e. $\left\{e^{\pm i\left(\frac{2r+1}{5}\right)\pi}\right\}, r = 0, 1, 2$

97. We have,

$$x^7 + 1 = 0$$

$$x = \cos\left(\frac{2r+1}{7}\right)\pi + i\sin\left(\frac{2r+1}{7}\right)\pi$$

where $r = 0, 1, 2, 3, 4, 5, 6.$

$$\left\{e^{\pm i\left(\frac{2r+1}{7}\right)\pi}\right\}, r = 0, 1, 2, 3$$

Hence, the solutions set is

$$\left\{-1, e^{\pm i\left(\frac{\pi}{7}\right)}, e^{\pm i\left(\frac{3\pi}{7}\right)}, e^{\pm i\left(\frac{5\pi}{7}\right)}\right\}$$

98. We have,

$$x^{10} - 1 = 0$$

$\Rightarrow$ $(x^5 + 1)(x^5 - 1) = 0$

$\Rightarrow$ $x^5 - 1 = 0$ and $x^5 + 1 = 0$

When $x^5 - 1 = 0$,

$$\Rightarrow \quad x = \cos\left(\frac{2r\pi}{5}\right) + i\sin\left(\frac{2r\pi}{5}\right) = e^{i\frac{2r\pi}{5}}$$

Where $r = 0, 1, 2, 3, 4$

$$\left\{e^{\pm i\frac{2r\pi}{5}}\right\} \text{ where } r = 0, 1, 2$$

When $x^5 + 1 = 0$

$$\Rightarrow \quad x = \cos\left(\frac{2r+1}{5}\right)\pi + i\sin\left(\frac{2r+1}{5}\right)\pi$$

$$\Rightarrow \quad x = e^{i\left(\frac{2r+1}{5}\right)\pi}, r = 0, 1, 2, 3, 4$$

Thus $\left\{e^{\pm i\frac{(2r+1)\pi}{5}}\right\}$ where $r = 0, 1, 2$

Hence, the solutions are

$$\left\{e^{\pm i\left(\frac{2r\pi}{5}\right)}, e^{\pm i\left(\frac{(2r+1)\pi}{5}\right)}\right\}, r = 0, 1, 2$$

99. We have,

$$x^{10} + x^5 + 1 = 0$$

$\Rightarrow$ $(x^5)^2 + (x^5) + 1 = 0$

$\Rightarrow$ $(x^5) = \omega, \omega^2$

When $(x^5) = \omega$

$$\Rightarrow \quad x^5 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$x^5 = \cos\left(2r\pi + \frac{2\pi}{3}\right) + i\sin\left(2r\pi + \frac{2\pi}{3}\right)$$

$$\Rightarrow \quad x = \cos\left(\frac{2r\pi + \frac{2\pi}{3}}{5}\right) + i\sin\left(\frac{2r\pi + \frac{2\pi}{3}}{5}\right)$$

$$= \cos\left(\frac{6r\pi + 2\pi}{15}\right) + i\sin\left(\frac{6r\pi + 2\pi}{15}\right)$$

where $r = 0, 1, 2, 3, 4$

$$\Rightarrow \quad x = e^{i\left(\frac{6r\pi + 2\pi}{15}\right)}, r = 0, 1, 2, 3, 4$$

Similarly, $(x^5) = \omega^2$ will provide us

$$x = \cos\left(\frac{(2r+1)\pi + \frac{4\pi}{3}}{5}\right) + i\sin\left(\frac{(2r+1)\pi + \frac{4\pi}{3}}{5}\right)$$

$$\Rightarrow \quad = \cos\left(\frac{3(2r+1)\pi + 4\pi}{15}\right) + i\sin\left(\frac{3(2r+1)\pi + 4\pi}{15}\right)$$

$$= \cos\left(\frac{6r\pi + 7\pi}{15}\right) + i\sin\left(\frac{6r\pi + 7\pi}{15}\right)$$

$$= e^{i\left(\frac{6r\pi + 7\pi}{15}\right)}, r = 0, 1, 2, 3, 4$$

Hence, the solutions set is

$$\left\{e^{i\left(\frac{6r\pi + 2\pi}{15}\right)}, e^{i\left(\frac{6r\pi + 7\pi}{15}\right)}\right\} r = 0, 1, 2, 3, 4$$

100. We have

$$x^{10} - x^9 + x^8 - x^7 + \ldots + x^2 - x + 1 = 0$$

$\Rightarrow$ $1 - x + x^2 - x^3 + \ldots - x^{10} = 0$

$$\Rightarrow \quad \frac{1 - (-x)^{11}}{1 - (-x)} = 0$$

$\Rightarrow$ $x^{11} = -1$

$$\Rightarrow \quad x = \cos\left(\frac{(2r+1)\pi}{11}\right) + i\sin\left(\frac{(2r+1)\pi}{11}\right)$$

$$= e^{i\left(\frac{(2r+1)\pi}{11}\right)}, r = 0, 1, 2, \ldots, 10$$

Hence , the solutions are

$$\left\{e^{\pm i\left(\frac{(2r+1)\pi}{11}\right)}\right\}, r = 0, 1, 2, 3, 4, 5$$

101. We have $z^5 + 1 = 0$

Hence, the solutions of z are

$$\left\{-1, e^{\pm i\frac{\pi}{5}}, e^{\pm i\frac{3\pi}{5}}\right\}$$

$= \{-1, \alpha, \bar{\alpha}, \beta, \bar{\beta}\}$, where $\alpha, \beta \in C$

Now, $\alpha + \bar{\alpha} = 2\cos\left(\frac{\pi}{5}\right), \alpha\cdot\bar{\alpha} = 1$

and $\beta + \bar{\beta} = 2\cos\left(\frac{3\pi}{5}\right), \beta\cdot\bar{\beta} = 1$

Thus, $z^5 + 1$

$= (z + 1)(z - \alpha)(z - \bar{\alpha})(z - \beta)(z - \bar{\beta})$

$= (z + 1)(z^2 - (\alpha + \bar{\alpha})z + \alpha\cdot\bar{\alpha})(z^2 - (\beta + \bar{\beta})z + \beta\cdot\bar{\beta})$

$$= (z+1)\left(z^2 - 2\cos\left(\frac{\pi}{5}\right)z + 1\right)\left(z^2 - 2\cos\left(\frac{3\pi}{5}\right)z + 1\right)$$

$$\Rightarrow \frac{z^5+1}{(z+1)} = \left(z^2 - 2\cos\left(\frac{\pi}{5}\right)z + 1\right)\left(z^2 - 2\cos\left(\frac{3\pi}{5}\right)z + 1\right)$$

Put $z = i$, we get,

$$\Rightarrow \frac{i+1}{(i+1)} = \left(-1 - 2\cos\left(\frac{\pi}{5}\right)i + 1\right)\left(-1 - 2\cos\left(\frac{3\pi}{5}\right)i + 1\right)$$

$$\Rightarrow \quad 1 = \left(-2\cos\left(\frac{\pi}{5}\right)i\right)\left(-2\cos\left(\frac{3\pi}{5}\right)\right)$$

$$= \left(-2\cos\left(\frac{\pi}{5}\right)i\right)\left(-2\cos\left(\frac{3\pi}{5}\right)i\right)$$

$$\Rightarrow \quad 4\cos\left(\frac{\pi}{5}\right)\cos\left(\frac{3\pi}{5}\right) = -1$$

$$\Rightarrow \quad 4\cos\left(\frac{\pi}{5}\right)\cos\left(\pi - \frac{2\pi}{5}\right) = -1$$

$$\Rightarrow \quad 4\cos\left(\frac{\pi}{5}\right)\cos\left(\frac{2p}{5}\right) = 1$$

$$\Rightarrow \quad 4\cos\left(\frac{\pi}{5}\right)\cos\left(\frac{\pi}{2} - \frac{\pi}{10}\right) = 1$$

$$\Rightarrow \quad 4\cos\left(\frac{\pi}{5}\right)\sin\left(\frac{\pi}{10}\right) = 1$$

Hence, the result.

102. We have,

$$z^7 - 1 = 0$$

$$\Rightarrow \quad z = \cos\left(\frac{2r\pi}{7}\right) + i\sin\left(\frac{2r\pi}{7}\right)$$

where $r = 0, 1, 2, 3, 4, 5, 6$

$$\Rightarrow \quad z = e^{i\frac{2r\pi}{7}}, r = 0, 1, 2, 3, 4, 5, 6$$

Hence, the solution set is

$$\left\{e^{\pm i\frac{2r\pi}{7}}\right\}, \text{ where } r = 0, 1, 2, 3$$

$$= \left\{1, e^{\pm i\frac{2\pi}{7}}, e^{\pm i\frac{4\pi}{7}}, e^{\pm i\frac{6\pi}{7}}\right\}$$

$$= \{1, \alpha, \bar{\alpha}, \beta, \bar{\beta}, \gamma, \bar{\gamma}\}$$

Thus, $\alpha + \bar{\alpha} = 2\cos\left(\frac{2\pi}{7}\right), \alpha \cdot \bar{\alpha} = 1$

$$\beta + \bar{\beta} = 2\cos\left(\frac{4\pi}{7}\right), \beta \cdot \bar{\beta} = 1$$

and $\gamma + \bar{\gamma} = 2\cos\left(\frac{6\pi}{7}\right), \gamma \cdot \bar{\gamma} = 1$

Thus, $z^7 - 1$

$$= (z - 1)(z - \alpha)(z - \bar{\alpha})(z - \beta)(z - \bar{\beta})(z - \gamma)(z - \bar{\gamma})$$

$$\Rightarrow \quad \frac{z^7 - 1}{z - 1} = (z^2 - (\alpha + \bar{\alpha})z + \alpha \cdot \bar{\alpha})$$

$$\times (z^2 - (\beta - \bar{\beta})z + \beta\bar{\beta})(z^2 - (\gamma + \bar{\gamma})(z + \gamma \cdot \bar{\gamma})$$

$$= \left(z^2 - 2\cos\left(\frac{2\pi}{7}\right)z + 1\right)$$

$$\times \left(z^2 - 2\cos\left(\frac{4\pi}{7}\right)z + 1\right)\left(z^2 - 2\cos\left(\frac{6\pi}{7}\right)z + 1\right)$$

Put $z = i$, we get

$$\frac{-(i+1)}{i-1}$$

$$= \left(-2\cos\left(\frac{2\pi}{7}\right)i\right)\left(-2\cos\left(\frac{4\pi}{7}\right)i\right)\left(-2\cos\left(\frac{6\pi}{7}\right)i\right)$$

$$= \left(-8\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\cos\left(\frac{6\pi}{7}\right)i\right)$$

$$\Rightarrow \quad \frac{(i+1)}{i-1} = \left(8\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\cos\left(\frac{6\pi}{7}\right)i\right)$$

$$\Rightarrow \quad \frac{(i+1)^2}{(i)^2 - (1)^2} = 8\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\cos\left(\frac{6\pi}{7}\right)i$$

$$\Rightarrow \quad \frac{2i}{-2} = \left(8\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\cos\left(\frac{6\pi}{7}\right)i\right)$$

$$\Rightarrow \quad 8\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\cos\left(\frac{6\pi}{7}\right) = -1$$

$$\Rightarrow \quad 8\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\cos\left(\pi - \frac{\pi}{7}\right) = -1$$

$$\Rightarrow \quad 8\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\cos\left(\frac{\pi}{7}\right) = 1$$

$$\Rightarrow \quad \cos\left(\frac{\pi}{7}\right)\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right) = \frac{1}{8}$$

103. We have

$$(1 - i)^x = 2^x$$

$$\Rightarrow \quad |(1 - i)|^x = |2^x|$$

$$\Rightarrow \quad (\sqrt{2})^x = 2^x$$

$$\Rightarrow \quad 2^{\frac{x}{2}} = 2^x$$

$$\Rightarrow \quad 2^{x - \frac{x}{2}} = 1$$

$$\Rightarrow \quad 2^{\frac{x}{2}} = 1 = 2^0$$

$$\Rightarrow \quad \frac{x}{2} = 0$$

$$\Rightarrow \quad x = 0$$

Hence, the integral solution is $\{0\}$.

104. We have,

$$z = \left(\frac{\sqrt{3} - i}{2}\right)$$

$$= i\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = i\omega^2$$

Now,

$$z^{101} + z^{103} = i^{101}\omega^{202} + i^{103}z^{206}$$

$$= i\omega + i^3\omega^2$$

$$= i\omega - i\omega^2$$

$$= i(\omega - \omega^2)$$

$$= i\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2} - \frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

$$= i(i\sqrt{3})$$

$$= -\sqrt{3}$$

105. We have,

$$\bar{z}z^3 + z\overline{z^3} = 350$$

$$\Rightarrow \quad |z|^2 z^2 + |z|^2 \overline{z^2} = 350$$

$$\Rightarrow \quad |z|^2 (z^2 + \overline{z^2}) = 350$$

$$\Rightarrow \quad 2(x^2 + y^2)(x^2 - y^2) = 350$$

$$\Rightarrow \quad (x^2 + y^2)(x^2 - y^2) = 175$$

$$\Rightarrow \quad (x^2 + y^2)(x^2 - y^2) = 25 \times 7$$

$$\Rightarrow \quad (x^2 + y^2) = 25 \text{ and } (x^2 - y^2) = 7$$

$$\Rightarrow \quad x = 4 \text{ and } y = 3$$

Thus, the area of the rectangle $= 2x \times 2y$

$= 48$ sq. u.

106. We have,

$$\sum_{m=1}^{15} \text{Im}(z^{2m-1})$$

$$= \text{Im}(z + z^3 + z^5 + \ldots + z^{29})$$

$$= \sin\theta + \sin 3\theta + \sin 5\theta + \ldots + \sin 29\theta$$

$$= \frac{\sin(15\theta)}{\sin(\theta)} \times \sin(\theta + 14\theta)$$

$$= \frac{\sin^2(15\theta)}{\sin(\theta)}$$

$$= \frac{\sin^2(30°)}{\sin(2^{ï})}$$

$$= \frac{1}{4\sin(2^{ï})}$$

107. We have,

$$z^{p+q} - z^p - z^q + 1 = 0$$

$$\Rightarrow \quad z^p(z^q - 1) - 1(z^q - 1) = 0$$

$$\Rightarrow \quad (z^p - 1)(z^q - 1) = 0$$

$$\Rightarrow \quad \text{either } (z^p - 1) = 0 \text{ or } (z^q - 1) = 0$$

$$\Rightarrow \quad \text{either } (z - 1)(1 + z + z^2 + \ldots + z^{p-1}) = 0$$

$$\text{or} \quad (z - 1)(1 + z + z^2 + \ldots + z^{q-1}) = 0$$

$$\Rightarrow \quad \text{either } (1 + z + z^2 + \ldots + z^{p-1}) = 0$$

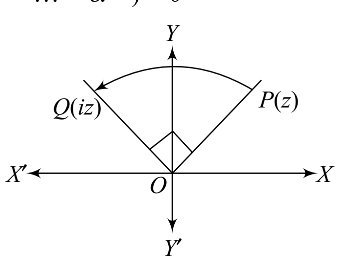
$$\text{or } (1 + z + z^2 + \ldots + z^{q-1}) = 0$$

$$\Rightarrow \quad \text{either } (1 + \alpha + \alpha^2 + \ldots + \alpha^{p-1}) = 0$$

$$\text{or} \quad (1 + \alpha + \alpha^2 + \ldots + \alpha^{q-1}) = 0$$

108. Let $z = 3 + 4i$.

If it is rotated through an angle of 90°, then it is converted to $ze^{i\frac{\pi}{2}} = iz$



Thus, $iz = i(3 + 4i) = 3i - 4 = -4 + 3i = (-4, 3)$.

Hence, the new position of P is $(-4, 3)$.

109. Let $z = 3 + 4i$.

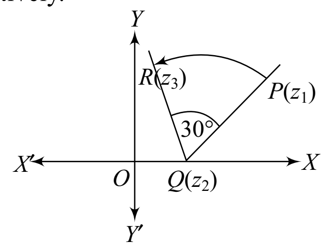
If it is rotated through an angle of 180°, then it is converted to $ze^{i\pi} = -z$

Thus, $-z = -(3 + 4i) = (-3 - 4i) = (-3, -4)$.

Hence, the new position of Q is $(-3, -4)$

110. Let the new position of P be R.

Also, let P, Q, R represent the complex numbers z_1, z_2, z_3 respectively.



By Coni method,

$$\left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \left|\frac{z_3 - z_2}{z_1 - z_2}\right| \times e^{i\frac{\pi}{6}}$$

$$\Rightarrow \quad \left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \frac{|z_3 - z_2|}{|z_1 - z_2|} \times e^{i\frac{\pi}{6}}$$

$$\Rightarrow \quad \left(\frac{z_3 - z_2}{z_1 - z_2}\right) = e^{i\frac{\pi}{6}} = \cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)$$

$$\Rightarrow \quad \left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \frac{\sqrt{3}}{2} + \frac{i}{2}$$

$$\Rightarrow \quad |z_3 - z_2| = 2(1 + 2i) \times \frac{1}{2}(\sqrt{3} + i)$$

$$\Rightarrow \quad = (1 + 2i)(\sqrt{3} + i)$$

$$\Rightarrow \quad = (\sqrt{3} + i + 2\sqrt{3}i - 2)$$

$$\Rightarrow \quad z_3 = z_2 + (\sqrt{3} - 2) + i(1 + 2\sqrt{3})$$

$$= 1 + (\sqrt{3} - 2) + i(1 + 2\sqrt{3})$$

$$= (\sqrt{3} - 1) + i(1 + 2\sqrt{3})$$

$$= ((\sqrt{3} - 1), (1 + 2\sqrt{3}))$$

Hence, the new position of P is

$$[(\sqrt{3} - 1), (1 + 2\sqrt{3})]$$

111. Let $z_1 = \sqrt{3} + i$ and $z_2 = -1 + i\sqrt{3}$

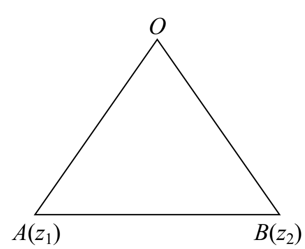
Now, $z_2 = -1 + i\sqrt{3} = i(i + \sqrt{3}) = iz_1$

Again, $i = 0 + i \cdot 1 = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = e^{i\frac{\pi}{2}}$

Thus, $\theta = \frac{\pi}{2}$

112. As we know that if z_1, z_2, z_3 are the vertices of an equilateral triangle, then

$$z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_2z_3 - z_3z_1 = 0$$



Putting $z_3 = 0$, we get,

$$z_1^2 + z_2^2 = z_1 z_2$$

Hence, the result.

113. Let z_1 and z_2 are the roots of $z^2 + az + b = 0$.

Then $z_1 + z_2 = -a$, $z_1 z_2 = b$

As we know that , if $0, z_1, z_2$ represent the vertices of an equilateral triangle, then

$$z_1^2 + z_2^2 = z_1 z_2$$

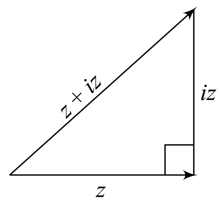
$\Rightarrow \quad (z_1 + z_2)^2 - 2z_1 z_2 = z_1 z_2$

$\Rightarrow \quad (z_1 + z_2)^2 = 3z_1 z_2$

$\Rightarrow \quad (-a)^2 = 3b$

$\Rightarrow \quad a^2 = 3b$

114. Clearly, it is a right-angled triangle.



Thus,

Area of the given triangle

$$= \frac{1}{2} \times |z| \times |iz|$$

$$= \frac{1}{2} \times |z|^2 \times |i|$$

$$= \frac{1}{2} \times |z|^2$$

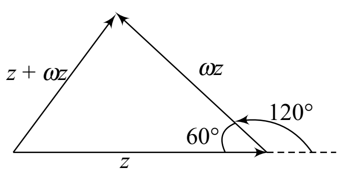
It is given that, $\frac{1}{2} \times |z|^2 = 50$

$\Rightarrow \quad |z|^2 = 100$

$\Rightarrow \quad |z| = 10$

Hence, the value of $|z|$ is 10.

115.



We have $|\omega z| = |\omega||z| = |z|$

Also, $|z + \omega z|$

$= |(1 + \omega)||z|$

$= |-\omega^2||z| = |z|$

Thus, it will form an equilateral triangle.

Hence,

$$\text{Area} = \frac{\sqrt{3}}{4} \times (\text{side})^2$$

$\Rightarrow \quad \frac{\sqrt{3}}{4} \times (\text{side})^2 = 16\sqrt{3}$

$\Rightarrow \quad \frac{\sqrt{3}}{4} \times |z|^2 = 16\sqrt{3}$

$\Rightarrow \quad |z|^2 = 64$

$\Rightarrow \quad |z| = 8$

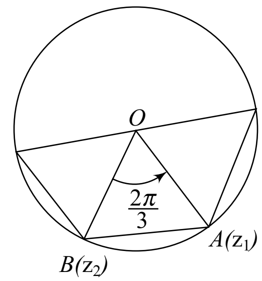
Thus, the value of

$(|z|^2 + |z| + 2) = 64 + 8 + 2 = 74$ sq. u.

116. Let $z = (1)^{1/n} = [\cos(2r\pi) + i \sin(2r\pi)]^{1/n}$

$$= \left(\cos\left(\frac{2r\pi}{n}\right) + i \sin\left(\frac{2r\pi}{n}\right)\right)$$

where $r = 0, 1, 2, 3, \ldots, (n - 1)$



Let

$$z_1 = 1 \text{ and } z_2 = e^{i\frac{2k\pi}{n}}.$$

It is given that,

$$(z_2 - 0) = (z_1 - 0)e^{i\frac{\pi}{2}}$$

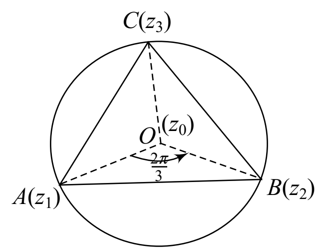
$\Rightarrow \quad e^{i\frac{2k\pi}{n}} = e^{i\frac{\pi}{2}}$

$\Rightarrow \quad \frac{2k\pi}{n} = \frac{\pi}{2}$

$\Rightarrow \quad n = 4k$

Hence, the result.

117.



We have,

$$\left(\frac{z_2 - z_0}{z_1 - z_0}\right) = \left|\frac{z_2 - z_0}{z_1 - z_0}\right| \times e^{i\frac{2\pi}{3}}$$

$\Rightarrow \quad \left(\frac{z_2 - z_0}{z_1 - z_0}\right) = \frac{|z_2 - z_0|}{|z_1 - z_0|} \times e^{i\frac{2\pi}{3}}$

$\Rightarrow \quad \left(\frac{z_2}{z_1}\right) = e^{i\frac{2\pi}{3}}$

$\Rightarrow \quad z_2 = z_1 e^{i\frac{2\pi}{3}}$

$\Rightarrow \quad z_2 = (1 + i\sqrt{3})\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$

$\Rightarrow \quad z_2 = \frac{1}{2}(1 + i\sqrt{3})(-1 + i\sqrt{3})$

$\Rightarrow \quad z_2 = -\frac{1}{2}(1 + 3) = -2$

Also, $\left(\frac{z_3 - z_0}{z_2 - z_0}\right) = \left|\frac{z_3 - z_0}{z_2 - z_0}\right| \times e^{i\frac{2\pi}{3}}$

$\Rightarrow \quad \left(\frac{z_3 - z_0}{z_2 - z_0}\right) = \frac{|z_3 - z_0|}{|z_2 - z_0|} \times e^{i\frac{2\pi}{3}}$

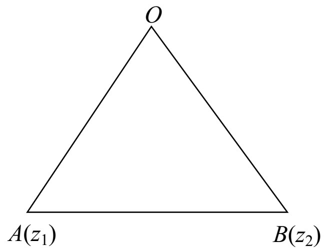
$\Rightarrow \quad \left(\frac{z_3}{z_2}\right) = e^{i\frac{2\pi}{3}}$

$\Rightarrow \quad z_3 = z_2 \times e^{i\frac{2\pi}{3}} = -2\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$

$\Rightarrow \quad z_3 = -2\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) = (1 - i\sqrt{3})$

118. As we know that, if z_1, z_2, 0 represent the vertices of an equilateral triangle, then

$$z_1^2 + z_2^2 = z_1 z_2$$



$\Rightarrow \quad (a+i)^2 + (1+ib)^2 = (a+i)(1+ib)$

$\Rightarrow \quad (a^2 - 1 + 2ia) + (1 - b^2 + 2ib) = (a-b) + i(1+ab)$

$\Rightarrow \quad (a^2 - b^2 + 2i(a-b)) = (a-b) + i(1+ab)$

Comparing the real and imaginary parts, we get,

$(a^2 - b^2) = (a-b)$ and $2(a+b) = (1+ab)$

$\Rightarrow \quad a = b$ and $2(a+b) = 1 + ab$

$\Rightarrow \quad a^2 - 4a + 1 = 0$

$\Rightarrow \quad (a-2)^2 = (\sqrt{3})^2$

$\Rightarrow \quad (a-2) = \pm\sqrt{3}$

$\Rightarrow \quad a = 2 \pm \sqrt{3}$

$\Rightarrow \quad a = 2 - \sqrt{3} = b$

119. Let $z = \cos\theta + i\sin\theta$

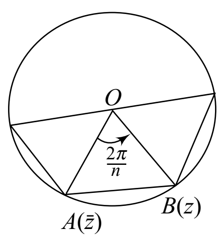
Then $\bar{z} = \cos\theta - i\sin\theta$

We have $\left(\dfrac{z - z_0}{\bar{z} - z_0}\right) = \left|\dfrac{z - z_0}{\bar{z} - z_0}\right| \times e^{i\frac{2\pi}{n}}$

$\Rightarrow \quad \left(\dfrac{z - z_0}{\bar{z} - z_0}\right) = \dfrac{|z - z_0|}{|\bar{z} - z_0|} \times e^{i\frac{2\pi}{n}}$

$\Rightarrow \quad \left(\dfrac{z}{\bar{z}}\right) = e^{i\frac{2\pi}{n}}$

$\Rightarrow \quad e^{i2\theta} = e^{i\frac{2\pi}{n}}$



$\Rightarrow \quad 2\theta = \dfrac{2\pi}{n}$

$\Rightarrow \quad \theta = \dfrac{\pi}{n}$ …(i)

Also, $\dfrac{\operatorname{Im}(z)}{\operatorname{Re}(z)} = \sqrt{2} - 1$

$\Rightarrow \quad \tan\theta = \sqrt{2} - 1 = \tan\left(\dfrac{\pi}{8}\right)$

$\Rightarrow \quad \theta = \dfrac{\pi}{8}$

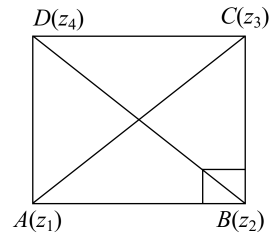
$\Rightarrow \quad \dfrac{\pi}{n} = \dfrac{\pi}{8}$ [from Eq. (i)]

$\Rightarrow \quad n = 8$

Hence, the value of n is 8.

120. Let $A(z_1)$, $B(z_2)$, $C(z_3)$ and $D(z_4)$ represent the vertices of a square $ABCD$ respectively.

We have $\left(\dfrac{z_3 - z_2}{z_1 - z_2}\right) = \left|\dfrac{z_3 - z_2}{z_1 - z_2}\right| \times e^{-i\frac{\pi}{2}}$



$\Rightarrow \quad \left(\dfrac{z_3 - z_2}{z_1 - z_2}\right) = e^{-i\frac{\pi}{2}}$

$\Rightarrow \quad \left(\dfrac{z_3 - z_2}{z_1 - z_2}\right) = -i$

$\Rightarrow \quad (z_3 - z_2) = -i(z_1 - z_2)$

$\Rightarrow \quad (1+i)z_2 = z_3 + iz_1$

$= 3 - 2i - 2 + 2i = 0$

$\Rightarrow \quad z_2 = \dfrac{0}{(1+i)} = 0 = (0, 0)$

The diagonals AC and BD bisect each other and they intersect at O, whose coordinates are $\left(\dfrac{5}{2}, \dfrac{1}{2}\right)$.

Let $z_4 = \alpha + i\beta = (\alpha, \beta)$

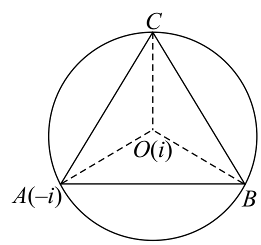
Now, $\dfrac{\alpha + 0}{2} = \dfrac{5}{2} \quad \Rightarrow \quad \alpha = 5$

and $\dfrac{\beta + 0}{2} = \dfrac{1}{2} \quad \Rightarrow \quad \beta = 1$

Thus, the coordinates of

$B = (0, 0)$ and $D = (5, 1)$

121. Let $A(z_1)$, $B(z_2)$, $C(z_3)$ represent the vertices of an equilateral triangle and its centre is $z_0 = i$.



We have,

$\left(\dfrac{z_2 - z_0}{z_1 - z_0}\right) = \left|\dfrac{z_2 - z_0}{z_1 - z_0}\right| \times e^{i\frac{2\pi}{3}}$

$\Rightarrow \quad \left(\dfrac{z_2 - z_0}{z_1 - z_0}\right) = \dfrac{|z_2 - z_0|}{|z_1 - z_0|} \times e^{i\frac{2\pi}{3}}$

$\Rightarrow \quad \left(\dfrac{z_2 - z_0}{z_1 - z_0}\right) = e^{i\frac{2\pi}{3}}$

$\Rightarrow \quad (z_2 - z_0) = (z_1 - z_0) \times e^{i\frac{2\pi}{3}}$

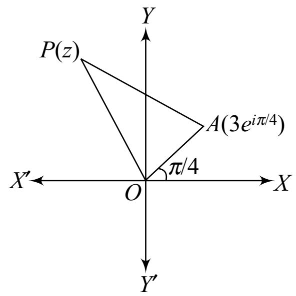
$\Rightarrow \quad z_2 = z_0 + (z_1 - z_0) \times e^{i\frac{2\pi}{3}}$

$\Rightarrow \quad z_2 = i + (-2i) \times \left(-\dfrac{1}{2} + \dfrac{i\sqrt{3}}{2}\right)$

$\Rightarrow \quad z_2 = i + i(1 - i\sqrt{3}) = 2i + \sqrt{3} = (\sqrt{3}, 2)$

Similarly, we can easily prove that $C = (-\sqrt{3}, 2)$.

122. Let $OA = 3$, so that the complex number A is $3e^{i\frac{\pi}{4}}$.



Let the point P be the complex number z.
Then by the rotation theorem, we have

$$\left(\frac{z-3e^{i\pi/4}}{0-3e^{i\pi/4}}\right)=\frac{4}{3}e^{-i\pi/2}=-\frac{4i}{3}$$

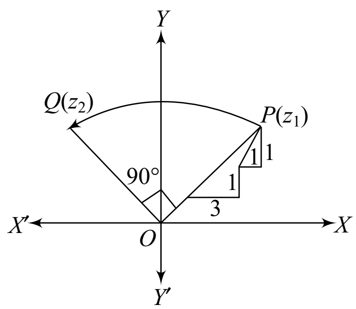
$\Rightarrow\quad 3(z-3e^{i\pi/4}) = -4i(-3e^{i\pi/4})$
$\qquad = 12ie^{i\pi/4}$

$\Rightarrow\quad 3z - 9e^{i\pi/4} = 12ie^{i\pi/4}$

$\Rightarrow\quad z - 3e^{i\pi/4} = 4ie^{i\pi/4}$

$\Rightarrow\quad z = (3+4i)e^{i\pi/4}$

123. Let the point P represents the complex number z_1 and Q be z_2.



Here, $z_1=\left(6+\sqrt{2}\cos\left(\frac{\pi}{4}\right), 5+\sqrt{2}\sin\left(\frac{\pi}{4}\right)\right)$
$= (7, 6)$

By rotation theorem, about the origin

$$\frac{z_2-0}{z_1-0}=\left|\frac{z_2-0}{z_1-0}\right|e^{i\pi/2}=e^{i\pi/2}=i$$

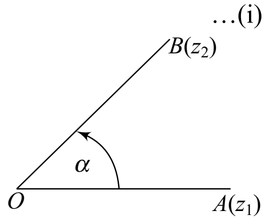
$\Rightarrow\quad z_2 = iz_1 = i(7+6i) = -6+7i = (-6, 7)$

124. Given $z^2 + pz + q = 0$

Let its roots are z_1, z_2.

$z_1 + z_2 = -p,\ z_1z_2 = q$ …(i)

$$\frac{z_2-0}{z_1-0}=\frac{OA}{OB}e^{i\alpha}=e^{i\alpha}$$



$\Rightarrow\quad \frac{z_2}{z_1}=e^{i\alpha}$

$\Rightarrow\quad z_2 = z_1e^{i\alpha}$ …(ii)

From Eqs (i) and (ii), we get

$$\frac{p^2e^{i\alpha}}{(1+e^{i\alpha})^2}=q$$

$$\Rightarrow\quad \frac{p^2e^{i\alpha}}{[(1+\cos\alpha)+i\sin\alpha]^2}=q$$

$$\Rightarrow\quad \frac{p^2e^{i\alpha}}{\left[2\cos\left(\frac{\alpha}{2}\right)\left(\cos\left(\frac{\alpha}{2}\right)+i\sin\left(\frac{\alpha}{2}\right)\right)\right]^2}=q$$

$$\Rightarrow\quad \frac{p^2e^{i\alpha}}{\left(2\cos\left(\frac{\alpha}{2}\right)\left(e^{i\frac{\alpha}{2}}\right)\right)^2}=q$$

$$\Rightarrow\quad \frac{p^2e^{i\alpha}}{\left[4\cos^2\left(\frac{\alpha}{2}\right)(e^{i\alpha})\right]}=q$$

$$\Rightarrow\quad p^2=4q\cos^2\left(\frac{\alpha}{2}\right)$$

Hence, the result.

125. As we know that, if $\operatorname{Arg}\left(\frac{z_2-z}{z_1-z}\right)=\alpha$, where $\alpha = 0, \pi$, the locus of z is a circle.

Hence, the locus of z in $\operatorname{Arg}\left(\frac{z-1}{z+1}\right)=\frac{\pi}{4}$ is a circle.

126. As we know that the locus of z is an ellipse, if $|z-z_1|+|z-z_2| = 2a$, where $2a > |z_1-z_2|$ and $a \in R^+$.

Hence, the locus of z in $|z-1|+|z+1| \le 4$ is an ellipse.

127. As we know that, the locus of z is a straight line if
$|z-z_1|+|z-z_2| = |z_1-z_2|$.

Hence, the locus of z in $|z-2|+|z+2| \le 4$ is a straight line.

128. Let $z = x + iy$.

Thus, $x+iy=t+5+i\sqrt{4-t^2}$

Comparing the real and imaginary parts, we get

$x=t+5,\ y=\sqrt{4-t^2}$

$\Rightarrow\quad x-5=t,\ y=\sqrt{4-t^2}$

Eliminating t, we get

$(x-5)^2+y^2=t^2+4-t^2$

$\Rightarrow\quad (x-5)^2+y^2=4$

Hence, the locus of z is a circle.

129. Let $z = x + iy$.

$$\text{Now, }\frac{(x+iy)^2}{x+iy-1}=\frac{(x^2-y^2+i\,2xy)}{(x-1)+iy}$$

$$=\frac{(x^2-y^2+i\,2xy)}{(x-1)+iy}\times\frac{(x-1)-iy}{(x-1)-iy}$$

$$=\frac{(x^2-y^2)(x-1)+4xy^2}{(x-1)^2+y^2}+i\,\frac{2(x-1)xy-y(x^2-y^2)}{(x-1)^2+y^2}$$

Since the given complex number is always real, so its imaginary part is zero.

Thus $\dfrac{(2(x-1)xy - y(x^2-y^2))}{(x-1)^2+y^2} = 0$

$\Rightarrow \quad 2(x-1)xy - y(x^2-y^2) = 0$

$\Rightarrow \quad 2(x-1)x - (x^2-y^2) = 0$

$\Rightarrow \quad 2x^2 - 2x - (x^2-y^2) = 0$

$\Rightarrow \quad x^2 + y^2 - 2x = 0$

$\Rightarrow \quad (x-1)^2 + y^2 = 1$

Thus, the locus of z is a circle.

130. Let $z = x + iy$.

Now,

$$\frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)}$$

$$= \frac{x-iy}{x^2+y^2}$$

$$= \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

It is given that, $\text{Re}\left(\dfrac{1}{z}\right) = c$

$\Rightarrow \quad \dfrac{x}{x^2+y^2} = c$

$\Rightarrow \quad (x^2+y^2) - \dfrac{x}{c} = 0$

Hence, the locus of z represents an equation of a circle.

131. Let $z = x + iy$.

Then $|(x+iy)^2 - 1| = (x^2+y^2) + 1$

$\Rightarrow \quad |(x^2-y^2-1) + i(2xy)| = (x^2+y^2) + 1$

$\Rightarrow \quad \sqrt{(x^2-y^2-1)^2 + 4x^2y^2} = ((x^2+y^2)+1)$

$\Rightarrow \quad (x^2-y^2-1)^2 + 4x^2y^2 = ((x^2+y^2)+1)^2$

$\Rightarrow \quad (x^2-y^2)^2 - 2(x^2-y^2) + 1 + 4x^2y^2$
$= ((x^2+y^2)^2 + 2(x^2+y^2) + 1)$

$\Rightarrow \quad ((x^2+y^2)^2 - 2(x^2-y^2) + 1)$
$= ((x^2+y^2)^2 + 2(x^2+y^2) + 1)$

$\Rightarrow \quad 4x^2 = 0$

$\Rightarrow \quad x = 0$

which represents a straight line.

Level III

1. Given $a = \dfrac{1+i}{\sqrt{2}}$

$\Rightarrow \quad a^2 = \dfrac{(1+i)^2}{2}$

$\Rightarrow \quad a^2 = \dfrac{1+i^2+2i}{2} = \dfrac{2i}{2} = i$

Now, $a^6 + a^4 + a^2 + 1 = (a^2)^3 + (a^2)^2 + a^2 + 1$
$= i^3 + i^2 + i + 1$
$= 0$

2. We have,

$x = -1 + 2i$

$\Rightarrow \quad (x+1) = 2i$

$\Rightarrow \quad (x+1)^4 = (2i)^4$

$\Rightarrow \quad x^4 + 4x^3 + 6x^2 + 4x + 1 = 16i^4$

$\Rightarrow \quad x^4 + 4x^3 + 6x^2 + 4x + 1 = 16$

$\Rightarrow \quad x^4 + 4x^3 + 6x^2 + 4x + 10 = 16 + 9 = 25$

3. Given $x = \dfrac{1+i\sqrt{3}}{2} = -\omega^2$

We have,

$y = x^4 - x^2 + 6x - 4$
$= (-\omega^2)^4 - (-\omega^2)^2 + 6(-\omega^2) - 4$
$= \omega^8 - \omega^4 - 6\omega^2 - 4$
$= \omega^2 - \omega - 6\omega^2 - 4$
$= -4 - \omega - 5\omega^2$
$= -(1 + \omega + \omega^2) - 3 - 4\omega^2$
$= -3 - 4\omega^2$
$= -3 - 4\left(-\dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}\right)$
$= -3 + 2 + i2\sqrt{3}$
$= -1 + i2\sqrt{3}$

4. We have $(i+i)^{6n} + (1-i)^{6n}$

Put $n = 1$, we get

$(i+1)^6 + (1-i)^6 = 2(1 + {}^6C_2 i^2 + {}^6C_4 i^4 + {}^6C_6 i^6)$
$= 2(1 - {}^6C_2 + {}^6C_4 - {}^6C_6)$
$= 2(1 - {}^6C_2 + {}^6C_2 - 1) = 0$

Put $n = 3$, we get

$(1+i)^{18} + (1-i)^{18}$
$= 2[1 + {}^{18}C_2 i^2 + {}^{18}C_4 i^4 + {}^{18}C_6 i^6 + {}^{18}C_{10} i^{10} + {}^{18}C_{12} i^{12} + {}^{18}C_{14} i^{14} + {}^{18}C_{16} i^{16} + {}^{18}C_{18} i^{18}]$
$= 2[1 - {}^{18}C_2 + {}^{18}C_4 - {}^{18}C_6 + {}^{18}C_8 - {}^{18}C_{10} + {}^{18}C_{12} - {}^{18}C_{14} + {}^{18}C_{16} - {}^{18}C_{18}]$
$= 2[1 - {}^{18}C_2 + {}^{18}C_4 - {}^{18}C_6 + {}^{18}C_8 - {}^{18}C_8 + {}^{18}C_6 - {}^{18}C_4 + {}^{18}C_2]$
$= 2 \times 0$

Hence, the value of $(1+i)^{6n} + (1-i)^{6n} = 0$

5. We have,

$$a + ib = \frac{3}{2 + \cos\theta + i\sin\theta}$$

$$= \frac{3}{(2+\cos\theta) + i\sin\theta}$$

$$= \frac{3}{(2+\cos\theta) + i\sin\theta} \times \frac{(2+\cos\theta) - i\sin\theta}{(2+\cos\theta) - i\sin\theta}$$

$$= \frac{3((2+\cos\theta) - i\sin\theta)}{(2+\cos\theta)^2 + \sin^2\theta}$$

$$= \frac{3((2+\cos\theta) + i\sin\theta)}{4 + 4\cos\theta + \cos^2\theta + \sin^2\theta}$$

$$= \frac{3[(2+\cos\theta) + i\sin\theta]}{5 + 4\cos\theta}$$

$$\Rightarrow \quad a = \frac{3(2+\cos\theta)}{5+4\cos\theta}, b = \frac{-3\sin\theta}{5+4\cos\theta}$$

$$\Rightarrow \quad a^2+b^2 = \frac{9(2+\cos\theta)^2+9\sin^2\theta}{(5+4\cos\theta)^2}$$

$$= \frac{9(5+4\cos\theta)}{(5+4\cos\theta)^2}$$

$$= \frac{9}{(5+4\cos\theta)}$$

$$\Rightarrow \quad a^2+b^2+3 = \frac{9}{(5+4\cos\theta)}+3$$

$$= \frac{9+15+12\cos\theta}{(5+4\cos\theta)}$$

$$= \frac{24+12\cos\theta}{(5+4\cos\theta)}$$

$$= \frac{12(2+\cos\theta)}{(5+4\cos\theta)}$$

$$= 4\cdot\frac{3(2+\cos\theta)}{(5+4\cos\theta)}$$

$$= 4a$$

$\Rightarrow \quad a^2+b^2 = 4a-3$

$\Rightarrow \quad M = 4, N = -3$

Thus, $M+N+10 = 4-3+10 = 11$

6. The given equation is

$$z^2-(5+2i)z+(21+i) = 0$$

$$\Rightarrow \quad z = \frac{(5+2i)\pm\sqrt{(5+2i)^2-4(21+i)}}{2}$$

$$= \frac{1}{2}[(5+2i)\pm\sqrt{16i-63}]$$

$$= \frac{1}{2}[(5+2i)\pm\sqrt{-(63-16i)}]$$

$$= \frac{1}{2}[(5+2i)\pm\sqrt{-(8-i)^2}]$$

$$= \frac{1}{2}[(5+2i)\pm i(8-i)]$$

$$= \frac{1}{2}[(5+2i)\pm(1+8i)]$$

$$= 3+5i, 2-3i.$$

7. Let the two roots of $x^2+cx+d = 0$ are α and $i\beta$

Then $\alpha+i\beta = -c$

$\Rightarrow \quad \overline{(\alpha+i\beta)} = \overline{-c}$

$\Rightarrow \quad (\alpha-i\beta) = \overline{-c}$

$\Rightarrow \quad 2\alpha = -(c+\overline{c})$

and $2i\beta = -(c-\overline{c})$

Therefore, $4i\alpha\beta = (c+\overline{c})(c-\overline{c}) = c^2-\overline{c^2}$

$\Rightarrow \quad 4d = c^2-\overline{c^2}$

$\Rightarrow \quad c^2-\overline{c^2} = 4d$

Hence, the result.

8. Given,

$$z\overline{z}^3+z^3\overline{z} = 350$$

$\Rightarrow \quad z\overline{z}(\overline{z}^2+z^2) = 350$

$\Rightarrow \quad 2(x^2-y^2)(x^2+y^2) = 350$

$\Rightarrow \quad (x^2-y^2)(x^2+y^2) = 175$

$= 7\times 25$

$\Rightarrow \quad (x^2-y^2) = 7$ and $(x^2+y^2) = 25$

Thus $x = 4, y = 3$

Hence, the area of a rectangle $= (2x.2y) = (4xy)$ $= 48$ sq. u.

9. It is given that

$$\overline{(\sin x+i\cos 2x)} = \cos x - i\sin 2x$$

$\Rightarrow \quad \sin x - i\cos 2x = \cos x - i\sin 2x$

$\Rightarrow \quad \sin x = \cos x$ and $\sin 2x = \cos 2x$

$\Rightarrow \quad \tan x = 1$ and $\tan 2x = 1$

$\Rightarrow \quad x = \varphi$

Thus no value of x satisfies the given equation.

10. Given,

$$z^3-iz^2-2iz-2 = 0$$

$\Rightarrow \quad z^2(z-i)-2i(z-i) = 0$

$\Rightarrow \quad (z-i)(z^2-2i) = 0$

$\Rightarrow \quad z = i$ and $z^2 = 2i$

$= i$ and $\pm 2\sqrt{i}$

11. Given $\frac{p}{a}+\frac{q}{b}+\frac{r}{c} = 1+i$

$$\Rightarrow \quad \left(\frac{p}{a}+\frac{q}{b}+\frac{r}{c}\right)^2 = (1+i)^2$$

$$\Rightarrow \quad \left(\frac{p}{a}+\frac{q}{b}+\frac{r}{c}\right)^2 = 1+i^2+2i = 2i$$

$$\Rightarrow \quad \frac{p^2}{a^2}+\frac{q^2}{b^2}+\frac{r^2}{c^2}+2\left(\frac{pq}{ab}+\frac{qr}{bc}+\frac{rp}{ca}\right) = 2i$$

$$\Rightarrow \quad \frac{p^2}{a^2}+\frac{q^2}{b^2}+\frac{r^2}{c^2}+2\frac{pqr}{abc}\left(\frac{c}{r}+\frac{a}{p}+\frac{b}{q}\right) = 2i$$

$$\Rightarrow \quad \frac{p^2}{a^2}+\frac{q^2}{b^2}+\frac{r^2}{c^2}+2\frac{pqr}{abc}\times 0 = 2i$$

$$\Rightarrow \quad \frac{p^2}{a^2}+\frac{q^2}{b^2}+\frac{r^2}{c^2} = 2i$$

12. Given $|z_2| = 2, |z_3| = 3$

$\Rightarrow \quad |z_2|^2 = 4, |z_3|^2 = 9$

$\Rightarrow \quad z_2\cdot\overline{z_2} = 4, z_3\cdot\overline{z_3} = 9$

$$\Rightarrow \quad \overline{z_2} = \frac{4}{z_2}, \overline{z_3} = \frac{9}{z_3}$$

Now, $|9z_1z_2+4z_1z_2+z_1z_2| = 12$

$$\Rightarrow \quad \left|z_1z_2z_3\left(\frac{9}{z_3}+\frac{4}{z_2}+\frac{1}{z_1}\right)\right| = 12$$

$\Rightarrow \quad |z_1 z_2 z_3|\left|\left(\frac{9}{z_3}+\frac{4}{z_2}+\frac{1}{z_1}\right)\right| = 12$

$\Rightarrow \quad |z_1 z_2 z_3||(\overline{z_3}+\overline{z_2}+\overline{z_1})| = 12$

$\Rightarrow \quad |z_1||z_2||z_3||\overline{z_1+z_2+z_3}| = 12$

$\Rightarrow \quad 6.|z_1+z_2+z_3| = 12$

$\Rightarrow \quad |z_1+z_2+z_3| = 2$

Hence, the value of $|z_1+z_2+z_3|$ is 2.

13. We have,

$(\sqrt{3}+i)^{100} = 2^{99}(a+ib)$

$\Rightarrow \quad \left(\frac{\sqrt{3}+i}{2}\right)^{100} = \frac{1}{2}(a+ib)$

$\Rightarrow \quad \left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^{100} = \frac{1}{2}(a+ib)$

$\Rightarrow \quad \left|\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^{100}\right| = \left|\left(\frac{a+ib}{2}\right)\right|$

$\Rightarrow \quad \left|\left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)\right|^{100} = \frac{1}{2}|(a+ib)|$

$\Rightarrow \quad 1 = \frac{1}{2}(\sqrt{a^2+b^2})$

$\Rightarrow \quad (a^2+b^2) = 4$

14. We have,

$(a+ib)^{2008} = (a-ib)$

$\Rightarrow \quad (a+ib)^{2008}\times(a-ib) = (a-ib)\times(a+ib)$

$\Rightarrow \quad (a+ib)^{2009} = (a^2+b^2)$

$\Rightarrow \quad (a+ib)^{2009} = \lambda, \lambda \in R^+$

$\Rightarrow \quad z^{2009} = \lambda, z = a+ib$

Clearly, it has 2010 roots.

Thus, the number of ordered pairs of (a, b) is 2010.

15. Given,

$|z+6| = |2z+3|$

$\Rightarrow \quad |x+iy+6| = 2|x+iy+3|$

$\Rightarrow \quad |(x+6)+iy| = |(2x+3)+i.2y|$

$\Rightarrow \quad |(x+6)+iy^2| = |(2x+3)+i.2y|^2$

$\Rightarrow \quad (x+6)^2+y^2 = (2x+3)+4y^2$

$\Rightarrow \quad x^2+12x+36+y^2 = 4x^2+12x+9+4y^2$

$\Rightarrow \quad 3x^2+3y^2 = 27$

$\Rightarrow \quad x^2+y^2 = 9$

Hence, the locus of z is $x^2+y^2 = 9$.

16. Given

$|z| = 1$

$\Rightarrow \quad |z|^2 = 1$

$\Rightarrow \quad z\cdot\bar{z} = 1$

$\Rightarrow \quad \bar{z} = \frac{1}{z}$

Now, $2\text{Re}(\omega) = (\omega+\bar{\omega})$

$$= \left(\frac{z-1}{z+1}\right)+\overline{\left(\frac{z-1}{z+1}\right)}$$

$$= \left(\frac{z-1}{z+1}\right)+\left(\frac{\bar{z}-1}{\bar{z}+1}\right)$$

$$= \left(\frac{z-1}{z+1}\right)+\left(\frac{(1/z)-1}{(1/z)+1}\right)$$

$$= \left(\frac{z-1}{z+1}\right)+\left(\frac{1-z}{1+z}\right)$$

$$= \left(\frac{z-1+1-z}{z+1}\right)$$

$$= 0$$

$\Rightarrow \quad \text{Re}(\omega) = 0$

17. Let $u = \frac{w-\bar{w}z}{1-z}$

Since u is purely real, so

$u = \bar{u}$

$\Rightarrow \quad \left(\frac{w-\bar{w}z}{1-z}\right) = \overline{\left(\frac{w-\bar{w}z}{1-z}\right)}$

$$= \left(\frac{\bar{w}-w\bar{z}}{1-\bar{z}}\right)$$

$\Rightarrow \quad (1-z)(w-\bar{w}z) = (1-z)(\bar{w}-w\bar{z})$

$\Rightarrow \quad w-\bar{w}z-wz+w\bar{w}z = \bar{w}-w\bar{z}-z\bar{w}+wz\bar{z}$

$\Rightarrow \quad (w-\bar{w})(z\cdot\bar{z}-1) = 0$

$\Rightarrow \quad (z\cdot\bar{z}-1) = 0 \qquad (\because w = \bar{w})$

$\Rightarrow \quad |z|^2-1 = 0$

$\Rightarrow \quad |z|^2 = 1$

$\Rightarrow \quad |z| = 1$

Thus, the set of values of $z = \{z : |z| = 1, z \neq 1\}$.

18. Given,

$|z| = 1 \Rightarrow |z|^2 = 1$

$\Rightarrow \quad z\cdot\bar{z} = 1$

We have,

$u = \frac{z}{1-z^2}$

$= \frac{z}{z\cdot\bar{z}-z^2}$

$= \frac{1}{\bar{z}-z}$

$= \frac{-1}{z-\bar{z}}$

$= \frac{-1}{2i\,\text{Im}(z)}$

$= \frac{-1}{2iy}$, where $z = x+iy$

$= \left(0, \frac{-1}{2y}\right)$

Thus, u lies on y-axis.

19. Let $z = r(\cos\theta + i\sin\theta)$

Also, $|z - 1| = 1$

$\Rightarrow |r(\cos\theta + i\sin\theta) - 1| = 1$

$\Rightarrow |(r\cos\theta - 1) + i(r\sin\theta)| = 1$

$\Rightarrow \sqrt{(r\cos\theta - 1)^2 + r^2\sin^2\theta} = 1$

$\Rightarrow (r\cos\theta - 1)^2 + r^2\sin^2\theta = 1$

$\Rightarrow r^2 - 2r\cos\theta = 0$

$\Rightarrow r = 2\cos\theta$

Now, $\dfrac{z-2}{z} = \dfrac{r(\cos\theta + i\sin\theta) - 2}{r(\cos\theta + i\sin\theta)}$

$= \dfrac{2\cos\theta(\cos\theta + i\sin\theta) - 2}{2\cos\theta(\cos\theta + i\sin\theta)}$

$= \dfrac{\cos\theta(\cos\theta + i\sin\theta) - 1}{(2\cos\theta(\cos\theta + i\sin\theta))}$

$= \dfrac{(2\cos^2\theta + i\,2\cos\theta\sin\theta) - 2}{2\cos\theta(\cos\theta + i\sin\theta)}$

$= \dfrac{-1 + \cos 2\theta + i\sin 2\theta}{[(1 + \cos 2\theta) + i\sin 2\theta]}$

$= \dfrac{i2\sin 2\theta}{2 + 2\cos 2\theta}$

$= \dfrac{i\sin 2\theta}{1 + \cos 2\theta}$

$= \dfrac{i\,2\sin\theta\cos\theta}{2\cos^2\theta}$

$= i\tan\theta$

$= i\tan(\text{Arg } z)$

Hence, the result.

20. Let $z = r_1(\cos\theta_1 + i\sin\theta_1)$

and $w = r_2(\cos\theta_2 + i\sin\theta_2)$

where $|z| = r_1$, $|w| = r_2$,

$\theta_1 = \text{Arg}(z)$, $\theta_2 = \text{Arg}(w)$

Now, $|z - w|^2$

$= (r_1\cos\theta_1 - r_2\cos\theta_2)^2 + (r_1\sin\theta_1 - r_2\sin\theta_2)^2$

$= r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_1 - \theta_2)$

$= r_1^2 + r_2^2 - 2r_1r_2 + 2r_1r_2 - 2r_1r_2\cos(\theta_1 - \theta_2)$

$= (r_1 - r_2)^2 + 2r_1r_2(1 - \cos(\theta_1 - \theta_2))$

$= (r_1 - r_2)^2 + 2r_1r_2 \cdot 2\sin^2\left(\dfrac{\theta_1 - \theta_2}{2}\right)$

$= (r_1 - r_2)^2 + 4r_1r_2\sin^2\left(\dfrac{\theta_1 - \theta_2}{2}\right)$

$\leq (r_1 - r_2)^2 + 4r_1r_2\left(\dfrac{\theta_1 - \theta_2}{2}\right)^2$

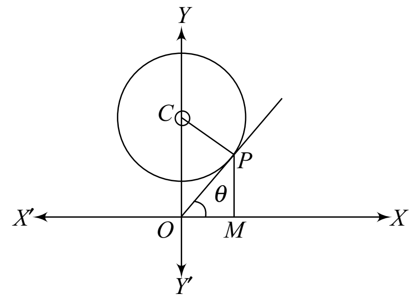
$= (r_1 - r_2)^2 + r_1r_2(\theta_1 - \theta_2)^2$

$(r_1 - r_2)^2 + (\theta_1 - \theta_2)^2 \quad (\because r_1, r_2 \leq 1)$

$= (|z| - |w|)^2 + [\text{Arg}(z) - \text{Arg}(w)]^2$

Hence, the result.

21.



Let $P = (x, y) = Z$

$\cos\theta = \dfrac{x}{2\sqrt{6}} \qquad \sin\theta = \dfrac{y}{2\sqrt{6}}$

$\Rightarrow \dfrac{1}{5} = \dfrac{x}{2\sqrt{6}} \qquad \dfrac{2\sqrt{6}}{5} = \dfrac{y}{2\sqrt{6}}$

$\Rightarrow x = \dfrac{2\sqrt{6}}{5} \qquad y = \dfrac{24}{5}$

Thus, the complex number is

$z = x + iy$

$= \dfrac{2\sqrt{6}}{5} + i\dfrac{24}{5}$

22. We have,

$\left|z - \dfrac{25}{z}\right| = 24$

$\Rightarrow |z^2 - 25| = 24|z|$

$\Rightarrow 24|z| = |z|^2 - 25| \geq |z^2| - 25$

$\Rightarrow |z^2| - 25 \leq 24|z|$

$\Rightarrow |z^2| - 24|z| - 25 \leq 0$

$\Rightarrow |z|^2 - 24|z| - 25 \leq 0$

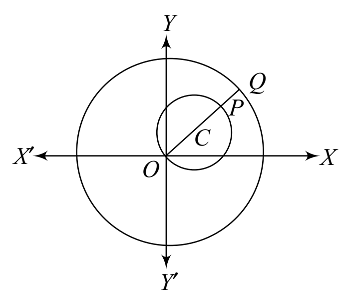
$\Rightarrow |z|^2 - 25|z| + |z| - 25 \leq 0$

$\Rightarrow (|z| - 25)(|z| + 1) \leq 0$

$\Rightarrow 1 \leq |z| \leq 25$

Thus, the greatest value of $|z|$ is 25.

23.



Here, $C = (3, 4)$, $CP = 5$, $OQ = 12$

Thus, the minimum value of

$|z_1 - z_2|$

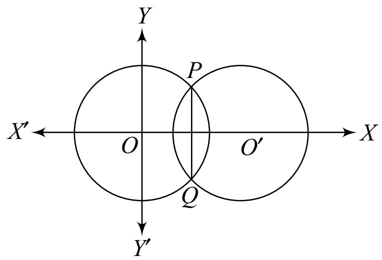
$= PQ$

$= OQ - OP$

$= 12 - 10$

$= 2$

24. Given curves are $|z-3| = 2$ and $|z| = 2$.



Let $z = x + iy$.

Given $|z| = 2$

$\Rightarrow \quad \sqrt{x^2+y^2} = 2$

$\Rightarrow \quad x^2 + y^2 = 4 \qquad \ldots(i)$

Also, $|z-3| = 2$

$\Rightarrow \quad |x + iy - 3| = 2$

$\Rightarrow \quad |(x-3) + iy| = 2$

$\Rightarrow \quad \sqrt{(x-3)^2+y^2} = 2$

$\Rightarrow \quad (x-3)^2 + y^2 = 4$

$\Rightarrow \quad x^2 - 6x + 9 + y^2 = 4$

$\Rightarrow \quad x^2 + y^2 - 6x + 9 = 4 \qquad \ldots(ii)$

From Eqs (i) and (ii), we get

$4 - 6x + 9 = 4$

$\Rightarrow \quad -6x = -9 \Rightarrow x = 3/2$

When $x = 3/2$, $y = \pm\dfrac{\sqrt{7}}{2}$

Hence, the points of intersection are

$$\left(\frac{3}{2}+i\frac{\sqrt{7}}{2}\right), \left(\frac{3}{2}-i\frac{\sqrt{7}}{2}\right)$$

Thus, the length of the common chord,

$= PQ$

$= 2\cdot\left(\dfrac{\sqrt{7}}{2}\right)$

$= \sqrt{7}$

25. We have,

$t^2 + t + 1 = 0$

$\Rightarrow \quad t = \omega, \omega^2$

When $t = \omega$,

$$\left(t+\frac{1}{t}\right)^2 = (\omega+\omega^2)^2 = 1$$

$$\left(t^2+\frac{1}{t^2}\right)^2 = (\omega+\omega^2)^2 = 1$$

$$\left(t^3+\frac{1}{t^3}\right)^2 = (1+1)^2 = 4$$

$$\left(t^4+\frac{1}{t^4}\right)^2 = (\omega+\omega^2)^2 = 1$$

$$\left(t^5+\frac{1}{t^5}\right)^2 = (\omega^2+\omega)^2 = 1$$

$$\left(t^6+\frac{1}{t^6}\right)^2 = (1+1)^2 = 4$$

$\vdots$

$$\left(t^{2014}+\frac{1}{t^{2014}}\right)^2 = (\omega+\omega^2)^2 = 1$$

Now, $\left(t+\dfrac{1}{t}\right)^2 + \left(t^2+\dfrac{1}{t^2}\right)^2 + \ldots + \left(t^{2014}+\dfrac{1}{t^{2014}}\right)^2$

$$= \left(t^3+\frac{1}{t^3}\right)^2 + \left(t^6+\frac{1}{t^6}\right)^2 + \left(t^9+\frac{1}{t^9}\right)^2 + \ldots + \left(t^{2013}+\frac{1}{t^{2013}}\right)^2$$

$$+ \left(t+\frac{1}{t}\right)^2 + \left(t^2+\frac{1}{t^2}\right)^2 + \left(t^4+\frac{1}{t^4}\right)^2 + \ldots + \left(t^{2014}+\frac{1}{t^{2014}}\right)^2$$

$= (4 + 4 + 4 + \ldots + 4)$ (671 times)

$+ (1 + 1 + 1 + \ldots + 1)$ [(2014 – 671) times]

$= 671 \times 4 + 1 \times 1343$

$= 2684 + 1343$

$= 4027$

26. We have,

$x + \dfrac{1}{x} = 1$

$\Rightarrow \quad x^2 + 1 = x$

$\Rightarrow \quad x^2 - x + 1 = 0$

$\Rightarrow \quad x = -\omega, -\omega^2$

When $x = -\omega$,

$x^{10} + x^{20} + x^{30} + \ldots + x^{100}$

$= (-\omega)^{10} + (-\omega)^{20} + (-\omega)^{30} + (-\omega)^{40} + \ldots + (-\omega)^{100}$

$= (\omega + \omega^2 + 1) + (\omega + \omega^2 + 1) + (\omega + \omega^2 + 1) + \omega$

$= \omega$

27. Let $\dfrac{1}{a+x} + \dfrac{1}{b+x} + \dfrac{1}{c+x} = \dfrac{2}{x}$

$$\Rightarrow \quad \frac{(b+x)(c+x)+(a+x)(c+x)+(a+x)(b+x)}{(a+x)(b+x)(c+x)} = \frac{2}{x}$$

$$\Rightarrow \quad \frac{3x^2+2(a+b+c)x+(ab+bc+ca)}{(a+x)(b+x)(c+x)} = \frac{2}{x}$$

$\Rightarrow \quad 3x^3 + 2(a+b+c)x^2 + (ab+bc+ca)x$

$= 2(a+x)(b+x)(c+x)$

$= 2(x^3 + (a+b+c)x^2 + (ab+bc+ca)x + abc)$

$\Rightarrow \quad x^3 - (ab+bc+ca)x - 2abc = 0$

which is a cubic in x, whose roots are ω, ω^2.

Let y is the third root.

Therefore, $\omega + \omega^2 + y = 0 \Rightarrow y = 1$

Thus, $\dfrac{1}{a+1} + \dfrac{1}{b+1} + \dfrac{1}{c+1} = \dfrac{2}{1} = 2$

28. We have,

$$\frac{1}{x-1} + \frac{1}{x\omega-1} + \frac{1}{x\omega^2-1}$$

$= \frac{1}{x-1} + \frac{\omega^2}{x-\omega^2} + \frac{\omega}{x-\omega}$

$= \frac{(x-\omega^2)(x-\omega) + \omega^2(x-1)(x-\omega) + \omega(x-1)(x-\omega^2)}{x-1}$

$= [(x^2 - (\omega + \omega^2)x + 1) + \omega^2(x^2 - (1 + \omega)x + \omega)$
$+ \omega(x^2 - (1 + \omega^2)x + \omega^2)] \div (x - 1)(x - \omega)(x - \omega^2)$
$= [(x^2 + x + 1) + \omega^2(x^2 + \omega^2 x$
$+ \omega) + \omega(x^2 + \omega x + \omega^2)] \div (x^3 - 1)$
$= ((1 + \omega^2 + \omega)x^2 + (1 + \omega + \omega^2)x + 3) \div (x^3 - 1)$
$= \frac{3}{(x^3-1)}$

Hence, the result.

29. We have,

$(1 + \omega^2)^n = (1 + \omega^4)^n$
$\Rightarrow (-\omega)^n = (-\omega^2)^n$
$\Rightarrow \left(\frac{-\omega}{-\omega^2}\right)^n = 1$
$\Rightarrow (\omega^2)^n = 1$
$\Rightarrow (\omega)^{2n} = 1$
$\Rightarrow n = 3$

30. We have,

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} z+1+\omega+\omega^2 & \omega & \omega^2 \\ z+1+\omega+\omega^2 & z+\omega^2 & 1 \\ z+1+\omega+\omega^2 & 1 & z+\omega \end{vmatrix} = 0$$

$(C_1 \to C_1 + C_2 + C_3)$

$$\Rightarrow (z+1+\omega+\omega^2)\begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & z+\omega^2 & 1 \\ 1 & 1 & z+\omega \end{vmatrix} = 0$$

$\Rightarrow (z + 1 + \omega + \omega^2) = 0$

and $\begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & z+\omega^2 & 1 \\ 1 & 1 & z+\omega \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} 1 & \omega & \omega^2 \\ 0 & z+\omega^2-\omega & 1-\omega^2 \\ 0 & 1-\omega & z+\omega-\omega^2 \end{vmatrix} = 0 \quad \begin{pmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$$

$$\Rightarrow \begin{vmatrix} z+(\omega^2-\omega) & 1-\omega^2 \\ 1-\omega & z-(\omega^2-\omega) \end{vmatrix} = 0$$

$\Rightarrow z^2 - (\omega^2 - \omega)^2 - (1 - \omega^2)(1 - \omega) = 0 =0$
$\Rightarrow z^2 - (\omega + \omega^2 - 2) - (1 - (\omega^2 + \omega) + 1) = 0$
$\Rightarrow z^2 - (-1 - 2) - (1 + 1 + 1) = 0$
$\Rightarrow z^2 + 3 - 3 = 0$
$\Rightarrow z = 0$

Thus, the number of roots is 1.

31. We have $|a + b\omega + c\omega^2|^2$

$= (a + b\omega + c\omega^2)\overline{(a + b\omega + c\omega^2)}$
$= (a + b\omega + c\omega^2)(a + b\overline{\omega} + c\overline{\omega^2})$
$= (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$
$= (a^2 + b^2 + c^2 - ab - bc - ca)$
$= \frac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2]$

$\Rightarrow |a + b\omega + c\omega^2| \geq 1$

This value is attained when $a = 2, b = 1, c = 1$.

32. We have $(1 + \omega)^7 = A + B\omega$

$\Rightarrow (-\omega^2)^7 = A + B\omega$
$\Rightarrow -\omega^{14} = A + B\omega$
$\Rightarrow -\omega^2 = A + B\omega$
$\Rightarrow A + B\omega + \omega^2 = 0$
$\Rightarrow A = 1 = B$

Thus, $A + B + 10 = 12$.

33. We have $x + \frac{1}{x} = 2\cos\alpha$

$\Rightarrow x^2 - 2x\cos\alpha + 1 = 0$
$\Rightarrow x = \frac{2\cos\alpha \pm \sqrt{4\cos^2\alpha - 4}}{2}$
$\Rightarrow x = \frac{2\cos\alpha \pm 2i\sin\alpha}{2} = \cos\alpha \pm i\sin\alpha$

similarly, $y = \cos\beta \pm i\sin\beta$

when $x = \cos\alpha + i\sin\alpha$, $y = \cos\beta + i\sin\beta$

Then

$\frac{x}{y} + \frac{y}{x} = \frac{\cos\alpha + i\sin\alpha}{\cos\beta + i\sin\beta} + \frac{\cos\beta + i\sin\beta}{\cos\alpha + i\sin\alpha}$
$= (\cos\alpha + i\sin\alpha)(\cos\beta - i\sin\beta)$
$+ (\cos\beta + i\sin\beta)(\cos\alpha - i\sin\alpha)$
$= (\cos(\alpha - \beta) + i\sin(\alpha - \beta))$
$+ (\cos(\beta - \alpha) + i\sin(\beta - \alpha))$
$= 2(\cos(\alpha - \beta))$

similarly, we can prove that

$\frac{x}{y} + \frac{y}{x} = 2\cos(\alpha + \beta)$

Hence, the result.

34. We have $x^4 + x^3 + x^2 + x + 1 = 0$

$\Rightarrow 1 + x + x^2 + x^3 + x^4 = 0$
$\Rightarrow \frac{1-x^5}{1-x} = 0$
$\Rightarrow x^5 = 1$
$\Rightarrow x = (\cos(2r\pi) + i\sin(2r\pi))^{1/5}$
$\Rightarrow x = \left(\cos\left(\frac{2r\pi}{5}\right) + i\sin\left(\frac{2r\pi}{5}\right)\right)$

where $r = 0, 1, 2, 3, 4$

Therefore the roots of x are

$1, \left(\cos\left(\frac{2\pi}{5}\right) + i\sin\left(\frac{2\pi}{5}\right)\right), \left(\cos\left(\frac{4\pi}{5}\right) + i\sin\left(\frac{4\pi}{5}\right)\right)$
$\left(\cos\left(\frac{6\pi}{5}\right) + i\sin\left(\frac{6\pi}{5}\right)\right), \left(\cos\left(\frac{8\pi}{5}\right) + i\sin\left(\frac{8\pi}{5}\right)\right)$

35. We have $x^{14} - x^{13} + x^{11} - x^{10} + \ldots - x + 1 = 0$

$\Rightarrow \quad 1 - x + x^2 - x^3 + \ldots + x^{14} = 0$

$\Rightarrow \quad 1 + (-x) + (-x)^2 + (-x)^3 + \ldots + (-x)^{14} = 0$

$$\Rightarrow \quad \frac{1-(-x)^{15}}{1-(-x)} = 0$$

$\Rightarrow \quad x^{15} = -1$

$\Rightarrow \quad x = (\cos(2r+1)\pi + i\sin(2r+1)\pi)^{1/15}$

$$\Rightarrow \quad x = \left(\cos\left(\frac{(2r+1)\pi}{15}\right) + i\sin\left(\frac{(2r+1)\pi}{15}\right)\right)$$

where $r = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7$

Thus, the roots of x are -1,

$$-1, \cos\left(\frac{(2r+1)\pi}{15}\right) \pm i\sin\left(\frac{(2r+1)\pi}{15}\right)$$

where $r = 1, 2, 3, 4, 5, 6, 7$

36. We have $x^{12} - x^{12} + x^{10} - x^9 + \ldots - x + 1 = 0$

$\Rightarrow \quad 1 - x + x^2 - x^3 + \ldots + x^{12} = 0$

$\Rightarrow \quad 1 + (-x) + (-x)^2 + (-x)^3 + \ldots + (-x)^{12} = 0$

$$\Rightarrow \quad \frac{1-(-x)^{13}}{1-(-x)} = 0$$

$\Rightarrow \quad x^{13} = -1$

$\Rightarrow \quad x = [\cos(2r+1)\pi + i\sin(2r+1)\pi]^{1/13}$

$$= \left[\cos\left(\frac{(2r+1)\pi}{13}\right) + i\sin\left(\frac{(2r+1)\pi}{13}\right)\right]$$

where $r = 0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6$

Thus, the roots of x are -1,

$$-1, \cos\left(\frac{(2r+1)\pi}{13}\right) \pm i\sin\left(\frac{(2r+1)\pi}{13}\right)$$

where $r = 1, 2, 3, 4, 5, 6$.

37. We have,

$$\sum_{k=1}^{10}\left[\sin\left(\frac{2\pi k}{11}\right) - i\cos\left(\frac{2\pi k}{11}\right)\right]$$

$$= \sum_{k=1}^{10}\left[-i^2\sin\left(\frac{2\pi k}{11}\right) - i\cos\left(\frac{2\pi k}{11}\right)\right]$$

$$= \sum_{k=1}^{10} -i\left[\cos\left(\frac{2\pi k}{11}\right) + i\sin\left(\frac{2\pi k}{11}\right)\right]$$

$$= \sum_{k=1}^{10} -ie^{i\left(\frac{2\pi k}{11}\right)}$$

$$= -i\left(\sum_{k=1}^{10} e^{i\left(\frac{2\pi k}{11}\right)}\right)$$

$$= -i\left(e^{i\left(\frac{2\pi}{11}\right)} + e^{i\left(\frac{4\pi}{11}\right)} + e^{i\left(\frac{6\pi}{11}\right)} + \ldots + e^{i\left(\frac{20\pi}{11}\right)}\right)$$

$$= -ie^{i\left(\frac{2\pi}{11}\right)}\left(1 + e^{i\left(\frac{2\pi}{11}\right)} + e^{i\left(\frac{4\pi}{11}\right)} + \ldots + e^{i\left(\frac{18\pi}{11}\right)}\right)$$

$$= -ie^{i\left(\frac{2\pi}{11}\right)}\left(\frac{1 - e^{i\left(\frac{20\pi}{11}\right)}}{1 - e^{i\left(\frac{2\pi}{11}\right)}}\right)$$

$$= -i\left(\frac{e^{i\left(\frac{2\pi}{11}\right)} - e^{i\left(\frac{22\pi}{11}\right)}}{1 - e^{i\left(\frac{2\pi}{11}\right)}}\right)$$

$$= -i\left(\frac{e^{i\left(\frac{2\pi}{11}\right)} - 1}{1 - e^{i\left(\frac{2\pi}{11}\right)}}\right)$$

$= -i \times -1$

$= i$

38. We have,

$z^8 + 1 = 0$

$\Rightarrow \quad z^8 = -1$

$\Rightarrow \quad z = [\cos(2r+1)\pi + i\sin(2r+1)\pi]^{1/8}$

$$= \left(\cos\left(\frac{(2r+1)\pi}{8}\right) + i\sin\left(\frac{(2r+1)\pi}{8}\right)\right)$$

where $r = 0, 1, 2, 3, \ldots, 7$

$$= \cos\left(\frac{\pi}{8}\right) \pm i\sin\left(\frac{\pi}{8}\right), \cos\left(\frac{3\pi}{8}\right) \pm i\sin\left(\frac{3\pi}{8}\right),$$

$$\cos\left(\frac{5\pi}{8}\right) \pm i\sin\left(\frac{5\pi}{8}\right), \cos\left(\frac{7\pi}{8}\right) \pm i\sin\left(\frac{7\pi}{8}\right)$$

39. We have,

$z^7 - 1 = 0$

$\Rightarrow \quad z^7 = 1$

$$\Rightarrow \quad z = \cos\left(\frac{2r\pi}{7}\right) + i\sin\left(\frac{2r\pi}{7}\right)$$

where $r = 0, 1, 2, \ldots, 6$

$$= 1, \cos\left(\frac{2\pi}{7}\right) \pm i\sin\left(\frac{2\pi}{7}\right)$$

$$\cos\left(\frac{4\pi}{7}\right) \pm i\sin\left(\frac{4\pi}{7}\right), \cos\left(\frac{6\pi}{7}\right) \pm i\sin\left(\frac{6\pi}{7}\right)$$

$$\Rightarrow \quad z^7 - 1 = (z-1)\left(z^2 - 2\cos\left(\frac{2\pi}{7}\right)z + 1\right)$$

$$\left(z^2 - 2\cos\left(\frac{4\pi}{7}\right)z + 1\right)$$

$$\left(z^2 - 2\cos\left(\frac{6\pi}{7}\right)z + 1\right)$$

Put $z = i$, then we get,

$$\Rightarrow \quad -(i+1) = (i-1)\left[\left(-2\cos\left(\frac{2\pi}{7}\right)i\right)\left(-2\cos\left(\frac{4\pi}{7}\right)i\right)\right]$$

$$\left[-2\cos\left(\frac{6\pi}{7}\right)i\right]$$

$$= -i^3(i-1)\left[8\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\cdot\cos\left(\frac{6\pi}{7}\right)\right]$$

$$\Rightarrow \quad \left(8\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\cdot\cos\left(\frac{6\pi}{7}\right)\right)$$

$$= \frac{(i+1)}{-i^3(i-1)} = \frac{(i+1)}{i(i-1)} = \frac{(i+1)}{(-1-i)} = -1$$

$$\Rightarrow \quad \cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\cdot\cos\left(\frac{6\pi}{7}\right) = -\frac{1}{8}$$

$$\Rightarrow \quad \cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\cdot\cos\left(\pi - \frac{\pi}{7}\right) = -\frac{1}{8}$$

$$\Rightarrow \quad -\cos\left(\frac{2\pi}{7}\right)\cos\left(\frac{4\pi}{7}\right)\cdot\cos\left(\frac{\pi}{7}\right) = -\frac{1}{8}$$

$$\Rightarrow \quad \cos\left(\frac{\pi}{7}\right)\cdot\cos\left(\frac{2\pi}{7}\right)\cdot\cos\left(\frac{4\pi}{7}\right) = \frac{1}{8}$$

Hence, the result.

40. We have,

$z^5 + 1 = 0.$

$\Rightarrow \quad z^5 = -1$

$\Rightarrow \quad z = (-1)^{1/5}$

$\Rightarrow \quad = [\cos(2r+1)\pi + i\sin(2r+1)\pi]^{1/5}$

$$\Rightarrow \quad = \left[\cos\left(\frac{(2r+1)\pi}{5}\right) + i\sin\left(\frac{(2r+1)\pi}{5}\right)\right]$$

where $r = 0, 1, 2, 3, 4$.

$$\Rightarrow \quad = -1, \cos\left(\frac{\pi}{5}\right) \pm i\sin\left(\frac{\pi}{5}\right), \cos\left(\frac{2\pi}{5}\right) \pm i\sin\left(\frac{2\pi}{5}\right)$$

Thus, $z^5 + 1$

$$= (z+1)\left[z^2 - 2\cos\left(\frac{\pi}{5}\right)z + 1\right]\left[z^2 - 2\cos\left(\frac{2\pi}{5}\right)z + 1\right]$$

Put $z = i$, then,

$i^5 + 1$

$$= (i+1)\left[\left(-2\cos\left(\frac{\pi}{5}\right)\right)\left(-2\cos\left(\frac{2\pi}{5}\right)\right)\right]$$

$$\Rightarrow \quad (i+1)$$

$$(i+1)\left[-2\cos\left(\frac{\pi}{5}\right)\right]\left[-2\cos\left(\frac{2\pi}{5}\right)\right]$$

$$\Rightarrow \quad 4\cos\left(\frac{\pi}{5}\right)\cos\left(\frac{2\pi}{5}\right) = 1$$

$$\Rightarrow \quad 4\cos\left(\frac{\pi}{5}\right)\sin\left(\frac{\pi}{2} - \frac{2\pi}{5}\right) = 1$$

$$\Rightarrow \quad 4\sin\left(\frac{\pi}{10}\right)\cos\left(\frac{\pi}{5}\right) = 1$$

Hence, the result.

41. We have,

$x^{10} + x^5 + 1 = 0$

$\Rightarrow \quad (x^5)^2 + x^5 + 1 = 0$

$\Rightarrow \quad x^5 = \omega, \omega^2$

When $x^5 = \omega$

$$\Rightarrow \quad x^5 = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow \quad x = \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{1/5}$$

$$\Rightarrow \quad x = \left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]^{1/5}$$

$$= \left[\cos\left(2r\pi + \frac{2\pi}{3}\right) + i\sin\left(2r\pi + \frac{2\pi}{3}\right)\right]^{1/5}$$

$$= \left[\cos\left(\frac{6r\pi + 2\pi}{3}\right) + i\sin\left(\frac{6r\pi + 2\pi}{3}\right)\right]^{1/5}$$

$$= \left[\cos\left(\frac{6r\pi + 2\pi}{15}\right) + i\sin\left(\frac{6r\pi + 2\pi}{15}\right)\right]$$

where $r = 0, 1, 2, 3, 4$.

$$\Rightarrow \quad x = \cos\left(\frac{2\pi}{15}\right) + i\sin\left(\frac{2\pi}{15}\right), \cos\left(\frac{8\pi}{15}\right) + i\sin\left(\frac{8\pi}{15}\right),$$

$$\cos\left(\frac{14\pi}{15}\right) + i\sin\left(\frac{14\pi}{15}\right), \cos\left(\frac{20\pi}{15}\right) + i$$

$$\sin\left(\frac{20\pi}{15}\right), \cos\left(\frac{26\pi}{15}\right) + i\sin\left(\frac{26\pi}{15}\right)$$

Similarly, we can easily find the other roots.

42. We have, $\omega^5 = 2$.

Now, $x = \omega + \omega^2$

$= (\omega + \omega^2)^5$

$= \omega^5 + 5\omega^6 + 10\omega^7 + 10\omega^8 + 5\omega^9 + \omega^{10}$

$= 2 + 10\omega + 20\omega^2 + 20\omega^3 + 10\omega^4 + 4$

$= 6 + 10(\omega^2 + 2\omega^3 + \omega^4) + 10(\omega^2 + \omega)$

$= 6 + 10(\omega + \omega^2)^2 + 10(\omega^2 + \omega)$

$= 6 + 10x^2 + 10x$

$x^5 - 10x^2 - 10x = 6$

44. Given $z = \cos\left(\dfrac{2\pi}{2n+1}\right) + i\sin\left(\dfrac{2\pi}{2n+1}\right), n \in I^+$

$\Rightarrow \quad z^{2n+1} = 1$

Now, $\alpha = z + z^3 + z^5 + \ldots + z^{2n-1}$

$$= z\left(\frac{1 - z^{2n}}{1 - z^2}\right)$$

$$= \left(\frac{z - z^{2n+1}}{1 - z^2}\right)$$

$$= \left(\frac{z - 1}{1 - z^2}\right)$$

$$= -\frac{1}{z+1}$$

Also, $\beta = z^2 + z^4 + z^6 + \ldots + z^{2n}$

$$= z^2\left(\frac{1-z^{2n}}{1-z^2}\right)$$

$$= z\left(\frac{z-z^{2n+1}}{1-z^2}\right)$$

$$= z\left(\frac{z-1}{1-z^2}\right)$$

$$= \left(\frac{1}{1+z}\right)$$

Now, $\alpha+\beta = -\frac{z+1}{z+1} = -1$

and $\alpha\cdot\beta = \frac{z}{(z+1)^2} = \frac{1}{z+\frac{1}{z}+2}$

Again,

$$z+\frac{1}{z} = (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta)$$

$$= 2\cos\theta, \quad \text{where } \theta = \left(\frac{2\pi}{2n+1}\right)$$

$$\Rightarrow z+\frac{1}{z}+2 = 2\cos\theta + 2 = 2(1+\cos\theta) = 4\cos^2\left(\frac{\theta}{2}\right)$$

$$= 4\cos^2\left(\frac{\pi}{2n+1}\right)$$

Hence, the required equation is

$$x^2 + x + \frac{1}{4\cos^2\left(\frac{\pi}{2n+1}\right)} = 0.$$

45. Let $z = (1)^{1/n} = (\cos(2r\pi) + i\sin(2r\pi))^{1/n}$

$$= \left[\cos\left(\frac{2r\pi}{n}\right) + i\sin\left(\frac{2r\pi}{n}\right)\right] = e^{i\frac{2r\pi}{n}}$$

where $r = 0, 1, 2, 3, \ldots, (n-1)$

Let $z_1 = 1$ and $z_2 = e^{i\frac{2k\pi}{n}}$

It is given that,

$$(z_2 - 0) = (z_1 - 0)e^{i\frac{\pi}{2}}$$

$$\Rightarrow \quad e^{i\frac{2k\pi}{n}} = e^{i\frac{\pi}{2}}$$

$$\Rightarrow \quad \frac{2k\pi}{n} = \frac{\pi}{2}$$

$$\Rightarrow \quad n = 4k$$

Hence, the result.

46. Given $\omega^5 = 1$

we have $\log_2 |1 + \omega + \omega^2 + \omega^3 - \omega^{-1}|$

$$= \log_2\left|1+\omega+\omega^2+\omega^3-\frac{1}{\omega}\right|$$

$$= \log_2\left|\frac{\omega+\omega^2+\omega^3+\omega^4-1}{\omega}\right|$$

$$= \log_2\left|\frac{1+\omega+\omega^2+\omega^3+\omega^4-2}{\omega}\right|$$

$$= \log_2\left|\frac{\left(\frac{1-\omega^5}{1-\omega}\right)-2}{\omega}\right|$$

$$= \log_2\left|\frac{0-2}{\omega}\right|$$

$$= \log_2\left(\frac{|-2|}{|\omega|}\right)$$

$= \log_2|2| - \log_2|\omega|$

$= \log_2 2 - \log_2 1$

$= 1 - 0$

$= 1$

47. Given $\beta^n = 1$

Let $S_n = 1 + 3\beta + 5\beta^2 + \ldots + (2n-1)\beta^{n-1}$

$\Rightarrow \quad S_n\beta = \beta + 3\beta^2 + \ldots + (2n-3)\beta^{n-1} + (2n-1)\beta^n$

Substracting we get,

$\Rightarrow \quad (1-\beta)S_n = 1 + 2\beta + 2\beta^2 + \ldots + 2\beta^{n-1} - (2n-1)\beta^n$

$= 1 + 2\beta + 2\beta^2 + \ldots + 2\beta^{n-1} - (2n-1)$

$= 2\beta + 2\beta^2 + \ldots + 2\beta^{n-1} - 2n$

$$= 2\left(\frac{1-\beta^n}{1-\beta}\right) - 2n$$

$= -2n$ ($\because \beta^n = 1$)

$$\Rightarrow \quad S_n = \frac{2n}{\beta-1}$$

Hence, the result.

48. Given $\omega^5 = 3$

We have,

$x = \omega + \omega^2$

$= (\omega + \omega^2)^5$

$= \omega^5 + 5\omega.\omega^2 + 10\omega^3.\omega^4 + 10\omega^2.\omega^6 + 5\omega.\omega^8 + 5\omega^{10}$

$= 3 + 15\omega + 30\omega^2 + 30\omega^3 + 15\omega^4 + 9$

$= 12 + 15\omega^2 + 30\omega^2 + 15\omega^4 + (15\omega^2 + 15\omega)$

$= 12 + 15(\omega^2 + 2\omega^2 + \omega^4) + 15(\omega^2 + \omega)$

$= 12 + 15(\omega + \omega^2)^2 + 15(\omega + \omega^2)$

$= 12 + 15x^2 + 15x$

$\Rightarrow \quad x^5 - 15x^2 - 15x = 12$

$\Rightarrow \quad x^5 - 15x^2 - 15x + 18 = 12 + 18 = 30$

49. We have

$(x^2 + 2ix) - (3x^2 + iy) = (3 - 5i) + (3x^2 + iy)$

$\Rightarrow \quad -2x^2 + i(2x - y) = 3(x^2 - 1) + i(y - 5)$

Comparing the real and imaginary parts, we get, $3(x^2 - 1) = -2x^2$ and $(2x - y) = (y - 5)$

$$\Rightarrow \quad 5x^2 = 3 \text{ and } (x-y) = -\frac{5}{2}$$

$$\Rightarrow \quad x^2 = \frac{3}{5} \text{ and } (x-y) = -\frac{5}{2}$$

$$\Rightarrow \quad x = \pm\sqrt{\frac{3}{5}} \text{ and } y = x + \frac{5}{2}$$

$$= \pm\sqrt{\frac{3}{5}} \text{ and } y = \pm\sqrt{\frac{3}{5}} + \frac{5}{2}$$

50. Given $\frac{A}{B} + \frac{B}{A} = 1$

$\Rightarrow \quad A^2 + B^2 - AB = 0$

$\Rightarrow \quad A^2 - AB + B^2 = 0$

$$\Rightarrow \quad A = \frac{B \pm \sqrt{B^2 - 4B^2}}{2}$$

$$= \frac{B \pm i\sqrt{3}B}{2} = \left(\frac{1 \pm i\sqrt{3}}{2}\right)B$$

$$= \left(\frac{1 + i\sqrt{3}}{2}\right)B, \left(\frac{1 - i\sqrt{3}}{2}\right)B$$

$$= -\omega^2 B, -\omega B$$

Let $z_1 = (0, 0), z_2 = A, z_3 = B$

When $A = -\omega B$

Now, $|z_1 - z_2| = |-(-\omega B)| = B$

$|z_1 - z_3| = |-B| = B$

$|z_2 - z_3| = |-wB - B|$

$= |(1 + \omega)||B| = |B|$

Thus, the complex numbers z_1, z_2 and z_3 form an equilateral triangle.

51. Put $z = x + iy$,

$(x^2 + y^2) - 2i(x + iy) + 2a(1 + i) = 0$

$\Rightarrow \quad (x^2 + y^2 + 2y + 2a) - i2(x - a) = 0$

Comparing the real and imaginary parts, we get

$(x^2 + y^2 + 2y + 2a) = 0, (x - a) = 0$

$\Rightarrow \quad (x^2 + y^2 + 2y + 2a) = 0, x = a$

$\Rightarrow \quad (a^2 + y^2 + 2y + 2a) = 0$

$\Rightarrow \quad (y^2 + 2y + a^2 + 2a) = 0$

For every real a,

$D \geq 0$

$\Rightarrow \quad 4 - 4(a^2 + 2a) \geq 0$

$\Rightarrow \quad 1 - (a^2 + 2a) \geq 0$

$\Rightarrow \quad (a^2 + 2a - 1) \leq 0$

$\Rightarrow \quad (a^2 + 1) \leq 2$

$\Rightarrow \quad |(a + 1)| \leq \sqrt{2}$

$\Rightarrow \quad -\sqrt{2} \leq (a + 1) \leq \sqrt{2}$

$\Rightarrow \quad 0 < a \leq \sqrt{2} - 1$, since $a > 0$.

Also, $(y^2 + 2y + a^2 + 2a) = 0$

$\Rightarrow \quad y^2 + 2y + 1 = 1 - 2a - a^2$

$\Rightarrow \quad (y + 1)^2 = 1 - 2a - a^2$

$\Rightarrow \quad y = -1 \pm \sqrt{1 - 2a - a^2}$

Hence, the complex numbers are

$a + \left(-1 \pm \sqrt{1 - 2a - a^2}\right).$

52. Given $z + a|z - 1| + 2i = 0$.

Put $z = x + iy$,

$(x + iy) + a|(x - 1) + iy| + 2i = 0$

$\Rightarrow \quad (x + iy) + a\sqrt{(x - 1)^2 + y^2} + 2i = 0$

$\Rightarrow \quad \left(x + a\sqrt{(x - 1)^2 + y^2}\right) + i(y + 2) = 0$

Comparing the real and imaginary part, we get

$\Rightarrow \quad \left(x + a\sqrt{(x - 1)^2 + y^2}\right) = 0, (y + 2) = 0$

$\left(x + a\sqrt{(x - 1)^2 + y^2}\right) = 0, y = -2$

$\Rightarrow \quad \left(x + a\sqrt{(x - 1)^2 + 4}\right) = 0$

$\Rightarrow \quad a\sqrt{(x - 1)^2 + 4} = -x$

$\Rightarrow \quad a^2((x - 1)^2 + 4) = x^2$

$\Rightarrow \quad a^2((x^2 - 2x + 5) = x^2$

$\Rightarrow \quad (a^2 - 1)x^2 - 2.a^2.x + 5a^2 = 0$

For $a \in R$,

$D \geq 0$

$\Rightarrow \quad 4a^4 - 20a^2(a^2 - 1) \geq 0$

$\Rightarrow \quad a^4 - 5a^2(a^2 - 1) \geq 0$

$\Rightarrow \quad -4a^4 + 5a^2 \geq 0$

$\Rightarrow \quad 4a^2 - 5 \leq 0$

$\Rightarrow \quad a^2 \leq \frac{5}{4}$

$\Rightarrow \quad |a| \leq \frac{\sqrt{5}}{2}$

$\Rightarrow \quad -\frac{\sqrt{5}}{2} \leq a \leq \frac{\sqrt{5}}{2}$

Hence, the range of a is

$$a \in \left[-\frac{\sqrt{5}}{2}, \frac{\sqrt{5}}{2}\right] - \{-1, 1\}$$

53. Given circles are $|z| = 1$ and $|z - 1| = 1$.

Let $|z - \alpha| = k$, where $\alpha = a + ib$, $a, b, k \in R$ cuts the circles $|z| = 1$ and $|z - 1| = 4$ orthogonally.

Therefore, $k^2 + 1 = |\alpha - 0|^2 = \alpha^2 = \alpha\bar{\alpha}$ and

$k^2 + 16 = |\alpha - 1|^2$

$= (\alpha - 1)(\bar{\alpha} - 1)$

$= \alpha\bar{\alpha} - \alpha - \bar{\alpha} + 1$

$= k^2 + 1 - \alpha - \bar{\alpha} + 1$

Thus, $\alpha + \bar{\alpha} = -14$

$\Rightarrow \quad 2a = -14$

$\Rightarrow \quad a = -7$

Therefore, $\alpha = a + ib = -7 + ib$

Also, $k^2 = |\alpha|^2 - 1 = 49 + b^2 - 1$

$\Rightarrow \quad k^2 = 48 + b^2$

$\Rightarrow \quad k = \sqrt{48 + b^2}$

Hence, the equation of the family of circles is

$|z - (-7 + ib)| = \sqrt{48 + b^2}$

54. Put $z = x + iy$

Now, $\left|\frac{z-4}{z-8}\right| = 1$

$\Rightarrow |z-4| = |z-8|$

$\Rightarrow |z-4|^2 = |z-8|^2$

$\Rightarrow |(x-4)+iy|^2 = |(x-8)+iy|^2$

$\Rightarrow (x-4)^2 + y^2 = (x-8)^2 + y^2$

$\Rightarrow (x-4)^2 = (x-8)^2$

$\Rightarrow x^2 - 8x + 16 = x^2 - 16x + 64$

$\Rightarrow 8x = 48$

$\Rightarrow x = 6$

Also, $\left|\frac{z-12}{z-8i}\right| = \frac{5}{3}$

$\Rightarrow 9|z-12|^2 = 25|z-8i|^2$

$\Rightarrow 9|(x-12)+iy|^2 = 25|x+i(y-8)|^2$

$\Rightarrow 9|(x-12)^2+y^2) = 25(x^2+(y-8)^2)$

$\Rightarrow 9(36+y^2) = 25(36+(y-8)^2)$

$\Rightarrow 25(y-8)^2 - 9y^2 = 324 - 900 = -576$

$\Rightarrow 16y^2 - 400y + 25.64 = -576$

$\Rightarrow y^2 - 25y + 25.4 = -36$

$\Rightarrow y^2 - 25y + 136 = 0$

$\Rightarrow (y-18)(y-8) = 0$

$\Rightarrow y = 8, 18$

Hence, the complex numbers are $6 + 8i$, $6 + 18i$.

55. Given equation is

$$2\sqrt{2}\,x^4 = (\sqrt{3}-1) + i(\sqrt{3}+1)$$

$$\Rightarrow x^4 = \frac{(\sqrt{3}-1)}{2\sqrt{2}} + i\frac{(\sqrt{3}+1)}{2\sqrt{2}}$$

$$\Rightarrow x^4 = \cos(15°) + i\sin(15°)$$

$$\Rightarrow x^4 = \cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right)$$

$$\Rightarrow x = \left[\cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right)\right]^{1/4}$$

$$= \left[\cos\left(2\pi k + \frac{\pi}{12}\right) + i\sin\left(2\pi k + \frac{\pi}{12}\right)\right]^{1/4}$$

$$= \left[\cos\left(\frac{\pi k}{2} + \frac{\pi}{48}\right) + i\sin\left(\frac{\pi k}{2} + \frac{\pi}{48}\right)\right]$$

where $k = 0, 1, 2, 3$.

56. Put $z = x + iy$

Now, $|z - 3 + i| = 3$

$\Rightarrow |x + iy - 3 + i| = 3$

$\Rightarrow |(x-3) + i(y+1)| = 3$

$\Rightarrow \sqrt{(x-3)^2 + (y+1)^2} = 3$

$\Rightarrow (x-3)^2 + (y+1)^2 = 9$

$\Rightarrow x^2 + y^2 - 6x + 2y + 1 = 0$ …(i)

Also, $\text{Arg}\left(\frac{3z-6-3i}{2z-8-6i}\right) = \frac{\pi}{4}$

$$\Rightarrow \text{Arg}\left(\frac{3x+i3y-6-3i}{2x+i2y-8-6i}\right) = \frac{\pi}{4}$$

$$\Rightarrow \text{Arg}\left(\frac{3x-6+i3(y-3)}{(2x-8)+i2(y-3)}\right) = \frac{\pi}{4}$$

$$\Rightarrow \text{Arg}\left(\frac{\{(3x-6)+i3(y-1)\}\times\{(2x-8)-2i(y-3)\}}{(2x-8)^2+4(y-3)^2}\right) = \frac{\pi}{4}$$

$$\Rightarrow \text{Arg}\left(\frac{(3x-6)(2x-8)+6(y-1)(y-3)+i\left\{\begin{matrix}3(y-1)(2x-8)\\-2(y-3)(x-6)\end{matrix}\right\}}{(2x-8)^2+4(y-3)^2}\right) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{3(y-1)(2x-8)-2(y-3)(x-6)}{(3x-6)(2x-8)+6(y-1)(y-3)}\right) = \frac{\pi}{4}$$

$$\Rightarrow \left(\frac{3(y-1)(2x-8)-2(y-3)(x-6)}{(3x-6)(2x-8)+6(y-1)(y-3)}\right) = 1$$

$$\Rightarrow \left(\frac{(y-1)(x-4)-(y-3)(x-2)}{(x-2)(x-4)+(y-1)(y-3)}\right) = 1$$

$\Rightarrow x^2 + y^2 - 8x - 2y + 13 = 0$ …(ii)

Subtracting Eqs (i) and (ii), we get

$x = 6 - 2y$

Put this the value of x in (i), we get

$\Rightarrow (6-2y)^2 + y^2 - 6(6-2y) + 2y + 1 = 0$

$\Rightarrow 5y^2 - 10y + 1 = 0$

$$\Rightarrow y^2 - 2y + 1 = 1 - \frac{1}{5} = \frac{4}{5}$$

$$\Rightarrow (y-1)^2 = \left(\frac{2}{\sqrt{5}}\right)^2$$

$$\Rightarrow y = 1 \pm \frac{2}{\sqrt{5}}$$

Thus, $x = 6 - 2y = 6 - 2\left(1 \pm \frac{2}{\sqrt{5}}\right)$

$$= \left(4 \mp \frac{4}{\sqrt{5}}\right)$$

Hence, the required complex numbers will be $\left(4 - \frac{4}{\sqrt{5}}\right) + i\left(1 + \frac{2}{\sqrt{5}}\right)$, $\left(4 + \frac{4}{\sqrt{5}}\right) + i\left(1 - \frac{2}{\sqrt{5}}\right)$

57. Given $2|z| + z^2 - 5 + i\sqrt{3} = 0$

Put $z = x + iy$,

$$2(x^2+y^2) + (x^2-y^2) + i\,2xy - 5 + i\sqrt{3} = 0$$

$$\Rightarrow (3x^2 + y^2 - 5) + i(2xy + \sqrt{3}) = 0$$

$\Rightarrow \quad (3x^2 + y^2 - 5) = 0, (2xy + \sqrt{3}) = 0$

$\Rightarrow \quad (3x^2 + y^2 - 5) = 0, y = -\dfrac{\sqrt{3}}{2x}$

$\Rightarrow \quad \left[3x^2 + \left(-\dfrac{\sqrt{3}}{2x}\right)^2 - 5\right] = 0$

$\Rightarrow \quad \left(3x^2 + \dfrac{3}{4x^2} - 5\right) = 0$

$\Rightarrow \quad 12x^4 - 20x^2 + 3 = 0$

$\Rightarrow \quad 12x^4 - 18x^2 - 2x^2 + 3 = 0$

$\Rightarrow \quad 6x^2(2x^2 - 3) - 1(2x^2 - 3) = 0$

$\Rightarrow \quad (6x^2 - 1)(2x^2 - 3) = 0$

$\Rightarrow \quad x = \pm\dfrac{1}{\sqrt{6}}, \pm\sqrt{\dfrac{3}{2}}$

Thus, $y = -\dfrac{\sqrt{3}}{2x} = \mp\dfrac{\sqrt{18}}{2}, \mp\dfrac{\sqrt{3}}{2} \times \dfrac{\sqrt{2}}{\sqrt{3}}$

$= \mp\dfrac{3}{\sqrt{2}}, \mp\dfrac{1}{\sqrt{2}}$

Hence, the complex numbers are

$\left(\pm\dfrac{1}{\sqrt{6}}, \mp\dfrac{3}{\sqrt{2}}\right), \left(\pm\sqrt{\dfrac{3}{2}}, \mp\dfrac{1}{\sqrt{2}}\right).$

58. Let $S = \sum_{q=1}^{10}\left[\sin\left(\dfrac{2q\pi}{11}\right) - i\cos\left(\dfrac{2q\pi}{11}\right)\right]$

$= \sum_{q=1}^{10} -\mathrm{i}\cos\left(\dfrac{2q\pi}{11}\right) + i\sin\left(\dfrac{2q\pi}{11}\right)$

$= \sum_{q=1}^{10} -ie^{i\frac{2q\pi}{11}}$

$= -i\sum_{q=1}^{10} e^{i\frac{2q\pi}{11}}$

$= -i\left(e^{i\frac{2\pi}{11}} + e^{i\frac{4\pi}{11}} + \ldots + e^{i\frac{20\pi}{11}}\right)$

$= -\mathrm{i}e^{i\frac{2\pi}{11}}\left(1 + e^{i\frac{2\pi}{11}} + \ldots + e^{i\frac{18\pi}{11}}\right)$

$= -ie^{i\frac{2\pi}{11}}\left(\dfrac{1 - e^{i\frac{20\pi}{11}}}{1 - e^{i\frac{2\pi}{11}}}\right)$

$= -\mathrm{i}\left(\dfrac{e^{i\frac{2\pi}{11}} - e^{i\frac{22\pi}{11}}}{1 - e^{i\frac{2\pi}{11}}}\right)$

$= -i\left(\dfrac{e^{i\frac{2\pi}{11}} - 1}{1 - e^{i\frac{2\pi}{11}}}\right)$

$= i$

Therefore, $\sum_{p=1}^{32}(3p+2)i^p$

$= 3\sum_{p=1}^{32} pi^p + 2\sum_{p=1}^{32} i^p$

$= 3\sum_{p=1}^{32}((p-1)+1)i^p + 2.0$

$= 3\sum_{p=1}^{32}(p-1)i^p + 3\sum_{p=1}^{32} i^p$

$= 3(i^2 + 2i^3 + 3i^4 + \ldots + 31i^{32}) + 0$

$= 3i^2\,(1 + 2i + 3i^2 + \ldots + 31i^{30})$

$= -3(1 + 2i + 3i^2 + \ldots + 31i^{30})$

$= -3(-16 + 16i)$

$= 48(1 - i)$

Let $S_1 = (1 + 2i + 3i^2 + \ldots + 31i^{30})$

$iS_1 = (i + 2i^2 + 3i^3 + \ldots + 31i^{31})$

Subjecting we get,

$(1 - i)S_1 = (1 + i^2 + i^3 + \ldots + i^{30} - 31i^{31})$

$= \left(\dfrac{1 - i^{31}}{1 - i} - 31i^{31}\right)$

$\Rightarrow \quad S_1 = \left(\dfrac{1 - i^{31}}{(1-\mathrm{i})^2} - \dfrac{31i^{31}}{(1-\mathrm{i})}\right)$

$= \left(\dfrac{1 - i^3}{(1-i)^2} - \dfrac{31i^3}{(1-i)}\right)$

$= \left(\dfrac{1 + i}{(1-i)^2} + \dfrac{31i}{(1-i)}\right)$

$= \left(\dfrac{1 + i + 31i(1-i)}{(1-i)^2}\right)$

$= \left(\dfrac{32 + 32i}{(1-i)^2}\right)$

$= \dfrac{32 + 32i}{-2i} = \dfrac{16 + 16i}{-i}$

$\Rightarrow \quad S_1 = (-16 + 16i)]$

59. Given $(3z - 1)^4 + (z - 2)^4 = 0$

$\Rightarrow \quad (3z - 1)^4 = -(z - 2)^4$

$\Rightarrow \quad \dfrac{(3z-1)^4}{(z-2)^4} = -1 = e^{i(2n+1)\pi}$

$\Rightarrow \quad \left(\dfrac{3z - 1}{z - 2}\right)^4 = e^{i(2n+1)\pi}$

$\Rightarrow \quad \left(\dfrac{3z - 1}{z - 2}\right) = e^{i\frac{(2n+1)\pi}{4}}$

$\Rightarrow \quad z = \dfrac{1 - 2e^{i\frac{(2n+1)\pi}{4}}}{3 - e^{i\frac{(2n+1)\pi}{4}}}$, where $n = 0, 1, 2, 3.$

Put $n = 0$,

$$\Rightarrow \quad z = \frac{1-2e^{i\frac{\pi}{4}}}{3-e^{i\frac{\pi}{4}}} = \frac{1-2\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)}{3-\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)}$$

$$= \frac{\sqrt{2}-2(1+i)}{3\sqrt{2}-(1+i)} = \frac{(\sqrt{2}-2)-2i}{(3\sqrt{2}-1)-i}$$

$$= \frac{[(\sqrt{2}-2)-2i)][(3\sqrt{2}-1)+i]}{(3\sqrt{2}-1)^2+1}$$

$$= \frac{10-7\sqrt{2}}{20-6\sqrt{2}} + i\frac{(-5\sqrt{2})}{20-6\sqrt{2}} = a+ib$$

Similarly for $n = 1, 2, 3$, we get three other roots of the given equation.

60. Given $\alpha = e^{i\frac{2\pi}{7}}$

$\Rightarrow \quad \alpha^7 = e^{i2\pi} = \cos(2\pi) + i\sin(2\pi)$

$\Rightarrow \quad \alpha^7 = 1$

Now, $f(x) + f(\alpha x) + f(\alpha^2 x) + \ldots + f(\alpha^6 x)$

$$= 7A_0 + \sum_{k=1}^{20} A_k x^k (1 + \alpha^k + \alpha^{2k} + \ldots + \alpha^{6k})$$

$$= 7A_0 + \sum_{k=1}^{20} A_k x^k [(1 + \alpha^k + (\alpha^k)^2 + \ldots + (\alpha^k)^6)]$$

$$= 7A_0 + \sum_{k=1}^{20} A_k x^k \left(\frac{1-\alpha^{7k}}{1-\alpha^k}\right)$$

$$= 7A_0 + \sum_{k=1}^{20} A_k x^k \left(\frac{1-(\alpha^7)^k}{1-\alpha^k}\right)$$

$$= 7A_0 + \sum_{k=1}^{20} A_k x^k \left(\frac{1-1}{1-\alpha^k}\right)$$

$$= 7A_0 + \sum_{k=1}^{20} A_k x^k \cdot 0$$

$$= 7A_0$$

61. Given $z = \cos\left(\frac{2\pi}{2n+1}\right) + i\sin\left(\frac{2\pi}{2n+1}\right), n \in I^+$

$\Rightarrow \quad z^{2n+1} = 1$

Now, $\alpha = z + z^3 + z^5 + \ldots + z^{2n-1}$

$$= z\left(\frac{1-z^{2n}}{1-z^2}\right)$$

$$= \left(\frac{z-z^{2n+1}}{1-z^2}\right)$$

$$= \left(\frac{z-1}{1-z^2}\right)$$

$$= -\frac{1}{z+1}$$

Also, $\alpha = z^2 + z^4 + z^6 + \ldots + z^{2n}$

$$= z^2\left(\frac{1-z^{2n}}{1-z^2}\right)$$

$$= z\left(\frac{z-z^{2n+1}}{1-z^2}\right)$$

$$= z\left(\frac{z-1}{1-z^2}\right)$$

$$= -\left(\frac{z}{1+z}\right)$$

Now, $\alpha + \beta = -\frac{z+1}{z+1} = -1$

and $\alpha \cdot \beta = \frac{z}{(z+1)^2} = \frac{1}{z+\frac{1}{z}+2}$

Again,

$$z + \frac{1}{z} = (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta)$$

$$= 2\cos\theta, \text{ where } \theta = \left(\frac{2\pi}{2n+1}\right)$$

$$\Rightarrow \quad z + \frac{1}{z} + 2 = 2\cos\theta + 2$$

$$= 2(1+\cos\theta) = 4\cos^2\left(\frac{\theta}{2}\right)$$

$$= 4\cos^2\left(\frac{\pi}{2n+1}\right)$$

Hence, the required equation is

$$x^2 + x + \frac{1}{4\cos^2\left(\frac{\pi}{2n+1}\right)} = 0$$

62. Given,

$z^{12} - 56z^6 - 512 = 0$

$\Rightarrow \quad (z^6)^2 - 56z^6 - 512 = 0$

$\Rightarrow \quad a^2 - 56a - 512 = 0, a = z^6$

$\Rightarrow \quad a^2 - 64a + 8a - 512 = 0$

$\Rightarrow \quad a(a-64) + 8(a-64) = 0$

$\Rightarrow \quad (a-64)(a+8) = 0$

$\Rightarrow \quad a = -8, 64$

When $a = -8$,

$z^6 = -8 = (i\sqrt{2})^6$

$\Rightarrow \quad z = \pm(i\sqrt{2})$

When $a = 64$,

$z^6 = 64 = 2^6$

$\Rightarrow \quad z = \pm 2$

Hence, the complex numbers are

$z = \pm 2, \pm i\sqrt{2}$

63. Let $z = x + iy$.

We have

$|z-4| = |z-8|$

$\Rightarrow \quad |(x+iy)-4| = |(x+iy)-8|$

$\Rightarrow \quad |(x-4)+iy| = |(x-8)+iy|$

$\Rightarrow \quad \sqrt{(x-4)^2+y^2} = \sqrt{(x-8)^2+y^2}$

$\Rightarrow \quad (x-4)^2+y^2 = (x-8)^2+y^2$

$\Rightarrow \quad (x-4)^2 = (x-8)^2$

$\Rightarrow \quad x^2-8x+16 = x^2-16x+64$

$\Rightarrow \quad 8x = 64-16 = 48$

$\Rightarrow \quad x = 6$

Also, $3|z-12| = 5|z-8i|$

$\Rightarrow \quad 3(x-12)+iy| = 5|x+i(y-8)|$

$\Rightarrow \quad 3\sqrt{(x-12)^2+y^2} = 5\sqrt{x^2+(y-8)^2}$

$\Rightarrow \quad 3\sqrt{y^2+36} = 5\sqrt{36+(y-8)^2}$

$\Rightarrow \quad 9(y^2+36) = 25(36+(y-8)^2)$

$\Rightarrow \quad 9(y^2+36) = 25(y^2-16y+100)$

$\Rightarrow \quad 16y^2-400y+2176 = 0$

$\Rightarrow \quad y^2-25y+136 = 0$

$\Rightarrow \quad (y-17)(y-8) = 0$

$\Rightarrow \quad y = 8, 17$

Hence, $\text{Im}(z) = y + 8, 17$

64. We have $z + \dfrac{1}{z} = 2\cos(2°)$

$\Rightarrow \quad z^2 - 2\cos(2°)z + 1 = 0$

$$\Rightarrow \quad z = \frac{2\cos(2°) \pm \sqrt{4\cos^2(2°)-4}}{2}$$

$$\Rightarrow \quad = \frac{2\cos(2°) \pm 2\sqrt{\cos^2(2°)-1}}{2}$$

$\Rightarrow \quad = \cos(2°) \pm i\sin(2°)$

$\Rightarrow \quad = \cos(2°) + i\sin(2°), \cos(2°) - i\sin(2°)$

$\Rightarrow \quad = e^{i(2°)}, e^{-i(2°)}$

Now,

$$z^{2010} + \frac{1}{z^{2010}} + 3 = e^{i(4020°)} + e^{-i(4020°)} + 3$$
$$= 2\cos(4020°) + 3$$
$$= 2\cos(90 \times 44 + 60°) + 3$$
$$= 2\cos(60°) + 3$$
$$= 2 \times \frac{1}{2} + 3$$
$$= 3$$

65. We have,

$(z^5+3)^2 = -16$

$\Rightarrow \quad (z^5+3) = \pm 4i$

$\Rightarrow \quad z^5 = -3 \pm 4i$

$\Rightarrow \quad |z^5| = |-3 \pm 4i| = 5$

$\Rightarrow \quad |z| = 5^{1/5}$

Hence, the value of $|z|$ is $5^{1/5}$.

66. Let $z = x + iy$

Given circles reduces to

$(x-2)^2 + y^2 = 9$ and $(x-2)^2 + (y-3)^2 = 16$

Hence, the equation of the radical axis is

$(x-2)^2 + (y-3)^2 - (x-2)^2 - y^2 = 16 - 9$

$\Rightarrow \quad (y-3)^2 - y^2 = 7$

$\Rightarrow \quad y^2 - 6y + 9 - y^2 = 7$

$\Rightarrow \quad 6y - 2 = 0$

$\Rightarrow \quad 3y - 1 = 0$

67. We have

$$\frac{x^3}{x^2} = \frac{2+11i}{3+4i}$$

$$\Rightarrow \quad x = \frac{2+11i}{3+4i}$$

$$\Rightarrow \quad x = \frac{(2+11i)(3-4i)}{(3+4i)(3-4i)}$$

$$\Rightarrow \quad x = \frac{6-8i+33i+44}{25}$$

$$\Rightarrow \quad x = \frac{50+25i}{25} = 2+i$$

$\Rightarrow \quad a = 2, b = 1$

Hence, the value of

$(a+b+2) = 2+1+2$
$= 5$

68. Let $z = x + iy$

We have $2016\,\text{Im}\left(\dfrac{1}{z}\right) - 1 = 0$

$$\Rightarrow \quad 2016\,\text{Im}\left(\frac{1}{x+iy}\right) = 1$$

$$\Rightarrow \quad 2016\,\text{Im}\left(\frac{x-iy}{x^2+y^2}\right) = 1$$

$$\Rightarrow \quad 2016\left(\frac{-y}{x^2+y^2}\right) = 1$$

$\Rightarrow \quad x^2 + y^2 = -2016y$

$\Rightarrow \quad x^2 + y^2 + 2016y = 0$

Hence, the locus of z is a circle.

69. Let $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$

We have

$z^{n-1} = \bar{z}$

$\Rightarrow \quad r^{n-1}\, e^{i(n-1)\theta} = re^{-i\theta}$

$\Rightarrow \quad r^{n-1}\, e^{i(n-1)\theta} - re^{-i\theta} = 0$

$\Rightarrow \quad re^{-i\theta}(r^{n-2}\, e^{in\theta} - 1) = 0$

$\Rightarrow \quad r = 0, (r^{n-2}\, e^{in\theta} - 1) = 0$

$\Rightarrow \quad re^{-i\theta}(r^{n-2}\, e^{in\theta} - 1) = 0$

$\Rightarrow \quad (r^{n-2}\, e^{in\theta} - 1) = 0$

$\Rightarrow \quad r^{n-2}\, e^{in\theta} = 1 = e^{i2k\pi}, k \in I$

Now, $r^{n-2} = 1$

$$\Rightarrow \quad r = \begin{cases} 1: & n \neq 2 \\ R - \{1\}: & n = 2 \end{cases}$$

and $\quad \theta = \dfrac{2k\pi}{n}, k \in I$

Hence, the required values of z are

$$z = \begin{cases} 0 \\ e^{i\frac{2k\pi}{n}} : & k \in I, n \neq 2 \\ re^{i(k\pi)} : & k \in I, r \in R - \{1\}, n = 2 \end{cases}$$

70. Let $S = \sum_{k=1}^{n-1} (n-k) \cos\left(\frac{2k\pi}{n}\right)$

$$= \sum_{k=1}^{n-1} (n-k)\,\omega^k$$

where $\omega = e^{i\frac{2\pi}{n}}, k = 1, 2, \ldots, n$ represents the nth root of unity.

Thus,

$$S = (n-1)\omega + (n-2)\omega^2 + (n-3)\omega^3 + \ldots + \omega^{n-1}$$

$\therefore \quad \omega S = (n-1)\omega^2 + (n-2)\omega^3 + \ldots + 2\omega^{n-1} + \omega^n$

Subtracting, we get

$$\Rightarrow \quad (1-\omega)S = (n-1)\omega - \omega^2 - \omega^3 - \ldots - 2\omega^{n-1} - \omega^n$$
$$= n\omega - (\omega + \omega^2 + \omega^3 + \ldots + \omega^{n-1} + \omega^n)$$
$$= n\omega - (\omega + \omega^2 + \omega^3 + \ldots + \omega^{n-1} + 1)$$
$$= n\omega - 0$$

$$\Rightarrow \quad S = \frac{n\omega}{(1-\omega)}$$

Now, we have

$$\frac{\omega}{(1-\omega)} = \frac{e^{i\theta}}{1-e^{i\theta}}$$
$$= \frac{e^{i\theta}}{(1-e^{i\theta})} \times \frac{(1-e^{-i\theta})}{(1-e^{-i\theta})}$$
$$= \frac{(e^{i\theta}-1)}{(1-(e^{i\theta}+e^{-i\theta})+1)}$$
$$= \frac{(\cos\theta - 1) + i\sin\theta}{(2 - 2\cos\theta)}$$
$$= \frac{(\cos\theta - 1) + i\sin\theta}{2(1-\cos\theta)}$$
$$= -\frac{1}{2} + \mathrm{i}\frac{\sin\theta}{2(1-\cos\theta)}$$

Therefore, $S = -\frac{n}{2} + i\frac{n\sin\theta}{2(1-\cos\theta)}$

$$\Rightarrow \quad \sum_{k=1}^{n-1} (n-k)e^{ik\theta} = -\frac{n}{2} + i\frac{n\sin\theta}{2(1-\cos\theta)}$$

Comparing the real parts, we get

$$\sum_{k=1}^{n-1} (n-k)\cos(k\theta) = -\frac{n}{2}$$

Hence, the result.

71. Given $z_1 + z_2 + z_3 = A$, …(i)

$z_1 + z_2\omega^2 + z_3\omega = B$ …(ii)

$z_1 + z_2\omega^2 + z_3\omega = C$ …(iii)

Adding Eqs (i), (ii) and (iii), we get,

$$3z_1 + z_2(1 + \omega + \omega^2) + z_3(1 + \omega + \omega^2) = A + B + C$$

Thus, $z_1 = \frac{1}{3}(A + B + C)$

Now multiplying Eqs (i), (ii) and (iii) by 1, ω^2 and ω respectively and adding, we get

$$z_1(1 + \omega^2 + \omega) + z_2(1 + \omega^3 + \omega) + z_3(1 + \omega^4 + \omega^2) = A + B\omega^2 + C\omega$$

Thus, $3z_2 = A + B\omega^2 + C\omega$

$$\Rightarrow \quad z_2 = \frac{1}{3}(A + B\omega^2 + C\omega)$$

Similarly, multiplying Eqs (i), (ii) and (iii) by 1, ω and ω^2 respectively and adding, we get

$$z_3 = \frac{1}{3}(A + B\omega + C\omega^2)$$

72. Let $z = e^{i\theta}$

We have $\frac{e^{i\theta}-1}{e^{i\theta}+1} = i\tan\left(\frac{\theta}{2}\right)$

Applying componendo and dividendo, we get,

$$\frac{2e^{i\theta}}{-2} = \frac{i\tan\left(\frac{\theta}{2}\right)+1}{i\tan\left(\frac{\theta}{2}\right)-1}$$

$$\Rightarrow \quad e^{i\theta} = \frac{1+i\tan\left(\frac{\theta}{2}\right)}{1-i\tan\left(\frac{\theta}{2}\right)}$$

$$\Rightarrow \quad z = \frac{1+i\tan\left(\frac{\theta}{2}\right)}{1-i\tan\left(\frac{\theta}{2}\right)}$$

$$= \frac{1+i\tan\left(\frac{\text{Arg } z}{2}\right)}{1-i\tan\left(\frac{\text{Arg } z}{2}\right)}$$

Hence, the result.

73. Let $a = \cos\theta + 2\cos\varphi + 3\cos\psi$

and $b = \sin\theta + 2\sin\varphi + 3\sin\psi$

Now, $a + ib$

$$= (\cos\theta + i\sin\theta) + 2(\cos\varphi + i\sin\varphi) + 3(\cos\psi + i\sin\psi)$$
$$= e^{i\theta} + 2e^{i\varphi} + 3e^{i\psi} = 0 + i.0 = 0$$

Let $z_1 = e^{i\theta}$, $2e^{i\varphi} + 3e^{i\psi} = 0 + i.0 = 0$

Thus, $z_1 + 2z_2 + 3z_3 = 0$

$$\Rightarrow \quad z_1^3 + 8z_2^3 + 27z_3^3 = 3(z_1)(2z_2)(3z_3)$$

$$\Rightarrow \quad e^{i3\theta} + 8e^{i3\varphi} + 27e^{i3\psi} = 18e^{i(\theta+\varphi+\psi)}$$

Comparing the real and imaginary parts, we get

$\cos 3\theta + 8\cos 3\varphi + 27\cos 3\psi = 18\cos(\theta + \varphi + \psi)$

$\sin 3\theta + 8\sin 3\varphi + 27\sin 3\psi = 18\sin(\theta + \varphi + \psi)$

Let $c = \cos(\theta + \psi) + 2\cos(\psi + \theta) + 3\cos(\theta + \varphi)$

and $d = \sin(\theta + \psi) + 2\sin(\psi + \theta) + 3\sin(\theta + \varphi)$

Now, $c + id = e^{i(\theta+\psi)} + 2e^{i(\psi+\theta)} + 3e^{i(\theta+\varphi)}$

$= z_2z_3 + 2z_2z_1 + 3z_1z_2$

$= z_1z_2z_3\left(\frac{1}{z_1} + \frac{2}{z_2} + \frac{3}{z_3}\right)$

As $|z_1| = 1 = |z_2| = |z_3|$

$\Rightarrow z_1\overline{z_1} = z_2\overline{z_2} = z_3\overline{z_3} = 1$

$\Rightarrow z_1z_2z_3(\overline{z_1} + 2\overline{z_2} + 3\overline{z_3})$

$\Rightarrow z_1z_2z_3(\overline{z_1 + 2z_2 + 3z_3})$

$\Rightarrow z_1z_2z_3(z_1 + 2z_2 + 3z_3)$

$\Rightarrow z_1z_2z_3 \times 0 = 0$

Thus $c + id = 0 + i.0$

$\Rightarrow d = 0$

$\Rightarrow \sin(\theta + \psi) + 2\sin(\psi + \theta) + 3\sin(\theta + \varphi) = 0$

74. We have

$x^5 - 1 = (x - 1)(x - \alpha_1)(x - \alpha_2)(x - \alpha_3)(x - \alpha_4)$

$\Rightarrow \frac{x^5 - 1}{(x-1)} = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3)(x - \alpha_4)$

$(x - 1)(x - \alpha_1)(x - \alpha_2)(x - \alpha_3)(x - \alpha_4)$

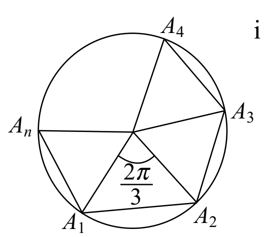
$= 1 + x + x^2 + x^3 + x^4$ …(i)

Putting $x = \omega$ and ω^2 in (i) and dividing, we get

$$\frac{\omega - \alpha_1}{\omega^2 - \alpha_1}\cdot\frac{\omega - \alpha_2}{\omega^2 - \alpha_2}\cdot\frac{\omega - \alpha_3}{\omega^2 - \alpha_3}\cdot\frac{\omega - \alpha_4}{\omega^2 - \alpha_4}$$

$$= \frac{1 + \omega + \omega^2 + \omega^3 + \omega^4}{1 + \omega^2 + \omega^4 + \omega^6 + \omega^8} = \frac{-\omega^2}{-\omega} = \omega$$

75. Assume that the circle is centred at the origin.



$OA_1 = e^{i\frac{2\pi}{n}}, OA_2 = e^{i\frac{4\pi}{n}}$

In general, $OA_p = e^{i\frac{2p\pi}{n}}$

Thus,

$|\overrightarrow{A_1A_p}| = |\overrightarrow{OA_p} - \overrightarrow{OA_1}|$

$= \left|e^{i\frac{2p\pi}{n}} - e^{i\frac{2\pi}{n}}\right|$

$= |e^{i\frac{2\pi}{n}}(e^{i(2p-1)\frac{\pi}{n}} - 1)|$

$= |(e^{i(2p-1)\frac{\pi}{n}} - 1)|$

$= \left|2i\sin\left\{(p-1)\frac{\pi}{n}\right\}e^{i(p-1)\frac{\pi}{n}}\right|$

$= 2\sin\left((p-1)\frac{\pi}{n}\right)$

Therefore,

$|A_1A_2|^2 + |A_1A_3|^2 + |A_1A_4|^2 + \ldots + |A_1A_n|^2$

$$= \sum_{p=1}^{n} 4\sin^2(p-1)\frac{\pi}{n}$$

$$= \sum_{p=1}^{n} 2\left(1 - \cos\left\{2(p-1)\frac{\pi}{n}\right\}\right)$$

$$= \sum_{p=2}^{n} 2\left(1 - \cos\left\{2(p-1)\frac{\pi}{n}\right\}\right)$$

$$= 2\left[(n-1) - \frac{\sin\left((n-1)\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)} \times \cos\left(\frac{2\left(\frac{2\pi}{n}\right) + 2(n-2)\frac{\pi}{n}}{2}\right)\right]$$

$$= 2\left[(n-1) - \frac{\sin\left(\pi - \frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)} \times \cos(\pi)\right]$$

$$= 2\left[(n-1) + \frac{\sin\left(\frac{\pi}{n}\right)}{\sin\left(\frac{\pi}{n}\right)}\right]$$

$= 2[(n - 1) + 1]$

$= 2n$

Level IV

1. Given $|z_1| = 1 = |z_2| = |z_3|$

$E = |z_1 - z_2|^2 + |z_2 - - z_3|^2 + |z_3 - z_1|^2$

$= 2(|z_1|^2 + 1 + |z_2|^2 + |z_3|^2)$

$-2(\text{Re}(z_1\overline{z_2}) + \text{Re}(z_2\overline{z_3}) + \text{Re}(z_3\overline{z_1}))$ …(i)

Also, $|z_1 + z_2 + z_3|^2$

$= |z_1|^2 + 1 + |z_2|^2 + |z_3|^2$

$+ 2(\text{Re}(z_1\overline{z}_2) + \text{Re}(z_2\overline{z}_3) + \text{Re}(z_3\overline{z}_1))$

$= 3 + 2[\text{Re}(z_1\overline{z}_2) + \text{Re}(z_2\overline{z}_3) + \text{Re}(z_3\overline{z}_1)]$

$\Rightarrow 3 + 2[\text{Re}(z_1\overline{z}_2) + \text{Re}(z_2\overline{z}_3) + \text{Re}(z_3\overline{z}_1)]$

$= |z_1 + z_2 + z_3|^2 \geq 0$

$\Rightarrow 2[\text{Re}(z_1\overline{z}_2) + \text{Re}(z_2\overline{z}_3) + \text{Re}(z_3\overline{z}_1)] \geq -3$ …(ii)

From Relations (i) and (ii), we get

$E = |z_1 - z_2|^2 + |z_2 - z_3|^2 + |z_3 - z_1|^2$

$\leq 2(|z_1|^2 + |z_1|^2 + |z_1|^2) + 3 = 2.3 + 3 = 9$

Hence, the maximum value of E is 9.

2. Let $z = x + iy$

Then the area of the triangle is

$$= \frac{1}{2}\begin{vmatrix} -x & -y & 1 \\ -y & x & 1 \\ x+y & y-x & 1 \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} -x & -y & 1 \\ -y & x & 1 \\ 0 & 0 & 3 \end{vmatrix} (R_3 \to R_3 + (R_1 + R_2))$$

$$= \frac{3}{2}\begin{vmatrix} -x & -y \\ -y & x \end{vmatrix}$$

$$= \frac{3}{2}(x^2 + y^2) = \frac{3}{2}|z|^2$$

3. Let $|z| = \frac{1}{2}$.

   Consider $z_1 = -1 + 4z$

   $$z = \frac{z_1 + 1}{4}$$

   $$|z| = \left|\frac{z_1 + 1}{4}\right|$$

   $$\left|\frac{z_1 + 1}{4}\right| = |z| = \frac{1}{2}$$

   $$|z_1 + 1| = 2$$

   Thus, z_1 lies on a circle with radius 2.

4. Given $|z_1| = 1$, $|z_2| = 2$, $|z_3| = 3$

   $\Rightarrow \quad |z_1|^2 = 1$

   $\Rightarrow \quad z_1 \cdot \overline{z_1} = 1$

   Similarly, $z_2 \cdot \overline{z_2} = 4$

   and $\quad z_3 \cdot \overline{z_3} = 9$

   Now, $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 12$

   $\Rightarrow \quad |z_1z_2z_3 \cdot \overline{z_3} + z_1z_2z_3 \cdot \overline{z_2} + z_1z_2z_3 \cdot \overline{z_1}| = 12$

   $\Rightarrow \quad |(z_1z_2z_3)(\overline{z_3} + \overline{z_2} + \overline{z_1})| = 12$

   $\Rightarrow \quad |z_1z_2z_3)||(\overline{z_3} + \overline{z_2} + \overline{z_1})| = 12$

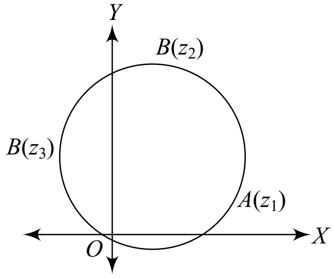
   $\Rightarrow \quad |z_1||z_2||z_2||\overline{(z_1 + z_2 + z_3)}| = 12$

   $\Rightarrow \quad 6|z_1 + z_2 + z_3| = 12$

   $\Rightarrow \quad |z_1 + z_2 + z_3| = 2$

   Hence, the result.

5.



Let angles at A and B be α and β.

Anti-clockwise rotation about A and C gives

$$-z_1 = (z_2 - z_1)e^{i\alpha} \text{ and } (z_2 - z_3) = -z_3e^{i\beta}$$

Multiplying, we get

$$z_1z_2 - z_1z_3 = (z_2z_3 - z_1z_3)e^{i(\alpha+\beta)}$$

If the points are concyclic, then $\alpha + \beta = \pi$

Thus, $z_1z_2 - z_1z_3 = (z_2z_3 - z_1z_3)(-1)$

$\Rightarrow \quad z_1z_3 - z_1z_3 = (-z_2z_3 + z_1z_3)$

$\Rightarrow \quad z_1z_2 + z_2z_3 = 2z_1z_3$

$\Rightarrow \quad \dfrac{z_1z_2}{z_1z_2z_3} + \dfrac{z_2z_3}{z_1z_2z_3} = \dfrac{2z_1z_3}{z_1z_2z_3}$

$\Rightarrow \quad \dfrac{1}{z_3} + \dfrac{1}{z_1} = \dfrac{2}{z_2}$

$\Rightarrow \quad \dfrac{1}{z_1} + \dfrac{1}{z_3} = \dfrac{2}{z_2}$

Hence, the result.

6. Do yourself

7. $N = (a + ib)^3 - 107i$

   $N + 107i = (a + ib)^3$

   $= a^3 + i3a^2b - 3ab^2 - ib^3$

   $= (a^3 - 3ab^2) + i(3a^2b - b^3)$

   Clearly, $b(3a^2 - b^2) = 107$

   $\Rightarrow \quad a = 6, b = 1$

   Thus, $N = (a^3 - 3ab^2)$

   $= a(a^2 - 3b^2)$

   $= 6(36 - 3.1)$

   $= 6.33$

   $= 198.$

8. We have

   $(1 + i)z^2 = (1 + i)(x + iy)^2$

   $= (1 + i)(x^2 - y^2 + i \cdot 2xy)$

   $= (x^2 - y^2 - 2xy) + i(x^2 - y^2 + 2xy)$

   Thus, $\text{Re}\{(1 + i)z^2\} = (x^2 - y^2 - 2xy)$

   $= (x - y)^2 - 2y^2$

   It is given that $(x - y)^2 - 2y^2 > 0$

   $\Rightarrow \quad (x - y + \sqrt{2}y)(x - y - \sqrt{2}y) > 0$

   $\Rightarrow \quad [(\sqrt{2} - 1)y + x][(\sqrt{2 + 1})y - x] < 0$

   i.e. $\quad y = -\dfrac{x}{(\sqrt{2} - 1)} = (\sqrt{2 + 1})x$

   and $\quad y = \dfrac{x}{(\sqrt{2} + 1)} = (\sqrt{2} - 1)x$

   Required set is constituted by the angles without their boundaries, whose sides are striaght lines $y = (\sqrt{2} - 1)x$, $y = -(\sqrt{2} + 1)x$ containing the x-axis

9. It is given that

   $f(z) = |z^3 - z = 2|$

   $f(\theta) = |(e^{i\theta})^3 - (e^{i\theta}) + 2|$

   $= f(\theta) = |(e^{i3\theta}) - (e^{i\theta}) + 2|$

   $= |\cos(3\theta) + i\sin(3\theta) - \cos\theta - \sin\theta + 2|$

$= |(\cos(3\theta) - \cos\theta + 2) + i(\sin(3\theta) - \sin\theta)|$

$= \sqrt{(\cos(3\theta) - \cos\theta + 2)^2 + (\sin(3\theta) - \sin\theta)^2}$

$= \sqrt{\begin{aligned}&\cos^2(3\theta) + \cos^2\theta + 4 - 2\cos 3\theta\cos\theta \\ &+ 4\cos 3\theta - 4\cos\theta + \sin^2(3\theta) + \sin^2\theta \\ &- 2\sin 3\theta\sin\theta\end{aligned}}$

$= \sqrt{6 - 2\cos 2\theta + 4(\cos 3\theta - \cos\theta)}$

$= \sqrt{4(\cos 3\theta - \cos\theta) - 2\cos 2\theta + 6}$

Let $g(\theta) = 4(\cos 3\theta - \cos\theta) - 2\cos\theta + 6$

$g'(\theta) = 4(-3\sin 3\theta + \sin\theta) + 4\sin 2\theta$

For max or min, g¢(q) = 0 gives

$4(-3\sin 3\theta + \sin\theta) + 4\sin 2\theta = 0$

$(-3\sin 3\theta + \sin\theta) + \sin 2\theta = 0$

$\sin 2\theta + \sin\theta = 3\sin 3\theta$

$2\sin\left(\frac{3\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = 6\sin\left(\frac{3\theta}{2}\right)\cos\left(\frac{3\theta}{2}\right)$

$\sin\left(\frac{3\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = 3\sin\left(\frac{3\theta}{2}\right)\cos\left(\frac{3\theta}{2}\right)$

$\sin\left(\frac{3\theta}{2}\right)\left(\cos\left(\frac{\theta}{2}\right) - 3\cos\left(\frac{3\theta}{2}\right)\right) = 0$

$\sin\left(\frac{3\theta}{2}\right) = 0 \left(\cos\left(\frac{\theta}{2}\right) - 3\cos\left(\frac{3\theta}{2}\right)\right) = 0$

$\sin\left(\frac{3\theta}{2}\right) = 0 = \sin(\pi)$

$\left(\frac{3\theta}{2}\right) = (\pi)$

$\theta = \frac{2\pi}{3}$

Hence, the maximum value of f

$= \sqrt{\begin{aligned}&4\left(\cos 3\left(\frac{2\pi}{3}\right) - \cos\left(\frac{2\pi}{3}\right)\right) \\ &- 2\cos 2\left(\frac{2\pi}{3}\right) + 6\end{aligned}}$

$= \sqrt{4\left(1 + \frac{1}{2}\right) + 2\cdot\frac{1}{2} + 6}$

$= \sqrt{6 + 1 + 6}$

$= \sqrt{13}$

10. It is given that, $|z| \le 2$

$\Rightarrow \sqrt{x^2 + y^2} \le 2$

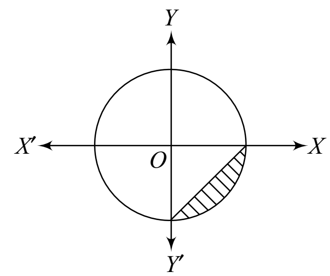
$\Rightarrow x^2 + y^2 \le 4$ ......(i)

Also, $(1 - i)z + (1 + i)\bar{z} \le 4$

$\Rightarrow (z + \bar{z}) + i(\bar{z} - z) \le 4$

$\Rightarrow (2x) - i(2y) \le 4$

$\Rightarrow x - iy \le 2$ ....(ii)



From Relations (i) and (ii), we get,

$A \cap B$ = area of the shaded part

$= \left(\frac{4\pi}{4} - \frac{1}{2}\times 2\times 2\right)$ sq.u

$= (\pi - 2)$ sq.u

11. Put $\left(\frac{1+i}{2}\right) = x$,

LHS $= (1 + x)(1 + x^2)(1 + x^4)\ldots(1 + x^{2n})$

$= \frac{(1-x)(1+x)(1+x^2)(1+x^4)\ldots(1+x^{2n})}{(1-x)}$

$= \frac{(1-x^2)(1+x^2)(1+x^4)\ldots(1+x^{2n})}{(1-x)}$

⋮

$= \frac{(1-x^{2n})(1+x^{2n})}{(1-x)}$

$= \frac{1-(x^2)^{2n}}{(1-x)}$

$= \frac{1-\left(\frac{i}{2}\right)^{2n}}{1-\left(\frac{1+i}{2}\right)}$

12. Given,

$z^3 + iz = 1$ .....(i)

$\Rightarrow \overline{z^3 + iz} = \bar{1}$

$\Rightarrow \overline{z^3} - i(\bar{z}) = 1$

$\Rightarrow (\bar{z})^3 - i(\bar{z}) = 1$ ...(ii)

From Eqs (i) and (ii), we get,

$z^3 - (\bar{z})^3 + i(z + \bar{z}) = 0$

$\Rightarrow (z - \bar{z})(z^2 + (\bar{z}) + z\bar{z}) + i(z + \bar{z}) = 0$

put $z = x + iy$,

$2iy(3x^2 - y^2) + 2ix = 0$

$\Rightarrow y(3x^2 - y^2) + x = 0$

which is always passing through the origin and never crosses the coordinates axes.

2nd part

Given $z^3 + iz = 1$

$\Rightarrow z^2 + i = \frac{1}{z}$

$= \frac{\bar{z}}{|z|^2}$

$\Rightarrow \quad (x+iy)^2 + i = \dfrac{x-iy}{x^2+y^2}$

$\Rightarrow \quad (x^2 - y^2 + i2xy + i) = \dfrac{x-iy}{x^2+y^2}$

$\Rightarrow \quad ((x^2 - y^2) + i(2xy+1)) = \dfrac{x-iy}{x^2+y^2}$

Comparing the imaginary part, we get

$2xy + 1 = -\dfrac{y}{x^2+y^2}$

$\Rightarrow \quad x^2 + y^2 = -\dfrac{y}{2xy+1}$

$\Rightarrow \quad |z|^2 = -\dfrac{\operatorname{Im}(z)}{2\operatorname{Re}(z)\operatorname{Im}(z)+1}$

$\Rightarrow \quad |z| = \sqrt{-\dfrac{\operatorname{Im}(z)}{2\operatorname{Re}(z)\operatorname{Im}(z)+1}}$

Hence, the result.

13. Do yourself

14. Given $\alpha = e^{i\frac{2\pi}{7}}$

$\Rightarrow \quad \alpha^7 = e^{i2\pi} = \cos(2\pi) + i\sin(2\pi)$
$= 1$

Now, $f(x) + f(\alpha x) + f(\alpha^2 x) + \ldots + f(\alpha^6 x)$

$= 7A_0 + \sum_{k=1}^{20} A_k x^k (1 + \alpha^k + \alpha^{2k} + \ldots + \alpha^{6k})$

$= 7A_0 + \sum_{k=1}^{20} A_k x^k [1 + \alpha^k + (\alpha^k)^2 + \ldots + (\alpha^k)^6]$

$= 7A_0 + \sum_{k=1}^{20} A_k x^k \left(\dfrac{1-\alpha^{7k}}{1-\alpha^k}\right)$

$= 7A_0 + \sum_{k=1}^{20} A_k x^k \left(\dfrac{1-(\alpha^7)^k}{1-\alpha^k}\right)$

$= 7A_0 + \sum_{k=1}^{20} A_k x^k \left(\dfrac{1-1}{1-\alpha^k}\right)$

$= 7A_0 + \sum_{k=1}^{20} A_k x^k \cdot 0$

$= 7A_0$

which is independent of α.

15. We have $z^7 - 1 = 0$

$\Rightarrow \quad z^7 = 1$

$\Rightarrow \quad z = \cos\left(\dfrac{2r\pi}{7}\right) + i\sin\left(\dfrac{2r\pi}{7}\right)$

where $r = 0, 1, 2, \ldots, 6$

$z = 1, \cos\left(\dfrac{2\pi}{7}\right) \pm i\sin\left(\dfrac{2\pi}{7}\right)$

$\Rightarrow \quad \cos\left(\dfrac{4\pi}{7}\right) \pm i\sin\left(\dfrac{4\pi}{7}\right), \cos\left(\dfrac{6\pi}{7}\right) \pm i\sin\left(\dfrac{6\pi}{7}\right)$

Thus, $z^7 - 1 = (z-1)\left[z^2 - 2\cos\left(\dfrac{2\pi}{7}\right)z + 1\right]$

$\left[z^2 - 2\cos\left(\dfrac{4\pi}{7}\right)z + 1\right]$

$\left[z^2 - 2\cos\left(\dfrac{6\pi}{7}\right)z + 1\right]$

16. We have,

$z^7 + 1 = 0$

$\Rightarrow \quad z^7 = -1$

$\Rightarrow \quad z = \cos\left(\dfrac{(2r+1)\pi}{7}\right) + i\sin\left(\dfrac{(2r+1)\pi}{7}\right)$

where $r = 0, 1, 2, \ldots, 6$

$z = -1, \cos\left(\dfrac{\pi}{7}\right) \pm i\sin\left(\dfrac{\pi}{7}\right)$

$\Rightarrow \quad \cos\left(\dfrac{3\pi}{7}\right) \pm i\sin\left(\dfrac{3\pi}{7}\right), \cos\left(\dfrac{5\pi}{7}\right) \pm i\sin\left(\dfrac{5\pi}{7}\right)$

$\Rightarrow \quad z^7 + 1 = (z+1)\left(z^2 - 2\cos\left(\dfrac{\pi}{7}\right)z + 1\right)$

$\left(z^2 - 2\cos\left(\dfrac{3\pi}{7}\right)z + 1\right)$

$\left(z^2 - 2\cos\left(\dfrac{5\pi}{7}\right)z + 1\right)$

Put $z = i$,

$\dfrac{i^7+1}{1+i} = \left[-2\cos\left(\dfrac{\pi}{7}\right)i\right]\left[-2\cos\left(\dfrac{3\pi}{7}\right)i\right]$

$\left(-2\cos\left(\dfrac{5\pi}{7}\right)i\right)$

$\Rightarrow \quad \dfrac{-i+1}{1+i} = 8\cos\left(\dfrac{\pi}{7}\right)\cos\left(\dfrac{3\pi}{7}\right)\cos\left(\dfrac{5\pi}{7}\right)i$

$\Rightarrow \quad \dfrac{-i+1}{1+i} \times \dfrac{1-i}{1-i} = 8i\cos\left(\dfrac{\pi}{7}\right)\cos\left(\dfrac{3\pi}{7}\right)\cos\left(\dfrac{5\pi}{7}\right)$

$\Rightarrow \quad \cos\left(\dfrac{\pi}{7}\right)\cos\left(\dfrac{3\pi}{7}\right)\cos\left(\dfrac{5\pi}{7}\right) = -\dfrac{1}{8}$

$\Rightarrow \quad 8i\cos\left(\dfrac{\pi}{7}\right)\cos\left(\dfrac{3\pi}{7}\right)\cos\left(\dfrac{5\pi}{7}\right) = -\dfrac{2i}{2}$

$\Rightarrow \quad \cos\left(\dfrac{\pi}{7}\right)\cos\left(\dfrac{3\pi}{7}\right)\cos\left(\dfrac{5\pi}{7}\right) = -\dfrac{1}{8}$

Hence, the result.

17. We have $|z_1 + \omega^k z_2|^2$

$= |z_1|^2 + |\omega^k z_2|^2 + z_1 \overline{\omega^k}\, \overline{z_2} + \overline{z_1}\omega^k z_2$

$= |z_1|^2 + |z_2|^2 + \overline{\omega^k}\, z_1 \overline{z_2} + \omega^k \overline{z_1} z_2$

Thus, $\sum_{k=0}^{n-1} |z_1 + \omega^k z_2|$

$$= n(|z_1|^2 + |z_2|^2) + z_1\overline{z_2}\sum_{k=0}^{n-1}\overline{\omega^k} + \overline{z_1}z_2\sum_{k=0}^{n-1}\omega^k$$

$$= n(|z_1|^2 + |z_2|^2) + z_1\overline{z_2}\cdot 0 + \overline{z_1}z_2\cdot 0$$

$$= n(|z_1|^2 + |z_2|^2)$$

18. Let $z = r(\cos\theta + i\sin\theta)$

Also, $|z - 1| = 1$

$\Rightarrow$ $|r(\cos\theta + i\sin\theta) - 1| = 1$

$\Rightarrow$ $|(r\cos\theta - 1) + i(r\sin\theta)| = 1$

$\Rightarrow$ $\sqrt{(r\cos\theta - 1)^2 + r^2\sin^2\theta} = 1$

$\Rightarrow$ $(r\cos\theta - 1)^2 + r^2\sin^2\theta = 1$

$\Rightarrow$ $r^2 - 2r\cos\theta = 0$

$\Rightarrow$ $r = 2\cos\theta$

Now, $\dfrac{z-2}{z} = \dfrac{r(\cos\theta + i\sin\theta) - 2}{r(\cos\theta + i\sin\theta)}$

$$= \frac{2\cos\theta(\cos\theta + i\sin\theta) - 2}{2\cos\theta(\cos\theta + i\sin\theta)}$$

$$= \frac{\cos\theta(\cos\theta + i\sin\theta) - 1}{\cos\theta(\cos\theta + i\sin\theta)}$$

$$= \frac{(2\cos^2\theta + i2\cos\theta\sin\theta) - 2}{2\cos\theta(\cos\theta + i\sin\theta)}$$

$$= \frac{-1 + \cos 2\theta + i\sin 2\theta}{(1 + \cos 2\theta) + i\sin 2\theta}$$

$$= \frac{i2\sin 2\theta}{2 + 2\cos 2\theta}$$

$$= \frac{\mathrm{i}\sin 2\theta}{1 + 1\cos 2\theta}$$

$$= \frac{i2\sin\theta\cos\theta}{2\cos^2\theta}$$

$$= i\tan\theta$$

$$= i\tan(\operatorname{Arg} z)$$

Hence, the result.

19. Given $z = \cos\left(\dfrac{2\pi}{2n+1}\right) + i\sin\left(\dfrac{2\pi}{2n+1}\right), n \in I^+$

$\Rightarrow$ $z^{2n+1} = 1.$

Now, $\alpha = z + z^3 + z^5 + \ldots + z^{2n-1}$

$$= z\left(\frac{1 - z^{2n}}{1 - z^2}\right)$$

$$= \left(\frac{z - z^{2n+1}}{1 - z^2}\right)$$

$$= \left(\frac{z - 1}{1 - z^2}\right)$$

$$= -\frac{1}{z+1}$$

Also, $\beta = z^2 + z^4 + z^6 + \ldots + z^{2n}$

$$= z^2\left(\frac{1 - z^{2n}}{1 - z^2}\right)$$

$$= z\left(\frac{z - z^{2n+1}}{1 - z^2}\right)$$

$$= z\left(\frac{z - 1}{1 - z^2}\right)$$

$$= \frac{1}{1+z}$$

Now, $\alpha + \beta = -\dfrac{z+1}{z+1} = -1$

and $\alpha\cdot\beta = \dfrac{z}{(z+1)^2} = \dfrac{1}{z + \dfrac{1}{z} + 2}$

Again,

$$z + \frac{1}{z} = (\cos\theta + i\sin\theta) + (\cos\theta - i\sin\theta)$$

$$= 2\cos\theta, \text{ where } \theta = \left(\frac{2\pi}{2n+1}\right)$$

$\Rightarrow$ $z + \dfrac{1}{z} + 2 = 2\cos\theta + 2 = 2(1 + \cos\theta) = 4\cos^2\left(\dfrac{\theta}{2}\right)$

$$= 4\cos^2\left(\frac{\pi}{2n+1}\right)$$

Hence, the required equation is

$$x^2 + x + \frac{1}{4\cos^2\left(\dfrac{\pi}{2n+1}\right)} = 0$$

20. It is given that ,

$$z + \frac{1}{z} = 2(\cos\theta + i\sin\theta) = 2e^{i\theta}$$

$$z^2 - 2(e^{i\theta})z + 1 = 0$$

$$z = \frac{2e^{i\theta} \pm \sqrt{4e^{i2\theta} - 4}}{2}$$

$$z = e^{i\theta} \pm \sqrt{e^{i2\theta} - 1}$$

$$z = (\cos\theta + i\sin\theta) \pm \sqrt{(\cos(2\theta) - 1) + i\sin(2\theta)}$$

$$= (\cos\theta + i\sin\theta) \pm \sqrt{-2\sin^2\theta + i\sin(2\theta)}$$

$$= (\cos\theta + i\sin\theta) \pm \sqrt{2\sin\theta}\sqrt{-\sin\theta + i\cos\theta}$$

$$= (\cos\theta + i\sin\theta) \pm \sqrt{2\sin\theta}\sqrt{\cos\left(\frac{\pi}{2} + \theta\right) + i\sin\left(\frac{\pi}{2} + \theta\right)}$$

$$= (\cos\theta + i\sin\theta) \pm \sqrt{2\sin\theta}\left(\cos\left(\frac{\pi}{4} + \frac{\theta}{2}\right) + i\sin\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right)$$

Let $\alpha = (\cos\theta + i\sin\theta)$

$$+\sqrt{2\sin\theta}\left(\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right)+i\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right)$$

$\beta = (\cos\theta + i\sin\theta)$

$$-\sqrt{2\sin\theta}\left(\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right)+i\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right)$$

Now, $(\alpha - i) = \left\{\cos\theta + \sqrt{2\sin\theta}\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right\}$

$$+i\left\{\sin\theta + \sqrt{2\sin\theta}\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)-1\right\}$$

$|(\alpha - i)|^2$

$$=\left\{\cos\theta + \sqrt{2\sin\theta}\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right\}^2$$

$$+\left\{\sin\theta + \sqrt{2\sin\theta}\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)-1\right\}^2$$

$= 2 + 2\sin\theta$

$$+2\sqrt{2\sin\theta}\left\{\begin{matrix}\cos\theta\cos\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\\ +\sin\theta\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\end{matrix}\right\}$$

$$-2\left\{\sin\theta + \sqrt{2\sin\theta}\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right\}$$

$$= 2 + 2\sqrt{2}\sin\theta\cos\left(\frac{\pi}{4}+\frac{\theta}{2}-\theta\right)$$

$$-2\sqrt{2\sin\theta}\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$$

$$= 2 + 2\sqrt{2\sin\theta}\left\{\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)-\sin\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right\}$$

$$= 2 + 2\sqrt{2\sin\theta}\left\{\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)-\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right\}$$

$= 2 + 0$

$= 2$

Similarly, $|\beta - i|^2 = 2$

Hence, $|\alpha - i| = |\beta - i|$

21. Do yourself

22. We have $A = \frac{2}{\sqrt{3}}e^{i\frac{\pi}{2}} = \frac{2i}{\sqrt{3}}$

$$B = \frac{2}{\sqrt{3}}e^{-i\frac{\pi}{6}} = \frac{2}{\sqrt{3}}\left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right) = \left(1-\frac{i}{\sqrt{3}}\right)$$

$$C = \frac{2}{\sqrt{3}}e^{-i\frac{5p}{6}} = \frac{2}{\sqrt{3}}\left(-\frac{\sqrt{3}}{2}-\frac{i}{2}\right) = \left(-1-\frac{i}{\sqrt{3}}\right)$$

Now, $AP^2 = x^2 + \left(y - \frac{2}{\sqrt{3}}\right)^2$

$$BP^2 = (x-1)^2 + \left(y + \frac{1}{\sqrt{3}}\right)^2$$

$$CP^2 = (x+1)^2 + \left(y + \frac{1}{\sqrt{3}}\right)^2$$

Thus, $AP^2 + BP^2 + CP^2$

$$= x^2 + \left(y - \frac{2}{\sqrt{3}}\right)^2 + (x-1)^2$$

$$+\left(y + \frac{1}{\sqrt{3}}\right)^2 + (x+1)^2 + \left(y + \frac{1}{\sqrt{3}}\right)^2$$

$$= 3(x^2+y^2) + \left(\frac{4}{3}+\frac{2}{3}+2\right)$$

$= 3(x^2+y^2) + 4$

$= 3\cdot\frac{1}{3} + 4$

$= 5$

23. The given equation s are

$\text{Re}(z) = |z - 2a|$ and $|z - 4a| = 3a$

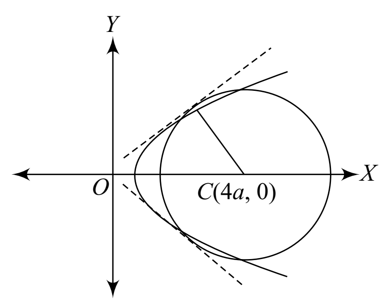
Put $z = x + iy$

The given equations are reduces to

$y^2 = 4a(x-a)$, $(x-4a)^2 + y^2 = 9a^2$

Equation of any tangent to the parabola

$y^2 = 4a(x-a)$ is $y = m(x-a) + \frac{a}{2m}$



$$mx - y + \frac{a}{2m} - \frac{am}{2} = 0$$

It will be a common tangent to circle also

Thus, the length of perpendicular from the centre to the tangent is equal to the radius of the circle

Now,

$$\left|\frac{4am + \frac{a}{m} - am}{\sqrt{m^2+1}}\right| = 3a$$

$$\left|\frac{3am + \frac{a}{m}}{\sqrt{m^2+1}}\right| = 3a$$

$$\left(3am + \frac{a}{m}\right)^2 = 9a^2(m^2+1)$$

$$9a^2m^2 + \frac{a^2}{m^2} + 6a^2 = 9a^2m^2 + 9a^2$$

$$\frac{a^2}{m^2} + 6a^2 = 9a^2$$

$$\frac{a^2}{m^2} = 9a^2 - 6a^2 = 3a^2$$

$$\frac{a^2}{m^2} = 3a^2$$

$$m^2 = \frac{1}{3}$$

$$m = \pm\sqrt{\frac{1}{3}}$$

Thus, the equations of the common tangents are

$$y = \pm\sqrt{\frac{1}{3}}(x-a) \pm a\sqrt{3}$$

24. Given $z\bar{z}^3 + z^3\bar{z} = 350$

$\Rightarrow$ $z\bar{z}(\bar{z}^2 + z^2) = 350$

$\Rightarrow$ $2(x^2 - y^2)(x^2 + y^2) = 350$

$\Rightarrow$ $(x^2 - y^2)(x^2 + y^2) = 175$

$\Rightarrow$ $(x^2 - y^2)(x^2 + y^2) = 7 \times 25$

$\Rightarrow$ $(x^2 - y^2) = 7, (x^2 + y^2) = 25$

Thus $x = 4, y = 3$

Hence, the area of a rectangle $= (2x \cdot 2y)$

$= (4xy)$

$= 48$ sq. u.

25. Given $\beta^n = 1$

Let $S_n = 1 + 3\beta + 5\beta^2 + \ldots + (2n-1)\beta^{n-1}$

$\Rightarrow$ $S_n\beta = \beta + 3\beta^2 + \ldots + (2n-3)\beta^{n-1} + (2n-1)\beta^n$

Subtracting, we get,

$(1-\beta)S_n = 1 + 2\beta + 2\beta^2 + \ldots + 2\beta^{n-1} - (2n-1)\beta^n$

$= 1 + 2\beta + 2\beta^2 + \ldots + 2\beta^{n-1} - (2n-1)$

$= 2\beta + 2\beta^2 + \ldots + 2\beta^{n-1} - 2n$

$= 2\left(\frac{1-\beta^n}{1-\beta}\right) - 2n$

$= -2n$ $\quad (\because \beta^n = 1)$

$\Rightarrow$ $S_n = \frac{2n}{\beta - 1}$

26. Given $x^3 = -9 + 46i$

$\Rightarrow$ $(a + ib)^3 = -9 + 46i$

$\Rightarrow$ $a^3 + i3a^2b - 3ab^2 - ib^3 = -9 + 46i$

$\Rightarrow$ $(a^3 - 3ab^2) + i(3a^2b - b^3) = -9 + 46i$

$\Rightarrow$ $(a^3 - 3ab^2) = -9, (3a^2b - b^3) = 46$

$\Rightarrow$ $a(a^2 - 3b^2) = -9, b(3a^2 - b^2) = 46$

$\Rightarrow$ $a = 3, b = 2$

Hence, the value of

$(a^3 + b^3) = 27 + 8 = 35.$

27. We have $\omega^5 = 2$

Now, $x = \omega + \omega^2$

$\Rightarrow$ $x^5 = (\omega + \omega^2)^5$

$\Rightarrow$ $x^5 = \omega^5 + 5\omega^6 + 10\omega^7 + 10\omega^8 + 5\omega^9 + \omega^{10}$

$\Rightarrow$ $x^5 = 2 + 10\omega + 20\omega^2 + 20\omega^3 + 10\omega^4 + 4$

$\Rightarrow$ $x^5 = 6 + 10(\omega^2 + 2\omega^3 + \omega^4) + 10(\omega^2 + \omega)$

$\Rightarrow$ $x^5 = 6 + 10(\omega + \omega^2)^2 + 10(\omega^2 + \omega)$

$\Rightarrow$ $x^5 = 6 + 10x^2 + 10x$

$\Rightarrow$ $x^5 - 10x^2 - 10x = 6.$

28. Given equation is

$z^2 - (3 + i)z + m + 2i = 0$

It will provide us the real solution only when

$z^2 - 3z + m = 0$ and $2 - z = 0$

$\Rightarrow$ $z^2 - 3z + m = 0$ and $z = 2$

Thus, $m = 3z - z^2$

$\Rightarrow$ $m = 3 \times 2 - 4 = 2$

Hence, the value of m is 2.

29. We have $P(2) = 0$

$\Rightarrow$ $32 + 8a + 4b + 2c + 3 = 0$

$\Rightarrow$ $8a + 4b + 2c = -35$ ...(i)

Also, $P(i) = 2i^4 + ai^3 + bi^2 + ci + 3 = 0$

$\Rightarrow$ $2 - ai - b + ci + 3 = 0$

$\Rightarrow$ $(5 - b) + i(c - a) = 0$

$\Rightarrow$ $(5 - b) = 0, (c - a) = 0$

$\Rightarrow$ $b = 5, c = a$ ...(ii)

From Relations (i) and (ii), we get

$8a + 4b + 2c = -35$

$\Rightarrow$ $8a + 20 + 2a = -35$

$\Rightarrow$ $10a = -55$

$\Rightarrow$ $a = -\frac{55}{10} = -\frac{11}{2}$

30. We have $z^5 + 1 = 0$

Hence, the solutions of z are

$\left\{-1, e^{\pm i\frac{\pi}{5}}, e^{\pm i\frac{3\pi}{5}}\right\}$

$= \{-1, \alpha, \bar{\alpha}, \beta, \bar{\beta}\}$, where $\alpha, \omega, \in C$

Now, $\alpha + \bar{\alpha} = 2\cos\left(\frac{\pi}{5}\right), \alpha \cdot \bar{\alpha} = 1$

and $\beta + \bar{\beta} = 2\cos\left(\frac{3}{5}\right), \beta \cdot \bar{\beta} = 1$

Thus, $z^5 + 1$

$= (z+1)(z-\alpha)(z-\bar{\alpha})(z-\beta)(z-\bar{\beta})$

$= (z+1)(z^2 - (\alpha + \bar{\alpha})z + \alpha \cdot \bar{\alpha})$

$[z^2 - (\beta + \bar{\beta})z + \beta \cdot \bar{\beta}]$

$= (z+1)\left[z^2 - 2\cos\left(\frac{\pi}{5}\right)z + 1\right]$

$\left[z^2 - 2\cos\left(\frac{3\pi}{5}\right)z + 1\right]$

$$\Rightarrow \quad = \frac{z^5+1}{z+1} = \left[z^2 - 2\cos\left(\frac{\pi}{5}\right)z + 1\right]\left[z^2 - 2\cos\left(\frac{3\pi}{5}\right)z + 1\right]$$

Put $z = i$,

$$\Rightarrow \quad \frac{i+1}{i+1} = \left[-1 - 2\cos\left(\frac{\pi}{5}\right)i + 1\right]\left[-1 - 2\cos\left(\frac{3\pi}{5}\right)i + 1\right]$$

$$\Rightarrow \quad 1 = \left[-2\cos\left(\frac{\pi}{5}\right)i\right]\left[-2\cos\left(\frac{3\pi}{5}\right)\right]$$

$$\Rightarrow \quad 1 = \left[-2\cos\left(\frac{\pi}{5}\right)i\right]\left[-2\cos\left(\frac{3\pi}{5}\right)i\right]$$

$$\Rightarrow \quad 4\cos\left(\frac{\pi}{5}\right)\cos\left(\frac{3\pi}{5}\right) = -1$$

$$\Rightarrow \quad 4\cos\left(\frac{\pi}{5}\right)\cos\left(\pi - \frac{2\pi}{5}\right) = -1$$

$$\Rightarrow \quad 4\cos\left(\frac{\pi}{5}\right)\cos\left(\frac{2\pi}{5}\right) = 1$$

$$\Rightarrow \quad 4\cos\left(\frac{\pi}{5}\right)\cos\left(\frac{\pi}{2} - \frac{\pi}{10}\right) = 1$$

$$\Rightarrow \quad 4\cos\left(\frac{\pi}{5}\right)\sin\left(\frac{\pi}{10}\right) = 1$$

Hence, the result.

31. We have,

$$(z+1)^7 + z^7 = 0$$

$$\Rightarrow \quad \left(\frac{z+1}{z}\right)^7 = -1$$

$$\Rightarrow \quad \left(1 + \frac{1}{z}\right)^7 = -1$$

$$\Rightarrow \quad \left(1 + \frac{1}{z}\right) = \cos\left(\frac{2r+1}{7}\right)\pi + i\sin\left(\frac{2r+1}{7}\right)\pi$$

$$\Rightarrow \quad \frac{1}{z} = -1 + \cos\left(\frac{2r+1}{7}\right)\pi + i\sin\left(\frac{2r+1}{7}\right)\pi$$

$$= -2\sin^2\left(\frac{2r+1}{14}\right)\pi + i\sin\left(\frac{2r+1}{7}\right)\pi$$

$$= -2\sin\left(\frac{2r+1}{14}\right)\pi\begin{pmatrix}\sin\left(\frac{2r+1}{14}\right)\pi + \\ i\cos\left(\frac{2r+1}{14}\right)\pi\end{pmatrix}$$

$$= 2i\sin\left(\frac{2r+1}{14}\right)\pi\begin{pmatrix}\cos\left(\frac{2r+1}{14}\right)\pi + \\ i\sin\left(\frac{2r+1}{14}\right)\pi\end{pmatrix}$$

$$\Rightarrow \quad z = \frac{\left(\cos\left(\frac{2r+1}{14}\right)\pi - i\sin\left(\frac{2r+1}{14}\right)\pi\right)}{2i\sin\left(\frac{2r+1}{14}\right)\pi}$$

$$\text{Re}(z) = -\frac{i\sin\left(\frac{2r+1}{14}\right)\pi}{2i\sin\left(\frac{2r+1}{14}\right)\pi} = -\frac{1}{2}$$

Thus, $\sum_{r=1}^{7}\text{Re}(Z_r) = -\frac{7}{2}$

Also, $z = \dfrac{\left(\cos\left(\frac{2r+1}{14}\right)\pi - i\sin\left(\frac{2r+1}{14}\right)\pi\right)}{2i\sin\left(\frac{2r+1}{14}\right)\pi}$

$$\Rightarrow \quad z = \frac{\sin\left(\frac{2r+1}{14}\right)\pi + i\cos\left(\frac{2r+1}{14}\right)\pi}{-2\sin\left(\frac{2r+1}{14}\right)\pi}$$

$$\text{Im}(z) = -\frac{1}{2}\frac{\cos\left(\frac{2r+1}{14}\right)\pi}{\sin\left(\frac{2r+1}{14}\right)\pi} = -\frac{1}{2}\cot\left(\frac{2r+1}{14}\right)\pi$$

Thus, $\sum_{r=1}^{7}\text{Im}(Z_r)$

$$= \sum_{r=1}^{7} -\frac{1}{2}\cot\left(\frac{2r+1}{14}\right)\pi$$

$$= -\frac{1}{2}\sum_{r=1}^{7}\cot\left(\frac{2r+1}{14}\right)\pi$$

$$= -\frac{1}{2}\left[\cot\left(\frac{3\pi}{14}\right) + \cot\left(\frac{5\pi}{14}\right) + \cot\left(\frac{7\pi}{14}\right) + \cot\left(\frac{9\pi}{14}\right) + \cot\left(\frac{11\pi}{14}\right) + \cot\left(\frac{13\pi}{14}\right) + \cot\left(\frac{15\pi}{14}\right)\right]$$

$$= -\frac{1}{2}\left[\cot\left(\frac{3\pi}{14}\right) + \cot\left(\frac{5\pi}{14}\right) + \cot\left(\frac{7\pi}{14}\right) - \cot\left(\frac{5\pi}{14}\right) - \cot\left(\frac{3\pi}{14}\right) - \cot\left(\frac{\pi}{14}\right) + \cot\left(\frac{\pi}{14}\right)\right]$$

$$= 0$$

32. Find all real values of the parameter 'a' or which the equation $(a-1)z^4 - 4z^4 - a + 2 = 0$ has only purely imaginary roots.

33. Given equation is

$$x^4 + ax^3 + bx^2 + cx + d = 0$$

Let its roots are

$$\alpha + i\beta, \alpha - i\beta, \gamma + i\delta, \gamma - i\delta$$

Given $\alpha + i\beta + \gamma + i\delta = 3 + 4i$

$$(\alpha+\gamma) + i(\beta+\delta) = 3 + 4i$$

$$(\alpha+\gamma) = 3, (\beta+\delta) = 4$$

Again, $(\alpha - i\beta)(\gamma - i\delta) = 13 + 11i$

$$(\alpha\gamma - \beta\delta) - i(\alpha\delta + \beta\gamma) = 13 + 11i$$

$$(\alpha\gamma - \beta\delta) = 13, (\alpha\delta + \beta\gamma) = -11$$

Now, $b = \Sigma(\alpha + i\beta)(\alpha - i\beta)$

$$= (\alpha+i\beta)(\alpha-i\beta) + (\alpha+i\beta)(\gamma+i\delta) + (\alpha+i\beta)(\gamma-i\delta) + (\alpha-i\beta)(\gamma+i\delta) + (\alpha-i\beta)(\gamma-i\delta) + (\gamma+i\delta)(\gamma-i\delta)$$

$$= \alpha^2+\beta^2+\gamma^2+\delta^2 + (a+i\beta)(2\gamma) + (\alpha-i\beta)(2\gamma)$$

$$= \alpha^2+\beta^2+\gamma^2+\delta^2 + 4\alpha\gamma$$

$$= (\alpha^2+\gamma^2) + (\beta^2+\delta^2) + 4\alpha\gamma$$

$$= (\alpha+\gamma)^2 + (\beta+\delta)^2 + 2(\alpha\gamma - \beta\delta)$$

$= 9 + 16 + 2 \times 13$

$= 25 + 26$

$= 51$

34. We have $Z^{2m} + Z^{2m-1} + Z^{2m-2} + \ldots + Z + 1 = 0$

$$\Rightarrow \quad \frac{z^{2m+1}-1}{z-1} = 0$$

$$\Rightarrow \quad z^{2m+1} - 1 = 0$$

Again Z_r, $r = 1, 2, 3, \ldots, 2m$, $m \in N$ are the roots of the equation $(z^{2m+1} - 1) = 0$

So,

$$(z^{2m+1} - 1) = (z-1)(z-z_1)(z-z_2)\ldots(z-z_m)$$

$$\Rightarrow \quad \left[\frac{z^{2m+1}-1}{(z-1)}\right] = (z-z_1)(z-z_2)\ldots(z-z_m)$$

$$\Rightarrow \quad (z-z_1)(z-z_2)\ldots(z-z_m) = 1 + z + z^2 + \ldots + z^{2m}$$

Taking log of both the sides, we get

$$\log\{(z-z_1)(z-z_2)\ldots(z-z_m)\} = \log(1 + z + z^2 + \ldots + z^{2m})$$

Differentiating both sides $w.\ r.\ t.\ z$, we get

$$\Rightarrow \quad \frac{1}{z-z_1} + \frac{1}{z-z_2} + \ldots + \frac{1}{z-z_m} = \frac{1 + 2z + 3z^2 + \ldots + 2mz^{2m-1}}{1 + z + z^2 + z^3 + \ldots + z^{2m}}$$

Put $z = 1$,

$$\frac{1}{1-z_1} + \frac{1}{1-z_2} + \frac{1}{1-z_3} + \ldots + \frac{1}{1-z_{2m}} = \frac{1+2+3+\ldots+2m}{1+1+1+\ldots+1\ (2m \text{ times})}$$

$$= \frac{2m(1+2m)}{2} \times \frac{1}{(2m+1)}$$

$$= m$$

Thus, $\dfrac{1}{z-z_1} + \dfrac{1}{z-z_2} + \ldots + \dfrac{1}{z-z_m} = m$

$$\Rightarrow \quad \sum_{r=1}^{2m}\left(\frac{1}{Z_r - 1}\right) = -m$$

Integer Type Questions

1. Let the equation be

$$\frac{1}{(a+x)} + \frac{1}{(b+x)} + \frac{1}{(c+x)} = \frac{2}{x}$$

Clearly, $x = \omega, \omega^2$ satisfying the given equation

Put $x = 1$, we get,

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} = 2$$

2. Given expression is

$$1 + \cos\left[\left\{(1-\omega)(1-\omega^2) + \ldots + (10-\omega)(10-\omega^2)\right\}\frac{\pi}{900}\right]$$

$$= 1 + \cos\left[\left\{\begin{matrix}(1^2+2^2+\ldots+10^2) \\ +(1+2+\ldots+10)+10\end{matrix}\right\}\frac{\pi}{900}\right]$$

$$= 1 + \cos\left[\{385 + 55 + 10\}\frac{\pi}{900}\right]$$

$$= 1 + \cos\left(450 \times \frac{\pi}{900}\right)$$

$$= 1 + \cos\left(\frac{\pi}{2}\right)$$

$= 1 + 0$

$= 1$

3. We have
$|z - z_1|^2 + |z - z_2|^2 = |z_1 - z_2|^2$
$= \left(2 \times \frac{\sqrt{3}}{2}\right)^2$
$= 3$

4. The given equation is
$z^7 + (z+1)^7 = 0$
Clearly, $\sum_{r=1}^{7} \text{Re}(z_r) = -\frac{7}{2}$
Now, $2\left(\sum_{r=1}^{7} \text{Re}(z_r)\right) = -2 \times \frac{7}{2} = -7$
$\Rightarrow \lambda = -7$
$\Rightarrow \lambda + 10 = -7 + 10 = 3$

5. It is given that, $x + \frac{1}{x} = 1$
$x = -\omega, -\omega^2$
Now,
$1 + x^{20} + x^{30} + x^{40} = 1 + \omega^2 + 1 + \omega = 1$
$2 + x^{50} + x^{60} + x^{70} = 2 + \omega^2 + 1 + \omega = 2$
$3 + x^{80} + x^{90} + x^{100} = 3 + \omega^2 + 1 + \omega = 3$
Thus, the product = 1.2.3 = 6

6. We have,
$\frac{\text{Im}(z_1)}{\text{Re}(z_1)} = (\sqrt{2} - 1)$
$\Rightarrow \tan\theta = (\sqrt{2} - 1) = \tan\left(\frac{\pi}{8}\right)$
$\Rightarrow \theta = \left(\frac{\pi}{8}\right)$
$\Rightarrow \frac{\pi}{n} = \left(\frac{\pi}{8}\right)$
$\Rightarrow n = 8$

8. We have
$$\frac{i+\sqrt{3}}{2} = i\left(\frac{1}{2} - \frac{i\sqrt{3}}{2}\right) = -i\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -i\,\omega$$
$$\frac{i-\sqrt{3}}{2} = i\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) = -i\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = -i\,\omega^2$$
Now, $\left(\frac{i+\sqrt{3}}{2}\right)^{2016} + \left(\frac{i-\sqrt{3}}{2}\right)^{2016} + 3$
$= (-i\omega)^{2016} + (-i\omega^2)^{2016} + 3$
$= \omega^{2016} + \omega^{4032} + 3$
$= 1 + 1 + 3$
$= 5$

9. We have,
$x^4 - 1 = 0$
$\Rightarrow (x^2 - 1)(x^2 + 1) = 0$
$\Rightarrow x = \pm 1, \pm i$
Also, $x^5 - x^3 + x^2 - 1 = 0$
$\Rightarrow x^5 + x^2 - x^3 - 1 = 0$
$\Rightarrow x^2(x^3 + 1) - (x^3 + 1) = 0$
$\Rightarrow (x^3 + 1)(x^2 - 1) = 0$
$\Rightarrow (x + 1)(x^2 - x + 1)(x^2 - 1) = 0$
$\Rightarrow x = \pm 1, -\omega, -\omega^2$
Thus, the number of common roots = 2

10. We have,
$(1 - i)^x = 2^x$
$\Rightarrow \left(\frac{1-i}{2}\right)^x = 1$
It is possible only when, $x = 0$
Thus, the number of solution is 1
Therefore, $m = 1$
Clearly, the number of common roots of
$$\begin{cases} 1 + z^{100} + z^{1985} = 0 \\ 1 + 2z + 2z^2 z^3 = 0 \end{cases} \text{ is 2}$$
So $n = 2$
Hence, the value of $(m + n + 3)$ is 6

11. Clearly, $|z_1| = 1 = |z_2| = |z_3|$
Now, $|z_1 - z_2|^2 + |z_2 - z_3|^2 + |z_3 - z_1|^2$
$= 2(|z_1|^2 + |z_2|^2 + |z_3|^2) - 2(\text{Re}(z_1\bar{z}_2) + \text{Re}(z_2\bar{z}_3) + \text{Re}(z_3\bar{z}_1))$
$= 6 - 2(\text{Re}(z_1\bar{z}_2) + \text{Re}(z_2\bar{z}_3) + \text{Re}(z_3\bar{z}_1))$
$\leq 6 + 3 = 9$
where
$3 + 2\left(\text{Re}(z_1\bar{z}_2) + \text{Re}(z_2\bar{z}_3) + \text{Re}(z_3\bar{z}_1)\right)$
$= |z_1 + z_2 + z_3|^2 \geq 0$
$\Rightarrow 2\left(\text{Re}(z_1\bar{z}_2) + \text{Re}(z_2\bar{z}_3) + \text{Re}(z_3\bar{z}_1)\right) \geq -3$
$\Rightarrow -2\left(\text{Re}(z_1\overline{z_2}) + \text{Re}(z_2\overline{z_3}) + \text{Re}(z_3\overline{z_1})\right) \leq 3$

12. We have,
$|z_1| = 1, |z_2| = 2, |z_3| = 3$
$|z_1|^2 = 1, |z_2|^2 = 4, |z_3|^2 = 9$
$\Rightarrow z_1 \cdot \bar{z}_1 = 1, z_2 \cdot \bar{z}_2 = 4, z_3 \cdot \bar{z}_3 = 9$
$\Rightarrow \bar{z}_1 = \frac{1}{z_1}, \bar{z}_2 = \frac{4}{z_2}, \bar{z}_3 = \frac{9}{z_3}$
Now, $|9z_1z_2 + 4z_1z_3 + z_2z_3| = 12$
$\Rightarrow \left|z_1 z_2 z_3\left(\frac{9}{z_3} + \frac{4}{z_2} + \frac{1}{z_1}\right)\right| = 12$
$\Rightarrow |z_1 z_2 z_3(\bar{z}_3 + \bar{z}_2 + \bar{z}_1)| = 12$
$\Rightarrow |z_1||z_2||z_3|\,|(\bar{z}_3 + \bar{z}_2 + \bar{z}_1)| = 12$
$\Rightarrow |z_1||z_2||z_3|\,|\overline{(z_1 + z_2 + z_3)}| = 12$
$\Rightarrow 1 \cdot 2 \cdot 3|(z_1 + z_2 + z_3)| = 12$
$\Rightarrow |(z_1 + z_2 + z_3)| = 2$

13. $1+\sum_{k=0}^{12} e^{i\left(\frac{2k+1}{13}\right)}$

$= 1+\left(e^{\frac{i}{13}}+e^{\frac{i3}{13}}+e^{\frac{i5}{13}}+\ldots+e^{\frac{i25}{13}}\right)$

$= 1+e^{\frac{i}{13}}\left(1+e^{\frac{i2}{13}}+e^{\frac{i4}{13}}+\ldots+e^{\frac{i24}{13}}\right)$

$= 1+e^{\frac{i}{13}}\left(\dfrac{1-e^{\frac{i26}{13}}}{1-e^{\frac{i2}{13}}}\right)$

$= 1+e^{\frac{i}{13}}\left(\dfrac{1-1}{1-e^{\frac{i2}{13}}}\right)$

$= 1 + 0$

$= 1$

14. See the solution of Q. 22 (Level - IV)

15. We have,

$x^2 + x + 1 = 0$

$x = \omega, \omega^2$

Now, $x+\dfrac{1}{x}=\omega+\omega^2=-1$

$x^2+\dfrac{1}{x^2}=\left(x+\dfrac{1}{x}\right)^2-2=1-2=-1$

$x^3+\dfrac{1}{x^3}=1+1=2$

$x^4+\dfrac{1}{x^4}=\left(x^2+\dfrac{1}{x^2}\right)^2-2=1-2=-1$

$x^5+\dfrac{1}{x^5}=\left(x^4+\dfrac{1}{x^4}\right)\left(x+\dfrac{1}{x}\right)-\left(x^3+\dfrac{1}{x^3}\right)$

$=1-2=-1$

$x^6+\dfrac{1}{x^6}=\left(x^3+\dfrac{1}{x^3}\right)^2-2=4-2=2$

Hence, the sum of the product

$=\dfrac{1}{9}(9\times 4+18)$

$=\dfrac{1}{9}\times 54\times 6.$

Previous Years' JEE-Advanced Examinations

1. Given $x = a + b$, $y = a\alpha + b\beta$ and $z = a\beta + b\alpha$

Thus,

$x\,y\,z = (a + b)(a\alpha + b\beta)(a\beta + b\alpha)$

$= (a + b)(a^2\alpha\beta + ab\alpha^2 + ab\beta^2 + b^2\alpha\beta)$

$= (a + b)[\alpha\beta(a^2 + b^2) + ab(\alpha^2 + \beta^2)]$

$= (a + b)[(\alpha^2 + \beta^2) - ab]$

$(\because\ \alpha\beta = 1,\ \alpha^2 + \beta^2 = -1)$

$= (a + b)(a^2 - ab + b^2)$

$= (a^3 + b^3)$

2. Given,

$(x - 1)^3 + 8 = 0$

$\Rightarrow\ (x - 1)^3 = -8 = (-2)^3$

$\Rightarrow\ x - 1 = -2, -2\omega, -2\omega^2$

$\Rightarrow\ x = 1 - 2, 1 - 2\omega, 1 - 2\omega^2$

$\Rightarrow\ x = -1, 1 - 2\omega, 1 - 2\omega^2$

3. Given,

$\left(\dfrac{1+i}{1-i}\right)^n=1$

$\Rightarrow\ (i)^n = 1$

$\Rightarrow\ n = 4$

Hence, the smallest positive integer = 4.

4. Given,

$\left|\dfrac{z-5i}{z+5i}\right|=1$

$\Rightarrow\ |z - 5i| = |z + 5i|$

$\Rightarrow\ |z - 5i|^2 = |z + 5i|^2$

$\Rightarrow\ (z-5i)\overline{(z-5i)}=(z+5i)\overline{(z+5i)}$

$\Rightarrow\ (z-5i)(\bar{z}+5i)=(z+5i)(\bar{z}-5i)$

$\Rightarrow\ (z\cdot\bar{z}+5iz-5i\bar{z}+25)$

$=(z\cdot\bar{z}-5iz+5i\bar{z}+25)$

$\Rightarrow\ 10iz=10i\bar{z}$

$\Rightarrow\ z=\bar{z}$

$\Rightarrow\ x + iy = x - iy$

$\Rightarrow\ 2\,iy = 0$

$\Rightarrow\ y = 0.$

Thus, the complex number z satisfies the x-axis.

5. Given,

$|z + 2| < |z - 2|$

$\Rightarrow\ |z + 2|^2 < |z - 2|^2$

$\Rightarrow\ (z+2)(\bar{z}+2)<(z-2)(\bar{z}-2)$

$\Rightarrow\ (z\cdot\bar{z}+2(z+\bar{z})+4)<(z\cdot\bar{z}-2(z+\bar{z})+4)$

$\Rightarrow\ 4(z+\bar{z})<0$

$\Rightarrow\ (z+\bar{z})<0$

$\Rightarrow\ x + iy + x - iy < 0$

$\Rightarrow\ 2x < 0$

$\Rightarrow\ \mathrm{Re}(z) < 0$

6. Given,

$z=\left(\dfrac{\sqrt{3}}{2}+\dfrac{i}{2}\right)^5+\left(\dfrac{\sqrt{3}}{2}-\dfrac{i}{2}\right)^5$

$=\left[\cos\left(\dfrac{\pi}{6}\right)+i\sin\left(\dfrac{\pi}{6}\right)\right]^5+\left[\cos\left(\dfrac{\pi}{6}\right)-i\sin\left(\dfrac{\pi}{6}\right)\right]^5$

$=\left[\cos\left(\dfrac{5\pi}{6}\right)+i\sin\left(\dfrac{5\pi}{6}\right)\right]+\left[\cos\left(\dfrac{5\pi}{6}\right)-i\sin\left(\dfrac{5\pi}{6}\right)\right]$

$=2\cos\left(\dfrac{5\pi}{6}\right)$

Thus, $\mathrm{Im}(z) = 0$

7. Given,

$$\omega = \left(\frac{1-iz}{z-\text{i}}\right)$$

$$= \left(\frac{-i^2-iz}{z-i}\right)$$

$$= (-i)\left(\frac{i+z}{z-i}\right)$$

$$\Rightarrow \quad \left|(-i)\left(\frac{i+z}{z-i}\right)\right| = |\omega|$$

$$\Rightarrow \quad |(-i)|\left|\left(\frac{i+z}{z-i}\right)\right| = |\omega|$$

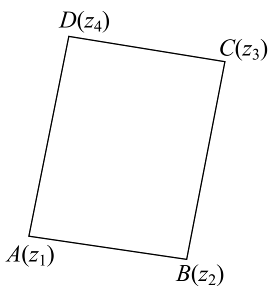
$$\Rightarrow \quad \left|\left(\frac{i+z}{z-i}\right)\right| = 1$$

$\Rightarrow \quad |z+i| = |z-i|$

$\Rightarrow \quad z$ is equidistant from $-i$ and i

$\Rightarrow \quad z$ lies on real axis.

8. As we know that, the diagonals of a parallelogram bisect each other.



Thus, mid-point of AC = Mid-point of BD

$$\Rightarrow \quad \frac{z_1+z_3}{2} = \frac{z_2+z_4}{2}$$

$\Rightarrow \quad (z_1+z_3) = (z_2+z_4)$

9. Let $|z_1| = |z_1| = |z_1| = k$

$\Rightarrow \quad z_1, z_2, z_3$ lie on a circle with centre at the origin and the radius k.

As z_1, z_2, z_3 are the vertices of an equilateral triangle, the circumcentre and centroid of the triangle coincide.

Thus, $\frac{z_1+z_2+z_3}{3} = 0$

$\Rightarrow \quad z_1+z_2+z_3 = 0$

10. Let $z_1, z_2, z_3 \in$ AP

$\Rightarrow \quad 2z_2 = z_1 + z_3$

$$\Rightarrow \quad z_2 = \frac{z_1+z_3}{2}$$

Thus, z_2 is the mid-point of z_1 and z_3

Hence, z_1, z_2, z_3 lie on a striaght line.

11. Since $\frac{|c-a|}{|w-u|} = \frac{|r(b-a)|}{|r(v-u)|} = \frac{|(b-a)|}{|(v-u)|}$

and $\frac{|c-b|}{|w-v|} = \frac{|(1-r)(a-b)|}{|(1-r)(u-v)|} = \frac{|(a-b)|}{|(u-v)|}$

Thus, the two triangles are similar.

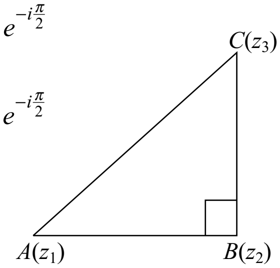
12. Clearly, it is a right-angled triangle.

Thus, the area of the given triangle $= \frac{1}{2} \times |z| \times |iz|$

$$= \frac{1}{2} \times |z|^2 \times |i|$$

$$= \frac{1}{2} \times |z|^2$$

13. We have,



$$\left(\frac{z_3-z_2}{z_1-z_2}\right) = \left|\frac{z_3-z_2}{z_1-z_2}\right| \times e^{-i\frac{\pi}{2}}$$

$$\Rightarrow \quad \left(\frac{z_3-z_2}{z_1-z_2}\right) = \frac{|z_3-z_2|}{|z_1-z_2|} \times e^{-i\frac{\pi}{2}}$$

$$\Rightarrow \quad \left(\frac{z_3-z_2}{z_1-z_2}\right) = e^{-\iota\frac{\pi}{2}} = -i$$

$\Rightarrow \quad (z_3-z_2) = -i(z_1-z_2)$

$\Rightarrow \quad (z_3-z_2)^2 = -(z_1-z_2)^2$

$\Rightarrow \quad z_3^2 + z_2^2 - 2z_2z_3 = -(z_1^2 + z_2^2 - 2z_1z_2)$

$\Rightarrow \quad z_1^2 + z_2^2 - 2z_1z_2 = 2z_1z_3 + 2z_2z_3 - 2z_1z_2 - 2z_3^2$

$\Rightarrow \quad (z_1-z_2)^2 = 2(z_1-z_3)(z_3-z_2)$

14. We have,

$|z_1+z_2| = |z_1| + |z_2|$

$\Rightarrow \quad |z_1+z_2|^2 = (|z_1| + |z_2|)^2$

$\Rightarrow \quad |z_1|^2 + |z_2|^2 + 2|z_1||z_1| \cos(\theta_1 - \theta_2)$

$= (|z_1|^2 + |z_2|^2 + 2|z_1||z_2|)$

$\Rightarrow \quad 2|z_1||z_2| \cos(\theta_1 - \theta_2) = 2|z_1||z_2|$

$\Rightarrow \quad \cos(\theta_1 - \theta_2) = 1$

$\Rightarrow \quad \cos(\theta_1 - \theta_2) = \cos(0)$

$\Rightarrow \quad \theta_1 = \theta_2$

$\Rightarrow \quad \text{Arg}(z_1) = \text{Arg}(z_2)$

$\Rightarrow \quad \text{Arg}(z_1) - \text{Arg}(z_2) = 0$

15. Given,

$$\sum_{k=1}^{6}\left[\sin\left(\frac{2\pi k}{7}\right) - i\cos\left(\frac{2\pi k}{7}\right)\right]$$

$$= \sum_{k=1}^{6} i\left[\cos\left(\frac{2\pi k}{7}\right) - i\sin\left(\frac{2\pi k}{7}\right)\right]$$

$$= \sum_{k=1}^{6} ie^{-i\frac{2\pi k}{7}}$$

$$= i\left(e^{-i\frac{2\pi}{7}} + e^{-i\frac{4\pi}{7}} + \ldots + e^{-i\frac{12\pi}{7}}\right)$$

$$= ie^{-i\frac{2\pi}{7}}\left(1 + e^{-i\frac{2\pi}{7}} + \ldots + e^{-i\frac{10\pi}{7}}\right)$$

$$= ie^{-i\frac{2\pi}{7}}\left(\frac{1-e^{-i\frac{12\pi}{7}}}{1-e^{-i\frac{2\pi}{7}}}\right)$$

$$= i\left(\frac{e^{-\frac{12\pi}{7}} - e^{-\frac{i14\pi}{7}}}{1-e^{-i\frac{2\pi}{7}}}\right)$$

$$= i\left(\frac{e^{-\frac{i2\pi}{7}} - 1}{1-e^{-i\frac{2\pi}{7}}}\right) = -i$$

16. We have,

$$\sin x - i\cos 2x = \cos x - i\sin 2x$$
$$\Rightarrow \quad \sin x = \cos x, \cos 2x = \sin 2x$$
$$\Rightarrow \quad \tan x = 1, \tan 2x = 1$$

It is not possible for any real value of x.
Thus, $x = \varphi$
No questions asked in 1989.

17. Given,

$$\text{Arg}\left(\frac{z - z_1}{z - z_2}\right) = \frac{\pi}{4}$$
$$\Rightarrow \quad \text{Arg}(z - z_1) - \text{Arg}(z - z_2) = \frac{\pi}{4}$$
$$\Rightarrow \quad \text{Arg}(x + iy - (10 + 6i)) - \text{Arg}(x + iy - (4 + 6i)) = \frac{\pi}{4}$$
$$\Rightarrow \quad \text{Arg}(x - 10) + i(y - 6)) - \text{Arg}(x - 4) + i(y - 6)) = \frac{\pi}{4}$$
$$\Rightarrow \quad \tan^{-1}\left(\frac{y-6}{x-10}\right) - \tan^{-1}\left(\frac{y-6}{x-4}\right) = \frac{\pi}{4}$$
$$\Rightarrow \quad \tan^{-1}\left(\frac{\left(\frac{y-6}{x-10}\right) - \left(\frac{y-6}{x-4}\right)}{1 + \left(\frac{y-6}{x-10}\right)\cdot\left(\frac{y-6}{x-4}\right)}\right) = \frac{\pi}{4}$$
$$\Rightarrow \quad \left(\frac{\left(\frac{y-6}{x-10}\right) - \left(\frac{y-6}{x-4}\right)}{1 + \left(\frac{y-6}{x-10}\right)\cdot\left(\frac{y-6}{x-4}\right)}\right) = 1$$
$$\Rightarrow \quad \frac{(y-6)(x-4) - (y-6)(x-10)}{(x-10)(x-4) + (y-6)^2} = 1$$
$$\Rightarrow \quad x^2 - 14x + 40 + (y - 6)^2$$
$$= 24 - 4y - 6x + 10y + 6x - 60$$
$$= 6y - 36$$
$$\Rightarrow \quad x^2 - 14x + 40 + (y - 6)^2 = 6y - 36$$
$$\Rightarrow \quad x^2 + y^2 - 14x - 12y + 76 = 6y - 36$$
$$\Rightarrow \quad x^2 + y^2 - 14x - 18y + 112 = 0$$
$$\Rightarrow \quad (x^2 - 14x) + (y^2 - 18y) + 112 = 0$$
$$\Rightarrow \quad (x - 7)^2 + (y - 9)^2 = 49 + 81 - 112$$
$$\Rightarrow \quad (x-7)^2 + (y-9)^2 = 18 = (3\sqrt{2})^2$$
$$\Rightarrow \quad |z - (7 + 9i)| = 3\sqrt{2}$$

18. We have $\left|\frac{z-1}{z+1}\right| = 1$

$$\Rightarrow \quad |z - 1| = |z + 1|$$
$$\Rightarrow \quad |z - 1|^2 = |z + 1|^2$$
$$\Rightarrow \quad (z-1)(\overline{z}-1) = (z+1)(\overline{z}+1)$$
$$\Rightarrow \quad z\overline{z} - (z + \overline{z}) + 1 = z\overline{z} + (z + \overline{z}) + 1$$
$$\Rightarrow \quad 2(z + \overline{z}) = 0$$
$$\Rightarrow \quad 2 \times 2\text{Re}(z) = 0$$
$$\Rightarrow \quad \text{Re}(z) = 0$$

$\Rightarrow \quad x = 0$, where $z = x + iy$.

19. Given,

$$z = -1 \Rightarrow \theta = \pi$$

Now, $\text{Arg}(z^{2/3}) = \frac{2}{3}\text{Arg}(z)$

$$= \frac{2\pi}{3}$$

20. Let $z = x + iy$
Given, $x = 0$
Now, $z^2 = (x + iy)^2$

$$= x^2 + i^2xy - y^2$$
$$= -y^2$$

Thus, $\text{Re}(z^2) = -y^2$
Clearly, $\text{Im}(z^2) = 0$

21. We have,

$$\left(\frac{1+2i}{1-i}\right) = \frac{(1+2i)(1+i)}{(1-i)(1+i)}$$
$$= \frac{1 + 2i + i - 2}{2}$$
$$= \frac{-1 + 3i}{2}$$

Clearly, the given complex number lies in fourth quadrant.

22. Given,

$$|\beta| = 1 \Rightarrow |\beta|^2 = 1 \Rightarrow \beta\cdot\overline{\beta} = 1$$

Now, $\left|\frac{\beta - \alpha}{1 - \overline{\alpha}\beta}\right| = \left|\frac{\beta - \alpha}{1 - \overline{\alpha}\cdot\frac{1}{\overline{\beta}}}\right|$

$$= \left|\frac{\overline{\beta}(\beta - \alpha)}{\overline{\beta} - \overline{\alpha}}\right|$$
$$= |\overline{\beta}|\left|\frac{(\beta - \alpha)}{\overline{\beta} - \overline{\alpha}}\right|$$
$$= |\overline{\beta}|\frac{|\beta - \alpha|}{|\overline{\beta} - \overline{\alpha}|}$$
$$= |\overline{\beta}|\frac{|\beta - \alpha|}{|\overline{\beta - \alpha}|}$$
$$= |\overline{\beta}|, \qquad (\because |z| = |\overline{z}|)$$
$$= 1$$

23. Given,

$$(1 + \omega)^3 - (1 + \omega^2)^3 = (-\omega^2)^3 - (-\omega)^3$$
$$= -\omega^6 + \omega^3$$
$$= -1 + 1 = 0$$

24. Given lines are $\alpha\overline{z} + \overline{\alpha}z + 1 = 0$ ...(i)
and $\beta\overline{z} + \overline{\beta}z - 1 = 0$ ...(ii)
From Eq. (i), we get

$$\alpha(x - iy) + \overline{\alpha}(x + iy) + 1 = 0$$
$$\Rightarrow \quad (\alpha + \overline{\alpha})x + iy(\alpha - \overline{\alpha}) + 1 = 0$$

Slope = $\left(\dfrac{\alpha+\bar{\alpha}}{\alpha-\bar{\alpha}}\right)$

Similarly, slope of the line (ii) = $\left(\dfrac{\beta+\bar{\beta}}{\beta-\bar{\beta}}\right)$

Since both the lines are mutually perpendicular to each other, so

$$\left(\frac{\alpha+\bar{\alpha}}{\alpha-\bar{\alpha}}\right)\times\left(\frac{\beta+\bar{\beta}}{\beta-\bar{\beta}}\right)=-1$$

$\Rightarrow\quad (\alpha+\bar{\alpha})(\beta+\bar{\beta})=-(\alpha-\bar{\alpha})(\beta-\bar{\beta})$

$\Rightarrow\quad \alpha\beta+\alpha\bar{\beta}+\bar{\alpha}\beta+\bar{\alpha}\bar{\beta}=-\alpha\beta+\alpha\bar{\beta}+\bar{\alpha}\beta-\bar{\alpha}\bar{\beta}$

$\Rightarrow\quad 2(\alpha\beta+\bar{\alpha}\bar{\beta})=0$

$\Rightarrow\quad (\alpha\beta+\bar{\alpha}\bar{\beta})=0$

25. We have,

$$\left(\frac{z_2-z_0}{z_1-z_0}\right)=\left|\frac{z_2-z_0}{z_1-z_0}\right|\times e^{i\frac{2\pi}{3}}$$

$\Rightarrow\quad \left(\dfrac{z_2-z_0}{z_1-z_0}\right)=\dfrac{|z_2-z_0|}{|z_1-z_0|}\times e^{i\frac{2\pi}{3}}$

$\Rightarrow\quad \left(\dfrac{z_2}{z_1}\right)=e^{i\frac{2\pi}{3}}$

$\Rightarrow\quad z_2=z_1e^{i\frac{2\pi}{3}}$

$=(1+i\sqrt{3})\left(-\dfrac{1}{2}+\dfrac{i\sqrt{3}}{2}\right)$

$=\dfrac{1}{2}(1+i\sqrt{3})(-1+i\sqrt{3})$

$=-\dfrac{1}{2}(1+3)=-2$

Also, $\left(\dfrac{z_3-z_0}{z_2-z_0}\right)=\left|\dfrac{z_3-z_0}{z_2-z_0}\right|\times e^{i\frac{2\pi}{3}}$

$\Rightarrow\quad \left(\dfrac{z_3-z_0}{z_2-z_0}\right)=\dfrac{|z_3-z_0|}{|z_2-z_0|}\times e^{i\frac{2\pi}{3}}$

$\Rightarrow\quad \left(\dfrac{z_3}{z_2}\right)=e^{i\frac{2\pi}{3}}$

$\Rightarrow\quad z_3=z_2\times e^{i\frac{2\pi}{3}}=-2\left(-\dfrac{1}{2}+\dfrac{i\sqrt{3}}{2}\right)$

$=-2\left(-\dfrac{1}{2}+\dfrac{i\sqrt{3}}{2}\right)=(1-i\sqrt{3})$

26. Given

$(1+\omega)^7=A+B\omega$

$\Rightarrow\quad (-\omega^2)^7=A+B\omega$

$\Rightarrow\quad -\omega^{14}=A+B\omega$

$\Rightarrow\quad -\omega^2=A+B\omega$

$\Rightarrow\quad A+B\omega+\omega^2=0$

$\Rightarrow\quad A=1, B=1$

27. Let $\text{Arg}(\omega)=\theta$

Then $\text{Arg}(z)=\pi-\theta$

Now, $z=|z|e^{i(\pi-\theta)}=|\omega|e^{i(\pi-\theta)}$

$=|\omega|e^{i\pi}\cdot e^{-i\theta}=-|\omega|\cdot e^{-i\theta}=-\bar{\omega}$

28. Given $|z+i\omega|=|z-i\bar{\omega}|$

29. Given $iz^3+z^2-z+i=0$

$\Rightarrow\quad iz^3-z+z^2+i=0$

$\Rightarrow\quad iz(z^2+i)+(z^2+i)=0$

$\Rightarrow\quad (z^2+i)(iz+1)=0$

$\Rightarrow\quad (z^2+i)=0, (iz+1)=0$

$\Rightarrow\quad z=-\dfrac{1}{i}, z^2=-i$

$\Rightarrow\quad |z|=\left|-\dfrac{1}{\text{i}}\right|, |z^2|=|-i|$

$\Rightarrow\quad |z|=1, |z|^2=1$

$\Rightarrow\quad |z|=1$

30. We have $|z_1-z_2|^2$

$=|z_1|^2+|z_1|^2-2|z_1||z_2|\cos(\theta_1-\theta_2)$

$=r_1^2+r_2^2-2r_1r_2\cos(\theta_1-\theta_2)$

$=r_1^2+r_2^2-2r_1r_2+2r_1r_2-2r_1r_2\cos(\theta_1-\theta_2)$

$=(r_1-r_2)^2+2r_1r_2(1-\cos(\theta_1-\theta_2))$

$=(r_1-r_2)^2+2r_1r_2\,2\sin^2\left(\dfrac{\theta_1-\theta_2}{2}\right)$

$\le(r_1-r_2)^2+2r_1r_2\,2\cdot\left(\dfrac{\theta_1-\theta_2}{2}\right)^2$

$=(r_1-r_2)^2+r_1r_2(\theta_1-\theta_2)^2$

$\le(r_1-r_2)^2+(\theta_1-\theta_2)^2$

$\Rightarrow\quad |z_1-z_2|^2\le(|z_1|^2-|z_2|^2)+(\text{Arg}(z_1)-\text{Arg}(z_2))^2$

Hence, the result.

31.

32. We have,

$\bar{z}=iz^2$

$\Rightarrow\quad (x-iy)=i(x+iy)^2$

$\Rightarrow\quad (x-iy)=i(x^2-y^2+i2xy)$

$\Rightarrow\quad (x-iy)=(-2xy+i(x^2-y^2))$

$\Rightarrow\quad x=-2xy, y=y^2-x^2$

When $x=-2xy$

$\Rightarrow\quad x(1+2y)=0$

$\Rightarrow\quad x=0, y=-1/2$

When $x=0$, then $y^2=y$

$\Rightarrow\quad y=0, 1$

Thus, the complex numbers are (0, 0), (0, 1).

When $y=-1/2$, then

$-\dfrac{1}{2}=\dfrac{1}{4}-x^2$

$\Rightarrow\quad x^2=\dfrac{1}{4}+\dfrac{1}{2}=\dfrac{3}{4}$

$\Rightarrow \quad x = \pm\dfrac{\sqrt{3}}{2}$

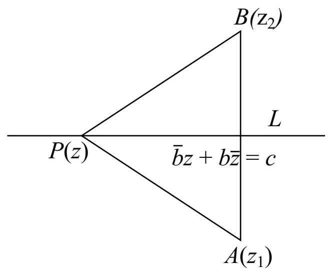
Thus, the complex numbers are

$$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right), \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$$

Hence, the solutions are (0, 0), (0, 1), i.e.

$$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right) \text{and} \left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$$

33. Let P be the point which represents z.



Here $P(z)$ lies on the line, so $AP = BP$

$\Rightarrow \quad |z - z_1| = |z - z_2|$

$\Rightarrow \quad |z - z_1|^2 = |z - z_2|^2$

$\Rightarrow \quad (z - z_1)(\bar{z} - \bar{z}_1) = (z - z_1)(\bar{z} - \bar{z}_2)$

$\Rightarrow \quad z \cdot \bar{z} - z \cdot \bar{z}_1 - z_1\bar{z} + z_1\bar{z}_1$

$\qquad = z \cdot \bar{z} - z \cdot \bar{z}_2 - z_2\bar{z} + z_2\bar{z}_2$

$\Rightarrow \quad -z \cdot \bar{z}_1 - z_1\bar{z} + z_1\bar{z}_1 = -z \cdot \bar{z}_2 - z_2\bar{z} + z_2\bar{z}_2$

$\Rightarrow \quad (\bar{z}_2 - \bar{z}_1)z + (z_2 - z_1)\bar{z} = |z_2|^2 - |z_1|^2$

which is identical with $\bar{b}z + b\bar{z} = c$

$$\frac{(\bar{z}_2 - \bar{z}_1)}{\bar{b}} = \frac{(z_2 - z_1)}{b} = \frac{|z_2|^2 - |z_1|^2}{c} = k \text{ (say)}$$

Thus, $(\bar{z}_2 - \bar{z}_1) = \bar{b}k, (z_2 - z_1) = bk$

and $\quad (|z_2|^2 - |z_1|^2 = ck$

Now,

$$b\bar{z}_1 + b\bar{z}_2 = \frac{1}{k}[(z_2 - z_1)\bar{z}_1 + (\bar{z}_2 - \bar{z}_1)z_2]$$

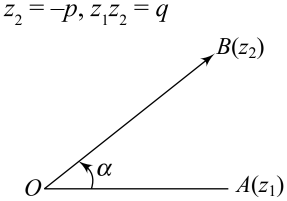
$$= \frac{1}{k}[|z_2|^2 - |z_2|^2] = c$$

Hence, the result.

34. Given equation is $z^2 + pz + q = 0$.

Let its roots are z_1 and z_2.

Thus, $z_1 + z_2 = -p, z_1z_2 = q$



We have,

$$\frac{z_2}{z_1} = \left|\frac{z_2}{z_1}\right| e^{i\alpha} = \frac{|z_2|}{|z_1|} e^{i\alpha} = e^{i\alpha}$$

$\Rightarrow \quad \dfrac{z_2}{z_1} = \dfrac{e^{i\alpha}}{1}$

$\Rightarrow \quad \dfrac{z_2 - z_1}{z_2 + z_1} = \dfrac{e^{i\alpha} - 1}{e^{i\alpha} + 1}$

$$= \frac{\cos\alpha + i\sin\alpha - 1}{\cos\alpha + i\sin\alpha + 1}$$

$$= -\frac{2\sin^2\left(\frac{\alpha}{2}\right) - \mathrm{i} \cdot 2\sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)}{2\cos^2\left(\frac{\alpha}{2}\right) + i2 \cdot \sin\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{2}\right)}$$

$$= \frac{i\,2\sin\left(\frac{\alpha}{2}\right)\left(\cos\left(\frac{\alpha}{2}\right) + i\sin\left(\frac{\alpha}{2}\right)\right)}{2\cos\left(\frac{\alpha}{2}\right)\left(\cos\left(\frac{\alpha}{2}\right) + i\sin\left(\frac{\alpha}{2}\right)\right)}$$

$$= \frac{i\sin\left(\frac{\alpha}{2}\right)}{\cos\left(\frac{\alpha}{2}\right)} = i\tan\left(\frac{\alpha}{2}\right)$$

$\Rightarrow \quad \left(\dfrac{z_2 - z_1}{z_2 + z_1}\right)^2 = -\tan^2\left(\dfrac{\alpha}{2}\right)$

$\Rightarrow \quad \dfrac{(z_2 - z_1)^2}{(z_2 + z_1)^2} = -\tan^2\left(\dfrac{\alpha}{2}\right)$

$\Rightarrow \quad \dfrac{(z_2 + z_1)^2 - 4z_1z_2}{(z_2 + z_1)^2} = -\tan^2\left(\dfrac{\alpha}{2}\right)$

$\Rightarrow \quad 1 - \dfrac{4z_1z_2}{(z_2 + z_1)^2} = -\tan^2\left(\dfrac{\alpha}{2}\right)$

$\Rightarrow \quad 1 - \dfrac{4q}{p^2} = -\tan^2\left(\dfrac{\alpha}{2}\right)$

$\Rightarrow \quad \dfrac{4q}{p^2} = 1 + \tan^2\left(\dfrac{\alpha}{2}\right)$

$\Rightarrow \quad \dfrac{4q}{p^2} = \sec^2\left(\dfrac{\alpha}{2}\right) = \dfrac{1}{\cos^2\left(\dfrac{\alpha}{2}\right)}$

$\Rightarrow \quad p^2 = 4q\cos^2\left(\dfrac{\alpha}{2}\right)$

35. Given $\displaystyle\sum_{k=1}^{n-1}(n - k)\cos\left(\frac{2k\pi}{n}\right)$

Let $\alpha = \displaystyle\sum_{k=1}^{n-1}(n - k)\cos\left(\frac{2k\pi}{n}\right)$

and $\beta = \displaystyle\sum_{k=1}^{n-1}(n - k)\sin\left(\frac{2k\pi}{n}\right)$

Now,

$$\alpha + i\beta = \sum_{k=1}^{n-1}(n-k)\left[\cos\left(\frac{2k\pi}{n}\right) + \sin\left(\frac{2k\pi}{n}\right)\right]$$

$$= \sum_{k=1}^{n-1}(n-k)\,z^k, \text{ where } z = e^{i\frac{2\pi}{n}}$$

$$= (n-1)z + (n-2)z^2 + \ldots + 2\cdot z^{n-2} + z^{n-1}$$

Let $S = (n-1)z + (n-2)z^2 + \ldots + 2\cdot z^{n-2} + z^{n-1}$

$\Rightarrow \quad zS = (n-1)z^2 + (n-2)z^3 + \ldots + 2\cdot z^{n-1} + z^n$

Subtracting, we get

$$(1-z)S = (1-n)z - (z^2 + z^3 + \ldots + z^n)$$

$$= (1-n)z - z^2(1 + z + \ldots + z^{n-2})$$

$$= (1-n)z - z^2\left(\frac{1-z^{n-1}}{1-z}\right)$$

$$\Rightarrow \quad S = (1-n)\frac{z}{(1-z)} - z^2\left(\frac{1-z^{n-1}}{(1-z)^2}\right)$$

$$= \frac{(1-n)z}{(1-z)} - \frac{(z^2-z)}{(1-z)} \qquad (\because z^n = 1)$$

$$= \frac{(1-n)z}{(1-z)} + z$$

$$= \frac{(1-n)z}{(1-z)} + z = \frac{nz}{1-z}$$

$$= \frac{ne^{\frac{i2\pi}{n}}}{1-e^{\frac{i2\pi}{n}}} = \frac{ne^{\frac{i\pi}{n}}}{e^{-\frac{i\pi}{n}} - e^{\frac{i\pi}{n}}}$$

$$= \frac{n\left[\cos\left(\frac{\pi}{n}\right) + i\sin\left(\frac{\pi}{n}\right)\right]}{-2i\sin\left(\frac{\pi}{n}\right)}$$

$$\Rightarrow \quad \alpha + i\beta = \frac{n\left[\cos\left(\frac{\pi}{n}\right) + i\sin\left(\frac{\pi}{n}\right)\right]}{-2i\sin\left(\frac{\pi}{n}\right)}$$

$$\Rightarrow \quad \alpha - i\beta = \frac{i\,n\cos\left(\frac{\pi}{n}\right) - n\sin\left(\frac{\pi}{n}\right)}{2\sin\left(\frac{\pi}{n}\right)}$$

Comparing the real and imaginary part, we get

$$\alpha = -\frac{n}{2}$$

36. Given,

$$(1 + \omega - \omega^2)^7 = (-\omega^2 - \omega^2)^7$$
$$= (-2\omega^2)^7$$
$$= (-2)^7(\omega^2)^7 = -128\omega^2$$

37. Given,

$$\sum_{n=1}^{13} i^n + i^{n+1} = (i + i^2) + (i^2 + i^3) + \ldots + (i^{13} + i^{14})$$
$$= (i + i^2 + i^3 + \ldots + i^{12} + i^{13})$$
$$+ (i + i^2 + i^3 + \ldots + i^{12} + i^{13} + i^{14})$$
$$= (i^{13} + i^{14})$$
$$= i - 1$$

38. Given,

$$4 + 5\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{334} + 3\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)^{365}$$
$$= 4 + 5\omega^{334} + 3\omega^{365}$$
$$= 4 + 5\omega + 3\omega^2$$
$$= 3(1 + \omega + \omega^2) + (1 + 2\omega)$$
$$= 3.0 + (1 + 2\omega)$$
$$= 1 + 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$
$$= 1 - 1 + i\sqrt{3}$$
$$= i\sqrt{3}$$

39. Given

$|z^2|\omega - |\omega^2|z = z - \omega$

$\Rightarrow \quad |z|^2\omega - |\omega|^2 z = z - w$

$\Rightarrow \quad |z|^2\omega + \omega = z + |\omega|^2 z$

$\Rightarrow \quad (|z|^2 + 1)\omega = (1 + |\omega|^2)z$

$$\Rightarrow \quad \frac{\omega}{z} = \frac{(1+|\omega|^2)}{(|z|^2+1)}$$

$$\Rightarrow \quad \frac{z}{\omega} = \frac{(|z|^2+1)}{(1+|\omega|^2)}$$

Thus, $\frac{z}{\omega}$ is a pure real.

$$\Rightarrow \quad \frac{z}{\omega} = \frac{\bar{z}}{\bar{\omega}}$$

$\Rightarrow \quad z\bar{\omega} = \omega\bar{z}$ ....(i)

Now, $|z^2|\omega - |\omega^2|z = z - \omega$

$\Rightarrow \quad |z|^2\omega - |\omega|^2 z = z - \omega$

$\Rightarrow \quad (z\cdot\bar{z})\omega - (\omega\cdot\bar{\omega})z = z - \omega$

$\Rightarrow \quad (\omega\cdot\bar{z})z - (z\cdot\bar{\omega})\omega = z - \omega$

$\Rightarrow \quad (z\cdot\bar{\omega})z - (z\cdot\bar{\omega})\omega = z - \omega$, from (i)

$\Rightarrow \quad (z\cdot\bar{\omega})(z - \omega) = z - \omega$

$\Rightarrow \quad (z - \omega)((z\cdot\bar{\omega}) - 1) = 0$

$\Rightarrow \quad (z - \omega) = 0, ((z\cdot\bar{\omega}) - 1) = 0$

$\Rightarrow \quad z = \omega, (z\cdot\bar{\omega}) = 1$

Hence, the result.

40. Given $(3z-1)^4 + (z-2)^4 = 0$

$\Rightarrow \quad (3z-1)^4 = -(z-2)^4$

$$\Rightarrow \quad \frac{(3z-1)^4}{(z-2)^4} = -1 = e^{i(2n+1)\pi}$$

$\Rightarrow \quad \left(\frac{3z-1}{z-2}\right)^4 = e^{i(2n+1)\pi}$

$\Rightarrow \quad \left(\frac{3z-1}{z-2}\right) = e^{i\frac{(2n+1)\pi}{4}}$

$\Rightarrow \quad z = \frac{1-2e^{i\frac{(2n+1)\pi}{4}}}{3-e^{i\frac{(2n+1)\pi}{4}}}$, where $n = 0, 1, 2, 3$.

Put $n = 0$, we get

$$\Rightarrow \quad z = \frac{1-2e^{i\frac{\pi}{4}}}{3-e^{i\frac{\pi}{4}}} = \frac{1-2\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)}{3-\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)}$$

$$= \frac{\sqrt{2}-2(1+i)}{3\sqrt{2}-(1+i)} = \frac{(\sqrt{2}-2)-2i}{(3\sqrt{2}-1)-i}$$

$$= \frac{((\sqrt{2}-2)-2i)(3\sqrt{2}-1)+i)}{(3\sqrt{2}-1)^2+1}$$

$$= \frac{10-7\sqrt{2}}{20-6\sqrt{2}} + i\frac{(-5\sqrt{2})}{20-6\sqrt{2}} = a+ib$$

Similarly for $n = 1, 2, 3$, we get three other roots of the given equation.

41. We have,

$\arg(-z) - \arg(z)$

$= \arg\left(\frac{-z}{z}\right)$

$= \arg(-1)$

$= \pi$

42. Given,

$|z_1| = 1$

$\Rightarrow \quad |z_1|^2 = 1$

$\Rightarrow \quad z_1\bar{z}_1 = 1$

$\Rightarrow \quad \bar{z}_1 = \frac{1}{z_1}$

Similarly, $\bar{z}_2 = \frac{1}{z_2}$,

and $\bar{z}_2 = \frac{1}{z_2}$,

Now, $\left|\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right| = 1$

$\Rightarrow \quad |\bar{z}_1+\bar{z}_2+\bar{z}_3| = 1$

$\Rightarrow \quad |\overline{z_1+z_2+z_3}| = 1$

$\Rightarrow \quad |z_1+z_2+z_3| = 1$

43. Let $z = (1)^{1/n} = (\cos(2r\pi) + i\sin(2r\pi))^{1/n}$

$\Rightarrow \quad z = \left(\cos\left(\frac{2r\pi}{n}\right)+i\sin\left(\frac{2r\pi}{n}\right)\right) = e^{i\frac{2r\pi}{n}}$

where $r = 0, 1, 2, 3, \ldots, (n-1)$

Let $z_1 = 1$ and $z_2 = e^{i\frac{2k\pi}{n}}$

It is given that,

$(z_2 - 0) = (z_1 - 0)e^{i\frac{\pi}{2}}$

$\Rightarrow \quad e^{i\frac{2k\pi}{n}} = e^{i\frac{\pi}{2}}$

$\Rightarrow \quad \frac{2k\pi}{n} = \frac{\pi}{2}$

$\Rightarrow \quad n = 4k$

Hence, the result.

44. We have

$\left(\frac{z_1-z_2}{z_2-z_3}\right) = \frac{1-i\sqrt{3}}{2}$

$= -\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$

$\Rightarrow \quad \left(\frac{z_2-z_1}{z_2-z_3}\right) = \left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)$

$= e^{i\frac{2\pi}{3}}$

So, z_1, z_2, z_3 are the vertices of an equilateral triangle.

45. Given

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & -2-\omega^2 & \omega^2-1 \\ 0 & \omega^2-1 & \omega-1 \end{vmatrix}$$

$$= \begin{vmatrix} -2-\omega^2 & \omega^2-1 \\ \omega^2-1 & \omega-1 \end{vmatrix}$$

$$= \begin{vmatrix} -1+\omega & \omega^2-1 \\ \omega^2-1 & \omega-1 \end{vmatrix}$$

$= (\omega-1)^2 - (\omega^2-1)^2$

$= (\omega^2 - 2\omega + 1) - (\omega^4 - 2\omega^2 + 1)$

$= (\omega^2 - 2\omega + 1) - (\omega - 2\omega^2 + 1)$

$= 3\omega^2 - 3\omega$

$= 3\omega(\omega - 1)$

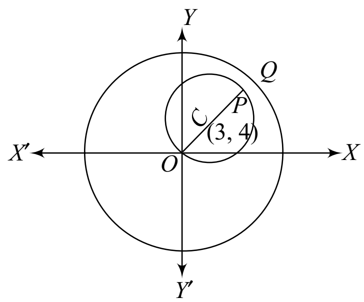
46. Hence, the minimum value of $|z_1 - z_2|$

$= PQ$

$= OQ - OP$

$= 12 - 10$

$= 2$



47. Given, $z^{p+q} - z^p - z^q + 1 = 0$

$\Rightarrow \quad z^p(z^q - 1) - 1(z^q - 1) = 0$

$\Rightarrow \quad (z^q - 1)(z^p - 1) = 0$

either $(z^p - 1) = 0$ or $(z^q - 1) = 0$

either $(\alpha^p - 1) = 0$ or $(\alpha^q - 1) = 0$

either $\dfrac{(\alpha^p - 1)}{(\alpha - 1)} = 0$ or $\dfrac{(\alpha^q - 1)}{(\alpha - 1)} = 0$

either $1 + \alpha + \alpha^2 + \ldots + \alpha^{p-1} = 0$

or $\quad 1 + \alpha + \alpha^2 + \ldots + \alpha^{q-1} = 0$

48. Given $|z| = 1$ and $\omega = \dfrac{z-1}{z+1}$

$\Rightarrow \quad z - 1 = \omega z + \omega$

$\Rightarrow \quad (1 - \omega)z = 1 + \omega$

$\Rightarrow \quad z = \dfrac{1+\omega}{1-\omega}$

$\Rightarrow \quad |z| = \left|\dfrac{1+\omega}{1-\omega}\right| = \dfrac{|1+\omega|}{|1-\omega|}$

$\Rightarrow \quad |1 + \omega| = |1 - \omega|$

$\Rightarrow \quad |1 + \omega|^2 = |1 - \omega|^2$

$\Rightarrow \quad (1+\omega)(1+\bar{\omega}) = (1-\omega)(1-\bar{\omega})$

$\Rightarrow \quad 1 + \omega + \bar{\omega} + |\omega|^2 = 1 - \omega - \bar{\omega} + |\omega|^2$

$\Rightarrow \quad \omega + \bar{\omega} = -\omega - \bar{\omega}$

$\Rightarrow \quad 2(\omega + \bar{\omega}) = 0$

$\Rightarrow \quad (\omega + \bar{\omega}) = 0$

$\Rightarrow \quad 2\text{Re}(\omega) = 0$

$\Rightarrow \quad \text{Re}(\omega) = 0$

49. Given, $|z_1| < 1$, $|z_2| > 1$

We have, $1 - \left|\dfrac{1 - z_1\bar{z}_2}{z_1 - z_2}\right|^2$

$$= \frac{|z_1 - z_2|^2 - |1 - z_1\bar{z}_2|^2}{|z_1 - z_2|^2}$$

$$= \frac{(z_1 - z_2)(\bar{z}_1 - \bar{z}_2) - (1 - z_1\bar{z}_2)(1 - \bar{z}_1 z_2)}{|z_1 - z_2|^2}$$

$$= \frac{1}{|z_1 - z_2|^2}[(|z_1|^2 + |z_2|^2 - z_1\bar{z}_2 - \bar{z}_1 z_2)$$
$$-(1 + |z_1|^2 \cdot |z_2|^2 - z_1\bar{z}_2 - \bar{z}_1 z_2)]$$

$$= \frac{(|z_1|^2 + |z_2|^2 - 1 - |z_1|^2 \cdot |z_2|^2)}{|z_1 - z_2|^2}$$

$$= \frac{(|z_1|^2 - 1)(1 - |z_2|^2)}{|z_1 - z_2|^2} > 0$$

Therefore, $\left|\dfrac{1 - z_1\bar{z}_2}{z_1 - z_2}\right|^2 < 1$

$\Rightarrow \quad \left|\dfrac{1 - z_1\bar{z}_2}{z_1 - z_2}\right| < 1$

50. We have $\displaystyle\sum_{r=1}^{n}(a_r z^r) = 1$

$\Rightarrow \quad 1 = \displaystyle\sum_{r=1}^{n}(a_r z^r)$

$\Rightarrow \quad 1 = \left|\displaystyle\sum_{r=1}^{n}(a_r z^r)\right| \le \displaystyle\sum_{r=1}^{n}|a_r||z|^n$

$\Rightarrow \quad 1 < \displaystyle\sum_{r=1}^{n}2\left(\frac{1}{3}\right)^n < \sum_{r=1}^{\infty}2\left(\frac{1}{3}\right)^n$

$\Rightarrow \quad 1 < \dfrac{\frac{2}{3}}{1 - \frac{1}{3}} = 1$

It is a contradiction.

Thus, there is no complex number z such that

$$|z| < \frac{1}{3} \text{ and } \sum_{r=1}^{n}(a_r z^r) = 1$$

51. Given

$(1 + \omega^2)^n = (1 + \omega^4)^n$

$\Rightarrow \quad \left(\dfrac{1+\omega^2}{1+\omega^4}\right)^n = 1$

$\Rightarrow \quad \left(\dfrac{-\omega}{-\omega^2}\right)^n = 1$

$\Rightarrow \quad \left(\dfrac{1}{\omega}\right)^n = 1$

$\Rightarrow \quad (\omega^2)^n = 1$

Thus, the least value of n is 3.

52. Given $\left|\dfrac{z-\alpha}{z-\beta}\right| = k$

$\Rightarrow \quad \dfrac{|z-\alpha|^2}{|z-\beta|^2} = k^2$

$\Rightarrow \quad \dfrac{(z-\alpha)(\bar{z}-\bar{\alpha})}{(z-\beta)(\bar{z}-\bar{\beta})} = k^2$

$\Rightarrow \quad (z-\alpha)(\bar{z}-\bar{\alpha}) = k^2(z-\beta)(\bar{z}-\bar{\beta})$

$\Rightarrow \quad |z|^2 - \alpha\bar{z} - \bar{\alpha}z + |\alpha|^2 = k^2(|z|^2 - \beta\bar{z} - \bar{\beta}z + |\beta|^2)$

$\Rightarrow \quad (1-k^2)|z|^2 - (\alpha - k^2\beta)\bar{z} - (\bar{\alpha} - \bar{\beta}k^2)z$
$\qquad + (|\alpha|^2 - k|\beta|^2) = 0$

$$\Rightarrow \quad |z|^2 - \frac{(\alpha - k^2\beta)}{(1-k^2)}\bar{z} - \frac{(\bar{\alpha} - \bar{\beta}k^2)}{(1-k^2)}z + \frac{(|\alpha|^2 - k|\beta|^2)}{(1-k^2)} = 0$$

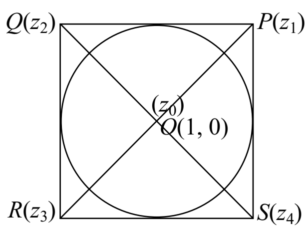
Thus, the centre of a circle is $\dfrac{(\alpha - k^2\beta)}{(1-k^2)}$.

and its radius $= \sqrt{\left|\dfrac{(\alpha - k^2\beta)}{(1-k^2)}\right|^2 - \left(\dfrac{|\alpha|^2 - k^2\beta\bar{\beta}}{(1-k^2)}\right)}$

$$= \sqrt{\left(\frac{(\alpha - k^2\beta)}{(1-k^2)}\right)\left(\frac{(\bar{\alpha} - k^2\bar{\beta})}{(1-k^2)}\right) - \left(\frac{|\alpha|^2 - k^2\beta\bar{\beta}}{(1-k^2)}\right)}$$

$$= \left|\frac{k(\alpha - \beta)}{1-k^2}\right|$$

53.



Since the centre of a square coincides with the centre of the circle, so $\dfrac{z_1 + z_3}{2} = 1$

$\Rightarrow \quad z_1 + z_3 = 2$

$\Rightarrow \quad z_3 = 2 - z_1 = 2 - (2 + i\sqrt{3}) = -i\sqrt{3}$

Here, $\angle z_1 z_0 z_2 = \dfrac{\pi}{2}$

By the rotation theorem,

$$\left(\frac{z_2 - z_0}{z_1 - z_0}\right) = \left|\left(\frac{z_2 - z_0}{z_1 - z_0}\right)\right| e^{i\pi/2} = i$$

$\Rightarrow \quad (z_2 - z_0) = i(z_1 - z_0)$

$\Rightarrow \quad z_2 = z_0 + i(z_1 - z_0)$

$\Rightarrow \quad z_2 = 1 + i(2 + i\sqrt{3} - 1) = 1 + i - \sqrt{3}$

$\quad = (1 - \sqrt{3}) + i$

Also, $z_4 = 2 - z_2 = 2 - (1 - \sqrt{3}) + i$

$\Rightarrow \quad z_4 = (1 + \sqrt{3}) - i$

54. The shaded region is outside the circle

$|z + 1| = 2$

Thus, $|z + 1| > 2$

Also, $\angle QPR = \dfrac{\pi}{2}$ as ΔQPR is a right triangle.

Therefore, $\text{Arg}(z + 1) < \dfrac{\pi}{2}$

55. Let $z = |a + b\omega + c\omega^2|$

$|z|^2 = |a + b\omega + c\omega^2|^2$

$= (a + b\omega + c\omega^2)(1 + b\bar{\omega} + c\bar{\omega}^2)$

$= a^2 + b^2 + c^2 - ab - bc - ca$

$= \dfrac{1}{2}[(a-b)^2 + (b-c)^2 + (c-a)^2]$

It is minimum only when $a = b$

and $(b - c)^2 = 1 = (c - a)^2$

Thus minimum of $|z|^2$ is 1

Therefore minimum of $|z|$ is 1.

56. Let $z_1 = \left(\dfrac{w - \bar{w}z}{1 - z}\right)$

It will be pure real if $z_1 = \bar{z}_1$

$\Rightarrow \quad \left(\dfrac{w - \bar{w}z}{1 - z}\right) = \left(\dfrac{\bar{w} - w\bar{z}}{1 - \bar{z}}\right)$

$\Rightarrow \quad (w - \bar{w}z)(1 - \bar{z}) = (\bar{w} - w\bar{z})(1 - z)$

$\Rightarrow \quad (w - \bar{w}z - w\bar{z} + \bar{w}z \cdot \bar{z}) = (\bar{w} - \bar{w}z - w\bar{z} + wz \cdot \bar{z})$

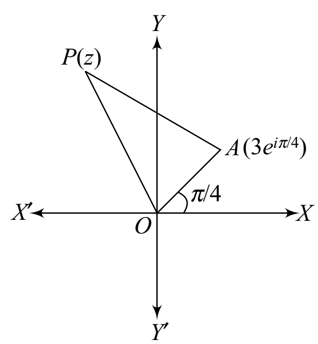
$\Rightarrow \quad (w - \bar{w}) + (\bar{w} - w)|z|^2 = 0$

$\Rightarrow \quad (w - \bar{w})(1 - |z|^2) = 0$

$\Rightarrow \quad (1 - |z|^2) = 0 \qquad (\because w \neq \bar{w})$

$\Rightarrow \quad |z| = 1$ and $z \neq 1$

57. Let $OA = 3$, so that the complex number A is $3e^{i\frac{\pi}{4}}$.



Let P be the complex number z.

Then by the rotation theorem, we have

$$\left(\frac{z - 3e^{i\pi/4}}{0 - 3e^{i\pi/4}}\right) = \frac{4}{3}e^{-i\pi/2} = -\frac{4i}{3}$$

$\Rightarrow \quad 3(z - 3e^{i\pi\pi/}) = -4i(-3e^{i\pi/4})$

$\quad = 12ie^{i\pi/4}$

$\Rightarrow \quad 3z - 9e^{i\pi/4} = 12ie^{i\pi/4}$

$\Rightarrow \quad z - 3e^{i\pi/4} = 4ie^{i\pi/4}$

$\Rightarrow \quad z = (3 + 4i)e^{i\pi/4}$

58. Let $z = \cos\theta + i\sin\theta$

$$\Rightarrow \quad \frac{z}{1-z^2} = \frac{\cos\theta + i\sin\theta}{1-(\cos\theta + i\sin\theta)^2}$$

$$= \frac{\cos\theta + i\sin\theta}{1-(\cos 2\theta + i\sin 2\theta)}$$

$$= \frac{\cos\theta + i\sin\theta}{(1-\cos 2\theta) - i\sin 2\theta}$$

$$= \frac{\cos\theta + i\sin\theta}{2\sin^2\theta - i\,2\sin\theta\cos\theta}$$

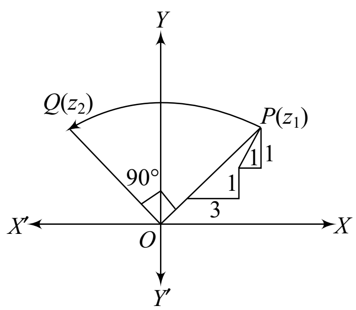
$$= \frac{(\cos\theta + i\sin\theta)}{-2\,i\sin\theta(\cos\theta + i\sin\theta)}$$

$$= \frac{1}{-2\,i\sin\theta}$$

$$= \frac{i}{2\sin\theta}$$

Thus, the complex number $\frac{z}{1-z^2}$ lies on the imaginary axis.

59.



Here, $z_1 = \left[6 + \sqrt{2}\cos\left(\frac{\pi}{4}\right), 5 + \sqrt{2}\sin\left(\frac{\pi}{4}\right)\right]$

$= (7, 6)$

By rotation theorem, about the origin

$$\frac{z_2 - 0}{z_1 - 0} = \left|\frac{z_2 - 0}{z_1 - 0}\right| e^{i\pi/2} = e^{i\pi/2} = i$$

$$z_2 = iz_1 = i(7 + 6i) = -6 + 7i = (-6, 7)$$

60. Given $\bar{z}z^3 + z\bar{z}^3 = 350$

$\Rightarrow \quad \bar{z}z \cdot z^2 + z\bar{z} \cdot \bar{z}^2 = 350$

$\Rightarrow \quad |z|^2 \cdot z^2 + |z|^2 \cdot \bar{z}^2 = 350$

$\Rightarrow \quad |z|^2 (z^2 + \bar{z}^2) = 350$

$\Rightarrow \quad (x^2 + y^2)(2x^2 - 2y^2) = 350$

$\Rightarrow \quad (x^2 + y^2)(x^2 - y^2) = 175$

$\Rightarrow \quad (x^2 + y^2)(x^2 - y^2) = 25 \times 7$

$\Rightarrow \quad (x^2 + y^2) = 25, (x^2 - y^2) = 7$

It is possible only when $x = 4, y = 3$.

Thus, the area of the rectangle $= (2 \times 4) \times (2 \times 3)$
$= 48$ sq unit.

61. Given,

$$\sum_{m=1}^{15} \text{Im}(z^{2m-1})$$

$$= \text{Im}(z) + \text{Im}(z^3) + \text{Im}(z^5) + \ldots + \text{Im}(z^{29})$$

$$= \sin\theta + \sin 3\theta + \sin 5\theta + \ldots + \sin 29\theta$$

$$= \sin\theta + \sin(\theta + 2\theta) + \sin(\theta + 2.2\theta) + \ldots + \sin(\theta + (15-1)2\theta)$$

$$= \frac{\sin\left(\frac{15.2\theta}{2}\right)}{\sin\left(\frac{2\theta}{2}\right)} \times \sin\left(\theta + (15-1)\left(\frac{2\theta}{2}\right)\right)$$

$$= \frac{\sin(15\theta)}{\sin(\theta)} \times \sin(15\theta)$$

$$= \frac{\sin^2(15\theta)}{\sin(\theta)}$$

$$= \frac{\sin^2(15 \times 2°)}{\sin(2°)}$$

$$= \frac{\sin^2(30°)}{\sin(2°)}$$

$$= \frac{1}{4\sin(2°)}$$

62. (A) $z = 0$ clealrly satisfies $|z - i|z|| = |z + i|z||$

For $z \neq 0$, we can write

$$\left|\frac{z}{|z|} - i\right| = \left|\frac{z}{|z|} + i\right|$$

$\Rightarrow \quad \frac{z}{|z|}$ is equidistant from $-i$ and i

$\Rightarrow \quad \frac{z}{|z|}$ lies on the real axis

$\Rightarrow \quad z$ lies on the real axis

$\Rightarrow \quad \text{Im}(z) = 0$ and $\text{Im}(z) \leq 1$.

(B) Given $|z + 4| + |z - 4| = 10$ represents an ellipse whose foci are -4 and 4 and the length of the major axis = 10.

Thus, $2ae = 8$ and $2a = 10$

$\Rightarrow \quad ae = 4$ and $a = 5$

$\Rightarrow \quad e = 4/5$.

(C) Given $|w| = 2$

$\Rightarrow \quad w = 2e^{i\theta} = 2(\cos\theta + i\sin\theta)$

Now, $z = w - \frac{1}{w} = 2e^{i\theta} - \frac{e^{-i\theta}}{2}$

$\Rightarrow \quad x = \frac{3}{2}\cos\theta,\ y = \frac{5}{2}\sin\theta$

$\Rightarrow \quad \frac{x^2}{9/4} + \frac{y^2}{25/4} = 1$

which represents an ellipse and its eccentricity

$$= \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{9}{25}} = \frac{4}{5}$$

(D) Given $|w| = 1$

$\Rightarrow \quad w = e^{i\theta}$

Now, $z = w + \frac{1}{w} = e^{i\theta} + e^{-i\theta} = 2\cos\theta$

As z is real, Im $(z) = 0$, $|\text{Im}(z)| \le 1$

Thus, $\text{Re}(z) = 2\cos\theta$

$\Rightarrow \quad |\text{Re}(z)| = |2\cos\theta| \le 2$

Finally, $z = 2\cos\theta$

$\Rightarrow \quad |z| = |2\cos\theta| \le 2 \le 3$

63. Given $z = (1 - t)z_1 + tz_2$

$z = z_1 + t(z_2 - z_1)$

$(z - z_1) = t(z_2 - z_1)$

$|(z - z_1)| = t|z_2 - z_1)|$

Also, $z - z_2 = (1 - t)z_1 + tz_2 - z_2$

$= (1 - t)z_1 + (t - 1)z_2$

$\Rightarrow \quad (z - z_2) = (1 - t)(z_1 - z_2)$

$\Rightarrow \quad |(z - z_2)| = (1 - t)\,|(z_1 - z_2)|$

Now, $|(z - z_1)| + |(z - z_2)|$

$= t|z_1 - z_2| + (1 - t)(z_1 - z_2)$

$= z_1 - z_2$

Again, $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = \begin{vmatrix} t(z_2 - z_1) & t(\bar{z}_2 - \bar{z}_1) \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix}$

$= (z_2 - z_1)(\bar{z}_2 - \bar{z}_1)\begin{vmatrix} t & t \\ 1 & 1 \end{vmatrix}$

$= 0$

Also, $\text{Arg}(z - 1) = \text{Arg}(z_2 - z_1)$, since z_1, z and z_2 lie on the same straight line and also lie on the same side of z_1.

64. Given

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} z+1+\omega+\omega^2 & \omega & \omega^2 \\ z+1+\omega+\omega^2 & z+\omega^2 & 1 \\ z+1+\omega+\omega^2 & 1 & z+\omega \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} z & \omega & \omega^2 \\ z & z+\omega^2 & 1 \\ z & 1 & z+\omega \end{vmatrix} = 0$$

$$\Rightarrow z\begin{vmatrix} 1 & \omega & \omega^2 \\ 1 & z+\omega^2 & 1 \\ 1 & 1 & z+\omega \end{vmatrix} = 0$$

$$\Rightarrow z\begin{vmatrix} 1 & \omega & \omega^2 \\ 0 & z+\omega^2-\omega & 1-\omega^2 \\ 0 & 1-\omega & z+\omega-\omega^2 \end{vmatrix} = 0$$

$$\Rightarrow z \times \begin{vmatrix} z+\omega^2-\omega & 1-\omega^2 \\ 1-\omega & z+\omega-\omega^2 \end{vmatrix} = 0$$

$\Rightarrow \quad z\{z^2 - (\omega - \omega^2)^2 - (1 - \omega^2 - \omega + \omega^3)\} = 0$

$\Rightarrow \quad z\{z^2 - (\omega^2 + \omega^4 - 2) - (2 - \omega^2 - \omega)\} = 0$

$\Rightarrow \quad z^3 = 0$

$\Rightarrow \quad z = 0$ is the only solution.

65. Given $|z - 3 - 2i| \le 2$

Now, $|2z - 6 + 5i| = |2(z - 3 - 2i) + 9i$

$\ge ||2(z - 3 - 2i)| - |9i|$

$= |2.2 - 9| = 5$

66. (i) $a + 8b + 7c = 0$, $9a + 2b + 3c = 0$,

and $\quad a + b + c = 0$

Solving, we get, $b = 6a$, $c = -7a$

Now, $\quad 2x + y + z = 1$

$\Rightarrow \quad 2a + b + c = 1$

$\Rightarrow \quad 2a + 6a - 7a = 1$

$\Rightarrow \quad a = 1$

Thus, $\quad b = 6$, $c = -7$

Therefore, $7a + b + c = 7 + 6 - 7$

$= 7$

(ii) $a = 2$, b and c satisfy (E)

$b = 12$ and $c = -14$

$\frac{3}{\omega^a} + \frac{1}{\omega^b} + \frac{3}{\omega^c} = \frac{3}{\omega^2} + \frac{1}{1} + \frac{3}{\omega^{-2}}$

$= 1 + \left(\frac{3}{\omega^2} + \frac{3}{\omega^{-2}}\right)$

$= 1 + 3(\omega + \omega^2)$

$= 1 - 3$

$= -2$

67. Given equation is

$z^2 + z + (1 - a) = 0$

Clearly the given equation do not have real root.

So, $\quad D < 0$

$\Rightarrow \quad 1 - 4(1 - a) < 0$

$\Rightarrow \quad 4a - 3 < 0$

$\Rightarrow \quad a < \frac{3}{4}$

68. Given $|z - z_0| = r$

and $|z - z_0| = 2r$

Thus, $|\alpha - z_0| = r$

and $\quad \left|\frac{1}{\bar{\alpha}} - z_0\right| = 2r$

$\Rightarrow \quad \left|\frac{\alpha}{|\alpha|^2} - z_0\right| = 2r \qquad (\because \alpha \cdot \bar{\alpha} = |\alpha|^2)$

Now, $(\alpha - z_0)(\bar{\alpha} - \bar{z}_0) = r^2$

$\Rightarrow \quad |\alpha|^2 - z_0\bar{\alpha} - \alpha\bar{z}_0 + |z_0|^2 = r^2$

Also, $\left(\frac{\alpha}{|\alpha|^2} - z_0\right)\left(\frac{\bar{\alpha}}{|\alpha|^2} - \bar{z}_0\right) = 4r^2$

$\Rightarrow \quad \frac{|\alpha|^2}{|\alpha|^4} - \frac{z_0\bar{\alpha}}{|\alpha|^2} - \frac{\bar{z}\alpha}{|\alpha|^2} + |z_0|^2 = 4r^2$

$\Rightarrow \quad 1 - z_0\bar{\alpha} - \bar{z}\alpha + |z_0|^2|\alpha|^2 = 4r^2\,|\alpha|^2$

$\Rightarrow \quad (|\alpha|^2 - 1) + |z_0|^2\,(1 - |\alpha|^2) = r^2(1 - 4\alpha^2)$

$\Rightarrow \quad (|\alpha|^2 - 1)\left(1 - \frac{r^2 + 2}{2}\right) = r^2(1 - 4\,|\alpha|^2)$

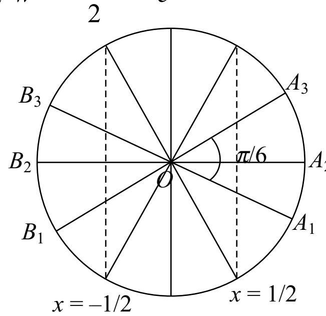
$\Rightarrow \quad (|\alpha|^2 - 1)\left(-\frac{r^2}{2}\right) = r^2(1 - 4\,|\alpha|^2)$

$\Rightarrow \quad (|\alpha|^2 - 1) = -2 + 8|\alpha|^2$

$\Rightarrow \quad 1 = 7|\alpha|^2$

$\Rightarrow \quad |\alpha| = \frac{1}{\sqrt{7}}$

69. We have $w = \frac{\sqrt{3} + i}{2} = e^{i\frac{\pi}{6}}$



So, $w^n = e^{i\left(\frac{n\pi}{6}\right)}$

Now, for z_1, $\cos\left(\frac{n\pi}{6}\right) > \frac{1}{2}$

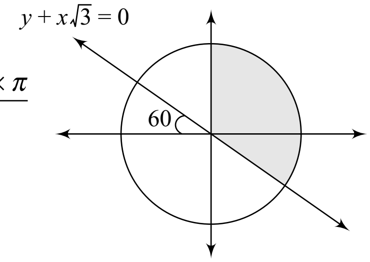
and for z_2, $\cos\left(\frac{n\pi}{6}\right) < -\frac{1}{2}$

Possible positions of z_1 are A_1, A_2, A_3 whereas of z_2 are B_1, B_2, B_3

So, the possible value of $\angle z_1 O z_2$, according to the given option is $\frac{2\pi}{3}$ or $\frac{5\pi}{6}$

70. (i) Area of the region $S_1 \cap S_2 \cap S_3$

= Shaded area



$= \frac{\pi \times 4^2}{4} + \frac{4^2 \times \pi}{6}$

$= 16\left(\frac{1}{4} + \frac{1}{6}\right)\pi$

$= \frac{20\pi}{3}$

(ii) Distance of (1, –3) from $y + x\sqrt{3} = 0$

$= \left|\frac{-3 + \sqrt{3} + 1}{\sqrt{1+3}}\right|$

$= \left|\frac{\sqrt{3} - 2}{2}\right|$

71. (P) z_k is the 10th root of unity

$\Rightarrow \quad \bar{z}_k$ is also the 10 th root of unity

$\Rightarrow \quad$ Take z_j is $\bar{z}_k$

(Q) $z_1 \neq 0$ take $z = \frac{z_k}{z_1}$

we can always find z.

(R) $(z^{10} - 1) = (z - 1)(z - z_1) \ldots (z - z_9)$

$\Rightarrow \quad (z - z_1)(z - z_2)\ldots(z - z_9) = \frac{(z^{10} - 1)}{(z - 1)}$

$= 1 + z + z^2 + \ldots + z^9$

Put $z = 1$, we get

$(1 - z_1)(1 - z_2) \ldots (1 - z_9) = 10$

$\Rightarrow \quad \frac{(1 - z_1)(1 - z_2)\ldots(1 - z_9)}{10} = 1$

$\Rightarrow \quad \frac{|(1 - z_1)||(1 - z_2)|\ldots|(1 - z_9)|}{10} = 1$

(S) $1 + z_1 + z_2 + \ldots + z_9 = 0$

$\Rightarrow \quad \text{Re}(1) + \text{Re}(z_1) + \text{Re}(z_2) + \ldots + \text{Re}(z_9) = 0$

$\Rightarrow \quad \text{Re}(z_1) + \text{Re}(z_2) + \ldots + \text{Re}(z_9) = -\text{Re}(1)$

$\Rightarrow \quad \text{Re}(z_1) + \text{Re}(z_2) + \ldots + \text{Re}(z_9) = -1$

$\Rightarrow \quad 1 - \sum_{k=1}^{9} \cos\left(\frac{2k\pi}{10}\right) = 1 - (-1) = 2\,.$

CHAPTER

5

Permutations and Combinations

CONCEPT BOOSTER

1. Factorial Notation

The continued product of first n natural numbers is called a factorial of n. It is denoted by $n!$ or n and is defined as

$$n! = 1 \cdot 2 \cdot 3 \ldots (n-2)(n-1)n$$
$$= n(n-1)(n-2) \ldots 3 \cdot 2 \cdot 1$$
$$= n \times (n-1)!$$
$$= n(n-1) \times (n-2)! \text{ and so on.}$$

(i) $(0)! = 1$
(ii) $(1)! = 1$

2. Exponent of a Prime p in $(n)!$

The exponent of a prime p in $(n)!$ is denoted as $E_p(n!)$ and is defined as

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots$$

where [,] = GIF.

3. Fundamental Principle of Counting

(i) Multiplication Rule

If an operation can be performed in m different ways, following which a second operation can be performed in n different ways, the whole operation can be performed in $m \times n$ different ways.

> ***Note:***
> 1. This can be extended to any finite number of operations.
> 2. Two operations are independent.

(ii) Addition Principle

If an operation can be performed in m different ways and another operation can be performed in n different ways, exactly one of the operations can be performed in $m + n$ ways.

4. Permutations

It is the different arrangements of a number of things taken some or all of them at a time.

The number of permutations of n different things taken r at a time is denoted as ${}^nP_r$ or $P(n, r)$ and is defined as

$${}^nP_r = \frac{n!}{(n-r)!}, \text{ where } r \le n, r \in W, n \in N.$$

5. Sum of Numbers

(i) For given n different digits $a_1, a_2, \ldots, a_n$, the sum of the digits in the unit places of all numbers formed $= (a_1 + a_2 + \cdots + a_n)(n-1)!$

(ii) The sum of the total numbers which can be formed with given n different digits $a_1, a_2, \ldots, a_n$,

$$= (a_1 + a_2 + \cdots + a_n) \times (n-1)! \times (1 + 10 + 10^2 + \cdots + 10^{n-1})$$

$$= (n-1)! \times (a_1 + a_2 + \cdots + a_n) \times \left(\frac{10^n - 1}{9}\right).$$

6. Permutation with Repetition

The number of permutations of n different things taken r at a time, when each can be repeated any number of times is n^r.

7. Permutations of Alike Objects

The number of permutation of n things taken all at a time, where p are alike of one kind, q are alike of second kind, r are alike of third kind and the rest are different is given by $\dfrac{n!}{p! \times q! \times r!}$.

8. Restricted Permutations

String Method

(i) Togetherness

The number of permutations of n different things taken all at a time when m specified things always come together is given as $(n - m + 1)! \times m!$.

(ii) Non-togetherness

The number of permutations of n different things taken all at a time when m specified things never come together is given as $n! - (n - m + 1)! \times m!$.

9. Rank of a Word in Dictionary

Rank of a word is the position of that word, when we arrange them in alphabetic order according to the English dictionary.

To understand the concept of the rank of a word we should remember the following steps.

Consider the three letters A, B and C.

The number of arrangement of the letters A, B and C is 3! and their order is

ABC—1st word
ACB—2nd word
BAC—3rd word
BCA—4th word
CAB—5th word
CBA—6th word

If I ask you, what is the rank of the word BCA? The answer is 4th.

10. Gap Method

The number of permutations of n different things taken all at a time when m specified things, no two of which are to occur together is ${}^{n+1}P_m \times n!$.

11. Circular Permutations

We have discussed earlier so far, the permutations of objects (for things) in a row. Such types of permutations are called **linear permutations** of linear arrangements. If n different things can be arranged in a row, the linear arrangements is $n!$, whereas every linear arrangements have a beginning and end but in **circular permutations**, there is neither beginning nor end.

When clockwise and anti-clockwise orders are taken as different, the number of circular permutations of n different things taken all at a time is $(n - 1)!$

But, when the clockwise and anti-clockwise orders are not different, i.e. the arrangements of beads in a necklace, arrangements of flowers in a garland, etc., the number of circular permutations of n different things $= \frac{1}{2} \times (n-1)!$.

12. Restricted Circular Permutations

(i) If clockwise and anti-clockwise arrangements are taken as different, the number of circular permutations of n different things, taken r at a time is given by

$$= \frac{{}^nP_r}{r}$$

(ii) If clockwise and anti-clockwise arrangements are not taken as different, the number of circular permutations of n different things, taken r at a time

$$= \frac{{}^nP_r}{2r}$$

13. Combinations

The different groups or selections of a number of things taken some or all of them at a time are called combinations.

The number of combinations of n different things taken r at a time is denoted by ${}^nC_r$ and is defined as ${}^nC_r = \frac{n!}{r! \times (n-r)!}$, where $r \le n$, $r \in W$, $n \in N$.

14. Some Important Results to Remember

(i) ${}^nC_r = nC_{n-r}$

(ii) ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$

(iii) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(iv) $\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{n-r+1}{r}$

(v) ${}^nC_r = \frac{n}{r} \times {}^{n-1}C_{r-1} = \frac{n(n-1)}{r(r-1)} \times {}^{n-2}C_{r-2}$

and so on.

(vi) ${}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n$

(vii) ${}^nC_0^2 + {}^nC_1^2 + {}^nC_2^2 + \cdots + {}^nC_n^2 = {}^{2n}C_n$

15. Restricted Combinations

(i) The number of selections of r objects out of n different objects when
  (a) k particular objects are always included $= {}^{n-k}C_{r-k}$
  (b) k particular objects are never included $= {}^{n-k}C_r$

(ii) The number of combinations of r things out of n different things such that k particular things are not included in any selection $= {}^nC_r - {}^{n-k}C_{r-k}$

(iii) The number of selections of r consecutive things out of n things in a row $= n - r + 1$

(iv) The number of selections of r consecutive things out of n things along a circle $= \begin{cases} n: & r < n \\ 1: & r = n \end{cases}$

(v) The number of combinations of n objects out of which k are identical, taken r at a time

$$=\begin{cases} {}^{n-k}C_r + {}^{n-k}C_{r-1} + {}^{n-k}C_{r-2} + \cdots + {}^{n-k}C_0: & r \le k \\ {}^{n-k}C_r + {}^{n-k}C_{r-1} + {}^{n-k}C_{r-2} + \cdots + {}^{n-k}C_{r-k}: & r \ge k \end{cases}$$

(vi) The number of combinations of n different objects taking r at a time when p particular objects are always included and q particular things are always excluded = ${}^{n-p-q}C_{r-p}$.

16. Combinations from Distinct Objects

(i) The number of selections of zero or more things out of n different things,

$${}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2n$$

(ii) The number of selections of one or more things out of n different things,

$${}^nC_1 + {}^nC_2 + {}^nC_3 + \cdots + {}^nC_n = 2^n - 1$$

17. Combinations from Identical Objects

(i) The number of selections of r objects out of n identical objects = 1.

(ii) The number of selections of zero or more objects from n identical objects = $n + 1$

(iii) The total number of selections of at least one out of $(a_1 + a_2 + \ldots + a_n)$, where a_1 are alike of one kind, a_2 are alike of second kind, …, a_n are alike of nth kind,

$$[(a_1 + 1)(a_2 + 1)(a_3 + 1) \ldots (a_n + 1)] - 1$$

18. Combinations when Identical and Distinct Objects are Present

The number of selections of one or more objects each of n objects $(a_1 + a_2 + \ldots + a_n)$, where a_1 are alike of one kind, a_2 are alike of 2nd kind, …, a_n are alike of nth kind and k are distinct = $[(a_1 + 1)(a_2 + 1)(a_3 + 1) \ldots (a_n + 1)2^k - 1]$.

19. Divisor of a Given Natural Number

Let $\alpha \in N$, where $\alpha = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \ldots p_k^{\alpha_k}$, $p_1, p_2, \ldots, p_k$ are different prime numbers and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are natural numbers. Then

(i) The total number of divisors of α including 1 and $\alpha = (\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1)\ldots(\alpha_k + 1)$

(ii) The total number of divisors of α excluding 1 and $\alpha = (\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1)\ldots(\alpha_k + 1) - 2$.

(iii) The total number of divisors of α excluding either 1 or $\alpha = (\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1)\ldots(\alpha_k + 1) - 1$

(iv) Sum of the divisors

$$= (p_1^0 + p_1^2 + \cdots + p_1^{\alpha_1})(p_2^0 + p_2^1 + \cdots + p_2^{\alpha_2}) \ldots (p_k^0 + p_k^1 + \cdots + p_k^{\alpha_k})$$

$$= \left(\frac{1 - p_1^{\alpha_1+1}}{1 - p_1}\right)\left(\frac{1 - p_2^{\alpha_2+1}}{1 - p_2}\right)\cdots\left(\frac{1 - p_k^{\alpha_k+1}}{1 - p_k}\right)$$

(v) The number of ways in which α can be resolved as a product of two factors

$$=\begin{cases} \dfrac{1}{2}\displaystyle\prod_{r=1}^{k}(\alpha_r + 1): & \text{if } \alpha \text{ is not a perfect square} \\ \dfrac{1}{2}\left[\dfrac{1}{2}\displaystyle\prod_{r=1}^{k}(\alpha_r + 1) + 1\right]: & \text{if } \alpha \text{ is a perfect square} \end{cases}$$

(vi) The number of ways in which composite number α can be resolved into two factors which are relatively prime (or co-prime) to each other is equal to 2^{k-1}, where k is the number of different factors or different primes in α.

(vii) The number of ordered pairs (x, y) such that the LCM of x and y is a composite number of α is $(2\alpha_1 + 1)(2\alpha_2 + 1)(2\alpha_3 + 1)\ldots(2\alpha_k + 1)$.

20. Distribution into Group of Unequal Size among Sets (Groups) and Persons

(i) The number of ways in which $(m + n + p)$ different things can be divided into unequal groups which contain m, n, p things, respectively

$$= {}^{m+n+p}C_m \times {}^{n+p}C_n \times {}^pC_p$$

$$= \frac{(m + n + p)!}{m! \times n! \times p!}$$

(ii) The number of ways to distribute $(m + n + p)$ things among three persons in the groups containing m, n and p things, respectively

= (Number of ways of divide) × (Number of groups)

$$= \frac{(m + n + p)!}{m! \times n! \times p!} \times 3!$$

21. Distribution into Group of Equal Size among Sets (Groups) and Persons

(i) The number of ways in which mn different things can be divided equally in n groups

$$= \left(\frac{(mn)!}{(m!)^n}\right) \times \frac{1}{(n!)}$$

(ii) The number of ways in which mn different things can be distributed equally among n different groups

$$= \left(\frac{(mn)!}{(m!)^n}\right)$$

22. Arrangement into Groups

(i) The number of ways in which n different things can be arranged in r different groups
$= {}^{n+r-1}P_n$, when blank groups are permitted

(ii) The number of ways in which n different things can be arranged in r different groups is
$= {}^{n-1}C_{r-1} \times n!$, when blank groups are not permitted.

(iii) The number of ways in which n different things can be distributed into r different groups
$= r^n - {}^rC_1(r-1)^n + {}^rC_2(r-2)^n - \ldots + (-1)^{r-1}\,{}^rC_{r-1}$,
where blank groups are permitted.

(iv) The number of ways in which n identical things can be distributed into r different groups is
$= {}^{n-1}C_{r-1}$, where blank groups are not allowed.

(v) The number of ways in which n identical things can be distributed into r different groups is
$= {}^{n+r-1}C_{r-1}$, where blank groups are allowed.

(vi) If a group has n things in which r are identical, the number of ways of selecting k things from a group is

$$= \begin{cases} \displaystyle\sum_{k=0}^{r} {}^{n-r}C_k : & k \le r \\ \displaystyle\sum_{r=k}^{k} {}^{n-r}C_k : & k > r \end{cases}$$

23. De-arrangement

Assume $x_1, x_2, x_3, \ldots, x_n$ be n distinct objects such that their positions are fixed in a row. If we now re-arrange $x_1, x_2, x_3, \ldots, x_n$ in such a way that no one occupy its original positions, such an arrangement is called de-arrangement.

For example, a de-arrangement of

is

1	2	3	4	5	6
2	3	5	1	6	4

A de-arrangement of $x_1, x_2, x_3, \ldots, x_n$ is a bijection. Thus $f:\{x_1, x_2, \ldots, x_n\} \to \{x_1, x_2, \ldots, x_n\}$ is a bijective function such that $f(x_i) \ne x_i$, $i = 1, 2, \ldots, n$.

If n things are arranged in a row, the number of ways in which they can de-rangement in such a way that no one of them occupied as its original positions is denoted by $D(n)$ and is defined as $D(n) = n! \times \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + (-1)^n \frac{1}{n!}\right)$

When r things goes to wrong place out of n things, $(n - r)$ things goes to its original place.

Then $D_n = {}^{n-r}C_r \times D_r$, where D_n be the number of group, if all n things goes to the wrong place and D_r be the number of ways, if r things goes to the wrong place.

24. Multinomial Theorem

24.1 Combination with Repetitions

(i) The number of combinations of r things out of n things of which p alike of one kind, q alike of second kind, s alike of third kind and rest $(n - p - q - s)$ things are all different

$= \text{Co-efficient of } x^r \text{ in}$
$(1 + x + \ldots + x^p)(1 + x + \ldots + x^q)(1 + x + \ldots + x^s)$
$(1 + x)(1 + x) \ldots \text{ to } (n - p - q - s) \text{ times}$

$= \text{Co-efficient of } x^r \text{ in}$

$$\left(\frac{1-x^{p+1}}{1-x}\right)\left(\frac{1-x^{q+1}}{1-x}\right)\left(\frac{1-x^{s+1}}{1-x}\right)(1+x)^{n-p-q-s}$$

(ii) The number of combinations of r things out of n things of which p alike of one kind, q alike of second kind, s alike of third kind, when each things is taken at least once

$= \text{Co-efficient of } x^r \text{ in}$
$(x + x^2 + \ldots + x^p)(x + x^2 + \ldots + x^q)(x + x^2 + \ldots + x^s)$
$= \text{Co-efficient of } x^{(r-p-q-s)} \text{ in}$

$$\left(\frac{1-x^{p+1}}{1-x}\right)\left(\frac{1-x^{q+1}}{1-x}\right)\left(\frac{1-x^{s+1}}{1-x}\right)$$

24.2 Permutations with Repitions

(i) The number of permutations of r things out of n things in which p alike of one kind, q alike of second kind, s alike of third kind and so on.

$= \text{Co-efficient of } x^r \text{ in}$

$$\left\{r! \times \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^p}{p!}\right)\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^q}{q!}\right)\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^s}{s!}\right)\cdots\right\}$$

(ii) The number of ways in which n identical things can be distributed into r groups so that no group contains less than m things and more than k things $(m < k)$ is Co-efficient of x^n in

$(x^m + x^{m+1} + x^{m+2} + \cdots + x^k)^r$
$= \text{Co-efficient of } x^{n-mr} \text{ in}$
$(1 + x + x^2 + \cdots + x^{k-m})^r$

$$= \text{Co-efficient of } x^{n-mr} \text{ in } \left(\frac{1-x^{k-m+1}}{1-x}\right)^r \times (1-x)^{-r}$$

25. Solutions of the Equation with the Help of Multinomial Theorem

(i) The number of positive integral solutions of the equation $x_1 + x_2 + \;\; + x_m = n$.
Hence, the required number of solutions

$= \text{Co-efficient of } x^n \text{ in } (x + x^2 + \cdots + x^n)^m$
$= \text{Co-efficient of } x^{n-m} \text{ in}$
$(1 + x + x^2 + \cdots + x^{n-1})^m$

$$= \text{Co-efficient of } x^{n-m} \text{ in } \left(\frac{1-x^n}{1-x}\right)^m$$

$= \text{Co-efficient of } x^{n-m} \text{ in } (1 - x^n)^m \times (1 - x)^{-m}$

$= \text{Co-efficient of } x^{n-m} \text{ in } (1 - mx^n + \cdots) \times (1-x)^{-m}$
$= {}^{m+n-m-1}C_{n-m}$
$= {}^{n-1}C_{m-1}$

(ii) The number of negative integral solutions of the equation $x_1 + x_2 + \ldots + x_m = n$.
Hence, the required number of solutions
$= \text{Co-efficient of } x^n \text{ in } (1 + x + x^2 + \cdots + x^n)^m$
$$= \text{Co-efficient of } x^n \text{ in } \left(\frac{1-x^{n+1}}{1-x}\right)^m$$
$= \text{Co-efficient of } x^n \text{ in } (1 - x^{n+1})^m \times (1-x)^{-m}$
$= \text{Co-efficient of } x^n\ (1 - mx^{n+1} + \cdots) \times (1-x)^{-m}$
$= {}^{m+n-1}C_{m-1}$

(iii) The number of integral solutions of the equation $x_1 + x_2 + \cdots + x_m = n$, when $x_1 \ge c_1, x_2 \ge c_2, \ldots, x_m \ge c_m$
Let $y_1 = x_1 - c_1, y_2 = x_2 - c_2, \ldots, y_n = x_n - c_n$
Then the given equation can be written as
$$y_1 + y_2 + y_3 + \cdots + y_m = n - (c_1 + c_2 + c_3 + \cdots + c_m)$$
where $y_i \ge 0$, $i = 1, 2, 3, \ldots, n$
Hence, the required number of solutions
= the number of solutions of the negative integers
$$= \left(n - \sum_{i=1}^{m} c_i\right) + {}^{n-1}C_{m-1}$$

26. Selection of Squares

Let there be m rows, whereas 1st row has m_1 squares, 2nd row has m_2 squares, 3rd row has m_3 squares and so on.

Now, if we placed n x's in the squares such that each row contain at least one x, the number of the possible ways

$= \text{Co-efficient of } x^n \text{ in}$
$$({}^{m_1}C_1 x + {}^{m_1}C_2 x^2 + \cdots + {}^{m_1}C_{m_1} x^{m_1})$$
$$\times ({}^{m_2}C_1 x + {}^{m_2}C_2 x^2 + \cdots + {}^{m_2}C_{m_2}\, x^{m_2})$$
$$\times ({}^{m_3}C_1 x + {}^{m_3}C_2 x^2 + \cdots + {}^{m_3}C_{m_3}\, x^{m_3})$$
$$\times \cdots$$

27. Geometrical Problems

(i) If there are n points in a plane of which m ($< n$) are colinear, the total number of straight lines obtained by joining these n points is ${}^nC_2 - {}^mC_2 + 1$.

(ii) If there are n points in a plane of which m ($< n$) are colinear, the total number of triangles formed by joining these n points is ${}^nC_3 - {}^mC_3$.

(iii) The number of diagonals in a polygon of n sides
$$= {}^nC_2 - 2 = \frac{n(n-3)}{2}.$$

(iv) If n straight lines are drawn in a plane such that no two lines are parallel and no three lines are concurrent, the number of parts into which these lines divided the plane is $1 + \Sigma n$.

(v) If m parallel lines in a plane are intersected by a family of lines other n parallel lines, the total number of parallelograms so formed
$$= {}^mC_2 \times {}^nC_2$$
$$= \frac{mn(m-1)(n-1)}{4}$$

(vi) **Number of squares:** The number of squares of any size $n \times p$ that can be formed
$$= np + (n-1)(p-1) + (n-2)(p-2) + \ldots + [n(n-1)(p-(n-1))]$$
$$= \begin{cases} \sum_{r=1}^{n} (n-r+1)(p-r+1): & n < p \\ \sum_{r=1}^{n} r^2: & n = p \end{cases}$$

(vii) **Number of rectangles:** The number of rectangles of any size $n \times p$ that can be formed
$$= {}^{n+1}C_2 \times {}^{p+1}C_2$$
$$= \begin{cases} \frac{np(n+1)(p+1)}{4}: & n < p \\ \left(\frac{n(n+1)}{2}\right)^2: & n = p \end{cases}$$

Exercises

Level I

(Questions based on Fundamentals)

FACTORIAL NOTATION

1. Find x, if $\frac{x}{5!} + \frac{x}{6!} = \frac{1}{7!}$.
2. Simplify: $\frac{(2n)!}{n!}$.
3. Simplify: $\frac{3 \cdot 6 \cdot 9 \cdot 12 \ldots (3n-3)3n}{3^n}$.
4. Find the unit digit of $1! + 2! + 3! + 4! + \ldots + (10)!$.
5. If $N = a! + b! + c! + d! + e!$ be a two-digit number, find the value of N, where $a, b, c, d, e \in I$.
6. If the product of factorials of n consecutive positive integers be a single-digit number, find the maximum value of n.

7. If the sum of the factorials of N consecutive natural numbers be a three-digit number, find the maximum value of N.
8. If the unit digit of $N = a! + b! + c!$ is 9, where a, b and c are positive integers, find the product of $\{a! \times b! \times c!\}$.
9. If N be the sum of factorials of all the prime numbers less than 100, find the last two digits of N.
10. A three-digit number xyz such that $x + y + z = 25$. Find the sum of all possible values of $x!y!z!$.
11. Find the sum of $1.1! + 2.2! + 3.3! + 4.4! + \ldots + n \cdot n!$.
12. Find the sum of
$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+\cdots+\frac{99}{100!}.$$

EXPONENT OF A PRIME *P* IN (*N*)!

13. Find the exponent of 3 in (10)!
14. Find the exponent of 5 in (50)!
15. Find the highest power of 10 in 50! + 60! + 70!
16. Find the highest power of 10 in $50! \times 60! \times 70!$
17. Find the highest power of 10 in 10! + 20! + 30! + 40! + … + (2020)!
18. Prove that 33! is divisible by 2^{19}.
19. Find the number of zeroes at the end of (100)!.
20. If the highest power of 10 in $N!$ be 16, find the highest power of 10 in $(N + 1)!$
21. If $m(n)!$ be completely divisible by 5^{11}, where m is a single-digit number, find the minimum value of n.
22. If N be the product of the first 100 multiples of 5, find the highest power of 10 in N.

FUNDAMENTAL PRINCIPLE OF COUNTING

23. Suppose, there are 3 different ways you can come from your home to Sakchi and 4 different ways you can come from Sakchi to MIIT-JEE Institute. In how many ways you can come from your home to MIIT-JEE Institute.
24. There are 5 doors in your classroom. In how many ways a student can enter the classroom and leave by different door?
25. In how many different ways can three students queue up at a bus stop?
26. 5 students compete in a race. In how many ways first three prizes be given?
27. A question paper consists of two sections A and B respectively. There are 5 and 7 questions in section A and B, respectively. In how many ways a student can attempt the questions?
28. A student has 5 different books. In how many ways he can arrange the books in a self of his almirah?
29. How many distinct integers are there in between 100 and 1000?
30. Find the number of terms in the product $(a + b)(c + d + e)(f + g + h + i)(j + k + l + m + n)$.
31. Find the total number of 3-digit numbers which can be made from the digits 1, 2, 3, 4 and 5.
32. Find the number of positive integral solutions of $x + y = 10$.
33. A question paper has two sections A and B respectively. Section A has 7 questions whereas section B has 6 questions, respectively. In how many ways a student can attempt for a single question either from section A or section B?
34. In a class, there are 15 boys and 10 girls. In how many ways your class teacher can select a class monitor?
35. How many 3-digit numbers can be made by the digits either 3 or 4 or 5?
36. How many distinct positive numbers can be made by the digits 1, 2, 3, 4, and 5?
37. How many three-digit distinct numbers can be made by the digits 1, 2, 3, 4 and 5 in which all are either even or odd numbers?
38. Find the number of positive integral solutions of $a + b + c = 6$.

PERMUTATIONS

39. Suppose 8 people enter an event in a swim meet. In how many ways could the gold, silver and bronze medals be awarded?
40. How many integers between 100 and 999 (inclusive) consist of odd digits?
41. How many 3-digits even numbers can be formed by using the digits 1, 2, 3, 4 and 5?
42. Find the number of ways 6 flags of different colors are to be used to form a signal of 4 flags.
43. A flag containing 4 strips of different colours to be designed. If 6 colours can be used, find the number of possible flags.
44. In an experiment on social interaction, 6 people can sit in 6 seats in a row. In how many ways this can be done?
45. In how many ways 3 men stay in a hotel room, if 8 rooms are available?
46. In a club with 15 members, how many ways 3 officers—president, vice-president and treasurer can be chosen?
47. Find the number of permutations of the letters of the word TABLE.
48. If a be the permutations of the letters of the word MAIN, b be the permutations of the letters of the word EXAM, c be the permutations of the letters of the word LOVE, find the value of $a + b + c + 10$.
49. How many 4-digit numbers can be made by the digits 1, 2, 3, 4 and 5, where repetition is not allowed?
50. How many 4-digit distinct odd numbers can be made from the digits 3, 4, 5, 6 and 7?
51. How many numbers lying in between 100 and 1000 can be made from the digits 0, 1, 2, 3, 4 and 5, where repetitions of the digits is not allowed?

52. How many numbers in between 300 and 1000 can be made from the digits 0, 2, 3, 4, 5 and 6, where repetition is not allowed?
53. How many 4-digit even numbers can be made by the digits 0, 2, 3, 4, 5, 6 and 7?
54. How many 5-digit even numbers can be formed from the digits 1, 2, 3, 4 and 5?
55. How many 5-digit odd numbers divisible by 5 can be made by the odd digits?
56. How many 5-digit even numbers divisible by 5 can be made by the digits 0, 1, 2, 3, 4, 5, 6, 7 and 8?
57. How many distinct positive numbers can be made from the digits 0, 1, 2, 3 and 4, where repetition is not allowed?
58. How many numbers of 4-digits greater than 2000 can be made from the digits 0, 1, 2, 3, 4 and 5, where the repetition is not allowed?
59. Suppose two sets A and B consist of 4 and 7 elements, respectively. How many one one functions you can make from A to B?
60. Consider two sets A and B have 5 elements respectively. How many one one onto functions you can make from A to B?
61. Suppose two sets A and B have 4 elements, respectively. How many inverse functions are exist from A to B?
62. How many onto functions can be defined from a set A to a set B, if the sets A and B are consisting of 5 and 3 elements, respectively.
63. Find the number of onto functions between two sets A and B having 5 and 4 elements respectively.

SUM OF NUMBERS

64. Find the sum of the digits in the unit place of all numbers made by the digits 2, 3, 4 and 5.
65. Find the sum of all four-digit numbers made by the digits 1, 2, 3 and 4.
66. Find the sum of all four-digit numbers that can be made from the digits 0, 1, 2 and 3.
67. Find the sum of all 4-digit numbers that can be made by the digits 2, 3, 3 and 4.

PERMUTATION WITH REPETITIONS

68. How many different outcomes are there in an experiment consisting of n tosses of a coin?
69. In how many ways 4 delegates can be put in 5 hotels, if there is no restriction?
70. In how many ways 3 friends can be put up in a 5 hotel of a town, if
    (i) there is no restriction?
    (ii) no two friends can stay together?
71. In how many ways 5 rings can be worn on the four fingers of one hand?
72. If m is the number of ways, a child put 3 marbles in his 4 pockets and n is the number of ways 3-digit distinct numbers can be made from the digits 1, 2, 3, 4 and 5, find the value of $m + n + 10$.
73. In how many ways 5 prizes can be given to 10 students, if each student can get any number of prizes?
74. How many ATM pin codes can be made by the digits of a bank?
75. How many 3 letters code words can be made from the word I N T E X (where repetition is allowed)?
76. In how many ways can a ten questions multiple choice answered in JEE Main examination, if there are four choices a, b, c and d to each question?
77. A student of MIIT-JEE Institute wants to appear in JEE advanced mock examination, which contains 10 multiple choice questions. Each question has four choices a, b, c and d, respectively and more than one correct answers are given. In how many ways the student can give the answer?

PERMUTATIONS OF ALIKE OBJECTS

78. Find the number of permutations of the letters of the word IIT.
79. In how many ways can you permute the letters of the word AIEEE?
80. In how many ways can you permute the letters of the word MATHEMATICS?
81. In how many ways can you permute the letters of the word CONSTITUTION?
82. How many seven-digit numbers can be formed with the digits 1, 2, 2, 2, 3, 3 and 5?
83. Find the sum of all the numbers that can be made by the digits 2, 3, 3, 4, 4 and 4?
84. If m be permutation of the letters of the word JEE-MAIN and n be the permutations of the letters of the word IIT, find the value of $m + n + 10$.
85. Find the number of ways in which 10 students can be arranged in a row such that A is always ahead of B.
86. Find the number of ways in which four particular persons A, B, C, D and six more persons can stand in a queue so that A always stands before B, B before C and C before D.
87. Find the number of permutations of the letters a, b, c and d such that b does not follow a, c does not follow b and d does not follow c.

RESTRICTED PERMUTATIONS

88. In how many ways, 10 boys can be seated in a row so that 2 boys always sit together?
89. In how many ways, 6 boys and 5 girls can be seated in a row so that all boys and girls are sit together?
90. Find the number of ways in which the letters of the word FRACTION be arranged so that all vowels are together.
91. Find the number of ways in which the letters of the word EQUATION be arranged so that all vowels are together.

92. Find the number of ways the letters of the word LAUGH can be arranged so that all vowels and consonants are together.
93. How many different words can be made by the letters of the word UNIVERSITY, so that all vowels are together?
94. In how many ways, 10 students can be seated in a row so that three students A, B and C are always together and C is always ahead of A and B?
95. Find the number of words can be made by the letters of the word ALGEBRA so that all vowels do not come together.
96. Find the number of words that can be made out of the letters of the word INTEGER so that I and N are never come together.
97. Find the number of arrangements that can be made out of the letters of the word SUCCESS so that all Ss do not come together.
98. Find the number of arrangements of the letters of the word BANANA in which two Ns do not appear adjacently.
99. Find the number of arrangements of the letters of the word PARALLEL so that all Ls do not come together but all As come together.
100. How many words, can be made by the letters of the word INTERMEDIATE so that no vowels in between two consonants?
101. In how many ways, can the letters of the word NINETEEN be arranged so that neither all the vowels nor all the consonants are together?

RANK OF A WORD IN DICTIONARY

102. Find the rank of the word TOUGH in the English dictionary.
103. Find the rank of the word IIT in the English dictionary.
104. Find the rank of the word AIEEE in the English dictionary.
105. Find the rank of the word ANNA in the English dictionary.
106. Find the rank of the word PATNA in the English dictionary.
107. Find the rank of the word SURITI according to the English dictionary.
108. Find the rank of the word SANIA in the English dictionary.

GAP METHOD

109. Find the number of ways in which the letters of the word FRACTION be arranged so that no two vowels are together.
110. Find the number of ways in which the letters of the word TRIANGLE be arranged so that no two vowels are together.
111. All the letters of the word EAMCET are arranged in all possible ways. Find the number of such arrangements in which no two vowels are adjacent to each other.
112 In how many ways, letters of the word NINETEEN be arranged so that no two Ns are together?
113. In how many ways, letters of the word NINETEEN be arranged so that no two vowels as well as no two consonants are together?
114. In how many ways, 5 girls and 6 boys in a class can be seated in a row so that the boys and girls are alternate?
115. In how many ways, 5 boys and 5 girls are seated in a row so that they are alternate?
116. How many different nine-digit numbers can be formed from the number 223355888 by re-arranging its digit so that odd digits occupy even positions?
117. How many words can be made from the letters of the word MATHEMATICS without changing the relative order of vowels and consonants?
118. How many words can be made from the letters of the word ACCIDENT without changing the relative order of vowels?
119. How many words can be made from the letters of the word INEFFECTIVE keeping the position of each vowel is fixed?

CIRCULAR PERMUTATIONS

120. In how many ways, 6 Indians and 5 Englishmen can be seated in a round table if
    (i) there is no restriction?
    (ii) all the 5 Englishmen sit together?
    (iii) all the 5 Englishmen do not sit together?
    (iv) no two Englishmen sit together?
121. In how many ways, can we arrange 6 different flowers in a circle?
122. In how many ways, can 50 different pearls be arranged to form a necklace?
123. In how many ways, 3 ladies and 5 gentlemen arrange themselves around a round table so that every gentlemen may have at least one lady?
124. In how many ways, 20 persons can sit around a round table in such a way that there is exactly one person between A and B?
125. In how many ways, 7 boys and 8 girls can be seated around a circular table?
126. In how many ways, 7 boys and 8 girls can be seated around a circular table having 15 chairs numbered from 1 to 15?
127. In how many ways, 10 students can be seated around a circular table in which 3 students A, B and C are always together and B is always between A and C?
128. In how many ways, 20 persons can be seated around two circular tables having 10 seats each?
129. In how many ways, 20 persons be seated around a round table if there are 10 seats are available there?

130. In how many ways, 20 different pearls of two colors can be set alternately on a necklace, there being 10 pearls of each color?
131. How many necklace of 20 beads each can be made from 10 beads of various colors?

COMBINATIONS

132. In how many ways, 7 different books be given to 2 students—one gets 4 books and other gets 3 books?
133. In how many ways, Master Rohan can invites 4 friends out of his 10 friends in a dinner?
134. In how many ways, I can select the best 5 mathematics students out of 12 students for an Olympiad examination?
135. In how many ways, a team of 3 boys and 4 girls can be selected out of 7 boys and 9 girls?
136. In how many ways, 5 vowels and 5 consonants can be selected from 26 letters in the English alphabet?
137. If a bus conductor has 190 different tickets, find the number of stoppage that the bus has consider only one way journey?
138. On a new year day, every student of a class sends a card to every other student. The postman delivers 600 cards. How many students are there in the class?
139. A man has 10 children to take them in a circus. He takes 4 of them at a time to the circus as often as he can without taking the same 4 children together more than once. How many times will he has to go to circus?
140. In an election, three wards of a town are canvassed by 4, 5 and 8 men, respectively. If there are 20 volunteers, in how many ways can they have allotted to different wards?
141. In how many ways, 4 cards are choosing from a pack of 52 cards in which all are of the same suit?
142. In how many ways, 4 cards are choosing from a pack of 52 cards in which all are of different suits?
143. Out of 6 gentlemen and 4 ladies, a committee of 5 is to be formed. In how many ways can this be done so as to include at least one lady in each committee?
144. Find the domain of the function $f(x) = {}^{2x-5}C_{x-2}$.
145. Find the range of the function $f(x) = {}^{3x-7}C_{x-1}$.
146. If ${}^{10}C_r = {}^{10}C_{r+2}$, find r.
147. If ${}^{n}C_{n-4} = 15$, find the value of n
148. If ${}^{15}C_r : {}^{15}C_{r-1} = 11 : 5$, find r.
149. Find the value of ${}^{20}C_{13} + {}^{20}C_{15} + {}^{20}C_9 - {}^{20}C_7 - {}^{20}C_5 - {}^{20}C_{11}$.
150. If ${}^{n-1}C_3 + {}^{n-1}C_4 > {}^{n}C_3$, prove that $n > 7$.
151. Find the value of ${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$.
152. If ${}^{n}C_{r-1} = 36$, ${}^{n}C_r = 84$ and ${}^{n}C_{r+1} = 126$, find n and r.
153. Solve for x: ${}^{x+3}P_3 = {}^{x+2}C_3 + 20$.
154. If $m = {}^{n}C_2$, find the value of ${}^{m}C_2$.
155. If $\frac{1}{{}^{4}C_n} = \frac{1}{{}^{5}C_n} + \frac{1}{{}^{6}C_n}$, find n.

RESTRICTED COMBINATIONS

156. In how many ways, can we select the best 20 students for JEE-Advanced examination from a batch of 100 students, if
    (i) two particular students are always chosen?
    (ii) five particular students are never chosen?
157. A group of students consists of 4 boys and 5 girls. Find the number of ways of selecting a team of at least 3 boys and 4 girls.
158. How many ways are there choosing a hand of 6 cards containing an ace and a king of the same suit from a pack of 52 cards?
159. 10 cards are taken out from a well-shuffle pack of 52 cards. In how many ways of choosing a hand of 10 cards consists of at least one ace?
160. A bag contains 5 black and 5 red balls, all balls being different. Find the number of ways in which 2 black and 3 red balls can be selected.
161. Out of 6 gentleman and 4 ladies, a committee of 5 is to be formed. In how many ways, can this be done so as to include at least one lady in each committee?
162. From 6 boys and 7 girls, a committee of 5 is to be formed so as to include at least one girl. Find the number of ways this can be done?
163. A candidate is required to answer 7 questions out of 12 questions, which are divided into two groups containing 6 questions each. He is not permitted to attempt more than 5 from either group. In how many different ways, can be chosen the seven questions?
164. In an examination, the question paper contains three different sections A, B and C containing 4, 5 and 6 questions respectively. In how many ways, a candidate can make a selection of 7 questions, selecting at least two questions from each section.
165. In an election for 3 seats, there are 6 candidates. A voter cannot vote for more than 3 candidates. In how many ways can he vote?

COMBINATIONS FROM DISTINCT OBJECTS

166. Master Amit has 8 friends. In how many ways he can invite one or more of them in his birthday?
167. In how many ways, Master Roshan can give answer at least one question out of 10 questions in his class test?
168. In an examination, a student will pass if he/she passes all 5 subjects in his class. In how many ways he/she will fail the examination?
169. A questions paper consists of two sections having 3 and 5 questions respectively. One question from each section is compulsory. In how many ways can a student select the question?
170. A bag contains 2 white and 3 red cubes, all of different sizes. In how many ways can 3 cubes be selected from the bag if

(i) at least one cube is white?
(ii) at least one cube is red?

171. There are n students in a class and a group of three is taken by class teacher to zoo. The same set of three children is not taken more than once. It is found that he goes to zoo 84 times more than the visits of a particular child, find the value of n.
172. In a club election, the number of contestants is one more than the maximum number of the candidates which a voter can vote for. If the total number of ways a voter can vote is 126, find the number of contestants.
173. In a chess tournament, the participants were to play one game with one another. Two players fell ill having played 6 games each and without playing with each other. If the total number of games played is 117, find the number of participants at the beginning.
174. A person is permitted to select at least 1 and atmost n coins from a collection of $(2n + 1)$ distinct coins. If the total number of ways in which it can be done is 255, find the value of n.
175. A student is allowed to select atmost n books from a $(2n + 1)$ books. If the total number ways he can select at least one book is 63, find the value of n.

COMBINATIONS FROM IDENTICAL OBJECTS

176. In how many ways, Master Hemant can select 10 red balls out of 20 red balls?
177. In how many ways, Master Kalicharan can select zero or more balls from 12 green balls?
178. In how many ways, Master Bheem can select at least 1 ball out of 10 red balls, 15 green balls, 12 blue balls and 5 black balls?
179. Out of 6 apples, 5 mangoes and 4 bananas, how many selections of fruits can be made?
180. Out of 6 apples, 5 mangoes and 7 bananas, how many selections of fruits can be made such that at least one fruit of each type is always included?
181. Out of 10 apples, 6 mangoes and 4 bananas, how many selections of fruits can be made such that at least one mongo is always included?
182. In a fruit basket, 4 mangoes, 5 bananas and 3 different types of fruits are kept. In how many ways one can select fruits from this fruit basket?
183. Find the number of ways in which one or more letters be selected from the letter of the word ABRACADABRA.
184. Find the number of ways in which one or more letters be selected from the letter of the word MATHEMATICS.
185. Find the number of ways in which one or more letters be selected from the letters of the sequence DADDY DID A DEADLY DEED.
186. Find the number of ways in which a selection of 4 letters can be made from the letters of the word PROPORTION.
187. Find the number of ways in which an arrangement of 4 letters can be made from the letters of the word PROPORTION.
188. Find the number of ways in which a selection of 4 letters can be made from the letters of the word MATHEMATICS.
189. Find the number of ways in which an arrangement of 4 letters can be made from the letters of the word MATHEMATICS.
190. Find the number of ways in which a selection of 4 letters can be made from the letters of the word PASSPORT.
191. Find the number of ways in which an arrangement of 4 letters can be made from the letters of the word PASSPORT.
192. Find the number of ways in which we can select 5 letters of the word INTERNATIONAL.
193. Find the number of ways in which an arrangement of 5 letters can be made from the letters of the word INTERNATIONAL.

DIVISOR OF A GIVEN NATURAL NUMBER

194. If $\alpha = 10800$, find the
   (i) total number of divisors of α
   (ii) the number of even divisors of α
   (iii) the number of divisors α of the form $(4m + 2)$
   (iv) the number of divisors α, which are multiple of 15.
195. In how many ways, the number 18900 can be split into two factors which are relatively prime or co-prime?
196. Find the number of ordered pairs (p, q) such that LCM of p and q is 2520.
197. Find the sum of all divisors of 2520.
198. Find the sum of all odd divisors of 360.
199. Find the sum of all even divisors of 38808.
200. Find the sum of all factors of 7200 that ends with 5.
201. How many number less than 10000 has only 3 factors?
202. In how many ways 8100 can be resolved into product of two factors?
203. In how many ways 10800 can be resolved as a product of two factors?

DISTRIBUTION INTO GROUPS

204. In how many ways 4 different things can be divided P, Q, R and S into 2 sets one having 1 thing and second set having 3 things respectively?
205. In how many ways 4 different things can be distributed among 2 persons such that one having 3 things and other gets 1 thing respectively?
206. In how many ways 4 different things can be divided P, Q, R and S into 2 sets such that each having 2 things respectively?
207. In how many ways 4 different things can be distributed between 2 persons such that each gets 2 things respectively?

ARRANGEMENT INTO GROUPS

208. In how many different ways, 5 different balls can be arranged into 3 different boxes so that no box remains empty?
209. Five balls of different colors are to be placed in three boxes of different sizes. Each box can hold all five balls. In how many different ways can we place the balls so that no box remains empty?
210. In how many ways, 4 different balls can be distributed into 2 boxes so that no box remains empty?
211. In how many ways, 5 different balls can be distributed into 3 boxes so that no box remains empty?
212. In how many ways, 6 different balls can be distributed into 3 boxes so that no box remains empty?
213. In how many ways, can 10 apples can given to 3 students if each child should get at least one apple?
214. In how many ways, 5 identical balls can be distributed into 3 different boxes so that no box remains empty?
215. Find the number of ways of distributing 4 red balls between two persons when each person get at least one ball.
216. In how many ways, 20 apples can be distributed to 4 students in such a way that each boy can receive any number of apples?
217. A box contains 25 balls in which 10 are identical. Find the number of ways of selecting 12 balls from the box.
218. A bag contains 15 balls of which 6 are identical. Find the number of ways of selecting
    (i) 8 balls.
    (ii) 4 balls.
219. Find the number of ways to put 2 letters in 2 addressed envelopes so that all are in wrong envelopes.
220. Find the number of ways to put 3 letters in 3 addressed envelopes so that all are in wrong envelopes.
221. Find the number of ways to put 4 letters in 4 addressed envelopes so that all are in wrong envelopes.
222. In how many ways, we can put 5 different coloured balls into 5 boxes of colours same as that of the balls so that no ball goes to the box of same colour?
223. Let two sets A and B having {1, 2, 3, 4, 5} elements respectively such that $f(i) \neq i$. Find the number of functions between two sets.
224. In how many ways, 5 pair of hand shocks can be distributed among 5 persons such that every person gets an odd pair?
225. There are 5 letters and 5 directed envelopes. In how many ways can all the letters be placed in which 2 are rightly placed?
226. There are 5 boxes of 5 different colours. Also there are 5 balls of colours same as those of the boxes. In how many ways, we can place 5 balls in 5 boxes such that at least 2 balls are placed in boxes of the same colour?
227. In how many ways 6 letters can be placed in 6 envelopes such that atmost 3 letters are placed in wrong envelopes?
228. Six friends go to a birthday party. They leave their coats in the lounge and pick them while returning back. In how many ways can they pick the coats so that exactly one of them picks his own coat?

MULTINOMIAL THEOREM

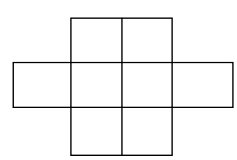
229. Find the number of selections and arrangements of 4 letters taken from the word ENGINEERING.
230. Find the total number of ways of selecting 5 letters of the word INDEPENDENT.
231. In how many ways, can three persons each throwing a single die once, make a sum of 15?
232. In how many ways, 16 identical things can be distributed among 4 persons if each person gets at least 3 things?

SOLUTIONS OF THE EQUATION WITH THE HELP OF MULTINOMIAL THEOREM

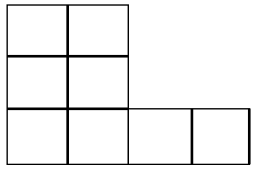
233. Find the number of positive solutions of $x + y + z + w = 20$ such that
    (i) zero value are included
    (ii) zero value are excluded.
234. Find the number of non-negative solutions of $3x + y + z = 24$
235. Find the number of non-negative integral solutions of $2x + 2y + z = 10$.
236. Find the number of non-negative solutions of the equation $x + y + z + t = 29$, where $x \geq 1, \geq 2, z \geq 3, t \geq 4$.
237. Find the number of non-negative integral solutions to the system of equations $x + y + z + t + u = 20$, $x + y + z = 5$.
238. Find the number of ordered triplets of positive integers which satisfy the inequality $20 \leq x + y + z \leq 50$.
239. Find the number of points (x, y, z) in space whose each co-ordinate is a negative integer such that $x + y + z + 12 = 20$
240. Find the number of positive integral solutions of $x_1x_2x_3 = 30$
241. Find the number of positive integral solutions of $x_1x_2x_3x_4 = 210$
242. In how many ways, can a die be thrown thrice by a person to make a sum of 12?
243. In how many ways, your teacher can distribute 10 chocolates into 4 boys if every boy can get at least one chocolate?

SELECTION OF SQUARES

244. Six Xs have to be placed in the squares of the adjoining figure, such that each row contains at least one X. In how many different ways can this be done?



245. In how many ways, the letters of the word PERSON can be placed in the squares of the adjoining figure so that no row remains empty?



GEOMETRICAL PROBLEMS

246. There are 12 points in a plane. Out of these 5 points are in a straight line and with the exception of these 4 points no other 3 points are in the same straight line. Find
   (i) the number of straight lines formed.
   (ii) the number of triangles formed
   (iii) the number of quadrilaterals formed by joining these 12 points.
247. Find the number of diagonals of a polygon of decagon.
248. If the number of diagonals of a polygon is 35, find the number of sides of the polygon.
249. Find the number of triangles whose vertices are at the vertices of an octagon but none of whose sides happen to come from octagon.
250. 10 parallel lines in a plane are intersected by a family of 8 parallel lines. Find the number of parallelogram so formed.
251. Let T_n denotes the number of triangles which can be formed using the vertices of a regular polygon of n sides. If $T_{n+1} - T_n = 21$, find n.
252. Find the number of rectangles of size 7×4.
253. Find the number of rectangles excluding squares from a rectangle of size 9×6.
254. Find the number of points of intersection of 5 circles.
255. Find the number of points of intersection of 8 lines and 4 circles.
256. A rectangle with sides $(2n - 1)$ and $(2m - 1)$ is divided into square of unit length. Find the number of rectangles which can be formed with sides of odd length.
257. Let $n \geq 2$ be an integer. Take n distinct points on a circle and join each pair of points by a line segment. Colour the line segment joining every pair of adjacent points by blue and the rest by red. If the number of red and blue line segments are equal, find the value of n.
258. The sides AB, BC and CA of a triangle ABC have 3, 4 and 5 interior points, respectively, on them. Find the number of triangles that can be constructed using these interior points.
259. Find the number of ways 2 squares can be chosen out of 8×8 chess board such that they have only one corner in common.

Level II

(Mixed Problems)

1. The exponent of 2 in $20 \times 19 \times \cdots \times 11$ is
   (a) 8 (b) 12 (c) 10 (d) 14
2. If (250)! is divisible by q_p, the maximum value of p is
   (a) 61 (b) 62 (c) 63 (d) 64
3. The number of zeroes at the end of (60)! is
   (a) 14 (b) 15 (c) 16 (d) 20
4. The value of the expression
$$\frac{(2n)!}{\{1 \cdot 3 \cdot 5 \ldots (2n-1)\} \times (n)!} \text{ is}$$
   (a) 2^{n-1} (b) 2^n (c) 2^{n+1} (d) 2^{n-2}
5. If a set A has 4 elements and another set B has 5 elements respectively, find the number of one one functions from A to B is
   (a) 12 (b) 108 (c) 120 (d) 625
6. If a set A has 2 elements and another set B has 3 elements, find the number of functions from A to B is
   (a) 8 (b) 27 (c) 27 (d) 9
7. If a set A has 4 elements and another set B has 5 elements, the number of relations from A to B is
   (a) 2^9 (b) 2^{10} (c) 2^{20} (d) 2^{30}
8. The number of ways 4 friends can stay in 10 hotels such that all friends do not stay in a same hotel is
   (a) $4^{10} - 4$ (b) $10^4 - 10$
   (c) $10^4 - 4$ (d) ${}^{10}P_4 - 4$
9. Let the number of elements in a set A is m and the number of elements of another set B is n. If the number of one one functions from A to B is 240, then $m - n$ is
   (a) 4 (b) 6 (c) 8 (d) –3
10. The rank of the word TACKLE according to the English dictionary is
   (a) 602 (b) 603 (c) 604 (d) 605
11. The number of ways in which 8 distinct toys can be distributed among 5 children is
   (a) 5^8 (b) 8^5 (c) ${}^8P_5$ (d) 40
12. The number of ways in which one can post 5 letters in 7 different boxes is
   (a) 35 (b) ${}^7P_5$ (c) 7^5 (d) 5^7
13. Three dice are rolled. The number of possible outcomes in which at least one dice shows 5 is
   (a) 215 (b) 36 (c) 125 (d) 91
14. If all the permutations of the letters of the word AGAIN are arranged as in dictionary, then 50th word is
   (a) NAAGI (b) NAGAI
   (c) NAAIG (d) NAIAG
15. The value of ${}^nC_0^2 + {}^nC_1^2 + {}^nC_2^2 + \cdots + {}^nC_n^2$ is
   (a) ${}^{2n}C_n$ (b) ${}^{2n}C_{n+1}$
   (c) ${}^{2n}C_{n-1}$ (d) ${}^{2n}C_{n+2}$
16. The number of arrangements which can be made using all the letters of the word LAUGH if vowels occur together is
   (a) 96 (b) 72 (c) 48 (d) 24
17. The number of ways in which 12 identical balls can be put into 5 different boxes so that no box is remain empty, is
   (a) 660 (b) 110 (c) 165 (d) 330.

18. The rank of the word IIT according to English dictionary is
(a) 2 (b) 1 (c) 3 (d) 4
19. In how many ways, can the letters of the word ARRANGE be arranged so that two Rs are never together is
(a) 800 (b) 600 (c) 900 (d) 700
20. The number of ways in which 10 candidates $A_1, A_2, \ldots, A_{10}$ can be ranked so that A_1 always above A_2 is
(a) (10)! (b) $\frac{(10)!}{2}$
(c) (9)! (d) $\frac{(9)!}{2}$
21. There are 4 letters and 4 directed envelopes. The number of ways in which all the letters can be put in the wrong envelope is
(a) 8 (b) 9 (c) 16 (d) 20
22. 6 boys and 6 girls sit along a line alternately in x ways and along a circle in y ways, then
(a) $x = y$ (b) $y = 12x$
(c) $x = 10y$ (d) $x = 12y$
23. The number of ways in which 7 persons can be arranged at a round table if two particular persons may not sit together is
(a) 480 (b) 120 (c) 80 (d) 100
24. A round table conference is to be held between 20 delegates of 20 countries. The number of ways can they be seated if two particular delegates sit together is
(a) $2 \times (17)!$ (b) $2 \times (19)!$
(c) $2 \times (18)!$ (d) $2 \times (20)!$
25. The number of positive integral solutions of $xyz = 20$ is
(a) 9 (b) 27 (c) 81 (d) 243
26. The number of positive integral solutions of $x + y + z = 10$ is
(a) 33 (b) 55 (c) 66 (d) 44
27. The number of positive integral solutions of $2x + 2y + z = 10$ is
(a) 33 (b) 55 (c) 66 (d) 44
28. The number of points of intersection of 5 circles is
(a) 30 (b) 20 (c) 66 (d) 40
29. The number of points of intersection of 8 lines and 4 circles is
(a) 32 (b) 64 (c) 76 (d) 104
30. The number of rectangles of size 6×4 is
(a) 110 (b) 160 (c) 210 (d) 150
31. The number of rectangles of any size in a chess board is
(a) 216 (b) 1296 (c) 64 (d) 1024.
32. The number of squares of any size in a rectangle of size 8×5 is
(a) 150 (b) 130 (c) 100 (d) 160
33. The number of zeroes at the end of (2013)! is
(a) 400 (b) 501 (c) 250 (d) 160
34. The number of ways of 52 cards can be equally divided among 4 persons is
(a) $\frac{(52)!}{(13!)^4 \times 4}$ (b) $\frac{(52)!}{(13!)^4}$
(c) $\frac{(52)!}{(13!)^4 \times 4!}$ (d) $\frac{(52)!}{(13!)^4 \times 2!}$
35. The number of ways 12 balls can be divided between 2 boys, one receiving 5 and the other gets 7 is
(a) 1584 (b) 1284 (c) 1384 (d) 1484
36. The number of non-negative integral solutions of $3x + y + z = 24i$
(a) 115 (b) 117 (c) 119 (d) 121
37. The number of solutions of $x + y + z = 6$, where $x, y, z \in N$, is
(a) 180 (b) 280 (c) 380 (d) 480
38. The number of ways 5 different balls can be arranged into 3 different boxes, so that no box remains empty, is
(a) 420 (b) 620 (c) 720 (d) 520
39. The number of ways 5 identical balls can be distributed into 3 different boxes, so that no box remains empty, is
(a) 5 (b) 6 (c) 10 (d) 15
40. The number of ways 16 identical balls can be distributed among 4 persons, if each person get at least 3 things is
(a) 55 (b) 25 (c) 35 (d) 45
41. The sum of all odd divisors of 360 is
(a) 75 (b) 76 (c) 78 (d) 88
42. The number of ways the number 18900 can be resolved as a product of two factors is
(a) 25 (b) 30 (c) 36 (d) 40
43. The number of ways 18900 can be split into two factors, which are relatively prime, is
(a) 8 (b) 10 (c) 12 (d) 6
44. The number of ways to put 5 letters in 5 addressed envelopes so that all are in wrong envelopes is
(a) 22 (b) 33 (c) 44 (d) 55
45. The number of ways to put 6 letters in 6 addressed envelopes so that all are in wrong envelopes is
(a) 262 (b) 363 (c) 265 (d) 565
46. The number of permutations of the letters a, b, c and d such that b does not follow a, c does not follow b, and d does not follow c, is
(a) 12 (b) 11 (c) 14 (d) 13
47. The number of function f from the set $\{1, 2, 3, 4, 5\}$ into the set $\{1, 2, 3, 4, 5\}$ such that $f(i) \neq i$, is
(a) 30 (b) 44 (c) 9 (d) 63
48. The number of ways distributing 12 identical oranges among 4 children so that every child gets at least one and no child gets more than 4 is
(a) 31 (b) 32 (c) 35 (d) 42
49. In how many ways can 20 oranges can be given to four children if each child get at least one orange?
(a) 969 (b) 691 (c) 891 (d) 979

50. Four dice are rolled. The number of possible outcomes, in which at least one dice shows 2, is
(a) 625 (b) 671 (c) 1296 (d) 216

51. The number of ways in which a selection of four letters can be made from the letters of the word MATHEMATICS, is
(a) 136 (b) ${}^{9}C_4$ (c) ${}^{11}C_4$ (d) $\frac{(11)!}{(2!)^3}$

Level III

(Problems for JEE-Advanced)

1. Find the number of signals that can be made by five flags of different colours when any number of them may be used.
2. According to the English dictionary, if the rank of the word is TOUGH is m and the rank of the word IIT is n, find the value of $m + n + 10$.
3. There is a polygon of n sides ($n > 5$). Triangles are formed by joining the vertices of the polygon. Find the number of triangles which have no side common with any of the sides of the polygon.
4. There are 12 intermediate stations between two places A and B. In how many ways, can a train be made to stop at 4 of these 12 intermediate stations so that no two of which are consecutive.
5. A straight is a five card hand containing consecutive values. How many different straight there?
6. If each m points on one straight line be joined to each of m points on the other straight line terminated by the points, excluding the points on the given two lines. Find the number of points of intersection of these lines.
7. John has x children by his first wife. Mary has $x + 11$ children by his first husband. They marry and have children their own. The whole family has 24 children. Assuming that two children of the same parents do not fight. Prove that the maximum number of fights that can be taken place is 191.
8. Five balls of different colours are to be placed in three boxes of different sizes. Each box can hold all five balls. In how many different ways, can we place the balls so that no box remains empty.
9. In how many ways, 16 identical balls can be distributed among 4 persons if each person gets at least 3 things?
10. In how many ways, 30 marks can be allotted to 8 questions if each question carries at least 2 marks?
11. In an examination, the maximum marks for each of the three papers are 50 each. Maximum marks for the 4th paper is 100. Find the number of ways in which the candidate can score 60% marks in the aggregate.
12. Find the number of positive unequal integral solutions of $a + b + c + d = 20$.
13. Find the number of selections of n letters together out of $3n$ letters of which n are a and n are b and the rest are unlike.
14. Find the the total number of ways of selecting 5 letters from the letters of the word INDEPENDENT.
15. Find the number of permutations of the letters of the word PARALLEL taken 4 at a time.
16. How many three-digit numbers are of the form xyz with $x < y > z$ and $x \neq 0$.
17. In how many ways, can 7 plus signs and 5 minus signs be arranged in a row so that no two minus signs are together?
18. Find the number of positive integral solutions of $x + 2y + 3z = 10$.
19. India and Pakistan will play an international series until one team wins 4 matches. No match ends in a draw. In how many ways the series can be won?
20. Find the number of quintuples (x, y, z, u, v) of positive integers satisfying both—$x + y + z + u = 30$ and $x + y + z + u = 27$.
21. Find the number of non-negative integral solutions of $x + y + z + 4t = 30$.
22. Find the number of non-negative integral solutions of $3x + y + z = 24$.
23. In how many ways, can a dice be thrown thrice by a person to make a sum of 12?
24. In how many ways, can 20 persons sit around a round table such that there is exactly one person between A and B?
25. An examination consists of 4 papers, each paper has a maximum of m marks. Find the number of ways a student get $2m$ marks.
26. Find the last two digits in $x = \sum_{k=1}^{2016} (k!)$
27. Let A be the set of 4-digit numbers $abcd$, where $a > b > c > d$, find $n(A)$.
28. In how many ways can the letters AAABBCD be arranged so that
(i) the two Bs are together but not two As are together.
(ii) no two Bs and no two As are together.
29. Let P_n denotes the number of ways of selecting 3 people out of n sitting in a row, if no two of them are consecutive and Q_n is the corresponding figure when they are in a circle. If $P_n - Q_n = 6$, find the value of n.
30. Let A and B two sets such that $A = \{0, 1, 2\}$ and $B = \{0, 1, 2, 3, 4, 5, 6, 7\}$. Find the number of functions from A into B, where $i < j \Rightarrow f(i) \leq f(j)$, $i, j \in A$.

Integer Type Questions

1. If ${}^{n+1}C_{r+1} : {}^{n}C_{r} : {}^{n-1}C_{r-1} = 11 : 6 : 3$, find the value of r.
2. If $\frac{{}^{n}C_r}{{}^{n}C_{r-1}} = \frac{7}{3}$ and $\frac{{}^{n}C_{r+1}}{{}^{n}C_r} = \frac{3}{2}$, find the value of n.

3. Find the number of positive integral values of x for which $4({}^{x-1}C_4 - {}^{x-1}C_3) < 5(x-2)(x-3)$.
4. If m is the number of ways, where 4 boys and 4 girls are seated in a row so that they are alternate and n is the number of ways, where 5 boys and 4 girls are seated in a round table so that they are alternate, find the value of $\left(\frac{m}{n}+2\right)$.
5. Let P_n denotes the number of ways in which three people can be selected out of n people sitting in a row so that no two of them are consecutive.
If $P_{n+1} - P_n = 15$, find n.
6. If $\frac{{}^nP_{r-1}}{a} = \frac{{}^nP_r}{b} = \frac{{}^nP_{r+1}}{c}$, find the value of $\left(\frac{b^2}{a(b+c)}+2\right)$.
7. If m be the number of ways, in which 7 persons can be seated in a round table if 2 particular persons may not sit together and n the number of arrangement of letters of the word DELHI, where E always comes before I, find the value of $\left(\frac{m}{n}-3\right)$.
8. If the number of ordered triplets of positive integers which satisfy the inequality $11 \le (a+b+c) \le 50$ is $({}^xC_3 - {}^yC_3)$, find the value of $\left(\frac{x}{y}+2\right)$.
9. If m be the number of positive integral solutions of $xyz = 30$ and n the number of integral solutions of $abcd = 210$ such that $m = 3^p$ and $n = 2^q$, find the value of $(q-2p)$.
10. In how many ways, 5 identical balls can be distributed into 3 different boxes so that no box remain empty?
11. If m be the number of ways in which a score of 11 can be made from a throw by three persons, each throwing a single die once and n is the number of ways 5 apples be distributed among the 3 students, so that each can get any number of apples, find the value of $(m-n)$.
12. If m be the number of different words that can be made from the word BHARAT in which B and H are never together and n be the number of words that can be made from the letter of the word LAUGH if vowels occur together, find the value of $\left(\frac{m}{n}+3\right)$.
13. If the number of ways in which n distinct objects can be put into two identical boxes, so that no box remains empty, is 127, find n.
14. Find the number of values of n, for which $\sum_{k=1}^{n}(k!)$ is the square of an integer.
15. In a certain test, there are n questions. In this test, 2^{n-k} students gave wrong answers for at least k questions, where $k = 1, 2, 3, \ldots, n$. If the total number of wrong answers given is 511, find the value of n.
16. Everybody in a room shakes hands with everybody else. If the total number of handshakes is 36, find the number of people in the room.
17. In a bakery shop, four types of biscuits are available. If a person can buy 10 biscuits, if he decides to take at least one biscuit of each variety is ${}^xC_y$ ways, find the value of $(x-y-2)$.
18. In a JEE-Advanced mock test, there are n questions. In this test, 3^{n-k} students gave wrong answers for at least k questions, where $k = 1, 2, 3, \ldots, n$. If the total number of wrong answers is 3280, find the value of n.
19. There are 4 pairs of hand gloves of 4 different colours. In how many ways can they be paired off so that a left handed glove and a right handed glove are not of the same colour?
20. If the number of possible outcomes in a throw of n ordinary dice in which at least one of the dice shows an odd number is 189, find the value of n.

Comprehensive Link Passages

Passage I

Let p be a prime number and n a positive integer, the exponent of a prime p in $n!$ is $E_p(n!)$ and is given by

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \cdots + \left[\frac{n}{p^k}\right]$$

where $p^k < n < p^{k+1}$, and [] = GIF

If we isolate the power of each prime contained in any number N, N can be written as $N = 2^{\alpha_1} \cdot 3^{\alpha_2} \cdot 5^{\alpha_3} \cdot 7^{\alpha_4}$, where α_i are whole numbers.

On the basis of the above information, answer the following questions.

1. The exponent of 7 in ${}^{100}C_{50}$ is
(a) 0 (b) 1 (c) 2 (d) 3
2. The number of zeroes at the end of 108! is
(a) 10 (b) 13 (c) 25 (d) 26
3. The last non-zero digit in 20! must be equal to
(a) 2 (b) 4 (c) 6 (d) 8
4. The exponent of 12 in 100! is
(a) 32 (b) 48 (c) 97 (d) none
5. The number of prime numbers among the numbers (105)! + 2, (105)! + 3, (105)! + 4, …, (105)! + 104 is
(a) 30 (b) 32 (c) 33 (d) none

Passage II

Suppose a lot of n objects contains n_1 objects of one kind, n_2 objects of second kind, n_3 objects of third kind, …, n_k objects of kth kind, such that $n_1 + n_2 + \ldots + n_k = r$.

The number of possible arrangements of r objects out of this lot is the co-efficient of x^r in the expansion of $(r)! \times \prod\left(\sum_{\lambda=0}^{n_1}\left(\frac{x^\lambda}{(\lambda)!}\right)\right)$.

On the basis of the above information, answers the following questions.

1. The number of permutations of the letters of the word INDIA, taken three at a time, must be
(a) 27 (b) 30 (c) 33 (d) 57
2. If $n_1 = n_2 = \ldots = n_k = 1$, the permutations of the r objects must be
(a) ${}^nP_r$ (b) ${}^nC_r$ (c) ${}^kP_r$ (d) ${}^kC_r$
3. The number of permutations of the letters of the word INEFFECTIVE, taken four at a time, must be
(a) 2214 (b) 1422 (c) 5424 (d) 2454
4. If $n_1 + n_2 + \ldots + n_k = r$, the number of permutation must be
(a) ${}^nC_r$ (b) ${}^nP_r$
(c) $(k + r)!$ (d) $\frac{(r)!}{n_1! \times n_2! \times \ldots \times n_k!}$
5. Five-letter words are to be formed out of the letters of the word INFINITESIMAL, then the number of permutations must be
(a) 16995 (b) 5665
(c) 11330 (d) 22660

Passage III

Different words are being formed by arranging the letters of the word SUCCESS. All the words obtained by written in the form of a dictionary.

On the basis of the above information, answer the following questions.

1. The number of words in which two C are together, but no two S are together is
(a) 120 (b) 96 (c) 24 (d) 420
2. The number of words in which no two C and no two S are together is
(a) 120 (b) 96 (c) 24 (d) 420
3. The number of words in which the consonants appear in alphabetical order is
(a) 42 (b) 40 (c) 420 (d) 280
4. The rank of the word SUCCESS in the dictionary is
(a) 328 (b) 329 (c) 330 (d) 331
5. The number of words in which the relative positions of vowels and consonants unaltered is
(a) 20 (b) 60 (c) 180 (d) 540.

Passage IV

Different words are being formed by arranging the letters of the word ARRANGE. All the words obtained are written in the form of a dictionary.

On the basis of the above information, answer the following questions.

1. The number of words in which two R are not together is
(a) 1260 (b) 660 (c) 900 (d) 240
2. The number of words in which neither two R nor two A come together is
(a) 1260 (b) 660 (c) 1160 (d) 540
3. The number of words in which the consonants appear in alphabetical order is
(a) 100 (b) 105 (c) 360 (d) 240
4. The rank of the word ARRANGE in the dictionary is
(a) 340 (b) 341 (c) 342 (d) 343.
5. The number of words in which at least one vowel is in between two consonants is
(a) 18 (b) 36 (c) 624 (d) 836.

Matrix Match

1. Match the following columns.

Column I		Column II	
(A)	The number of positive integral solutions of the equation $x_1x_2x_3x_4x_5 = 1050$ is λ, then λ is divisible by	(P)	3
(B)	Let y be the element of the set $A = \{1, 2, 3, 5, 6, 10, 15, 30\}$ and integers x_1, x_2, x_3 such that $x_1 \cdot x_2 \cdot x_3 = y$. If λ be the number of integral solutions of $x_1 \cdot x_2 \cdot x_3 = y$, then λ is divisible by	(Q)	4
		(R)	5
		(S)	8
(C)	Let a be a factor of 120. If λ be the number of positive integral solutions of $x_1 \cdot x_2 \cdot x_3 = a$, then λ is divisible by	(T)	16

2. Match the following columns.

Column I		Column II	
(A)	If λ be the number of ways in which 6 boys and 5 girls can be arranged in a line so that they are alternate, λ is divisible by	(P)	5 !
		(Q)	6 !
(B)	If λ be the number of ways in which 6 boys and 5 girls can be seated in a row such that two girls are never together, λ is divisible by	(R)	7 !
(C)	If λ be the number of ways in which 6 boys and 5 girls can be seated around a round table if all the five girls do not sit together, λ is divisible by	(S)	$5! \times 6!$
		(T)	$6! \times 7!$

3. Match the following columns.

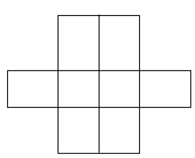
Column I		Column II	
(A)	If n be the number of ways in which 12 different books can be distributed in 3 groups, $\frac{(4)!}{(12)!}\times n$ is divisible by	(P)	4
		(Q)	6
(B)	If n be the number of ways in which 12 different things can be distributed in 3 groups, $\frac{(4!)^3\times(3!)^2}{(12)!}\times n$ is divisible by	(R)	12
		(S)	24
(C)	If n be the number of ways in which 12 different things can be distributed is 5 sets of 2, 2, 2, 3, 3 things, $\frac{(3!)^3\times(2!)^4\times 5}{(12)!}\times n$ is divisible by	(T)	30

4. Match the following columns.

Column I		Column II	
(A)	If λ_1 be the number of positive integral solutions of $x + y + z + u = 10$, λ_1 is divisible by	(P)	3
		(Q)	4
(B)	If λ_2 be the non-negative integral solutions of $x + y + z = 6$, λ_2 is divisible by	(R)	5
(C)	If λ_3 be the number of solutions of $x + y + z = 15$ such that $x \geq 1$, $y \geq 2$ and $z \geq 3$, λ_3 is divisible by	(S)	7
		(T)	11

Questions asked in Previous Years' JEE-Advanced Examinations

1. Six Xs have to place in the squares of the given figure in such a way that each row contains at least one X. In how many different ways can this be done? **[IIT-JEE, 1978]**



2. In how many ways, can pack of 52 cards be divided equally amongst four players in order? **[IIT-JEE, 1979]**
3. In how many ways, can you divide these cards in 4 sets, three of them having 17 cards each and fourth one just 1 card? **[IIT-JEE, 1979]**
4. Ten different letters of an alphabet are given. Words with five letters are formed from these given letters. The number of words, which have at least one letter repeated, is
   (a) 69760 (b) 30240 (c) 99748 (d) None **[IIT-JEE, 1980]**
5. Consider the set A of all determinants of order 3 with entries 0 and 1. Let B be the subset of A constituting of all determinants with value 1. Let C be the subset of A consisting all determinants with value –1. Then
   (a) C is empty.
   (b) B has as many elements as C.
   (c) $A = B \cup C$
   (d) B has twice as many elements in C.
   [IIT-JEE, 1980]
6. The value of ${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$ is
   (a) ${}^{47}C_5$ (b) ${}^{52}C_5$ (c) ${}^{52}C_5$ (d) none **[IIT-JEE, 1980]**
7. Five balls of different colours are to be placed in three boxes of different sizes. Each box can hold all five. In how many different ways, can we place the balls so that no box remains empty? **[IIT-JEE, 1981]**
8. Eight chairs are numbered 1 to 8. 2 women and 3 men wish to occupy one chair each. First the women choose the chairs from amongst the chairs marked 1 to 4 and then the men select the chairs from amongst the remaining. The number of possible arrangements is
   (a) ${}^6C_3 \times {}^4C_2$ (b) ${}^4P_2 \times {}^4P_3$
   (c) ${}^4C_2 + {}^4P_3$ (d) none
   [IIT-JEE, 1982]
9. In a certain test, a_i students gave wrong answers to at least i questions where $i = 1, 2, \ldots, k$. No students gave more than k wrong answers. The total number of wrong answers given is..... **[IIT-JEE, 1982]**
10. m men and n women are to be seated in a row so that no two women sit together. If $m > n$, show that the number of ways in which they can be seated is $\frac{m!\times(m+1)!}{(m-n+1)!}$. **[IIT-JEE, 1983]**
11. The sides AB, BC and CA of a triangle ABC have 3, 4 and 5 interior points respectively on them. The number of triangles that can be constructed using these interior points as vertices is...... **[IIT-JEE, 1984]**
12. A man has 7 relatives, 4 of them are ladies and 3 gentlemen and his wife has also 7 relatives, 3 of them are ladies and 4 are gentlemen. In how many ways, can they invite a dinner party of 3 ladies and 3 gentlemen so that there are 3 of man's relative and 3 of wife's relative. **[IIT-JEE, 1985]**
13. A box contains 2 white balls, 3 black balls and 4 red balls. In how many ways, can 3 balls be drawn from the box if at least one ball is included in the draw? **[IIT-JEE, 1986]**
14. A student is allowed to select atmost n books from a collection of $2n + 1$ books. If the total number of ways in which he can select at least one book is 63, find the value of n. **[IIT-JEE, 1987]**
15 Find the total number of ways in which 6 '+' and 4 '–' signs can be arranged in a line such that no two '–' signs occur together. **[IIT-JEE, 1988]**

16. A 5-digit number divisible by 3 is to be formed using the numbers 0, 1, 2, 3, 4 and 5 without repetition. Find the total number of ways this can be done.
[IIT-JEE, 1989]

17. In an examination, the maximum marks for each three papers are 50 each. Maximum marks for 4th paper are 100. Find the number of ways in which the candidate can score 60 % marks in the aggregate.
[IIT-JEE, 1989]

No questions asked in 1990.

18. 18 guests have to be seated, half on each side of a long table. 4 particular guests desire to sit on one particular side and three others on the other side. Determine the number of ways in which the sitting arrangements can be made?
[IIT-JEE, 1991]

19. There are four balls of different colours and four boxes of colours same as those of different colours. Find the number of ways in which the balls, one each in a box, could be placed such that a ball does not go to a box of its own colour. **[IIT-JEE, 1992]**

No questions asked in 1993.

20. A committee of 12 is to be formed from 9 women and 8 men. In how many ways this can be done if at least 5 women have to be included in a committee
(i) the women are in majority?
(ii) the men are in majority?
[IIT-JEE, 1994]

No questions asked in 1995.

21. Let n and k be positive integers such that $n \geq \frac{k(k+1)}{2}$.
The number of solutions $(x_1, x_2, \ldots, x_k)$, $x_1 \geq 1$, $x_2 \geq 2$, $\ldots$, $x_k \geq k$ all integers, satisfying $x_1 + x_2 + \ldots + x_k = n$, is...
[IIT-JEE, 1996]

22. Find the number of ways of selecting 5 letters from the letters of the word INDEPENDENT.
[IIT-JEE, 1997]

23. The number of divisors of the form $4n + 2$ (≥ 0) of the integer 240 is
(a) 4 (b) 8 (c) 10 (d) 3
[IIT-JEE, 1998]

24. An n-digit number is a positive number with exactly n digits. Nine hundred distinct n-digit numbers are to be formed using only the three digits 2, 5 and 7. The smallest value of n for which this is possible is
(a) 6 (b) 7 (c) 8 (d) 9
[IIT-JEE, 1998]

25. In a college of 300 students, every students reads 5 newspapers and every newspaper is read by 60 students. The number of newspapers is
(a) at least 30 (b) atmost 20
(c) exactly 25 (d) none
[IIT-JEE, 1998]

26. The number of ways in which 5 male and 2 female members of a committee can be seated around a round table so that the two females are not seated together is
(a) 480 (b) 600 (c) 720 (d) 840.
[IIT-JEE, 1999]

27. How many different nine-digit numbers can be formed from the number 223355888 by re-arranging its digits so that the odd digits occupy even positions?
(a) 16 (b) 36 (c) 60 (d) 80
[IIT-JEE, 2000]

28. For $2 \leq r \leq n$, $\binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2}$
(a) $\binom{n+1}{r-1}$ (b) $2\binom{n+1}{r+1}$
(c) $\binom{n+2}{r}$ (d) $2\binom{n+2}{r}$
[IIT-JEE, 2000]

29. Let T_n denote the number of triangles which can be formed using the vertices of a regular polygon of n sides.
If $T_{n+1} - T_n = 21$, then n equals
(a) 5 (b) 7 (c) 6 (d) 4
[IIT-JEE, 2001]

30. The arrangement of the letters of the word BANANA in which the two N's do not appear adjacently.
(a) 40 (b) 60 (c) 80 (d) 100
[IIT-JEE, 2002]

No questions asked in 2003.

31. If ${}^{n-1}C_r = (k^2 - 3)\,{}^{n}C_{r+1}$, then the value of k is
(a) $(-\infty, 2]$ (b) $(2, \infty)$
(c) $[-\sqrt{3}, \sqrt{3}]$ (d) $(\sqrt{3}, 2]$
[IIT-JEE, 2004]

32. A rectangle with sides $(2n - 1)$ and $(2m - 1)$ is divided into square of unit length. The number of rectangles which can be formed with sides of odd length is
(a) $(m + n - 1)^2$ (b) 4^{m-n-1}
(c) m^2n^2 (d) $mn(m + 1)(n + 1)$
[IIT-JEE, 2005]

33. If the L.C.M of p, q is $r^2t^4s^2$, where r, s, t are prime numbers and p, q are the +ve integers, the number of ordered pairs (p, q) is
(a) 252 (b) 254 (c) 225 (d) 224
[IIT-JEE, 2006]

34. The letters of the word COCHIN are permuted and all the permutations are arranged in an alphabetical order as in an english dictionary. The number of words that appear before the word COCHIN.
(a) 360 (b) 192 (c) 96 (d) 48
[IIT-JEE, 2007]

35. Consider all possible permutations of the letters of the word E N D E A N O E L.
Match the statements expressions in column-I with the statements/expressions in column-II

Column-I		Column-II	
(A)	The number of permutations containing the word ENDEA	(P)	5 !
(B)	The number of permutations in which the letter E occurs in the first and the last position is	(Q)	$5! \times 2$
(C)	The number of permutations in which none of the *N* occurs in the first five positions is	(R)	$5! \times 7$
(D)	The number of permutations in which the letters A, E, O occur only in odd positions is	(S)	$5! \times 2!$

[IIT-JEE, 2008]

36. The number of seven digits integers with sum of the digits equal to 10 and formed by using the digits 1, 2 and 3 only is
(a) 55 (b) 66 (c) 77 (d) 88
[IIT-JEE, 2009]

37. Let S = {1, 2, 3, 4}. The total number of unordered pairs of disjoint subsets of *S* is equal to
(a) 25 (b) 34 (c) 42 (d) 41
[IIT-JEE, 2010]

No question asked in 2011.

38. The total number of ways of 5 balls of different colors can be distributed among 3 persons so that each person get aleast one ball is
(a) 75 (b) 150 (c) 210 (d) 240
[IIT-JEE, 2014]

39. Comprehension.
Let a_n be denote the number of all *n*-digit positive integers formed by the digits 0, 1 or both such that no consecutive digits in them are 0. Let b_n = the number of such *n* digit integers ending with digit 1 and c_n = the number of such *n*-digit integers ending with digit 0.
(i) the value of b_n is
(a) 7 (b) 8 (c) 9 (d) 11
(ii) Which of the following is correct?
(a) $a_{17} = a_{16} + a_{15}$
(b) $c_{17} \neq c_{16} + c_{15}$
(c) $b_{17} \neq b_{16} + b_{15}$
(d) $a_{17} = b_{16} + c_{17}$
[IIT-JEE, 2012]

40. Consider the set of eight vectors
$V = \{a\hat{i} + b\hat{j} + c\hat{k} : a, b, c \in \{-1, 1\}\}$
Three non-coplanar vectors can be chosen from *V* in 2^p ways. Then *p* is......
[IIT-JEE, 2013]

41. Let $n_1 < n_2 < n_3 < n_4 < n_5$ be positive integers such that $n_1 + n_2 + n_3 + n_4 + n_5 = 20$. Then the number of such distinct arrangements $(n_1, n_2, n_3, n_4, n_5)$ is.......
[IIT-JEE, 2014]

42. Let $n \geq 2$ be an integer. Take *n* distinct points on a circle and join each pair of points by a line segment. Colour the line segment joining every pair of adjacent points by blue and the rest by red. If the number of red and blue line segments are equal, then the value of *n* is........
[IIT-JEE, 2014]

Answers

Level I

1. $x = \frac{1}{49}$
2. $2^n \times \{1 \cdot 3 \cdot 5 \ldots (2n-1\}$
3. $(n)!$
4. 3
5. 34
6. 2
7. 6
8. 12
9. 68
10. $51 \times 9! \times 8! \times 7!$
11. $(n+1)! - 1$
12. $\left(1 - \frac{1}{100!}\right)$
13. 4
14. 12
15. 12
16. 42
17. 2
18. 2^{19}
19. 24
20. 18
21. 45
22. 97
23. 12
24. 20
25. 6
26. 60
27. 35
28. 120

29. 648
30. 120
31. 120
32. 9
33. 13
34. 25
35. 9
36. 325
37. 60
38. 10
39. 336
40. 60
41. 24
42. 360
43. 360
44. 720
45. 336
46. 2730
47. 120
48. 120
49. 82
50. 72
51. 100
52. 82
53. 420
54. 48
55. 24
56. 1680
57. 260
58. 120
59. 540
60. 120
61. 24
62. 150
63. 240
64. 84
65. 66660
66. 38664
67. 39996
68. 2^n
69. 625
70. (i) 125 (ii) 60
71. 625
72. 134
73. 10^5
74. 9000
75. 125
76. 4^{10}
77. 15^{10}
78. 3
79. 20
80. $\dfrac{11!}{2!\times 2!\times 2!}$
81. $\dfrac{12!}{2!\times 3!\times 2!\times 2!}$
82. $\dfrac{7!}{3!\times 2!}$
83. 22222200
84. 2533
85. 1814400
86. 453600
87. 11
88. $9!\times 2!$
89. $6!\times 5!\times 2!$
90. $6!\times 3!$
91. $4!\times 5!$
92. 24
93. $12!\times 7!$
94. 13440
95. 2160
96. 2140
97. 360
98. 40
99. 720
100. $\dfrac{7!}{2!}\times\dfrac{5!}{2!}$
101. 992
102. 89
103. 1
104. 4
105. 3
106. 42
107. 236
108. 54
109. 14400
110. 14400
111. 72
112. 400
113. 32
114. $6!\times 5!$
115. 28800
116. 60
117. 15120
118. 3360
119. 360
120. (i) 10! (ii) $6!\cdot 5!$ (iii) $10! - 6!5!$ (iv) $6!\times 5!$
121. 120
122. $\dfrac{49!}{2}$
123. 720
124. $18\times 17!\times 2!$
125. 14!
126. 15!
127. 1680
128. ${}^{20}C_{10}\times 9!\times 9!$
129. $\dfrac{{}^{20}P_{10}}{20}$

130. $5! \times (9!)^2$
131. $\frac{1}{2} \times \frac{^{30}P_{20}}{20}$
132. 140
133. 210
134. 792
135. 4410
136. $^{21}C_5$
137. $n = 20, -19$
138. 24
139. 56
140. $\frac{20!}{4!5!3!8!}$
141. $9! \times (^{13}C_4)$
142. $(13)^4$
143. 246
144. $D_f = \{3\}$
145. 1
146. 4
147. 6
148. 5
149. 0
150. $n > 7$
151. $^{52}C_4$
152. $n = 9$ and $r = 3$
153. $x = 3$
154. $\frac{n(n-1)(n-2)(n+1)}{8}$
155. $n = 2$
156. (i) $^{98}C_{18}$ (ii) $^{95}C_{20}$
157. 30
158. $44 \times 43 \times 14 \times 41$
159. $^4C_1 \times {}^{48}C_9 + {}^4C_2 \times {}^{48}C_8 + {}^4C_3 \times {}^{48}C_7 + {}^4C_4 \times {}^{48}C_6$
160. $^5C_2 \times {}^6C_3$
161. 246
162. 1283
163. 780
164. 2700
165. 41
166. 255
167. 1023
168. 31
169. 217
170. (i) 9 (ii) 9
171. $n = 10$
172. 6
173. 17
174. 4
175. 3
176. 1
177. 13
178. 13727
179. 209
180. 210
181. 330
182. 241
183. 215
184. 197
185. 1919
186. 53
187. 758
188. 136
189. 2454
190. 26
191. 486
192. 256
193. 16650
194. (i) 60 (ii) 48 (iii) 12 (iv) 30
195. 8
196. 315
197. 9360
198. 78
199. 15
200. 403
201. 100
202. 23
203. 30
204. 4
205. 8
206. 3
207. 6
208. 720
209. 300
210. 14
211. 150
212. 540
213. 36
214. 6
215. 3
216. 1771
217. $2^{15} - 121$
218. (i) 501 (ii) 256
219. 1
220. 2
221. 9
222. 44
223. 44
224. 5280
225. 20
226. 31
227. 56
228. 264
229. 476
230. 72
231. 10
232. 35
233. (i) 1771 (ii) 969

234. 117
235. 66
236. 1540
237 336
238. ${}^{50}C_3 - {}^{19}C_3$
239. 55
240 27
241. 2048
242. 25
243. 84
244. 26
245. 18720
246. (i) 57 (ii) 210 (iii) 420
247. 35
248. 10
249. 16
250. 1260
251. 7
252. 280
253. 791
254. 20
255. 104
256. m^2n^2
257. $n = 5$
258. 258.205
259. 98

Level II

1. (c)	2. (a)	3. (a)	4. (b)	5. (c)
6. (d)	7. (c)	8. (b)	9. (d)	10. (c)
11. (a)	12. (c)	13. (c)	14. (c)	15. (a)
16. (c)	17. (d)	18. (b)	19. (c)	20. (b)
21. (b)	22. (d)	23. (a)	24. (c)	25. (b)
26. (c)	27. (c)	28. (b)	29. (d)	30. (c)
31. (b)	32. (c)	33. (b)	34. (b)	35. (a)
36. (b)	37. (b)	38. (c)	39. (b)	40. (c)
41. (c)	42. (c)	43. (a)	44. (c)	45. (c)
46. (b)	47. (b)	48. (a)	49. (a)	50. (b)
51. (a)				

Level III

1. 325
2. 100
3. $\frac{1}{6}n(n-4)(n-5)$
4. 126
5. 9216
6.
7. 191
8. 150
9. 35
10. 116280
11. 110556
12. 552
13. $(n+2)2^{n-1}$
14. 72
15. 286
16. 240
17. 56
18. 4
19. 70
20. 2600
21. 536
22. 117
23. 25
24. $(2)! \times (18)!$
25. $\frac{1}{3}(m+1)(2m^2+4m+3)$
26. 13
27. 210
28. (i) 24 (ii) 96
29. 10
30. 120

INTEGER TYPE QUESTIONS

1. $r = 5$
2. $n = 9$
3. 6, where $x = 5, 6, 7, 8, 9, 10$
4. 4
5. $n = 8$
6. 3
7. 5
8. 7
9. 5
10. 6
11. 6
12. 8
13. 8
14. 2
15. 9
16. 8
17. 4
18, 8
19. 9
20. 3

COMPREHENSIVE LINK PASSAGES

Passage I :	1. (a)	2. (c)	3. (b)	4. (b)	5. (d)
Passage II :	1. (c)	2. (b)	3. (a)	4. (d)	5. (a)
Passage III:	1. (c)	2. (a)	3. (b)	4. (d)	5. (d)
Passage IV :	1. (c)	2. (b)	3. (b)	4. (c)	5. (b)

MATRIX MATCH

1. A → (P, R); B → (Q, S, T); C → (Q, R, S, T)
2. A → (P, Q, S); B → (P, Q, R, S, T); C → (P, Q, S)
3. A → (P, Q, R, S); B → (Q); C → (P, Q, R, S, T)
4. A → (P, Q, S), B → (Q, S); C → (R, T)

Hints and Solutions

Level I

1. We have,

$$\frac{x}{5!}+\frac{x}{6!}=\frac{1}{7!}$$

$$\Rightarrow \frac{x}{5!}+\frac{x}{6\times 5!}=\frac{1}{7\times 6\times 5!}$$

$$\Rightarrow x+\frac{x}{6}=\frac{1}{7\times 6}$$

$$\Rightarrow \frac{7}{6}x=\frac{1}{7\times 6}$$

$$\Rightarrow x=\frac{1}{49}.$$

2. We have,

$$\frac{(2n)!}{n!}$$

$$=\frac{2n(2n-1)(2n-2)\ldots 6.5.4.3.2.1}{n!}$$

$$=\frac{\{2.4.6\ldots(2n-2)2n\}\times\{1.3.5\ldots(2n-1)\}}{n!}$$

$$=\frac{2^n\{1.2.3\ldots(n-1)\cdot n\}\times\{1.3.5\ldots(2n-1)\}}{n!}$$

$$=\frac{2^n\times(n)!\times\{1.3.5\ldots(2n-1)\}}{n!}$$

$$=2^n\times\{1.3.5\ldots(2n-1)\}$$

3. We have,

$$\frac{3.6.9.12\ldots(3n-3)3n}{3^n}$$

$$=\frac{3^n\{1.2.3\ldots(n-1)n\}}{3^n}$$

$$=\frac{3^n\times(n)!}{3^n}$$

$$=(n)!$$

4. Since the unit digit of a factorial more than 4 is zero, so the unit digit of the given expression
   = 1! + 2! + 3! + 4! + … + (10)!
   = Unit digit of 1! + 2! + 3! + 4!
   = Unit digit of (1 + 2 + 6 + 24)
   = Unit digit of 33
   = 3

5. As we know that 5! = 120 and 4! = 24
   Thus, e must be 4
   Therefore, $N = a! + b! + c! + d! + e!$
   = 0! + 1! + 2! + 3! + 4!
   = 1 + 1 + 2 + 6 + 24
   = 34

6. We have 1! × 2! = 2
   Thus, the maximum value of n is 2.

7. We have,
   N = 1! + 2! + 3! + 4! + 5! + 6!
   = 1 + 2 + 6 + 24 + 120 + 720
   = 873
   Thus, the maximum value of N is 6.

8. We have $N = a! + b! + c!$
   Since the unit digit of N is 9, then
   $a = 1$, $b = 2$ and $c = 3$
   Therefore,
   $\{a!\times b!\times c!\} = \{1!\times 2!\times 3!\}$
   $= \{1\times 1\times 2\times 1\times 2\times 3\}$
   $= 12$

9. Since the last two digits of all the factorials more than 9 is 00, so the sum of all prime factorials
   = 2! + 3! + 5! + 7!
   = 2 + 6 + 120 + 5040
   = 5168
   Thus, the last two digits is 68.

10. Since $x + y + z = 25$, so the possible values of xyz = 9, 9, 7 and 9, 8, 8. Hence,
    the required sum
    $= 3\times 9!\times 9!\times 7! + 3\times 9!\times 8!\times 8!$
    $= 3\times 9!\times 9\times 8!\times 7! + 3\times 9!\times 8!\times 8\times 7!$
    $= 3\times 9!\times 8!\times 7!(9 + 8)$
    $= 3\times (9 + 8)\times 9!\times 8!\times 7!$
    $= 51\times 9!\times 8!\times 7!$

11. We have,
    $1.1! + 2.2! + 3.3! + 4.4! \ldots + n.n!$
    $= (2-1).1! + (3-1).2! + (4-1).3! + \ldots + (n+1).n!$
    $= (2! - 1!) + (3! - 2!) + (4! - 3!) + \ldots + ((n+1)! - n!)$
    $= ((n+1)! - 1!)$
    $= (n+1)! - 1$

12. We have,

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\frac{4}{5!}+\cdots+\frac{99}{100!}$$

$$=\frac{(2-1)}{2!}+\frac{(3-1)}{3!}+\frac{(4-1)}{4!}+\frac{(5-1)}{5!}+\cdots+\frac{(100-1)}{100!}$$

$$=\left(1-\frac{1}{2!}\right)+\left(\frac{1}{2!}-\frac{1}{3!}\right)+\left(\frac{1}{3!}-\frac{1}{4!}\right)+\cdots+\left(\frac{1}{99!}-\frac{1}{100!}\right)$$

$$=\left(1-\frac{1}{100!}\right)$$

13. We have,

$$E_3(10!) = \left[\frac{10}{3}\right] + \left[\frac{10}{3^2}\right] + \left[\frac{10}{3^3}\right]$$
$= 3 + 1 + 0$
$= 4$

Thus, the exponent of 3 in (10)! is 4.

14. We have,

$$E_5((50)!) = \left[\frac{50}{5}\right] + \left[\frac{50}{5^2}\right] + \left[\frac{50}{5^3}\right]$$
$= 10 + 2 + 0$
$= 12$

Thus, the exponent of 5 in (50)! is 12.

15. We have,

Highest power of 10 in 50! + 60! + 70!
= Highest power of 5 in 50! + 60! + 70!

$$= \left[\frac{50}{5}\right] + \left[\frac{50}{5^2}\right] + \left[\frac{50}{5^3}\right]$$
$= 10 + 2 + 0$
$= 12$

16. Highest power of 10 in 50! × 60! × 70!
= Highest power of 5 in 50! × 60! × 70!
= Highest power of 5 in 50!
+ the highest power of 5 in 60!
+ the highest power of 5 in 70!

$$= \left[\frac{50}{5}\right] + \left[\frac{50}{5^2}\right] + \left[\frac{50}{5^3}\right] + \left[\frac{60}{5}\right] + \left[\frac{60}{5^2}\right] + \left[\frac{60}{5^3}\right] + \left[\frac{70}{5}\right] + \left[\frac{70}{5^2}\right] + \left[\frac{70}{5^3}\right]$$
$= (10 + 2 + 0) + (12 + 2 + 0) + (14 + 2 + 0)$
$= 42$

17. Highest power of 10 in
10! + 20! + 30! + 40! + ⋯ + (2020)!
= Highest power of 10 in 10!
= Highest power of 5 in 10!
= Exponent of 5 in 10!

$$= \left[\frac{10}{5}\right] + \left[\frac{10}{5^2}\right]$$
$= 2 + 0$
$= 2$

18. Now,
Highest power of 2 in 33! = the exponent of 2 in 33!

$$= \left[\frac{33}{2}\right] + \left[\frac{33}{2^2}\right] + \left[\frac{33}{2^3}\right] + \left[\frac{33}{2^4}\right] + \left[\frac{33}{2^5}\right] + \left[\frac{33}{2^6}\right]$$
$= 16 + 8 + 4 + 2 + 1 = 0$
$= 31$

Clearly, 33! is divisible by 2^{31}.
So, it is divisible by 2^{19}.

19. The number of zeroes at the end of (100)!
= Highest power of 10 in (100)!
= Highest power of 5 in (100)!
= Exponent of 5 in (100)!

$$= \left[\frac{100}{5}\right] + \left[\frac{100}{5^2}\right] + \left[\frac{100}{5^3}\right]$$
$= 20 + 4 + 0$
$= 24$

Thus, the number of zeroes at the end of (100)! is 24.

20. It is given that the highest power of 10 in N! is 16.

Thus, $\left[\frac{N}{5}\right] + \left[\frac{N}{5^2}\right] = 16$

⇒ $70 \le N \le 74$
⇒ $N = 70, 71, 72, 73, 74$.

When $N = 74$,
Highest power of 10 in (74 + 1)! = 75!
= Highest power of 5 in 75!
= Exponent of 5 in 75!

$$= \left[\frac{75}{5}\right] + \left[\frac{75}{5^2}\right] + \left[\frac{75}{5^3}\right]$$
$= 15 + 3 + 0$
$= 18$

21. As we know that, the highest power of 5 in (50)!

$$= \left[\frac{50}{5}\right] + \left[\frac{50}{5^2}\right] + \left[\frac{50}{5^3}\right]$$
$= 10 + 2 + 0$
$= 12$

So, the highest power of 5 in (45)!

$$= \left[\frac{45}{5}\right] + \left[\frac{45}{5^2}\right] + \left[\frac{45}{5^3}\right]$$
$= 9 + 1 + 0$
$= 10$

Thus, m should be 5.
Therefore, the minimum value of n is 45.

22. Given $N = 1.10.15 \ldots 500$
$= 5^{100} \times \{1.2.3.4 \ldots 100\}$
$= 5^{100} \times (100)!$

Now, Highest power of 10 in N
= Highest power of 2 in N
= Exponent of 2 in N
= Exponent of 2 in (100)!

$$= \left[\frac{100}{2}\right] + \left[\frac{100}{2^2}\right] + \left[\frac{100}{2^3}\right] + \left[\frac{100}{2^4}\right] + \left[\frac{100}{2^5}\right] + \left[\frac{100}{2^6}\right] + \left[\frac{100}{2^7}\right]$$
$= 50 + 25 + 12 + 6 + 3 + 1 + 0$
$= 97$

Also, the exponent of 5 in (100)!

$$= \left[\frac{100}{5}\right] + \left[\frac{100}{5^2}\right] + \left[\frac{100}{5^3}\right]$$
$= 20 + 4 + 0 = 24$

Thus, the highest power of 5 in $5^{100} \times (100)!$
$= 100 + 24$
$= 124.$
Therefore, the highest power of 10 in N is
= min of {exp. of 5, exp. of 2}
$= 97$

23. There are 3 different ways you can come from your home to Sakchi and 4 different ways you can come from Sakchi to MIIT-JEE Institute.
Thus, there are $3 \times 4 = 12$ ways you can come from your home to MIIT-JEE Institute.
24. There are 5 different ways the student can enter the class room and 4 different ways, the student can leave the class room.
Thus, there are $5 \times 4 = 20$ different ways the student can enter the room and leave the room.
25. There are three places in the queue to fill one for each student.
First place can be fill 3 different ways, whereas the second and third places can be filled in 2 and 1 different ways.
Thus, there are $3 \times 2 \times 1 = 6$ different ways the queued could be formed.
26. The first prizes can be given in 5 ways, whereas second and third prizes can be given in 4 and 3 ways.
Thus, the total number of ways first three prizes can be given in $5 \times 4 \times 3 = 60$ ways.
27. A student can give answers in 5 and 7 different ways in section A and B respectively.
Thus, the total possible ways a student can give the answers $= 5 \times 7 = 35$ different ways.
28. The student first place in his shelf can be filled in 5 different ways, second place can be filled in 4 different ways, third place can be filled in 3 different ways, fourth place can be filled in 2 different ways, whereas the last place can be filled in 1 different way.
Thus, the total different ways he can arrange his books in his shelf
$= 5 \times 4 \times 3 \times 2 \times 1$
$= 120$
29. Clearly, there are 999 3-digit numbers lying between 100 and 1000.
The first place can be filled in 9 different ways, whereas the second and third place can be filled in 9 and 8 different ways respectively.
Therefore, the number of three digit integers
$= 9 \times 9 \times 8$
$= 648$
30. Therefore, the number of terms in the given product
$= 2 \times 3 \times 4 \times 5$
$= 120$
31. The first place can be filled in 5 different ways, second place can be filled in 4 different ways, whereas the last place can be filled in 3 different ways.
Therefore, total numbers can be made
$= 5 \times 4 \times 3$
$= 120$
32. Case I: When $x < y$
The possible solutions are (1, 9), (2, 8), (3, 7), (4, 6).
Case II: When $x > y$
The possible solutions are (9, 1), (8, 2), (7, 3), (6, 4).
Case III: When $x = y$
The possible solution is only (5, 5).
Hence, the total positive integral solutions
$= 4 + 4 + 1$
$= 9$
33. The number of ways a student can attempt a question either from section A or section B
$= 7 + 6$
$= 13$
34. A class monitor can be selected from boys in 15 different ways, whereas from girls 10 different ways.
Thus, the number of ways a class monitor can be selected either from a boy or from a girl
$= 15 + 10 = 25$
35. Total 3-digit number made by the digits 3 or 4 or 5
$= 3 + 3 + 3$
$= 9$
36 Number of 1-digit number = 5
Number of 2-digit numbers $= 5 \times 4 = 20$
Number of 3-digit numbers $= 5 \times 4 \times 3 = 60$
Number of 4-digit numbers $= 5 \times 4 \times 3 \times 2 = 120$
Number of 5-digit numbers $= 5 \times 4 \times 3 \times 2 \times 1 = 120$
Therefore, total positive numbers
$= 5 + 20 + 60 + 120 + 120$
$= 325$
37. Case I: When numbers are odd
Here, the unit digit can be filled in 3 different ways.
Number of odd numbers $= 3 \times (4 \times 3) = 36$
Case II: When numbers are even
Here, the unit digit can be filled in 2 different ways.
Number of even numbers $= 2 \times (4 \times 3) = 24$
Thus, the total 3-digit distinct numbers
$= 36 + 24 = 60$
38. Thus, the positive integral solutions $= 3! + \frac{3!}{2!} + \frac{3!}{3!}$
$= 6 + 3 + 1$
$= 10$
39. The required number of possible ways $= {}^8P_3$
$= \frac{8!}{5!}$
$= 8.7.6$
$= 336$
40. The odd digits are 1, 3, 5, 7, 9.
The 3-digit numbers between 100 and 999
$= {}^5P_3 = 5.4.3 = 60$

41. The given digits are 1, 2, 3, 4 and 5.
A number will be even, if the unit digit is either even or zero.
Thus, the digit can filled in 2 different ways
Hence, the total even numbers $= 2 \times {}^4P_2$
$= 2.4.3$
$= 24$

42. The total number of possible ways 6 flags of different colours to form a signal of 4 colours $= {}^6P_4$
$= 6.5.4.3$
$= 360$

43. The total number of possible flags $= {}^6P_4$
$= 6.5.4.3$
$= 360$

44. The total number of possible ways they can take their seats $= {}^6P_6$
$= 6!$
$= 6.5.4.3.2.1$
$= 720$

45. The total number of ways 3 men can stay in the hotel room $= {}^8P_3$
$= 8.7.6$
$= 336$

46. The total number of possible ways
$= {}^{15}P_3$
$= 15.14.13 = 2730$

47. The total number of arrangements of the letters of the word TABLE $= {}^5P_5$
$= 5!$
$= 120$

48. We have, $a = {}^4P_4 = 4! = 24$
Similarly, $b = {}^4P_4 = 4! = 24$
and $c = {}^4P_4 = 4! = 24$
Thus, $a + b + c + 10 = 24 + 24 + 24 + 10$
$= 82$

49. The total number of 4-digit numbers
$= 5 \times 4 \times 3 \times 2$
$= 120$

50. The given digits are 3, 4, 5, 6 and 7.
The unit place can be filled in 3 different ways.
Hence, the total odd numbers can be made from the given digits $= 3 \times {}^4P_3$
$= 3.4.3.2.1$
$= 72$

51. Here, the first place can be filled in 5 different ways.
Hence, the total numbers $= 5 \times {}^5P_2$
$$= \frac{5 \times 5!}{3!}$$
$= 5 \times 20$
$= 100$

52. Here, the first place can be filled in 4 different ways.
Thus, the total 3-digit numbers $= 4 \times {}^5P_2$
$= 4.5.4$
$= 80$

53. Case I: When unit digit is 0
So, the unit place can be filled in 1 way
Thus, the number of numbers $= 6 \times 5 \times 4 \times 1$
$= 120$
Case II: When unit digit is not 0
So, the unit place can be filled in 3 different ways and thousand's place can be filled in 5 different ways, whereas hundreds and ten's place can be filled in 5 and 4 different ways.
Thus, the number of numbers $= 5 \times 5 \times 4 \times 3$
$= 300$
Hence, the total 4-digit numbers $= 120 + 300$
$= 420$

54. Here, unit place can be filled in 2 different ways. Therefore, the number of even numbers $= 2 \times {}^4P_4$
$= 2 \times 4!$
$= 2 \times 24$
$= 48$

55. Given digits are 1, 3, 5, 7 and 9.
The unit place can be filled in only 1 way.
Hence, the number of odd numbers divisible by 5
$= 1 \times {}^4P_4$
$= 1 \times 4! = 24$

56. Here, we put the unit place only zero (0).
The unit place can be filled in only 1 way.
Hence, the number of even numbers divisible by 5
$= 1 \times {}^8P_4$
$$= \frac{8!}{4!} = 8.7.6.5 = 1680.$$

57. Number of 1-digit numbers = 4
Number of 2-digit numbers $= 4 \times {}^4P_1 = 4 \times 4 = 16$
Number of 3-digit numbers $= 4 \times {}^4P_2 = 4 \times 4 \times 3$
$= 48$
Number of 4-digit numbers $= 4 \times {}^4P_3$
$= 4 \times 4 \times 3 \times 2 = 96$
Number of 5-digit numbers $= 4 \times {}^4P_4$
$= 4 \times 4! = 4 \times 24 = 96$
Hence, the total positive distinct numbers
$= 4 + 16 + 48 + 96 + 96$
$= 260$

58. Case I: When 2 occurs at the first place
The first place can be filled in 1 way.
Thus, the total numbers
$= 1 \times {}^4P_3 = 1 \times 4! = 24$
Case II: When 3, 4, 5 and 6 occur at the first place
The first place can be filled in 4 different ways.
Thus, the total numbers $= 4 \times {}^4P_3 = 4 \times 4! = 96$
Hence, the total number of numbers $= 24 + 96$
$= 120.$

59. The number of one one functions from A to $B = {}^7P_4$

$= \frac{7!}{3!} = 7 \times 5!$

$= 840$

60. Hence, total number of one one onto functions

$= {}^5P_5 = 5! = 120$

61. As we know that, the inverse of a function exists only when that function is one one and onto.

So, the number of inverse functions

= the number of one one onto functions

$= {}^4P_4 = 4! = 24$

62. The number of functions from A to $B = 3^5$

So, the number of into functions from A to B

$= 2^5 + 2^5 + 2^5 - 3$

$= 32 + 32 + 32 - 3$

$= 96 - 3$

$= 93$

The number of onto functions from A to B

= Total functions – into functions

$= 3^5 - 93$

$= 243 - 93$

$= 150$

63. Number of onto functions between two sets = Number of distribution of 5 balls into 4 boxes where no ball remains empty.

Here, we can make a quadruplet on 5 is $5 \to (1, 1, 1, 4)$

Thus, the number of onto functions

$= \frac{5!}{1!1!1!2!} \times \frac{4!}{3!}$

$= \frac{5!}{2!} \times \frac{4!}{3!}$

$= 60 \times 4$

$= 240$

64. The sum of the digits in the unit place

$= (4-1)! \times (2 + 3 + 4 + 5)$

$= 6 \times 14 = 84$

65. The required sum

$= (4-1)! \times (1+2+3+4) \times \left(\frac{10^4 - 1}{9}\right)$

$= 6 \times 10 \times 1111$

$= 66660$

66. The required sum

$= (4-1)! \times (0+1+2+3) \times \left(\frac{10^4 - 1}{9}\right)$

$-(3-1)! \times (1+2+3) \times \left(\frac{10^3 - 1}{9}\right)$

$= 6 \times 6 \times 1111 - 2 \times 6 \times 111$

$= 36 \times 1111 - 12 \times 111$

$= 39996 - 1332$

$= 38664$

67. Hence, the required sum is

$= \frac{(4-1)!}{2} \times (2+3+3+4) \times \left(\frac{10^4 - 1}{9}\right)$

$= 3 \times 12 \times 1111$

$= 36 \times 1111$

$= 39996$

68. The total number of possible ways

$= 2 \times 2 \times 2 \times \cdots \times 2$ (n times)

$= 2^n$

69. The first delegate can stay in any one of 5 hotels. So, it can be possible in 5 different ways.

So, the 2nd, 3rd and 4th delegates have also 5 choices.

Hence, the total number of possible ways

$= 5 \times 5 \times 5 \times 5$

$= 5^4 = 625$

70. (i) The total number of ways, 3 friends can stay in a 5 hotel of a two n

$= 5 \times 5 \times 5 \times 5^3 = 125$

(ii) The total number of ways, when no two friends stay in a same hotel

$= 5 \times 4 \times 3 = 60$

71. The total possible ways, 5 rings can be worn on the four fingers of one hand

$= 5 \times 5 \times 5 \times 5 = 5^4 = 625$

72. Here, m = the number of ways a child put 3 marbles in his 4 pockets

$= 4 \times 4 \times 4 = 64$

and n = the number of 3-digit distinct number

$= 5 \times 4 \times 3 = 60$

Therefore

$m + n + 10 = 64 + 60 + 10$

$= 134$

73. The total number of possible ways

$= 10 \times 10 \times 10 \times 10 \times 10 = 10^5$

74. The total number of possible ways

$= 9 \times 10 \times 10 \times 10$

$= 9 \times 10^3$

$= 9000$

75. The 3 letters code words can be made from the word INTEX

$= 5 \times 5 \times 5 = 125$

76. The total number of possible ways

$= 4 \times 4 \ \ldots \times 4$ (10 times)

$= 4^{10}$

77. Given that, each question has four choices a, b, c and d respectively.

Here, the option a is either correct or incorrect.

So, there are 2 ways, the student can give the answer for the option a.

Similarly, for the options b, c and d.

Thus, a particular questions have $2 \times 2 \times 2 \times 2 = 16$ possible answers.

It will be considered only when all the options are incorrect.

Thus, the total number of possible answers = 15.

Since the total number of questions is 10, so the required number of possible answers

$= 15 \times 15 \times 15 \times \cdots \times 15$ (10 times)

$= 15^{10}$

78. The required number of permutations of the letters of the word $= \frac{3!}{2!} = \frac{6}{2} = 3$

79. The required number of permutations of the letters of the word $= \frac{5!}{3!} = \frac{120}{6} = 20$

80. There are 2 Ms, 2 As, 2 Ts and 1 H, 1 I, 1 C, 1 S and 1 E.

Hence, the required permutations of the letters of the word

$= \frac{11!}{2! \times 2! \times 2!}$.

81. There are 2 Os, 3 Ts, 2 Ns, 2 Is and 1 C, 1 S and 1 U.

Hence, the required permutations of the letters of the word $= \frac{12!}{2! \times 3! \times 2! \times 2!}$

82. The required number of seven-digit numbers

$= \frac{7!}{3! \times 2!}$

83. The required sum

$= \frac{(6-1)!}{2! \times 3!} \times (2+3+3+4+4+4) \times \left(\frac{10^6 - 1}{9}\right)$

$= \frac{120}{12} \times 20 \times 111111$

$= 22222200$

84. We have, $m = \frac{7!}{2!} = \frac{5040}{2} = 2520$

and $n = \frac{3!}{2!} = \frac{6}{2} = 3$

Hence, the value of

$m + n + 10$

$= 2520 + 3 + 10$

$= 2533$

85. We have 10 students can be arranged themselves in 10! ways.

Out of them 1/2 ways A is ahead of B and another 1/2 ways B is ahead of A.

Hence, the required number of ways $= \frac{10!}{2}$

$= 1814400$

86. 10 persons can be arranged in 10 ! ways.

Out of them 1/2 ways A before B, 1/2 ways B before C and also out of them 1/2 ways C before D.

Hence, the required number of possible ways

$= \frac{10!}{8}$

$= 453600$

87. Number of permutations of $a, b, c, d = 4! = 24$

Number of permutations of $ab, c, d = 3! = 6$

Number of permutations of $a, bc, d = 3! = 6$

Number of permutations of $a, b, cd = 3! = 6$

Number of permutations of b and $c = 2! = 2$

Number of permutations of c and $d = 2! = 2$

Number of permutations of d and $b = 2! = 2$

Number of permutations of b, c and $d = 1$

Hence, the required number of permutations

$= 4! - 3! - 3! - 3! + 2! + 2! + 2! - 1!$

$= 11$

88. Here, 2 boys can be tied by a string and consider 1 thing

Total number of things = (10 – 2 + 1) = 9

Hence, the total number of ways they can sit $= 9! \times 2!$.

89. Here, 6 boys can be tied by a string and consider 1 thing. In a similar way 5 girls are tied by a string and also consider as 1 thing.

Thus, total number of things = 2

Hence, the total number of ways boys and girls can sit together $= 6! \times 5! \times 2!$

90. There are 3 vowels (A, I, O) and 5 consonants (F, R, C, T, N) in the given word.

3 vowels can be tied by a string and consider 1 thing.

Total number of things = 5 + 1 = 6

Hence, the number of ways, it can be arranged $= 6! \times 3!$

91. There are 5 vowels (A, E, I, O, U) and 3 consonants (Q, T, N) in the given word.

5 vowels can be tied by a string and consider 1 thing.

Total number of things = 3 + 1 = 4

Hence, the number of ways, it can be arranged $= 4! \times 5!$

92. There are 2 vowels (A, U) and 3 consonants (L, G, H) in the given word.

2 vowels can be tied by a string and consider 1 thing, similarly 3 consonants will consider as 1.

Total number of things = 1 + 1 = 2

Hence, the number of ways, it can be arranged

$= 2 \times 2! \times 3!$

$= 24$

93. There are 4 vowels (U, I, E, I) and 6 consonants (N, V, R, S, T, Y).

When 4 vowels are together, consider the 4 vowels as 1 thing.

Total number of things = 6 + 1 = 7

Hence, the number of ways, it can arranged

$= 7! \times \frac{4!}{2!}$

$= 7! \times 12$

$= 12 \times 7!$

94. A, B and C can be arranged themselves in 3! ways in which C is ahead of A and B in 2 ways
Hence, the number of possible arrangements
$$= (8)! \times \frac{2}{3!}$$
$$= 40320 \times \frac{1}{3}$$
$$= 13440$$

95. When all vowels come together, the possible arrangements $= \frac{5! \times 3!}{2}$
The number of arrangements of the word ALGEBRA, without any restriction $= \frac{7!}{2}$
Hence, the number of ways, all vowels do not come together = Total number of possible arrangements without any restrictions – number of possible arrangements when all vowels come together
$$= \frac{7!}{2} - \frac{5! \times 3!}{2}$$
$$= 2520 - 360$$
$$= 2160$$

96. When I and N are together, total number of things = 5 + 1 = 6
So, the possible arrangements $= \frac{6!}{2} = 380$
The number of arrangements of the word INTEGER, without any restriction $= \frac{7!}{2}$
Hence, the number of ways, I and N are never come together
= Total arrangements without restrictions – together
$$= \frac{7!}{2} - \frac{6!}{2}$$
$$= 2520 - 380$$
$$= 2140$$

97. The number of arrangements of the word SUCCESS, without any restriction
$$= \frac{7!}{2! \times 3!}.$$
$$= 420$$
When all Ss come together, the possible arrangements
$$= \frac{5!}{2!} \times \frac{3!}{3!} = \frac{5!}{2!} = 60$$
Hence, the number of ways, all Ss do not come together
= Total – together
= 420 – 60
= 360

98. When two Ns come together, the possible arrangements are
$$= \frac{5!}{3!} \times \frac{2!}{2!} = \frac{5!}{3!} = 20$$
The number of arrangements of the word BANANA, without any restriction
$$= \frac{6!}{3! \times 2!} = \frac{120}{2} = 60$$
Hence, the number of ways, all Ns do not appear adjacently
= Total – together
= 60 – 20
= 40

99. When both A are together, consider them as 1 unit, so the possible arrangements are $= \frac{7!}{3!}$
When both A and all three Ls are together, consider them as two separate unit.
So, the possible arrangements
$$= 5! \times \frac{2!}{2!} \times \frac{3!}{3!} = 5! = 120$$
Hence, the number of ways all Ls do not come together but all As come together.
$$= \frac{7!}{3!} - 5!$$
$$= 7 \times 5! - 5!$$
$$= (7 - 1) \times 5!$$
$$= 6 \times 120$$
$$= 720$$

100 There are 2 Is, 3 Es, 2 Ts, 1 N, 1 R, 1 M, 1 A and 1 D in the given word.
So, there are 6 vowels and 6 consonants.
As per the given conditions, all the consonants are together.
Consider all the consonants as 1 thing.
Total number of things = 6 + 1 = 7 things
Hence, the number of ways it can be done
$$= \frac{7!}{2! \times 3!} \times \frac{6!}{2!}$$
$$= \frac{7!}{2!} \times \frac{5!}{2!}$$

101. There are 3 Ns, 3 Es, 1 I and 1 T in the given word.
Case I: When there is no restriction, the number of arrangements
$$= \frac{8!}{3! \times 3!} = 1120$$
Case II: When all vowels are together, the number of arrangements
$$= \frac{5!}{3!} \times \frac{4!}{3!} = 80$$

Case III: When all consonants are together, the number of arrangements

$$= \frac{5!}{3!} \times \frac{4!}{3!} = 80$$

Case IV: When vowels and consonants are together, the number of arrangements

$$= 2! \times \frac{4!}{3!} \times \frac{4!}{3!}$$

$$= 32$$

Hence, the number of ways it can be done

= 1120 – (80 + 80 – 32)
= 1120 – 128
= 992

102. The alphabetical order of the word TOUGH is G, H, O, T, U.
The number of words beginning with G is 4!
The number of words beginning with H is 4!
The number of words beginning with O is 4!
The number of words beginning with TG is 3!
The number of words beginning with TH is 3!
The number of words beginning with TOG is 2!
The number of words beginning with TOH is 2!
The number of words beginning with TOU is TOUGH
Hence, the rank of the word TOUGH is
= 4! + 4! + 4! + 3! + 3! + 2! + 1! = 1
= 72 + 12 + 5
= 89

103. The alphabetical order of IIT is I, I, T
So, the rank of the word IIT is 1.

104. The alphabetical order of AIEEE is A, E, E, E, I.
The number of words beginning with AE is $\frac{3!}{2!} = 3$
The number of words beginning with AI is AIEEE.
Hence, the rank of the word AIEEE is = 3 + 1 = 4

105. The alphabetic order of the word ANNA is A, A, N, N.
The number of words beginning with AA is 1
The number of words beginning with AN is 1
The number of words beginning with ANN is ANNA
Hence, the rank of the word ANNA = 1 + 1 + 1 = 3

106. The alphabetic order of the word PATNA is A, A, N, P, T.
The number of words beginning with A is 4!
The number of words beginning with N is $\frac{4!}{2!}$
The number of words beginning with PAA is 2!
The number of words beginning with PAN is 2!
The number of words beginning with PATA is PATAN
The number of words beginning with PATN is PATNA
Hence, the rank of the word PATNA

$$= 4! + \frac{4!}{2!} + 2! + 2! + 1 + 1$$

$$= 42$$

107. The alphabetic order of the word SURITI is I, I, R, S, T, U.
The number of words beginning with I is 5!
The number of words beginning with R is $\frac{5!}{2!}$
The number of words beginning with SI is 4!
The number of words beginning with SR is $\frac{4!}{2!}$
The number of words beginning with ST is $\frac{4!}{2!}$
The number of words beginning with SUI is 3!
The number of words beginning with SURI is 1
The number of words beginning with SURIT is SURITI
Hence, the rank of the word SURITI is

$$= 5! + \frac{5!}{2!} + 4! + \frac{4!}{2!} + \frac{4!}{2!} + 3! + 1 + 1$$

= 120 + 60 + 24 + 12 + 12 + 6 + 1 + 1
= 180 + 56
= 236

108. The alphabetic order of the word SANIA is A, A, I, N, S.
The number of words beginning with A is 4!
The number of words beginning with I is $\frac{4!}{2!}$
The number of words beginning with N is $\frac{4!}{2!}$
The number of words beginning with SAA is 2!
The number of words beginning with SAI is 2!
The number of words beginning with SANA is SANAI
The number of words beginning with SANI is SANIA
Hence, the rank of the word SANIA

$$= 4! + \frac{4!}{2!} + \frac{4!}{2!} + 2! + 2! + 1 + 1$$

= 48 + 6
= 54

109. There are 3 vowels (A, I, O) and 5 consonants (F, R, C, T, N)
Thus, there are 6 gaps in between 5 consonants
Here, we place 3 vowels in between 6 gaps in ${}^6P_3$ ways, whereas 5 consonants can be arranged themselves in 5! ways.
Thus, the total number of ways it can be done

$$= {}^6P_3 \times 5!$$

$$= \frac{6!}{3!} \times 5!$$

= 5! × 5!
= 120 × 120
= 14400

110. There are 3 vowels (A, I, E) and 5 consonants (T, R, N, G, L).

So, we place 3 vowels in between 6 gaps of 5 consonants.
Hence, the number of ways it can be done
$= {}^6P_3 \times 5!$
$= 120 \times 120$
$= 14400.$

111. There are 3 vowels (A, E, E) and 3 consonants (M, C, T)
So, we place 3 vowels (2 are alike) in between of 4 gaps of 3 consonants.
Hence, the number of ways it can be done
$$= \frac{{}^4P_3}{2!} \times 3!$$
$$= \frac{4!}{2!} \times 3!$$
$$= 12 \times 6 = 72$$

112. There are 3 Ns, 3 Es, 1 I and 1 T in the given word NINETEEN.
So, we place 3 Ns in between of 6 gaps of 3 Es, 1 I and 1 T.
Hence, the number of ways, it can be done
$$= \frac{{}^6P_3}{3!} \times \frac{5!}{3!} \times \frac{3!}{3!}$$
$$= \frac{6!}{3! \times 3!} \times \frac{5!}{3!}$$
$$= \frac{5!}{3!} \times \frac{5!}{3!}$$
$= 20 \times 20$
$= 400$

113. There are 3 Ns, 3 Es, 1 I and 1 T in the given word NINETEEN.
So, there are 4 vowels and 4 consonants.
Case I: When word starts with vowel
Then vowels occupy the odd places whereas consonants take even places.
Hence, the number of ways it can be done
$$= \frac{4!}{3!} \times \frac{4!}{3!}$$
$= 4 \times 4$
$= 16$
Case II: When word starts with consonants
Then consonants occupy the odd places, whereas the vowels take even places.
Hence, the number of ways, also it can be done
$$= \frac{4!}{3!} \times \frac{4!}{3!}$$
$= 4 \times 4$
$= 16$
Thus, the total number of ways it can be done
$= 16 + 16$
$= 32$

114. The arrangements will be of the form
B G B G B G B G B G B
6 boys can be arranged themselves in 6! ways whereas 5 girls can be arranged them in 5! ways.
Hence, the number of ways it can be done
$= 6! \times 5!$

115. Case I: When first place occupies by a boy
The seating arrangement can be of the form
B G B G B G B G B G
Hence, it can be done
$= 5! \times 5!$
Case II: When the first place occupies by a girls
The seating arrangement can be of the form
G B G B G B G B G B
Hence, it can be done
$= 5! \times 5!$
Thus, the total number of ways, it can be done
$= 5! \times 5! + 5! \times 5!$
$= 2 \times 5! \times 5!$
$= 2 \times 120 \times 120$
$= 28800$

116. Odd digits are 3, 3, 5, 5 and even digits are 2, 2, 8, 8, 8.
Odd digits take places 2nd, 4th, 6th and 8th positions whereas even digits take places 1st, 3rd, 5th, 7th and 9th positions.
Hence, the number of nine-digit numbers can be formed from the given number
$$= \frac{4!}{2! \times 2!} \times \frac{5!}{2! \times 3!}$$
$= 6 \times 10$
$= 60$

117. There are 2 Ms, 2 As, 2 Ts, 1 H, 1 E, 1 I, 1 C and 1 S in the given word.
So, there are 4 vowels and 7 consonants.
Hence, the number of possible arrangements
$$= \frac{4!}{2!} \times \frac{7!}{2! \times 2!}$$
$= 3 \times 5040$
$= 15120$

118. There are 3 vowels (A, I, E) and 5 consonants (C, C, D, N, T) in the given word.
Now, 3 vowels can be arranged in 3! ways
So, there is only one way where they will be in given order.
Hence, the total number of such words $= \frac{8!}{2!} \times \frac{1}{3!}$
$= 3360$

119. There are 5 vowels (I, E, E, I, E) and 6 consonants (N, F, F, C, T, V).

Since the position of each vowel is fixed, then 6 consonants can be arranged themselves in $\frac{6!}{2!}$ ways

$$= \frac{720}{2} = 360$$

120. (i) Total number of persons = 6 + 5 = 11
Thus, 11 persons can be seated in a round table in (11 – 1)! = (10)! ways
(ii) Consider 5 Englishmen as a one person.
Total number of persons = 6 + 1 = 7
Thus, the total number of ways, they can be seated in a round table
$= (7 - 1)! \times 5! = 6! \times 5!$
(iii) when 5 Englishmen do not sit together, the possible seating arrangements
= Total – together
$= (10)! - 6! \times 5!$
(iv) 6 Indians can be seated around a round table in 5! ways.
If no two Englishmen do not sit together, we can fill them in between 6 Indians in ${}^6P_5$ ways.
Hence, the total number of possible seating arrangements $= {}^6P_5 \times 5!$
$= 6! \times 5!$

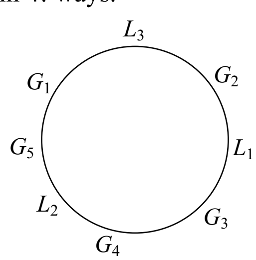
121. The number of circular arrangements of 6 different flowers = (6 – 1)! = 5! = 120

122. As we know that, to form a necklace, clockwise and anti-clockwise arrangements are the same.
Hence, the required number of arrangements

$$= \frac{1}{2} \times (50 - 1)!$$

$$= \frac{49!}{2}$$

123. The five gentlemen can arrange themselves around a round table in 4! ways.



If every gentlemen is to have at least one lady by his side, then two ladies must be in two of the adjacent gaps and the third lady must occupy only one gap.
Thus, for every arrangement of gentlemen, ladies can be arranged themselves in 5 × 3! ways = 30 ways (where two adjacent gaps may be chosen in 5 ways and the 3 ladies each time may be arranged in 3! ways).
Hence, the required number of ways it can be done
$= 5 \times 3! \times 4!$
$= 5 \times 6 \times 24$
$= 720$

124. If two persons be A and B, the remaining 18 persons would be $P_1, P_2, P_3, \ldots, P_{18}$.
Take one of 18 persons, say P_1, and make him sit between A and B, i.e. A, P_1, B.
In the similar way, we can place $P_2, P_3, \ldots, P_{18}$ in between A and B.
Thus, there are 18 ways of choosing one person between A and B.
Hence, the required number of possible ways
$= 18 \times 17! \times 2!$

125. Here, total number of persons = 7 + 8 = 15
Hence, the number of ways they can be seated around a circular table = (15 – 1)! = 14! ways.

126. Since each of the seat is given by the number and all are different numbers, so the arrangement of this seating is a linear arrangement.
Hence, the number of ways, this seating arrangement can be done = 15!

127. Three students A, B and C can be arranged themselves in 3! ways in which in 2 cases B is in between A and C.
So, total number of things = 7 + 1 = 8
Hence, the number of ways it can be done

$$= 7! \times \frac{2}{3!}$$

$= 7 \times 5! \times 2$
$= 14 \times 120$
$= 1680$

128. First, we will select 10 persons out of 20 persons.
It can be done in ${}^{20}C_{10}$ ways.
Around each table 10 persons can be seated in 9 ! ways.
Hence, the number of ways they can be seated
$= {}^{20}C^{10} \times 9! \times 9!$

129. The number of ways 20 persons can be seated around a round table

$$= \frac{{}^{20}P_{10}}{20}$$

130. The number of ways it can be done

$$= \frac{(10)!}{2} \times 9!$$

$$= \frac{10 \times 9!}{2} \times 9!$$

$$= 5 \times (9!)^2$$

131. The number of ways it can be done

$$= \frac{1}{2} \times \frac{{}^{30}P_{20}}{20}$$

132. The number of ways it can be distributed between two students
$= {}^7C_4 \times {}^4C_3$

$$= \frac{7!}{4! \times 3!} \times \frac{4!}{3! \times 1!}$$
$$= \frac{7.6.5.4}{6}$$
$= 140$

133. The number of ways Master Rohan can invite his friends

$= {}^{10}C_4$
$$= \frac{10!}{4! \times 6!}$$
$$= \frac{10.9.8.7}{24}$$
$= 210$

134. The number of ways I can select the students for the Olympiad examination

$= {}^{12}C_5$
$$= \frac{12!}{5! \times 7!}$$
$$= \frac{12.11.10.9.8}{120}$$
$= 792$

135. The number of ways a team of 3 boys and 4 girls are selected

$= {}^{7}C_3 \times {}^{9}C_4$
$$= \frac{7!}{3!4!} \times \frac{9!}{4!5!}$$
$$= \frac{7.6.5}{6} \times \frac{9.8.7.6}{24}$$
$= 35 \times 126$
$= 4410$

136. The number of total selections

$= {}^{5}C_3 \times {}^{2}C_5$
$= {}^{5}C_5 \times {}^{21}C_5$
$= {}^{21}C_5$

137. In general, we buy a ticket for journey of two stoppages (one place to another place).
Consider the number of stoppages is n.
Thus, ${}^{n}C_2 = 190$

$\Rightarrow \quad \frac{n(n-1)}{2} = 190$
$\Rightarrow \quad n(n - 1) = 380$
$\Rightarrow \quad n^2 - n - 380 = 0$
$\Rightarrow \quad (n - 20)(n + 19) = 0$
$\Rightarrow \quad n = 20, -19$

Hence, the number of stoppages is 20.

138. Let n be the number of students.
Number of pairs of the students = ${}^{n}C_2$
For ${}^{n}C_2$ pairs, number of cards sent = 2
Thus, $2 \times {}^{n}C_2 = 600$

$\Rightarrow \quad {}^{n}C_2 = 300$
$\Rightarrow \quad \frac{n(n-1)}{2} = 300$
$\Rightarrow \quad n^2 - n - 600 = 0$
$\Rightarrow \quad (n - 24)(n + 25) = 0$
$\Rightarrow \quad n = 24, -25$

Hence, the number of students = 24.

139. The number of times he has to go to the circus with his children

$= {}^{8}C_3$
$$= \frac{8!}{3!5!}$$
$$= \frac{8.7.6}{6}$$
$= 56$

140. The total number of ways they can be allotted to different wards

$= {}^{20}C_4 \times {}^{16}C_5 \times {}^{11}C_8$
$$= \frac{20!}{4!16!} \times \frac{16!}{5!11!} \times \frac{11!}{8!3!}$$
$$= \frac{20!}{4!5!3!8!}$$

141. In a pack of 52 cards, there are 4 suits, i.e. spades, clubs, diamond and hearts.
Hence, the number of possible ways of selecting 4 cards from a pack of 52 cards

$= {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4$
$= 4 \times ({}^{13}C_4)$

142. In a pack of 52 cards, there are 4 suits, i.e. spades, clubs, diamond and hearts.
Hence, the number of possible ways of selectting 4 cards from a pack of 52 cards

$= {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4$
$= ({}^{13}C_4)^4$
$= (13)^4$

143. Total number of persons = 6 + 4 = 10.
Hence, the number of ways it can be done

$= {}^{10}C_5 - {}^{6}C_5$
$$= \frac{10!}{5!5!} - \frac{6!}{5!1!}$$
$$= \frac{10.9.8.7.6}{120} - 6$$

$= 252 - 6$
$= 246$

144. As we know that ${}^nC_r$ is defined only when $r \in n, r \in W$, $n \in N$.
Thus, $2x - 5 \le x - 2,\ x - 2 \ge 0,\ 2x - 5 > 0$

$\Rightarrow \quad x \le 3, x \ge 2, x > \frac{5}{2}$

$\Rightarrow \quad x = 3$

Hence, the domain of the function $f(x) = D_f = \{3\}$.

145. As we know that, ${}^nC_r$ is defined only when $r \le n, r \in W$, $n \in N$.
Thus, $3x - 7 \le x - 1,\ 3x - 7 > 0,\ x - 1 \ge 0$

$\Rightarrow \quad 3x - x \le 7 - 1, x > \frac{7}{3}, x \ge 1$

$\Rightarrow \quad 2x \le 6, x > \frac{7}{3}, x \ge 1$

$\Rightarrow \quad x \le 3, x > \frac{7}{3}, x \ge 1$

$\Rightarrow \quad x = 3$

Thus, the domain of the function $f(x) = D_f = \{3\}$
Hence, the range of a function is $= {}^2C_2 = 1$

146. We have,

${}^{10}C_r = {}^{10}C_{r+2}$

$\Rightarrow \quad r + r + 2 = 10$

$\Rightarrow \quad 2r + 2 = 0$

$\Rightarrow \quad 2r = 8$

$\Rightarrow \quad r = 4$

Hence, the value of r is 4.

147. Given

${}^nC_{n-4} = 15$

$\Rightarrow \quad {}^nC_4 = 15$

$\Rightarrow \quad \frac{n!}{4!(n-4)!} = 15$

$\Rightarrow \quad \frac{n(n-1)(n-2)(n-3)}{24} = 15$

$\Rightarrow \quad n(n-1)(n-2)(n-3) = 15 \times 24$

$\Rightarrow \quad n(n-1)(n-2)(n-3) = 360$

$\Rightarrow \quad n(n-1)(n-2)(n-3) = 6 \times 5 \times 4 \times 3$

$\Rightarrow \quad n = 6$

148. We have,

${}^{15}C_r : {}^{15}C_{r-1} = 11 : 5$

$\Rightarrow \quad \frac{{}^{15}C_r}{{}^{15}C_{r-1}} = \frac{11}{5}$

$\Rightarrow \quad \left(\frac{15 - r + 1}{r}\right) = \frac{11}{5}$

$\Rightarrow \quad 11r = 75 - 5r + 5$

$\Rightarrow \quad 16r = 80$

$\Rightarrow \quad r = 5$

Hence, the value of r is 5.

149. We have,

${}^{20}C_{13} + {}^{20}C_{15} + {}^{20}C_9 - {}^{20}C_7 - {}^{20}C_5 - {}^{20}C_{11}$
$= {}^{20}C_{20-13} + {}^{20}C_{20-15} + {}^{20}C_9 - {}^{20}C_7 - {}^{20}C_5 - {}^{20}C_{11}$
$= {}^{20}C_7 + {}^{20}C_5 + {}^{20}C_{11} - {}^{20}C_7 - {}^{20}C_5 - {}^{20}C_{11}$
$= {}^{20}C_7 + {}^{20}C_5 + {}^{20}C_{11} - {}^{20}C_7 - {}^{20}C_5 - {}^{20}C_{11}$
$= 0$

150. We have,

${}^{n-1}C_3 + {}^{n-1}C_4 > {}^nC_3$

$\Rightarrow \quad {}^{n-1}C_4 + {}^{n-1}C_3 > {}^nC_3$

$\Rightarrow \quad {}^{n-1}C_4 + {}^{n-1}C_{4-1} > {}^nC_3$

$\Rightarrow \quad {}^nC_4 + {}^nC_3$

$\Rightarrow \quad n > 4 + 3$

$\Rightarrow \quad n > 7$

151. We have,

$${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$$

$= {}^{47}C_4 + ({}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{47}C_3)$
$= ({}^{47}C_4 + {}^{47}C_3) + ({}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3)$
$= ({}^{48}C_4 + {}^{48}C_3) + ({}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3)$
$= ({}^{49}C_4 + {}^{49}C_3) + ({}^{51}C_3 + {}^{50}C_3)$
$= ({}^{50}C_3 + {}^{50}C_3) + ({}^{51}C_3)$
$= ({}^{51}C_3 + {}^{51}C_3)$
$= {}^{52}C_4$

152. We have,

$$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{84}{36} \text{ and } \frac{{}^nC_{r+1}}{{}^nC_r} = \frac{126}{84}$$

$\Rightarrow \quad \frac{n - r + 1}{r} = \frac{7}{3} \text{ and } \frac{n - r}{r + 1} = \frac{3}{2}$

$\Rightarrow \quad 10r - 3n = 3 \text{ and } 10r - 4n = -6$

Solving, we get,

$n = 9$ and $r = 3$.

153. We have, ${}^{x+3}P_3 = {}^{x+2}C_3 + 20$

$\Rightarrow \quad \frac{(x+3)!}{(x+3-2)!} = \frac{(x+2)!}{3!(x+2-3)!} + 20$

$\Rightarrow \quad \frac{(x+3)!}{(x+1)!} = \frac{(x+2)!}{3!(x-1)!} + 20$

$\Rightarrow \quad (x+3)(x+2) = \frac{x(x+2)(x+1)}{6} + 20$

$\Rightarrow \quad 6(x+3)(x-2) = x(x+2)(x+1) + 10$

$\Rightarrow \quad 6(x^2 + 5x + 6) = x(x^2 + 3x + 2) + 120$

$\Rightarrow \quad x^3 + 3x^2 - 6x^2 + 2x - 30x + 120 - 36 = 0$

$\Rightarrow \quad x^3 - 3x^2 - 28x + 84 = 0$

$\Rightarrow \quad x^2(x - 3) - 28(x - 3) = 0$

$\Rightarrow \quad (x-3)(x^2-28)=0$

$\Rightarrow \quad x = 3, \pm\sqrt{28}$

Thus $x = 3$.

154. Given,

$m = {}^nC_2$

$\Rightarrow \quad m = \frac{n(n-1)}{2}$

Thus, ${}^mC_2 = \frac{m(m-1)}{2}$

$= \frac{\left(\frac{n^2-n}{2}\right)\left(\frac{n^2-n}{2}-1\right)}{2}$

$= \frac{(n^2-n)(n^2-n-2)}{8}$

$= \frac{n(n-1)(n-2)(n+1)}{8}$

155. We have,

$\frac{1}{{}^4C_n} = \frac{1}{{}^5C_n} + \frac{1}{{}^6C_n}$

$\Rightarrow \quad \frac{n!(4-n)!}{4!} = \frac{n!(5-n)!}{5!} + \frac{n!(6-n)!}{6!}$

$\Rightarrow \quad \frac{(4-n)!}{4!} = \frac{(5-n)!}{5!} + \frac{(6-n)!}{6!}$

$\Rightarrow \quad \frac{(5-n)!}{5!} + \frac{(6-n)!}{6!} = \frac{(4-n)!}{4!}$

$\Rightarrow \quad \frac{(5-n)}{5} + \frac{(6-n)(5-n)}{6\times 5} = 1$

$\Rightarrow \quad 6(5-n)+(6-n)(5-n)=30$

$\Rightarrow \quad (5-n)(12-n)=30$

$\Rightarrow \quad (n-5)(n-12)=30$

$\Rightarrow \quad n^2-17n+60-30=0$

$\Rightarrow \quad n^2-17n+30=0$

$\Rightarrow \quad (n-2)(n-15)=0$

$\Rightarrow \quad n = 2, 15$

Hence, the value of n is 2.

156. (i) Since two particular students are always chosen, the selection is needed for another 18 students from 98 students.
Hence, the required number of selections $= {}^{98}C_{18}$

(ii) Since five particular students are never chosen, the selection is needed for 20 students from 95 students.
Hence, the required number of selection is $= {}^{95}C_{20}$

157. The required number of possible selection

$= {}^4C_3 \times {}^5C_4 + {}^4C_4 \times {}^5C_4 + {}^4C_3 \times {}^5C_5 + {}^4C_4 \times {}^5C_5$

$= 4\times 5 + 1\times 5 + 4\times 1 + 1\times 1$

$= 20+5+4+1$

$= 30$

158. As we know that, a pack of cards consists of 52 cards in which 4 are aces and 4 are kings.
Hence, the required number of possible selections

$= {}^4C_1 \times {}^4C_1 \times {}^4C_{44}$

$= 4\times 4\times \frac{44!}{4!\times 40!}$

$= 4\times 4\times \frac{44\times 43\times 42\times 41}{4\times 3\times 2\times 1}$

$= 44\times 43\times 14\times 41$

159. The required number of possible selection

$= {}^4C_3 \times {}^{48}C_9 + {}^4C_2 \times {}^{48}C_8 + {}^4C_3 \times {}^{48}C_7 + {}^4C_4 \times {}^{48}C_6$

160. The required number of possible selection

$= {}^5C_2 \times {}^6C_3$

161. Total number of persons = 6 + 4 = 10
Hence, the total number of possible selection

$= {}^{10}C_5 - {}^6C_5$

$= \frac{10!}{5!\times 5!} - 6$

$= \frac{10\times 9\times 8\times 7\times 6}{120} - 6$

$= 63\times 4 - 6$

$= 252 - 6$

$= 246$

162. Total number of children = 6 + 7 = 13
Hence, the possible number of selection

$= {}^{13}C_5 - {}^6C_5$

$= \frac{13!}{5!\times 8!} - 6$

$= \frac{13\times 12\times 11\times 10\times 9}{120} - 6$

$= 13\times 11\times 9 - 6$

$= 1289 - 6$

$= 1283$

163. The possible number of selections

$= {}^6C_2 \times {}^6C_5 + {}^6C_3 \times {}^6C_4 + {}^6C_4 \times {}^6C_3 + {}^6C_5 \times {}^6C_2$

$= 15\times 6 + 20\times 15 + 15\times 20 + 6\times 15$

$= 90 + 300 + 300 + 90$

$= 780$

164. The triplet of 7 considered as (2, 2, 3).
Hence, the possible number of selections

$= {}^4C_2 \times {}^5C_2 \times {}^6C_3 + {}^4C_2 \times {}^5C_3 \times {}^6C_2 + {}^4C_3 \times {}^5C_2 \times {}^6C_2$

$= 6\times 10\times 20 + 6\times 10\times 15 + 4\times 10\times 15$

$= 1200 + 900 + 600$

$= 2700$

165. The possible ways he can vote $= {}^6C_1 + {}^6C_2 + {}^6C_3$

$= 6 + 15 + 20 = 41$

166. Hence, the number of possible ways Master Amit can invite his friends $= {}^8C_1 + {}^8C_2 + {}^8C_3 + \cdots + {}^8C_{10}$

$= 2^8 - 1$

$= 256 - 1$

$= 255$

167. The number of possible ways Master Roshan can give the answer is $= {}^{10}C_1 + {}^{10}C_2 + {}^{10}C_3 + \ldots + {}^{10}C_{10}$
$= 2^{10} - 1$
$= 1024 - 1$
$= 1023$

168. The student will fail if he/she fails in one or more subjects.
The total number of ways he/she can fail
$= {}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5$
$= 2^5 - 1$
$= 32 - 1 = 31$

169. The number of possible ways, the student select the questions
$= ({}^3C_1 + {}^3C_2 + {}^3C_3) \times ({}^5C_1 + {}^5C_2 + {}^5C_3 + {}^5C_4 + {}^5C_5)$
$= (2^3 - 1) \times (2^5 - 1)$
$= 7 \times 31 = 217$

170. (i) The number of possible selections, in which at least one cube is white, is
$= {}^2C_1 \times {}^3C_2 + {}^2C_2 \times {}^3C_1$
$= 2 \times 3 + 1 \times 3$
$= 6 + 3$
$= 9$

(ii) The number of possible selections, in which at least one cube is red, is
$= {}^3C_1 \times {}^3C_2 + {}^3C_2 \times {}^2C_1$
$= 3 \times 1 + 3 \times 2$
$= 3 + 6$
$= 9$

171. The number of visits of teacher
= Number of groups of 3 children out of n
$= {}^nC_3$
Number of visits of a particular child $= {}^{n-1}C_2$
According to the given condition,
$${}^nC_3 - {}^{n-1}C_2 = 84$$
$$\Rightarrow \frac{n(n-1)(n-2)}{6} - \frac{(n-1)(n-2)}{2} = 84$$
$$\Rightarrow \frac{(n-1)(n-2)}{2}\left(\frac{n}{3} - 1\right) = 84$$
$$\Rightarrow \frac{(n-1)(n-2)(n-3)}{6} = 84$$
$$\Rightarrow (n-1)(n-2)(n-3) = 504$$
$$\Rightarrow (n-1)(n-2)(n-3) = 9 \times 8 \times 7$$
$$\Rightarrow (n-1) = 9$$
$$\Rightarrow n = 10$$

172. Let the number of contestants be n.
According to the questions,
$${}^{n+1}C_1 + {}^{n+1}C_2 + {}^{n+1}C_3 + \cdots + {}^{n+1}C_n = 126$$
$$\Rightarrow 2^{n+1} - 2 = 126$$
$$\Rightarrow 2^{n+1} = 128$$
$$\Rightarrow 2^{n+1} = 2^7$$
$$\Rightarrow n + 1 = 7$$
$$\Rightarrow n = 6.$$
Thus, the number of contestants = 6.

173. Total number of games played by the remaining players = 117 – 12 = 105
Let the number of remaining participants be n.
Thus,
$${}^nC_2 = 105$$
$$\Rightarrow \frac{n(n-1)}{2} = 105$$
$$\Rightarrow n(n-1) = 210$$
$$\Rightarrow n^2 - n - 210 = 0$$
$$\Rightarrow (n-15)(n+14) = 0$$
$$\Rightarrow n = 15$$
Hence, the number of participants at the beginning
$= 15 + 2 = 17$

174. It is given that
$${}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + \cdots + {}^{2n+1}C_n = 256 \quad \ldots(i)$$
$${}^{2n+1}C_{2n+1} + {}^{2n+1}C_{2n} + {}^{2n+1}C_{2n-1} + \cdots + {}^{2n+1}C_{n+1} = 256 \quad \ldots(ii)$$
Adding Eqs (i) and (ii), we get
$${}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_{2n+1} = 512$$
$$\Rightarrow 2^{2n+1} = 512 = 2^9$$
$$\Rightarrow 2n + 1 = 9$$
$$\Rightarrow n = 4$$

175. It is given that
$${}^{2n+1}C_1 + {}^{2n+1}C_2 + \cdots + {}^{2n+1}C_n + {}^{2n+1}C_{n+1} \quad \ldots(i)$$
$${}^{2n+1}C_{2n} + {}^{2n+1}C_{2n-1} + {}^{2n+1}C_{2n-2} + \cdots + {}^{2n+1}C_{n+1} = 63 \quad \ldots(ii)$$
Adding Eqs (i) and (ii), we get
$${}^{2n+1}C_1 + {}^{2n+1}C_2 + \cdots + {}^{2n+1}C_n + {}^{2n+1}C_{n+1} + \cdots + {}^{2n+1}C_{2n-1} + {}^{2n+1}C_{2n} = 63 + 63$$
$$\Rightarrow 2^{2n+1} - 2 = 126$$
$$\Rightarrow 2^{2n+1} = 128$$
$$\Rightarrow 2^{2n+1} = 128 = 2^7$$
$$\Rightarrow 2n + 1 = 7$$
$$\Rightarrow n = 3$$
Hence, the value of n is 3.

176. Here, all balls are in red colours.
So, the objects are identical.
Hence, the total number of selections is 1.

177. Here, all balls are in green colours.
So, the objects are identical.
As we know that the number of selections of zero or more objects from n identical objects is $n + 1$.
Thus, the total number of possible selections = 12 + 1 = 13.

178. The total number of possible selections
$= (10 + 1)(15 + 1)(12 + 1)(5 + 1) - 1$
$= 11 \times 16 \times 13 \times 6 - 1$
$= 176\ \ 78 - 1$
$= 13728 - 1$
$= 13727$

179. The number of possible selections

$= (6 + 1)(5 + 1)(4 + 1) - 1$
$= 7 \times 6 \times 5 - 1$
$= 210 - 1$
$= 209$

180. The number of possible selections

$= 6 \times 5 \times 7$
$= 210$

181. The number of selections

$= (10 + 1) \times 6 \times (4 + 1)$
$= 11 \times 6 \times 5$
$= 66 \times 5$
$= 330$

182 The number of possible selections

$= (4 + 1)(5 + 1)2^3 - 1$
$= 5 \times 6 \times 8 - 1$
$= 240 - 1$
$= 239$

183. Given word is 'ABRACABDABRA' in which 5 As, 2 Bs, 2 Rs, 1 C and 1 D.

Hence, the number of selections

$= (5 + 1)(2 + 1)(2 + 1)2^2 - 1$
$= 54 \times 4 - 1$
$= 216 - 1$
$= 215$

184. Given word is MATHEMATICS in which 2 Ms, 2 As, 2 Ts and H, E, I, C, S are 1 each.

Hence, the number of selections

$= (2 + 1)(2 + 1)(2 + 1)2^5 - 1$
$= 9 \times 32 - 1$
$= 198 - 1$
$= 197$

185. There are 9 Ds, 3 As, 2 Ys, 3 Es, 1 I and 1 L.

Hence, the number of selections

$= (9 + 1)(3 + 1)(2 + 1)(3 + 1)2^2 - 1$
$= 10 \times 4 \times 3 \times 4 \times 4 - 1$
$= 10 \times 192 - 1$
$= 1920 - 1$
$= 1919$

186. There are 10 letters of six different types, namely, O, O, O; P, P; R, R; T; I; N.

i.e. 3 Os, 2 Ps, 2 Rs, 1 T, 1 I, 1 N.

Case I: When 3 alike and 1 different,
the number of selections $= {}^1C_1 \times {}^5C_1 = 1 \times 5 = 5$

Case II: When 2 alike of one kind and another 2 alike of 2nd kind, the number of selections $= {}^3C_2 = 3$

Case III: When 2 are alike and another 2 are different, the number of selections $= {}^3C_1 \times {}^5C_5 = 3 \times 10 = 30$

Case IV: When all 4 are different, the number of selections $= {}^6C_4 = \frac{6 \times 5}{2} = 15$

Therefore, the number of total selections

$= 5 + 3 + 30 + 15 = 53.$

187. The total number of possible arrangements

$$= 5 \times \frac{4!}{3!} + 3 \times \frac{4!}{2!2!} + 30 \times \frac{4!}{2!} + 15 \times 4!$$

$= 5 \times 4 + 3 \times 6 + 30 \times 12 + 15 \times 24$
$= 20 + 18 + 360 + 360$
$= 758$

188. There are 11 letters of 8 different types, namely, MM, AA, TT, H, E, I, C, S, i.e. 2 Ms, 2 As, 2 Ts, 1 H, 1 E, 1 I, 1 C, 1 S.

Case I: When 2 alike of one kind and another 2 are alike of 2nd kind, the number of selections $= {}^3C_2 = 3$

Case II: When 2 are alike and 2 are different, the number of selections

$$= {}^3C_1 \times {}^7C_2 = 3 \times \frac{7 \times 6}{2} = 63$$

Case III: When all 4 are different, the number of selections

$$= {}^8C_4 = \frac{8!}{4! \times 4!} = \frac{8.7.6.5}{24} = 70$$

Therefore, the number of total selections

$= 3 + 63 + 70$
$= 136$

189. Total number of possible arrangements

$$= 3 \times \frac{4!}{2!\,2!} + 63 \times \frac{4!}{2!} + 70 \times 4!$$

$= 3 \times 6 + 63 \times 12 + 70 \times 24$
$= 18 + 7560 + 1680$
$= 2454$

190. There are 8 letters of 6 different types, namely, PP, SS, A, O, R, T, i.e. 2 Ps, 2 Ss, 1 A, 1 O, 1 R, 1 T.

Case I: When 2 alike of one kind and another 2 are alike of 2nd kind, the number of selections $= {}^2C_2$

Case II: When 2 are alike and 2 are different, the number of selections

$= {}^2C_1 \times {}^5C_1 = 2 \times 5 = 10$

Case III: When all 4 are different, the number of selections

$$= {}^6C_4 = \frac{6 \times 5}{2} = 15$$

Therefore, the total number of selections

$= 1 + 10 + 15 = 26$

191. Total number of possible arrangements

$$= 1 \times \frac{4!}{2!\,2!} + 10 \times \frac{4!}{2!} + 15 \times 4!$$

$= 1 \times 6 + 10 \times 12 + 15 \times 24$
$= 6 + 120 + 360$
$= 486.$

192. There are 14 letters of 8 different types, namely, N, N, N; A, A; I, I; T, T; E; O; L; R, i.e. 3 Ns, 2 As, 2 Is, 2 Ts, 1 E, 1 O, 1 L, 1 R.

Case I: When 3 are alike and 2 are different, the number of selections

$= {}^1C_1 \times {}^7C_2 = 1 \times 21 = 21$

Case II: When 3 are alike of one kind and 2 are alike of 2nd kind, the number of selections

$= {}^1C_1 \times {}^3C_1 = 1 \times 3 = 3$

Case III: When 2 are alike and 3 are different, the number of selections

$= {}^4C_1 \times {}^7C_3 = 4 \times 35 = 140$

Case IV: When 2 sets are alike and 1 is different, the number of selections

$= {}^4C_2 \times {}^6C_1 = 6 \times 6 = 36$

Case V: When all are different, the number of selections

$$= {}^8C_5 = \frac{8 \times 7 \times 6}{6} = 56$$

Therefore, total number of selections

$= 21 + 3 + 140 + 36 + 56$

$= 256$

193. Total number of possible arrangements

$$= 21 \times \frac{5!}{3!} + 3 \times \frac{5!}{3!\,2!} + 140 \times \frac{5!}{2!} + 36 \times \frac{5!}{2!\,2!} + 56 \times 5!$$

$= 21 \times 20 + 3 \times 10 + 140 \times 60 + 36 \times 30 + 561 \times 20$

$= 420 + 30 + 8400 + 1080 + 6720$

$= 450 + 9480 + 6720$

$= 16650$

194. We have $\alpha = 10800 = 2^4 \times 3^3 \times 5^2$

(i) The total number of divisors

$= (4 + 1)(3 + 1)(2 + 1)$

$= 5 \times 4 \times 3 = 60$

(ii) The number of even divisors

$= 4 \times (3 + 1) \times (2 + 1)$

$= 4 \times 4 \times 3$

$= 48$

(iii) The number of divisors of the form $4m + 2$ is

= the number is of the form 6, 10

$= (3 + 1) \times (2 + 1)$

$= 12$

(iv) Now, $10800 = 2^4 \times 3^3 \times 5^2$

$= (3 \times 5)^2 \times 2^4 \times 3^1$

Thus, the number of divisors of multiple of 15 is

$= (2 + 1) \times (4 + 1) \times (1 + 1)$

$= 3 \times 5 \times 2$

$= 30$

195. We have $18900 = 2^2 \times 3^3 \times 5^2 \times 7$

Hence, the required number of possible ways

$= 2^{4-1} = 2^3 = 8$

196. We have $2520 = 2^3 \times 3^2 \times 5 \times 7$

Thus, the total number of possible ordered pairs (p, q) is

$= (2.3 - 1) \times (2.2 + 1) \times (2.1 + 1) \times (2.1 + 1)$

$= 7 \times 5 \times 3 \times 3$

$= 315$

197. We have $2520 = 2^3 \times 3^2 \times 5 \times 7$

Hence, the sum of all even divisors of 2520

$= (1 + 2 + 2^2 + 2^3) \times (1 + 3 + 3^2) \times (1 + 5) \times (1 + 7)$

$= 15 \times 13 \times 6 \times 8$

$= 9360$

198. We have $360 = 2^3 \times 3^2 \times 5$

Hence, the sum of all odd divisors of 360

$= (1 + 3 + 3^2) \times (1 + 5)$

$= 13 \times 6$

$= 78$

199. We have $38808 = 2^3 \times 3^2 \times 7^2 \times 1$

Hence, the sum of all even divisors

$= 1 + 2 + 2^2 + 2^3$

$= 15$

200. We have $7200 = 2^5 \times 3^2 \times 5^2$

Since the sum of all odd factors that ends with 5 is an odd factor, so all the factors will be $3^2 \times 5^2$.

Hence, the required sum of all factors ends with 5

$= (1 + 3 + 3^2) \times (1 + 5 + 5^2)$

$= 13 \times 31$

$= 403$

201. The numbers 1, p, p^2 has 3 factors only when p is a prime number.

As we know that, $100^2 = 10000$

Thus, there are 25 prime numbers that exist under 100.

Hence, the required number = 100.

202. We have, $8100 = 2^2 \times 3^4 \times 5^2$ = a perfect square

Hence, the number of possible ways

$$= \frac{1}{2}[(2+1)(4+1)(2+1)+1]$$

$= 23$

203. We have, $10800 = 2^4 \times 3^3 \times 5^2 \neq$ a perfect square.

Hence, the number of possible ways

$$= \frac{1}{2}[(4+1) \times (3+1) \times (2+1)]$$

$= 30$

204.

Set I	Set II	No. of ways
P	Q, R, S	
Q	P, R, S	${}^4C_1 \times {}^4C_3$
R	P, Q, S	
S	P, Q, R	

Hence, the number of possible ways

$= {}^4C_1 \times {}^4C_3$

$= 4$

205.

Person I	Person II	No. of ways
P	Q, R, S	
Q	P, R, S	
R	P, Q, S	
S	P, Q, R	${}^4C_3 \times {}^1C_1 \times 2!$

Q, R, S	P
P, R, S	Q
P, Q, S	R
P, Q, R	S

Hence, the number of possible ways

$= {}^4C_3 \times {}^1C_1 \times 2!$

$= 4 \times 2 = 8$

206.

Set I	Set II	No. of ways
P, Q	R, S	
P, R	Q, S	
P, S	Q, R	$\frac{{}^4C_3 \times {}^1C_1}{2!}$
Q, R	P, S	
Q, S	P, R	
R, S	P, Q	

Hence, the number of possible ways

$$= \frac{{}^4C_3 \times {}^1C_1}{2!}$$

$= 3$

207.

Person I	Person II	No. of ways
P, Q	R, S	
P, R	Q, S	
P, S	Q, R	
Q, R	P, S	${}^4C_2 \times {}^2C_2$
Q, S	P, R	
R, S	P, Q	

Hence, the number of possible ways

$= {}^4C_2 \times {}^2C_2$

$= 6$

208. The possible triplets of 5 are (1, 1, 3) and (1, 2, 2).

Hence, the number of possible ways

$$= 5! \times \frac{3!}{2!} + 5! \times \frac{3!}{2!}$$

$= 120 \times 3 + 120 \times 3$

$= 360 + 360$

$= 720$

Alternate method

The number of possible ways

$= {}^{5-1}C_{3-1} \times 5!$

$= {}^4C_2 \times 5!$

$= 6 \times 120$

$= 720$

209. The possible triplets of 5 are (1, 1, 3) and (1, 2, 2).

Hence, the number of possible ways

$$= \frac{5!}{1!1!3!} \times 3! + \frac{5!}{2!2!1!} \times 3!$$

$= 120 + 180$

$= 300$

210. The possible doublets of 4 are (1, 3) and (2, 2).

Hence, the number of possible ways

$$= \frac{4!}{1!3!} \times 2! + \frac{4!}{2!2!} \times \frac{2!}{2!}$$

$$= \frac{4!}{3} + \frac{4!}{2!2!}$$

$$= \frac{24}{3} + \frac{24}{4}$$

$= 8 + 6$

$= 14$

211. The possible triplets of 5 are (1, 1, 3) and (1, 2, 2).

Hence, the number of possible ways

$$= \frac{5!}{1!1!3!} \times \frac{3!}{2!} + \frac{5!}{2!2!1!} \times \frac{3!}{2!}$$

$$= \frac{5!}{2!} + \frac{5! \times 3!}{2!2!2!}$$

$= 60 + 90$

$= 150$

212. The possible triplets are (1, 2, 3), (1, 1, 4) and (2, 2, 2).

Hence, the number of possible ways

$$= \frac{6!}{1!2!3!} \times 3! + \frac{6!}{1!1!4!} \times \frac{3!}{2!} + \frac{6!}{2!2!2!} \times \frac{3!}{3!}$$

$$= \frac{6!}{2!} + \frac{6!}{4 \times 2!} + \frac{6!}{2!2!2!}$$

$$= \frac{720}{2} + \frac{720}{8} + \frac{720}{8}$$

$= 360 + 90 + 90$

$= 540$

213. The number of possible ways

$= {}^{10-1}C_{3-1}$

$= {}^9C_2$

$$= \frac{9 \times 8}{2}$$

$= 36$

Alternate method

Here, the possible triplets are (1, 3, 6), (1, 4, 5), (1, 2, 7), (2, 3, 5), (1, 1, 8), (2, 2, 6), (2, 4, 4) and (3, 3, 4)

Hence, the number of possible ways

$= (1 \times 3! + 1 \times 3! + 1 \times 3! + 1 \times 3!)$

$$+ \left(1 \times \frac{3!}{2!} + 1 \times \frac{3!}{2!} + 1 \times \frac{3!}{2!} + 1 \times \frac{3!}{2!}\right)$$

$= 6 \times 4 + 3 \times 4$

$= 24 + 12$

$= 36$

214. The number of possible ways

$= {}^{5-1}C_{3-1}$

$= {}^{4}C_2 = \frac{4 \times 3}{2} = 6$

Alternate method

Here, the possible triplets are (1, 1, 3) and (1, 2, 2).

Hence, the number of possible ways

$= 1 \times \frac{3!}{2!} + 1 \times \frac{3!}{2!}$

$= 3 + 3$

$= 6$

215. The number of possible ways

$= {}^{4-1}C_{2-1}$

$= {}^{3}C_1 = 3$

Alternate method

Here, the possible doublets are (1, 3) and (2, 2).

Hence, the number of possible ways

$= 1! \times 2! + 1! \times \frac{2!}{2!}$

$= 2 + 1$

$= 3$

216. The number of possible ways

$= {}^{20+4-1}C_{4-1}$

$= {}^{23}C_3$

$= \frac{23 \times 22 \times 21}{6}$

$= 23 \times 11 \times 7$

$= 253 \times 7$

$= 1771$

217. The number of possible selections is

$= \sum_{k=0}^{12} {}^{25-10}C_k$

$= \sum_{k=0}^{12} {}^{15}C_k$

$= {}^{15}C_0 + {}^{15}C_1 + {}^{15}C_2 + \cdots + {}^{15}C_{12}$

$= 215 - ({}^{15}C_{13} + {}^{15}C_{14} + {}^{15}C_{15})$

$= 2^{15} - (105 + 15 + 1)$

$= 2^{15} - 121$

218. (i) The number of possible selections

$= \sum_{k=8-6}^{8} {}^{9}C_k$

$= \sum_{k=2}^{8} {}^{9}C_k$

$= {}^{9}C_2 + {}^{9}C_3 + {}^{9}C_4 + {}^{9}C_5 + {}^{9}C_6 + {}^{9}C_7 + {}^{9}C_8$

$= 29 - ({}^{9}C_0 + {}^{9}C_1 + {}^{9}C_9)$

$= 512 - 11$

$= 501$

(ii) The number of possible selections

$\sum_{k=0}^{4} {}^{9}C_k = {}^{9}C_0 + {}^{9}C_1 + {}^{9}C_2 + {}^{9}C_3 + {}^{9}C_4$

$= 1 + 9 + 36 + 84 + 126$

$= 256$

219. The number of possible ways

$= 2! \times \left(1 - \frac{1}{1!} + \frac{1}{2!}\right)$

$= 1$

220. The number of possible ways

$= 3! \times \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!}\right)$

$= 3! \times \left(\frac{1}{2!} - \frac{1}{3!}\right)$

$= 3! \times \left(\frac{3-1}{3!}\right)$

$= 2$

221. The number of possible ways

$= 4! \times \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)$

$= 4! \times \left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)$

$= 4! \times \left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24}\right)$

$= 4! \times \frac{12 - 4 + 1}{24}$

$= 9$

222. The number of possible ways

$= 5! \times \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right)$

$= 5! \times \left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right)$

$= 5! \times \left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right)$

$= 5! \times \left(\frac{60 - 20 + 5 - 1}{120}\right)$

$= 44$

223. The number of total functions

$= 5! \times \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right)$

$= 44$

224. First of all, to distribute 5 right-handed shocks to 5 persons P_1, P_2, P_3, P_4 and P_5 respectively.

This can be done in 5! ways and for each distribution of right-handed shocks, left-handed shocks can be deranged in $D_5 = 44$ ways.

Hence, the number of possible ways

$= 5! \times 4$

$= 1320 \times 4$

$= 5280$

225. 2 letters can be selected out of 5 letters in ${}^5C_2$ ways. For each such selections, the remaining 3 letters placed in wrong envelopes in D_3 ways.

Hence, the number of possible selections

$= {}^5C_2 \times D_3$

$= 10 \times 2$

$= 20$

226. The number of ways to place at least 2 balls in boxes of the same color

= The number of ways without restriction – the number of ways all 5 balls are wrongly placed – the number of ways to select a ball in right box × number of ways in which remaining 4 balls in 4 wrong boxes

$= 5! - 44 - {}^5C_1 \times D_4$

$= 120 - 44 - 5 \times 9$

$= 120 - 44 - 45$

$= 120 - 89$

$= 31$

227. The number of possible ways

= 0 letter in wrong envelope and 6 are correct
+ 1 letter in wrong envelope and 5 are correct
+ 2 letters in wrong envelopes and 4 are correct
+ 3 letters in wrong envelopes and 3 are correct

$= 1 + 0$ (not possible) $+ {}^6C_4 \times D_2 + {}^6C_3 \times D_3$

$$= 1 + \frac{6 \times 5}{2} \times 1 + \frac{6 \times 5 \times 4}{6} \times 2$$

$= 1 + 15 + 40$

$= 56$

228. The number of possible ways = the number of ways pick up 1 right coat and 5 wrong coats

$= {}^6C_1 \times D_5$

$= 6 \times 44$

$= 264$

229. There are 11 letters in which 3 Es, 3 Ns, 2 Gs, 2 Is and 1 R.

= Co-efficient of x^4 in
$(1 + x + x^2 + x^3)^2 \times (1 + x + x^2) \times (1 + x)^3$

= Co-efficient of x^4 in

$$\left(\frac{1-x^4}{1-x}\right)^2 \times \left(\frac{1-x^3}{1-x}\right)^2 \times (1+x)$$

= Co-efficient of x^4 in
$\{(1 - x^4)(1 - x^3)\}^2\ (1 + x)(1 - x)^{-4}$

= Co-efficient of x^4 in
$(1 - x^3 - x^4)(1 + x)(1 - x)^{-4}$

= Co-efficient of x^4 in
$(1 - x - 2x^3 - 4x^4)\ (1 + {}^4C_1x + {}^5C_2x^2 + {}^6C_3x^3 + {}^7C_4x^4)$

$= {}^7C_4 + {}^6C_3 - 4 - 4.{}^2C_1$

$= 35 + 20 - 4 - 8$

$= 43$

The number of permutations of 4 letters is = Co-efficient of x^4 in

$$4! \times \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}\right)^2 \times \left(1 + \frac{x}{1!} + \frac{x^2}{2!}\right)^2 \times (1+x)$$

= Co-efficient of x^4 in

$$4! \times \left(1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{6}\right)^2 \times \left(1 + \frac{x}{1} + \frac{x^2}{2}\right)^2 \times (1+x)$$

= Co-efficient of x^4 in

$$4! \times \left(1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{7}{12}x^4\right) \times \left(1 + 3x + 4x^2 + 3x^3 + \frac{5}{4}x^4\right)$$

$$= 4! \times \left(\frac{7}{12} + \frac{5}{4} + (6+4) + 8\right)$$

$$= 24 \times \frac{22}{12} + 24 \times 18$$

$= 44 + 432$

$= 476$

230. There are 11 letters in which 3 Es. 3 Ns, 2 Ds, 1 I, 1 P and 1 T, respectively.

Hence, the total number of selections

= Co-efficient of x^5 in $(1 + x + x^2 + x^3)^2 \times (1 + x + x^2) \times (1 + x)^3$

= Co-efficient of x^5 in

$$\left(\frac{1-x^4}{1-x}\right)^2 \times \left(\frac{1-x^3}{1-x}\right) \times (1+x)^3$$

= Co-efficient of x^5 in
$(1 - 2x^4)(1 - x^3) \times (1 - x)^{-3} \times (1 + x)^3$

= Co-efficient of x^5 in
$(1 - x^3 - 2x^4) \times (1 + 3x + 3x^2 + x^3) \times (1 - x)^{-3}$

= Co-efficient of x^5 in
$(1 + 3x + 3x^2 - 5x^4 - 9x^5)$
$(1 + {}^3C_1x + {}^4C_2x^2 + {}^5C_3x^3 + {}^6C_4x^4 + {}^7C_5x^5)$

$= ({}^7C_5 - 9) + (3 \times {}^6C_4 - 15) + (3 \times {}^5C_3)$

$= 21 - 9 + 45 - 15 + 30$

$= 12 + 30 + 30$

$= 72$

231. The number of possible ways

= Co-efficient of x^{15} in $(x + x^2 + \cdots + x^6)^3$

= Co-efficient of x^{12} in $(x + \cdots + x^5)^3$

= Co-efficient of x^{12} in $\left(\frac{1-x^6}{1-x}\right)^3$

= Co-efficient of x^{12} in $(1 - x^6)^3 \times (1 - x)^{-3}$

= Co-efficient of x^{12} in $(1 - 3x^6 + 3x^{12})$
$(1 + {}^3C_1x + {}^4C_2x^2 + \cdots + {}^8C_6x^6 + \cdots + {}^{14}C_{12}x^{12})$

$= {}^{14}C_{12} - 3 \times {}^{8}C_{6} + 3$
$= 91 - 84 + 3$
$= 10$

Alternate method

The number of possible triplets to get 15 are
(5, 5, 5), (5, 6, 4) and (6, 6, 4)
Thus, the number of possible ways

$$= \frac{3!}{3!} + 3! + \frac{3!}{2!}$$

$= 1 + 6 + 3$
$= 10$

232. The number of possible ways
$= \text{Co-efficient of } x^{16} \text{ in } (x^3 + x^4 + \cdots + x^7)^4$
$= \text{Co-efficient of } x^4 \text{ in } (1 + x + x^2 + x^3 + x^4)^4$

$$= \text{Co-efficient of } x^4 \text{ in } \left(\frac{1 - x^5}{1 - x}\right)^5$$

$= \text{Co-efficient of } x^4 \text{ in } (1 - x^5)^4 \times (1 - x)^{-4}$
$= \text{Co-efficient of } x^4 \text{ in } (1 - x)^{-4}$
$= \text{Co-efficient of } x^4 \text{ in } (1 + {}^{4}C_1 x + {}^{5}C_2 x^2 + {}^{6}C_3 x^3 + {}^{7}C_4 x^4 + \cdots)$
$= {}^{7}C_4$
$= 35$

Alternate method

The number of possible quadruplets are
(3, 3, 3, 7), (3, 3, 5, 5), (3, 3, 4, 6), (3, 4, 4, 5) and (4, 4, 4, 4).
Hence, the number of possible ways

$$= \frac{4!}{3!} + \frac{4!}{2!2!} + \frac{4!}{2!} + \frac{4!}{2!} + \frac{4!}{2!}$$

$= 4 + 6 + 12 + 12 + 1$
$= 35$

233. (i) The total number of positive solutions
$= {}^{20+4-1}C_{4-1}$
$= {}^{23}C_3$

$$= \frac{23.22.21}{6}$$

$= 1771$

(ii) Here, $x \geq 1, y \geq 1, z \geq 1, w \geq 1$
$\Rightarrow \quad x - 1 \geq 0, y - 1 \geq 0, z - 1 \geq 0, w - 1 \geq 0$
Let $p = x - 1, q = y - 1, r = z - 1, s = w - 1$
Now, $x + y + z + w = 20$
$\Rightarrow \quad p + q + r + s + 4 = 20$
$\Rightarrow \quad p + q + r + s = 16$
Hence, the number of required solutions
$= {}^{16+4-1}C_{4-1}$
$= {}^{19}C_3$

$$= \frac{19.18.17}{6}$$

$= 969$

234. Given $3x + y + z = 24, x \geq 0, y \geq 0, z \geq 0$
Let $z = k$.
Then $y + z = 24 - 3k$
So, $0 \leq 24 - 3k \leq 24$
$\Rightarrow \quad -24 \leq -3k \leq 24$
$\Rightarrow \quad 0 \leq k \leq 8$
Thus, the number of integral solutions
$= {}^{24-3k+2-1}C_{2-1}$
$= {}^{25-3k}C_1 = (25 - 3k)$
Hence, the number of integral solutions

$$= \sum_{k=0}^{8} (25 - 3k)$$

$$= \sum_{k=0}^{8} 25 - 3\sum_{k=0}^{8} k$$

$$= 25 \times 9 - \frac{3 \times 8 \times 9}{2}$$

$= 225 - 108$
$= 117$

235. We have $2x + 2y + z = 10$

$$\Rightarrow \quad x + y = \frac{10 - z}{2}$$

So, $0 \leq x + y \leq 10$
Let $x + y + t = 10, x \geq 0, y \geq 0, z \geq 0$
Hence, the number of non-negative integral solutions
$= {}^{20+4-1}C_{4-1}$
$= {}^{23}C_3$

$$= \frac{12.11}{2} = 66$$

236. Here, $x - 1 \geq 0, y - 2 \geq 0, z - 3 \geq 0, t - 4 \geq 0$
Let $p = x - 1, q = y - 2, r = z - 3, s = t - 4$
$\Rightarrow x = p + 1, y = q + 2, z = r + 3, t = s + 4$
Now, $x + y + z + t = 29$
$\Rightarrow \quad p + q + r + s = 29 - 10 = 19$
Hence, the number of non-negative solutions
$= {}^{19+4-1}C_{4-1}$
$= {}^{22}C_3$

$$= \frac{22.21.20}{6}$$

$= 1540$

237. Given system of equations are
$x + y + z - t + u = 20, x + y + z = 5$
$x + y + z = 5$ …(i)
$\Rightarrow \quad t + u = 15$ …(ii)
Now, the number of non-negative integral solutions of
(i) $= {}^{5+3-1}C_{3-1}$
$= {}^{7}C_2 = 21$
Also, the number of non-negative integral solutions of (ii)
$= {}^{15+2-1}C_{2-1}$
$= {}^{16}C_1 = 16$

Hence, the number of total non-negative integral solutions = 21 × 16 = 336

238. The total numbers of ordered triplets
$= {}^{50}C_3 - {}^{19}C_3$

239. Let $x = -p, y = -q, z = -r$, where $p, q, r > 0$
Then $-p - q - r + 12 = 0$
$p + q + r = 12$
Hence, the required number of points in space
$= {}^{12-1}C_{3-1}$
$= {}^{11}C_2$
$= \frac{11 \times 10}{2} = 55$

240. We have $x_1x_2x_3 = 30 = 2 \times 3 \times 5$
Hence, the number of positive integral solutions
$= {}^{3+1-1}C_1 \times {}^{3+1-1}C_1 \times {}^{3+1-1}C_1$
$= {}^3C_1 \times {}^3C_1 \times {}^3C_1$
$= 3 \times 3 \times 3$
$= 27$

241. We have $x_1x_2x_3x_4 = 210$
$= 2 \times 3 \times 5 \times 7$
Hence, the number of positive integral solutions
$= ({}^{4+1-1}C_1 \times {}^{4+1-1}C_1 \times {}^{4+1-1}C_1 \times {}^{4+1-1}C_1) \times (1 + {}^4C_2 + {}^4C_4)$
(since all are positive or any two are negative or all four are negative)
$= ({}^4C_1 \times {}^4C_1 \times {}^4C_1 \times {}^4C_1) \times (1 + {}^4C_2 \times {}^4C_4)$
$= (4 \times 4 \times 4 \times 4) \times (1 + 6 + 1)$
$= 256 \times 8$
$= 2048$

242. Here, the possible triplets of 12 are (1, 5, 6), (2, 4, 6), (2, 5, 5), (3, 4, 5) (3, 3, 6) and (4, 4, 4)
Hence, the number of possible ways to get a sum 12
$= 3! + 3! + \frac{3!}{2!} + 3! + \frac{3!}{2!} + \frac{3!}{3!}$
$= 6 + 6 + 3 + 6 + 3 + 1$
$= 25$

243. Let four boys be x_1, x_2, x_3 and x_4.
Thus, $x_1 + x_2 + x_3 + x_4 = 10$,
where
$x_1 \ge 1, x_2 \ge 1, x_3 \ge 1, x_4 \ge 1$
Let $p = x_1 - 1, q = x_2 - 1, r_3 = x_3 - 1, s = x_4 - 1$
$\Rightarrow x_1 = p + 1, x_2 = q + 1, x_3 = r + 1, x_4 = s + 1$
Now, $x_1 + x_2 + x_3 + x_4 = 10$
$p + q + r + s + 4 = 0$
$p + q + r + s = 6$
Hence, the number of possible ways
= Number of non-negative solutions of the equation $p + q + r + s = 6$
$= {}^{6+4-1}C_{4-1}$
$= {}^9C_3$
$= \frac{9 \times 8 \times 7}{6}$
$= 84$

Alternate method
Here, the possible quadruplets of 10 are (1, 2, 3, 4), (1, 1, 4, 4), (2, 2, 2, 4), (1, 3, 3, 3), (1, 2, 2, 5), (1, 1, 3, 5), (1, 1, 2, 6) and (1, 1, 1, 7).
Hence, the number of possible ways
$= 4! + \frac{4!}{2!} + \frac{4!}{3!} + \frac{4!}{3!} + \frac{4!}{2!} + \frac{4!}{2!} + \frac{4!}{2!} + \frac{4!}{3!}$
$= 24 + 12 + 4 + 4 + 12 + 12 + 12 + 4$
$= 72 + 12$
$= 84$

244. The number of possible selections = Co-efficient of x^6 in the expansion of
$({}^2C_1x + {}^2C_2x^2) \times ({}^4C_1x + {}^4C_2x^2 + {}^4C_3x^3 + {}^4C_4x^4)$
$({}^2C_1x + {}^4C_2x^2)$
= Co-efficient of x^6 in
$({}^2C_1x + {}^2C_2x^2)^2 \times ({}^4C_1x + {}^4C_2x^2 + {}^4C_3x^3 + {}^4C_4x^4)$
= Co-efficient of x^6 in
$(2x + x^2)^2 \times (4x + 6x^2 + 4x^3 + x^4)$
= Co-efficient of x^6 in
$(2 + x)^2 \times (4 + 6x + 4x^2 + x^3)$
= Co-efficient of x^3 in
$(2 + 4x + x^2) \times (4 + 6x + 4x^2 + x^3)$
$= 2 + 16 + 6$
$= 26$

Alternate method
The number of possible selections
$= {}^8C_6 - 2$
$= 28 - 2$
$= 26$
Since either the first row or the third row may not have any x when the six xs are placed in the other two rows.

245. The number of arrangement of the letters P E R S O N
$= 6! = 720$
Also, the number of selections of the letter of the word PERSON in the given square = Co-efficient of x^6 in the expansion of
$({}^2C_1x + {}^2C_2x^2)^2 \times ({}^4C_1x + {}^4C_2x^2 + {}^4C_3x^3 + {}^4C_4x^4)$
$= 26$
Hence, the number of required possible ways
$= 26 \times 6!$
$= 26 \times 720$
$= 18720$

246. (i) The required number of straight lines
$= {}^{12}C_2 - {}^5C_2 + 1$
$= 66 - 10 + 1$
$= 57$

(ii) The number of required triangles

$= {}^{12}C_3 - {}^5C_3$
$= 220 - 10$
$= 210$

(iii) The number of quadrilaterals

$= {}^5C_0 \times {}^7C_4 + {}^5C_1 \times {}^7C_3 + {}^5C_2 \times {}^7C_2$
$= 35 + 175 + 210$
$= 420$

247. The number of diagonals

$= {}^{10}C_2 - 10$
$= 45 - 10$
$= 35$

248 Let the number of sides of the polygon be n.

Given ${}^nC_2 - 35$

$\Rightarrow \dfrac{n(n-1)}{2} - n = 35$

$\Rightarrow n_2 - 3n - 70 = 0$

$\Rightarrow (n-10)(n+7) = 0$

$\Rightarrow n = 10$

Hence, the number of sides of the polygon is 10.

249. The required number of triangles

= Total number of triangles – number of triangles having two sides common – number of triangles having one side common

$= {}^8C_3 - 8 - {}^8C_1 \times {}^4C_1$
$= 56 - 8 - 32$
$= 56 - 40$
$= 16$

250. Hence, the required number of parallelograms

$= {}^{10}C_2 \times {}^8C_2$
$= 45 \times 8$
$= 1260$

251. Given,

$T_{n+1} - T_n = 21$

$\Rightarrow {}^{n+1}C_3 - {}^nC_3 = 21$

$\Rightarrow \frac{1}{6}[n(n+1)(n-1) - n(n-1)(n-2)] = 21$

$\Rightarrow n(n-1)[(n+1) - (n-2)] = 126$

$\Rightarrow 3n(n-1) = 126$

$\Rightarrow n(n-1) = 42$

$\Rightarrow n^2 - n - 42 = 0$

$\Rightarrow (n-7)(n+6) = 0$

$\Rightarrow n = 7$

Hence, the value of n is 7.

252. The number of rectangles

$= {}^{7-1}C_2 \times {}^{4-1}C_2$
$= {}^8C_2 \times {}^5C_2$
$= 28 \times 10$
$= 280$

253. The number of squares

$= 6 \times 9 + 5 \times 8 + 4 \times 7 + 3 \times 6 + 2 \times 5 + 1 \times 4$
$= 54 + 40 + 28 + 18 + 10 + 4$
$= 108 + 46$
$= 154$

The number of rectangles

$= {}^{9+1}C_2 \times {}^{6+1}C_2$
$= {}^{10}C_2 \times {}^7C_2$
$= 45 \times 21$
$= 945$

Hence, the required number of rectangles excluding squares

$= 945 - 154$
$= 791$

254. The number of points of intersection of 5 circles

$= 2 \times {}^5C_2$
$= 2 \times 10$
$= 20$

255. The number of points of intersection of 8 lines and 4 circles

$= {}^8C_2 + 2 \times {}^4C_2 + 2 \times {}^8C_1 \times {}^4C_1$
$= 28 + 12 + 64$
$= 104$

256. Number of possible rectangles is

$= [1 + 3 + 5 + \ldots + (2m-1)] \times [1 + 3 + 5 + \ldots + (2n-1)]$
$= m^2n^2$

257. Number of red lines $= {}^nC_2$

Number of blue lines $= n$

Hence, ${}^nC_2 - n = n$

$\Rightarrow {}^nC_2 = 2n$

$\Rightarrow \dfrac{n(n-1)}{2} = 2n$

$\Rightarrow n^2 - n = 4n$

$\Rightarrow n^2 = 5n$

$\Rightarrow n = 0, 5$

Since $n \geq 2$, so $n = 5$.

258. The number of total triangles

$= {}^3C_1 \times {}^4C_1 \times {}^5C_1 + \{{}^3C_2({}^4C_1 + {}^5C_1)$
$+ {}^4C_2({}^3C_1 + {}^5C_1) + {}^5C_2({}^3C_1 + {}^4C_1)\}$
$= 60 + 3(4 + 5) + 6(3 + 5) + 10(3 + 4)$
$= 60 + 27 + 48 + 70$
$= 205$

259. In 8 × 8 chess board, 7 pairs of adjacent columns be chosen in which first square be chosen in 2 ways (one from top and one from bottom) and 1 square be chosen from the 2nd column.

$= 7(2 \times 1 + 6 \times 2)$
$= 7 \times 14$
$= 98$

Level III

1. No. of signals using 1 flag $= {}^5P_1$

No. of signals using 2 flag $= {}^5P_2$

No. of signals using 3 flag $= {}^5P_3$

No. of signals using 4 flag = $^5P_4$
No. of signals using 5 flag = $^5P_5$
Hence, the total number of ways
$= {}^5P_1 + {}^5P_2 + {}^5P_3 + {}^5P_4 + {}^5P_5$
$= 5 + 20 + 60 + 120 + 120$
$= 325$

2. The rank of the word TOUGH in dictionary,
$m = 4! + 4! + 4! + 3! + 3! + 2! + 2! + 1$
$= 89$
Clearly, the rank of the word IIT is 1
Thus, $n = 1$
Hence, the value of $m + n + 10$
$= 89 + 1 + 10$
$= 100$

3. The required number of triangles
$= {}^nC_3 - n - n(n-4)$
$$= \frac{n(n-1)(n-2)}{6} - n - n(n-4)$$
$$= \frac{n}{6}(n^2 - 3n + 2 - 6 - 6n + 24)$$
$$= \frac{n}{6}(n^2 - 9n + 20)$$
$$= \frac{n(n-4)(n-5)}{6}$$

4. Let $S_1, S_2, S_3, \ldots, S_8$ denote the stations where the train does not stop.
So, there are four stations where the train stops.
It can be possible in $^9C_4$ ways
$$= \frac{9\times 8\times 7\times 6}{24} = 126 \text{ ways.}$$

5. A pack of cards consists of 4 suits, namely, spade, club, diamond and hearts, respectively. Each suit consists of 13 cards. Number of ways of choosing 5 cards in 9 different ways.
But one card of any denominations can be selected from 4 suits in 4^5 ways.
Hence, the number of ways a hand containing 5 consecutive denominations
$= 9 \times 4^5 = 9 \times 1024 = 9216$

6. To get one point of intersection, we take 2 points on the first line and 2 points on the second line.
It can be done in $^mC_2 \times {}^nC_2$ ways
$$= \frac{m(m-1)}{2} \times \frac{n(n-1)}{2}$$
$$= \frac{mn(m-1)(n-1)}{4} \text{ ways.}$$

7. Let the number of children of John and his first wife be x and the number of children of John and Mary be y.
So, the number of children of Mary and her first husband $= x + 1$.
Thus, $x + x + 1 + y = 24$
$2x + y = 23$
Total number of fights between two children
$= {}^{24}C_2 = 276.$
Now, the total number of fights between two children of same parents $= {}^{x+1}C_2 + {}^xC_2 + {}^yC_2$
$= ({}^xC_1 + {}^xC_3) + {}^xC_2 + {}^yC_2$
$= ({}^xC_1 + 2\,{}^xC_2) + {}^{23-2x}C_2$
$= 3x^2 - 45x + 253$
Therefore, the total number of fights, subject to the condition that any two children of the same parents do not fight.
$N = 276 - (3x^2 - 45x + 253)$
$\Rightarrow \quad N = 23 - 3x^2 + 45x$
$$\Rightarrow \quad \frac{dN}{dx} = -6x + 45$$
For maximum or minimum, $\dfrac{dN}{dx} = 0$
$\Rightarrow \quad -6x + 45 = 0$
$\Rightarrow \quad x = 7\cdot 5$
$\Rightarrow \quad x = 7$
Also, $\dfrac{d^2N}{dx^2} = -6 < 0$
Thus, N will be maximum, when $x = 7$.
Hence, the maximum number of fights
$= 23 - 3 \times 7^2 + 45 \times 7$
$= 23 - 147 + 315$
$= 191$

8. The possible triplets of 5 are (1, 2, 2) and (1, 1, 3).
Hence, the number of possible ways
$$= \frac{5!}{1!\times 2!\times 2!} \times \frac{3!}{2!} + \frac{5!}{1!\times 1!\times 3!} \times \frac{3!}{2!}$$
$= 30 \times 3 + 20 \times 3$
$= 90 + 60$
$= 150$

9. The number of possible ways
= Co-efficient of x^{16} in $(x^3 + x^4 + \cdots + x^7)^4$
= Co-efficient of x^{16} in $x^{12}(1 + x + \cdots + x^4)^4$
= Co-efficient of x^4 in $(1 + x + \cdots + x^4)^4$
= Co-efficient of x^4 in $\left(\dfrac{1-x^5}{1-x}\right)^4$
= Co-efficient of x^4 in $(1 - x^5)^4 \times (1 - x)^{-4}$
= Co-efficient of x^4 in $(1 - x)^{-4}$
= Co-efficient of x^4 in
$(1 + {}^4C_1x + {}^5C_2x^2 + {}^6C_3x^3 + {}^7C_4x^4 + \cdots)$
$$= {}^7C_4 = \frac{7\times 6\times 5\times 4}{24} = 35$$

10. The number of possible ways
= Co-efficient of x^{30} in $(x^2 + x^3 + \cdots + x^{16})^8$
= Co-efficient of x^{30} in $x^{16}(1 + x + \cdots + x^{14})^8$
= Co-efficient of x^{14} in $(1 + x + \cdots + x^{14})^8$

$= \text{Co-efficient of } x^{14} \text{ in} \left(\frac{1-x^{15}}{1-x}\right)^8$

$= \text{Co-efficient of } x^{14} \text{ in } (1-x)^{-8}$

$= \text{Co-efficient of } x^{14}$ in

$(1 + {}^8C_1 x + {}^9C_2 x^2 + \cdots + {}^{21}C_{14} x^{14} + \cdots)$

$= {}^{21}C_{14} = {}^{23}C_7$

11. Total Marks $= 50 \times 3 + 100 = 250$

Now, 60% of total marks

$$= \frac{60}{100} \times 250 = 150$$

Thus, the number of ways the candidate can get 150 marks in total

$= \text{Co-efficient of } x^{150} \text{ in } (1 + x + x^2 + \cdots + x^{50})^3 \times (1 + x + x^2 + \cdots + x^{150})$

$= \text{Co-efficients } x^{150} \text{ in} \left(\frac{1-x^{51}}{1-x}\right)^3 \times \left(\frac{1-x^{151}}{1-x}\right)$

$= \text{Co-efficients } x^{150} \text{ in } (1-x)^3 \times (1-x)^4$

$= \text{Co-efficients } x^{150} \text{ in } (1 - 3x^{51} + 3x^{102} - x^{153})$

$\times (1 + {}^4C_1 x + \cdots + {}^{50}C_{48} x^{48} + {}^{51}C_{49} x^{49} + \cdots + {}^{101}C_{99} x^{99} + \cdots)$

$= {}^{153}C_{150} - 3 \times {}^{102}C_{99} - {}^{52}C_{49} + 3.\ {}^{51}C_{48}$

$= 110556$

12. Consider $a < b < c < d$.

Put $x = a, y = b - a, z = c - b, u = d - c$

Thus, $x, y, z, u \geq 1$

So, $a = x, b = x + y, c = x + y + z$

and $d = x + y + z + u$

So, the given equation becomes

$4x + 3y + 2z + u = 20$

The number of positive integral solutions of the equation is

$= \text{Co-efficient of } x^{20}$ in

$(x^4 + x^8 + x^{12} + \cdots)(x^3 + x^6 + x^9 + \cdots) \times (x^2 + x^4 + x^6 + \cdots)(x + x^2 + x^3 + \cdots)$

$= \text{Co-efficient of } x^{10}$ in

$(1 + x^4 + x^8 + \cdots)(1 + x^3 + x^6 + \cdots) \times (1 + x^2 + x^4 + \cdots)(1 + x + x^2 + \cdots)$

$= \text{Co-efficient of } x^{10}$ in

$$\left(\frac{1}{1-x^4}\right)\left(\frac{1}{1-x^3}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x}\right)$$

$= \text{Co-efficient of } x^{10}$ in

$(1-x^4)^{-1} \times (1-x^3)^{-1} \times (1-x^2)^{-1} \times (1-x)^{-1}$

$= \text{Co-efficient of } x^{10}$ in

$(1 + x^4 + x^8) \times (1 + x^3 + x^6 + x^9) \times (1 + x^2 + x^4 + x^6 + x^8 + x^{10}) \times (1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + x^9 + x^{10})$

$= \text{Co-efficient of } x^{10}$ in

$(1 + x^3 + x^4 + x^6 + x^7 + x^8 + x^9 + x^{10}) \times (1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + 4x^6 + 4x^7 + 5x^8 + 5x^9 + 6x^{10})$

$= 6 + 4 + 4 + 3 + 2 + 2 + 1 + 1$

$= 23$

Also, a, b, c and d can be permuted in ${}^4P_4$ ways.

Hence, the required number of solutions

$= 23 \times {}^4P_4 = 23 \times 24 = 552$

13. The required number

$= \text{Co-efficient of } x^n$ in the expansion of

$(1 + x + x^2 + \cdots + x^n)^2 (1 + x)^n$

$= \text{Co-efficients of } x^n \text{ in } \left(\frac{1-x^{n+1}}{1-x}\right)^2 (1+x)^n$

$= \text{Co-efficients of } x^n \text{ in } (1 - x^{n+1})^2 (1-x)^{-2} (1+x)^n$

$= \text{Co-efficients of } x^n \text{ in } (1-x)^{-2}(1+x)^n$

$= \text{Co-efficients of } x^n \text{ in } (1-x)^{-2}(2 - (1-x))^n$

$= \text{Co-efficients of } x^n$ in

$(1-x)^{-2}(2^n - {}^nC_1 2^{n-1}(1-x) + {}^nC_2 2^{n-2}(1-x)^2 - \cdots)$

$= 2^n \times {}^{2+n-1}C_n - {}^nC_1 \times 2^{n-1} \times {}^{1+n-1}C_n$

$= 2^n \times {}^{n+1}C_n - {}^nC_1 \times 2^{n-1} \times {}^nC_n$

$= 2^n \times (n+1) - n \times 2^{n-1}$

$= 2^{n-1}(2n + 2 - n)$

$= (n+2)2^{n-1}$

14. Total number of letters = 11, in which 3 Es, 3 Ns, 2 Ds and I, P, T occur once.

Hence, the required number of selections of 5 letters

$= \text{Co-efficients of } x^5$ in

$(1 + x + x^2 + x^3)^2 (1 + x + x^2)(1 + x)^3$

$= (1+x)^2(1+x^2)^2(1 + x + x^2)(1+x)^3$

$= (1+x)^5(1+x^2)^2(1 + x + x^2)$

$= (1+x)^5(1 + 2x^2 + x^4)(1 + x + x^2)$

$= (1+x)^5(1 + x + 3x^2 + 2x^3 + 3x^4 + x^5)$

$= (1 + {}^5C_1 x + {}^5C_2 x^2 + {}^5C_3 x^3 + {}^5C_4 x^4 + {}^5C_5 x^5) \times (1 + x + 3x^2 + 2x^3 + 3x^4 + x^5)$

$= 1 + {}^5C_5 + 3 \times {}^5C_1 + {}^5C_4 + 3 \times {}^5C_3 + 2 \times {}^5C_2$

$= 1 + 1 + 15 + 5 + 30 + 20$

$= 72$

15. Total number of letters = 8 in which 2 As, 3 Ls and E, P, R occur once.

Hence, the required number of arrangement of 4 letters

$= \text{Co-efficients of } x^4$ in

$$4! \times \left(1 + \frac{x}{1!} + \frac{x^2}{2!}\right)\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!}\right)\left(1 + \frac{x}{1!}\right)^3$$

$= 286$

16. How many three digit numbers are of the form xyz with $x < y > z$ and $x \neq 0$.

17. The number of possible ways

$$= \frac{{}^8P_5}{5!} = \frac{8!}{3! \times 5!} = \frac{8 \times 7 \times 6}{6} = 56$$

18. The number of positive integral solutions

$= \text{Co-efficients of } x^{10}$ in

$(x + x^2 + x^3 + \cdots)(x^2 + x^4 + x^6 + \cdots)(x^3 + x^6 + \cdots)$

$= \text{Co-efficients of } x^4$ in

$(1 + x + x^2 + \cdots)(1 + x^2 + \cdots)(1 + x^3 + \cdots)$

$= \text{Co-efficients of } x^4$ in

$(1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}$
$= (1+x+x^2+x^3+x^4)(1+x^2+x^4)(1+x^3)$
$= (1+x+x^2+x^3+x^4)(1+x^2+x^3+x^4)$
$= 1+1+1+1$
$= 4$

19. Consider I for India and P for Pakistan.
We can arrange I and P to show wins for India and Pakistan respectively.
Thus, IPPPP means first match won by India and 4 matches won by Pakistan, respectively.
Consider Pakistan wins the series, the last match is always won by Pakistan

Wins of I	Wins of P	No. of ways
0	4	1
1	4	$\frac{4!}{3!}=4$
2	4	$\frac{5!}{2!3!}=10$
3	4	$\frac{6!}{3!3!}=20$

Thus, the total number of ways = 35
Similarly, the same number of ways India can win the series = 35
Therefore, total number of ways, the series can be won = 35 + 35 = 70.

20. Find the number of quintuples (x, y, z, u, v) of positive integers satisfying both.
$x+y+z+u=30$ and $x+y+z+v=27$

22. Given $3x+y+z=24$, $x \geq 0$, $y \geq 0$, $z \geq 0$
Let $z = k \Rightarrow y+z = 24-3k$
So, $0 \leq 24-3k \leq 24$
$\Rightarrow$ $-24 \leq -3k \leq 0$
$\Rightarrow$ $0 \leq k \leq 8$
Thus, the number of integral solutions
$= {}^{24-3k+2-1}C_{2-1}$
$= {}^{25-3k}C_1 = (25-3k)$
Hence, the number of integral solutions

$$= \sum_{k=0}^{8}(25-3k)$$
$$= \sum_{k=0}^{8}25 - 3\sum_{k=0}^{8}k$$
$$= 25\times 9 - \frac{3.8.9}{2}$$
$$= 225 - 108$$
$$= 117$$

23. The possible triplets to get 12 are (1, 5, 6), (2, 4, 6), (3, 4, 5), (3, 3, 6), (2, 5, 5) and (4, 4, 4)
Hence, the number of possible ways

$$= 3! + 3! + 3! + \frac{3!}{2!} + \frac{3!}{2!} + \frac{3!}{3!}$$
$= 18 + 6 + 1$
$= 25.$

24. The number of possible ways it can be done
$= 2! \times (18)!$
$= 2 \times (18)!$

25. Let the student get x_i marks in each paper.
Thus, $x_1 + x_2 + x_3 + x_4 = 2m$, $0 \leq x_i \leq m$
Therefore, the number of ways of getting $2m$ marks = the number of non-negative integral solutions of the equation.
= Co-efficients of x^{2m} in $(1+x+x^2+x^3+\cdots+x^m)^4$
= Co-efficients of x^{2m} in $(1-x^{m+1})^4 \times (1-x)^{-4}$
= Co-efficients of x^{2m} in
$(1-x^{m+1})^4 \times (1 + {}^4C_1x + {}^5C_2x^2 + {}^6C_3x^3 + \ldots)$
$= {}^{2m+3}C_{2m} - 4({}^{m+2}C_{m-1})$
$= {}^{2m+3}C_3 - 4({}^{m+2}C_3)$

$$= \frac{(2m+3)(2m+2)(2m+1)}{6} - \frac{4(m+2)(m+1)m}{6}$$
$$= \frac{(m+1)}{3}((2m+3)(2m+1) - 2m(m+2))$$
$$= \frac{(m+1)}{3}(4m^2+8m+3-2m^2-4m)$$
$$= \frac{1}{3}(m+1)(2m^2+4m+3)$$

26. We have $x = \sum_{k=1}^{2016}(k!)$
$= 1! + 2! + 3! + 4! + 5! + 6! + 7! + 8! + 9! + 10! + 11! + 12! + \cdots + (2016!)$
In 5! and after last digit is 0 and after (10!) last two digit is 00
So sum of the last two digits
$= 01 + 02 + 06 + 24 + 20 + 20 + 40 + 20 + 80$
$= 13$
Hence, the last two digits in $x = \sum_{k=1}^{2016}(k!)$ is 13.

27. We select 4 digits out of 10 in ${}^{10}C_4$ ways.
Since the order is fixed for arrangement, i.e. $a > b > c > d$.
Hence, the number of possible ways
= (Number of ways to select) × 1
$= {}^{10}C_4 \times 1$
$$= \frac{10\times 9\times 8\times 7}{24} = 210$$

28. (i) Consider two Bs as single object, the letters BB, C, D in 3! ways.
i.e. × BB × C × D ×
So there are four gaps. We can fill As in four gaps which is possible in ${}^4C_3$ ways.
Hence, the required number of arrangements
$= {}^4C^3 \times 3!$
$= 4 \times 6 = 24$

(ii) The letters 2Bs, 1 C, 1 D can be arranged in $\frac{4!}{2!} = \frac{24}{2} = 12$

Consider the arrangement is

$\times B \times B \times C \times D \times$

Clearly, there are 5 gaps. In these gaps, we will fill 3As in ${}^5C_3$ ways.

Hence, the required number of arrangements

$= ({}^5C_3 \times 12 - 24)$
$= (10 \times 12 - 24)$
$= 120 - 24$
$= 96$

29. Given,

$P_n - Q_n = 6$

$\Rightarrow {}^{n-2}C_3 - ({}^nC_3 - n(n-3)) = 6$

$\Rightarrow {}^{n-2}C_3 - {}^nC_3 + n(n-3) = 6$

$\Rightarrow ({}^{n-2}C_3 - {}^nC_3) + n(n-3) = 6$

$\Rightarrow \left(\frac{(n-2)(n-3)(n-4)}{6} - \frac{n(n-1)(n-2)}{6}\right) = n(n-3) = 6$

$\Rightarrow (n-2)[(n-3)(n-4) - n(n-1)] + 6n(n-3) = 36$

$\Rightarrow -(n-2)(6n-12) + 6n(n-3) = 36$

$\Rightarrow -(n-2)(n-2) + n(n-3) = 6$

$\Rightarrow -n^2 + 4n - 4 + n^2 - 3n = 6$

$\Rightarrow n - 4 = 6$

$\Rightarrow n = 10$

Hence, the value of n is 10.

30. A function $f: A \to B$ such that

$f(0) \le f(1) \le f(2)$

is defined as follows

Case I: When $f(0) < f(1) < f(2)$

Number of functions $= {}^8C_3$

Case II: When $f(0) = f(1) < f(2)$

Number of functions $= {}^8C_2$

Case III: When $f(0) < f(1) = f(2)$

Number of functions $= {}^8C_2$

Case IV: When $f(0) = f(1) = f(2)$

Number of functions $= {}^8C_1$

Hence, the total number of functions

$= {}^8C_3 + {}^8C_2 + {}^8C_2 + {}^8C_1$
$= 56 + 28 + 28 + 8 = 120$

Integer Type Questions

1. We have

${}^{n+1}C_{r+1} : {}^nC_r : {}^{n-1}C_{r-1} = 11 : 6 : 3$

$\Rightarrow \frac{{}^{n+1}C_{r+1}}{{}^nC_r} = \frac{11}{6}, \frac{{}^nC_r}{{}^{n-1}C_{r-1}} = \frac{6}{3}$

$\Rightarrow \left(\frac{n+1}{r+1}\right) = \frac{11}{6}, \frac{n}{r} = 2$

$\Rightarrow 11r + 11 = 6n + 6, n = 2r$

$\Rightarrow 11r + 11 = 12r + 6$

$\Rightarrow r = 5$

2. We have

$\frac{{}^nC_r}{{}^nC_{r-1}} = \frac{7}{3}$ and $\frac{{}^nC_{r+1}}{{}^nC_r} = \frac{3}{2}$

$\Rightarrow \frac{n-r+1}{r} = \frac{7}{3}$ and $\frac{n-r}{r+1} = \frac{3}{2}$

$\Rightarrow 10r - 3n = 3$ and $2n - 2r = 3r + 3$

$\Rightarrow 10r - 3n = 3$ and $2n = 5r + 3$

$\Rightarrow 10r - 3n = 3$ and $4n = 10r + 6$

$\Rightarrow 4n - 6 - 3n = 3$

$\Rightarrow n = 9$

Hence, the value of n is 9.

3. We have

$4({}^{x-1}C_4 - {}^{x-1}C_3) < 5(x-2)(x-3)$

$\Rightarrow 4(x-1)(x-2)(x-3)\left(\frac{(x-4)}{24} - \frac{1}{6}\right) < 5(x-2)(x-3)$

$\Rightarrow 4(x-1)(x-2)(x-3)\frac{(x-8)}{24} < 5(x-2)(x-3)$

$\Rightarrow (x-1)(x-3)(x-8) < 30(x-2)(x-3)$

$\Rightarrow (x-2)(x-3)\{(x-1)(x-8) - 30\} < 0$

$\Rightarrow (x-2)(x-3)\{(x^2 - 9x - 22) < 0$

$\Rightarrow (x-2)(x-3)(x-11)(x+2) < 0$

$\Rightarrow -2 < x < 2, 3 < x < 11$

$\therefore x = 1, 4, 5, 6, 7, 8, 9, 10$

$x = 5, 6, 7, 8, 9, 10$, since $x = 1$ and 4 do not satisfy the given in equation.

4. Clearly, $m = 2! \times 4! \times 4!$

and $n = 4! \times 4!$

Hence, the value of $\left(\frac{m}{n} + 2\right) = 4$.

5. We have,

${}^{n-1}C_3 - {}^{n-2}C_3 = 15$

$\Rightarrow \frac{(n-1)(n-2)(n-3)}{6} - \frac{(n-2)(n-3)(n-4)}{6} = 15$

$\Rightarrow \frac{(n-2)(n-3)}{6}(n-1-n+4) = 15$

$\Rightarrow (n-2)(n-3) = 30$

$\Rightarrow n^2 - 5n + 6 = 30$

$\Rightarrow n^2 - 5n - 24 = 0$

$\Rightarrow (n-8)(n+3) = 0$

$\Rightarrow n = 8, -3$

$\Rightarrow n = 8$

Hence, the value of n is 8.

6. We have

$\frac{{}^nP_{r-1}}{a} = \frac{{}^nP_r}{b} = \frac{{}^nP_{r+1}}{c}$

$\Rightarrow \frac{b}{a} = \frac{{}^nP_r}{{}^nP_{r-1}}, \frac{c}{b} = \frac{{}^nP_{r+1}}{{}^nP_r}$

$\Rightarrow \quad \frac{b}{a} = (n - r + 1), \frac{c}{b} = (n - r)$

$\Rightarrow \quad \frac{b}{a} = \left(\frac{c}{b} + 1\right)$

$\Rightarrow \quad b^2 = a(c + b)$

$\Rightarrow \quad \frac{b^2}{a(c+b)} = 1$

$\Rightarrow \quad \left(\frac{b^2}{a(c+b)} + 2\right) = 1 + 2 = 3$

Hence, the value of $\left(\frac{b^2}{a(c+b)} + 2\right)$ is 3.

7. Clearly,
 $m = 6! - 5! \times 2!$
 $= 6 \times 5! - 2 \times 5! = (6 - 2) \times 5 = 4 \times 5!$
 and $n = \frac{5!}{2}$
 Hence, the value of $\left(\frac{m}{n} - 3\right)$
 $= \frac{4 \times 5!}{5!/2} - 3 = 8 - 3 = 5$

8. The number of ordered triplets of positive integers
 $= {}^{50}C_3 - {}^{10}C_3$
 Thus, $x = 50$ and $y = 10$.
 Hence, the value of $\left(\frac{x}{y} + 2\right)$
 $= \left(\frac{50}{10} + 2\right) = 7$

9. Clearly, $m = {}^{3+1-1}C_1 \times {}^{3+1-1}C_1 \times {}^{3+1-1}C_1$
 $= {}^3C_1 \times {}^3C_1 \times {}^3C_1 = 27 = 3^3$
 and $n = 4^4(1 + {}^4C_2 + {}^4C_4)$
 $= 4^4(1 + 6 + 1) = 4^4 \times 8 = 2^{11}$
 Thus, $p = 3$, $q = 11$
 Hence, the value of $(q - 2p) = 11 - 6 = 5$.

10. The possible triplets of 5 are (1, 1, 3) and (1, 2, 2).
 Hence, the required number of possible ways
 $= \frac{3!}{1! \times 2!} \times 1 + \frac{3!}{1! \times 2!} \times 1$
 $= 3 + 3$
 $= 6$

11. m = Co-efficients of x^{11} in $(x + x^2 + \cdots + x^6)^3$
 = Co-efficients of x^8 in $(1 + x + x^2 + \cdots + x^5)^3$
 = Co-efficients of x^8 in $\left(\frac{1 - x^6}{1 - x}\right)^3$
 = Co-efficients of x^8 in $(1 - x^6)^3 \times (1 - x)^{-3}$
 = Co-efficients of x^8 in
 $(1 - 3x^6) \times (1 + {}^3C_1x + {}^4C_2x^2 + \cdots + {}^{10}C_8x^8 + \cdots)$
 $= ({}^{10}C_8 - 3x^4C_2)$
 $= 45 - 18 = 27$
 and n = Co-efficients of x^5 in $(1 + x + x^2 + \cdots)^3$
 = Co-efficients of x^5 in $(1 - x)^{-3}$
 = Co-efficients of x^5 in $(1 + {}^3C_1x + {}^4C_2x^2 + \cdots + {}^7C_5x^5 + \cdots)$
 $= {}^7C_5 = 21$
 Hence, the value of $(m - n)$ is
 $= 27 - 21 = 6$

12. Clearly, $m = \frac{6!}{2!} - \frac{5!}{2!} \times 2! = 360 - 120 = 240$
 and $n = 4! \times 2! = 48$
 Hence, the value of $\left(\frac{m}{n} + 3\right)$
 $= \left(\frac{240}{48} + 3\right) = 8$

13. It can be done in $(2^{n-1} - 1)$ ways.
 It is given that $(2^{n-1} - 1) = 127$
 $\Rightarrow \quad 2^{n-1} = 128 = 2^7$
 $\Rightarrow \quad n - 1 = 7$
 $\Rightarrow \quad n = 8$
 Hence, the value of n is 8.

14. It is possible only when $n = 1$ and $n = 3$
 When $n = 1$, then
 $\sum_{k=1}^{n} (k!) = 1 = (1)^2$
 when $n = 3$, then
 $\sum_{k=1}^{n} (k!)$
 $= \sum_{k=1}^{3} (k!)$
 $= (1!) + (2!) + (3!)$
 $= 1 + 2 + 6 = 9 = (3)^2$
 Hence, the number of values of n is 2.

15. Given,
 $\sum_{k=1}^{n} 2^{n-k} = 511$
 $\Rightarrow \quad (2^{n-1} + 2^{n-2} + \cdots + 2 + 1) = 511$
 $\Rightarrow \quad (1 + 2 + 2^2 + \cdots + 2^{n-1}) = 511$
 $\Rightarrow \quad \left(\frac{2^n - 1}{2 - 1}\right) = 511$
 $\Rightarrow \quad 2^n - 1 = 511$
 $\Rightarrow \quad 2^n = 512$
 $\Rightarrow \quad 2^n = 2^9$
 $\Rightarrow \quad n = 9$
 Hence, the value of n is 9.

16. Given ${}^nC_2 = 36$
 $\Rightarrow \quad \frac{n(n-1)}{2} = 36$
 $\Rightarrow \quad n(n - 1) = 72$
 $\Rightarrow \quad n^2 - n - 72 = 0$

$\Rightarrow \quad (n-9)(n+8)=0$
$\Rightarrow \quad n=9,-8$
$\Rightarrow \quad n=9$

Hence, the number persons in that room is 9.

17. The number of possible ways

$= \text{Co-efficients of } x^{10} \text{ in } (x+x^2+x^3+\cdots+x^7)^4$
$= \text{Co-efficients of } x^{6} \text{ in } (1+x+x^2+\cdots+x^6)^4$

$$= \text{Co-efficients of } x^6 \text{ in} \left(\frac{1-x^7}{1-x}\right)^4$$

$= \text{Co-efficients of } x^6 \text{ in } (1-x)^{-4}$
$= {}^{6-4-1}C_{4-1}$
$= {}^{9}C_3$

Thus, $x=9$ and $y=3$.

Hence, the value of

$(x-y-2)$
$=9-3-2$
$=4$

18. It is given that

$$\sum_{k=1}^{n} 3^{n-k} = 3280$$

$\Rightarrow \quad (3^{n-1}+3^{n-2}+\cdots+3+1)=3280$
$\Rightarrow \quad (1+3+3^2+\cdots+3^{n-2}+3^{n-1})=3280$

$$\Rightarrow \quad \left(\frac{3^n-1}{3-1}\right)=3280$$

$\Rightarrow \quad (3^n-1)=3280\times 2$
$\Rightarrow \quad 3^n=6561$
$\Rightarrow \quad 3^n=81\times 81=3^4\times 3^4=3^8$
$\Rightarrow \quad n=8$

Hence, the value of n is 8.

19. The number of possible ways it can be done

$$= 4!\left(1-\frac{1}{1!}+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right)$$

$$= 4!\left(\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}\right)$$

$$= 24\times\left(\frac{1}{2}-\frac{1}{6}+\frac{1}{24}\right)$$

$$= 24\times\left(\frac{12-4+1}{24}\right)$$

$=9$

20. Clearly, $6^n-3^n=189$

$\Rightarrow \quad 6^n-3^n=(216-27)$
$\Rightarrow \quad 6^n-3^n=6^3-3^3$
$\Rightarrow \quad n=3$

Hence, the value of n is 3.

Previous Years' JEE-Advanced Examinations

1. Method R_1 R_2 R_3 Number of ways

I	1	3	2	${}^2C_1\times{}^4C_3\times{}^2C_2=8$
II	1	4	1	${}^2C_1\times{}^4C_4\times{}^2C_1=4$
III	2	2	2	${}^2C_2\times{}^4C_2\times{}^2C_2=6$
IV	2	3	1	${}^2C_2\times{}^4C_3\times{}^2C_1=8$

Thus, the total possible ways

$=8+4+6+6$
$=26$

2. Each player will get 13 cards.

The number of ways of distributing 52 cards giving 13 cards to each player

$= {}^{52}C_{13}\times{}^{39}C_{13}\times{}^{26}C_{13}\times{}^{13}C_{13}$

$$= \frac{(52)!}{\{(13)!\}^4}$$

3. The number of possible ways of dividing cards into four groups

$$= \frac{{}^{52}C_{17}\times{}^{35}C_{17}\times{}^{18}C_{17}\times 1}{3!}$$

$$= \frac{(52)!}{3!\times\{(17)!\}^3}$$

4. Number of five letter words that can be formed from the 10 letters $=10\times10\times10\times10\times10=10^5$

Number of 5 letter words that have none of their letter repeated $= {}^{10}P_5 = 30240$

Thus, the number of words which have at least one of their letter repeated $=10^5-30240=69760$

5. Clearly B and C have the same number of elements.

6. $${}^{47}C_4+\sum_{j=1}^{5}{}^{52-j}C_3$$

$= {}^{47}C_4+{}^{51}C_3+{}^{50}C_3+{}^{49}C_3+{}^{48}C_3+{}^{47}C_3$
$= ({}^{47}C_4+{}^{47}C_3)+{}^{51}C_3+{}^{50}C_3+{}^{49}C_3+{}^{48}C_3$
$= ({}^{48}C_4+{}^{48}C_3)+{}^{51}C_3+{}^{50}C_3+{}^{49}C_3$
$= ({}^{49}C_4+{}^{49}C_3)+{}^{51}C_3+{}^{50}C_3$
$= ({}^{50}C_4+{}^{50}C_3)+{}^{51}C_3$
$= ({}^{51}C_4+{}^{51}C_3)$
$= {}^{52}C_4$

7. The total number of possible ways

= Total number of onto functions between two sets consists of 5 and 3 elements respectively = total number of possible ways to distribute 5 balls into 3 boxes, where no box is remain empty.

$$= \frac{5!}{1!1!3!}\times\frac{3!}{2!}+\frac{5!}{1!2!2!}\times\frac{3!}{2!}$$

$$= \frac{120}{2}+\frac{120\times 6}{8}$$

$=60+90$
$=150$

8. Two women can be seated in ${}^4P_2$ ways and 3 men can occupy the chairs in ${}^6P_3$ in ways.
   Thus, the total number of possible ways
   $= {}^4P_2 \times {}^6P_3$
9. As the number of students answering incorrect at least r questions is a_r.
   The number of students answering exactly r questions $[1 \le r \le (n-1)]$ questions incorrect is $a_r - a_{r-1}$
   Also, the number of students answering all questions wrong is a_k.
   Thus, the total number of possible ways, a student can give wrong answers is
   $= 1(a_1 - a_2) + 2(a_2 - a_3) + \cdots + (k-1)(a_{k-1} - a_k) + ka_k$
   $= a_1 + a_2 + \cdots + a_k$
10. m men can take their seat in $m!$ ways.
    After m men have taken their seats, the women can take their n seats out of $(m + 1)$ seats.
    It can be done in ${}^nC_{m+1}$ ways.
    ∴ Total possible ways
    $$= m! \times {}^nC_{m+1} \times n!$$
    $$= \frac{m! \times (m+1)!}{(m-n+1)!}$$
11.
12. Husband and wife can invite their relatives in (3M, 0F), (2M, 1F), (1M, 2F) and (0M, 3F) ways
    The total number of possible ways
    $= ({}^4C_3 \times {}^4C_3) + ({}^4C_2 \times {}^4C_1)({}^3C_1 \times {}^4C_2)$
    $+ ({}^4C_1 \times {}^3C_2)({}^3C_2 \times {}^4C_1) + ({}^3C_3 \times {}^3C_3)$
    $= 485$
13. The number of ways of choosing 3 balls out of 2 white, 3 black and 4 red balls $= {}^9C_3$
    $$= \frac{9 \times 8 \times 7}{6} = 84$$
    If no black ball is included, the number of possible ways $= {}^6C_3$
    $$= \frac{6 \times 5 \times 4}{6} = 20$$
    Thus, the number of ways at least one ball is included in the selection = 84 – 20
    = 64
14. We have
    ${}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + \cdots + {}^{2n+1}C_n = 3$
    $\Rightarrow \quad {}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + {}^{2n+1}C_3 + \cdots + {}^{2n+1}C_n = 63 + {}^{2n+1}C_0$
    $= 64$
    $\Rightarrow \quad \frac{1}{2} \times 2^{2n+1} = 64$
    $\Rightarrow \quad 2^{2n} = 64 = 2^6$
    $\Rightarrow \quad 2n = 6$
    $\Rightarrow \quad n = 3$
15. We can arrange 6 '+' signs in just one way.
    Then we will have 7 gaps.
    In these 7 gaps, 4 '–' can be placed in ${}^7P_4$ ways.
    Thus, the total possible ways
    $$= \frac{6!}{6!} \times {}^7C_4 \times \frac{4!}{4!}$$
    $= 35$
16. As 0 + 1 + 2 + 3 + 4 + 5 = 15 to form a 5-digit number divisible by 3, we must leave either 0 or 3.
    When 0 is left out, the numbers are ${}^5P_5$
    When 3 is left out, then numbers are ${}^5P_5 - {}^4P_4$
    Also, 0 cannot be used at extreme left.
    Thus, the required number of ways
    $= {}^5P_5 + {}^5P_5 - {}^4P_4$
    $= 120 + 120 - 24$
    $= 240 - 24$
    $= 216$
17.
18. Note that 4 + 3 = 7 guests have already selected the sides where they wish to sit. We can choose 5 guests for the particular side in ${}^{11}C_5$ ways.
    Now the 9 guests can be arranged on the particular side in ${}^9P_9$ ways and also 9 guests on either sides in ${}^9P_9$ ways.
    Thus, the total number of possible ways
    $= {}^{11}C_5 \times 9! \times 9!$
19. Hence, the number of possible ways
    $$= 4! \times \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)$$
    $$= 4! \times \left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right)$$
    $$= 4! \times \left(\frac{12 - 4 + 1}{24}\right)$$
    $$= 24 \times \left(\frac{9}{24}\right)$$
    $= 9$
20.

Method	Women	Men	Number of ways
I	5	7	${}^9C_5 \times {}^8C_7 = 1008$
II	6	6	${}^9C_6 \times {}^8C_6 = 2352$
III	7	5	${}^9C_7 \times {}^8C_5 = 2016$
IV	8	4	${}^9C_8 \times {}^8C_4 = 630$
V	9	3	${}^9C_9 \times {}^8C_3 = 56$

    Thus, the women in majority in III, IV and V.
    So, the possible ways the women in majority
    = 2016 + 630 + 56
    = 2702
    Also, the men in majority in I
    Thus, the total possible ways = 1008.
21.
22.

23. We have $240 = 2^4 \times 3 \times 5$
Also, $4n + 2 = 2(2n + 1)$
As $(2n + 1)$ is odd, $(2n + 1) = 1, 3, 5, 15$.
Thus, there are 4 such divisors.

24. The number of n digit distinct numbers that can be formed by using digits 2, 5 and 7 is 3^n.
Therefore $3^n \geq 900$
$\Rightarrow 3^{n-2} \geq 100$
$\Rightarrow 3^{n-2} \geq 100 \geq 3^5$
$\Rightarrow n - 2 \geq 5$
$\Rightarrow n \geq 2 + 5 = 7$

25. Let the number of newspapers $= n$
Then $60 \times n = 300 \times 5$
$\Rightarrow n = 25$

26. Total number of possible ways
$= 4! \times {}^5P_2 = 24 \times 20 = 480$

27. Total number of possible ways
$$= \frac{4!}{2! \times 2!} \times \frac{5!}{3! \times 2!}$$
$= 6 \times 10$
$= 60$

28. $\binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2}$
$= {}^nC_r + {}^{2n}C_{r-1} + {}^nC_{r-2}$
$= ({}^nC_r + {}^nC_{r-1}) + ({}^nC_{r-1} + {}^nC_{r-2})$
$= ({}^{n+1}C_r + {}^{n+1}C_{r-1})$
$= {}^{n+2}C_r$
$$= \binom{n+2}{r}$$

29. Given,
$T_{n+1} - T_n = 21$
$\Rightarrow {}^{n+1}C_3 - {}^nC_3 = 21$
$$\Rightarrow \frac{(n+1)(n)(n-1)}{6} - \frac{n(n-1)(n-2)}{6} = 21$$
$\Rightarrow n(n^2 - 1) - n(n^2 - 3n + 2) = 126$
$\Rightarrow n^3 - n - n^3 + 3n^2 - 2n = 126$
$\Rightarrow 3n^2 - 3n = 126$
$\Rightarrow n^2 - n = 42$
$\Rightarrow n^2 - n - 42 = 0$
$\Rightarrow (n - 7)(n + 6) = 0$
$\Rightarrow n = 7, -6$
Hence, the value of n is 7.

30. The total number of possible ways
= Total – together
$$= \frac{6!}{3! \times 2!} - \frac{5!}{3!}$$
$$= \frac{5!}{2!} - \frac{5!}{3!}$$
$$= \frac{120}{2} - \frac{120}{6}$$
$= 60 - 20$
$= 40$

31. We have ${}^nC_{r+1} = (k^2 - 3)\, {}^{n-1}C_r$
$$\Rightarrow (k^2 - 3) = \frac{{}^nC_{r+1}}{{}^{n-1}C_r} = \frac{n}{r+1}$$
Here, $1 \leq r \leq n + 1$
$$\Rightarrow 0 < \frac{n}{n+1} \leq \frac{r}{n+1} \leq 1$$
Thus, $0 < k^2 - 3 \leq 1$
$\Rightarrow 3 < k^2 \leq 4$
$\Rightarrow \sqrt{3} < k \leq 2$

32. Number of possible rectangles is
$= [1 + 3 + 5 + \ldots + (2n - 1)] \times [1 + 3 + 5 + \ldots + (2n - 1)]$
$= m^2 n^2$

33. Required number of ordered pair (p, q) is
$= (2 \times 3 - 1)(2 \times 5 - 1)(2 \times 3 - 1) - 1$
$= 5 \times 9 \times 5 - 1$
$= 225 - 1$
$= 224$

34. COCHIN
The second place can be filled in ${}^4C_1$ ways and the remaining four alphabets can be arranged in 4! ways in four different places.
The next 97th word will be COCHIN.
Hence, there are 96 words before COCHIN.

35. (A) ENDEA, N, O E, L are five different letters
So, the different arrangement = 5!.

(B) If E is in the first and the last position, the number of permutations
$$= \frac{(9-2)!}{2!} = \frac{7!}{2!} = \frac{7 \times 6 \times 5!}{2} = 21 \times 5!$$

(C) The permutations of first 4 letters
$$= \frac{4!}{2!}$$
The permutations of last 5 letters
$$= \frac{5!}{3!}$$
Total permutations $= \frac{4!}{2!} \times \frac{5!}{3!}$
$= 2 \times 5!$

(D) For A, E and O, the permutations $= \frac{5!}{3!}$
and for others, the permutations $= \frac{4!}{2!}$
$\therefore$ Total permutations $= \frac{4!}{2!} \times \frac{5!}{3!}$
$= 2 \times 5!$

36. We find the co-efficient of x^{10} in the expansion of $(x + x^2 + x^3)^7$
= the co-efficient of x^3 in $(1 + x + x^2)^7$
= the co-efficient of x^3 in $(1 - x^3)^7(1 - x)^{-7}$

$= {}^{7+3-1}C_3 - 7$

$= {}^9C_3 - 7$

$= \frac{9 \times 8 \times 7}{6} - 7$

$= 84 - 7$

$= 77$

37. The total number of unordered pairs of disjoint subsets

$= \frac{3^4 + 1}{2}$

$= 41$

38. The total number of possible ways

= Total number of onto functions between two sets consists of 5 and 3 elements, respectively

= Total number of possible ways to distribute 5 balls into 3 boxes, where no box remains empty.

$= \frac{5!}{1! \times 1! \times 3!} \times \frac{3!}{2!} + \frac{5!}{1! \times 2! \times 2!} \times \frac{3!}{2!}$

$= \frac{120}{2} + \frac{120 \times 6}{8}$

$= 60 + 90$

$= 150$

39. (i) We have,

$a_n = b_n + c_n$,

$b_n = a_{n-1}$ and $c_n = a_{n-2}$

$\Rightarrow \quad a_n = a_{n-1} + a_{n-2}$

As $a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4, a_5 = 8$

Thus, $b_6 = a_5 = 8$

(ii) As $a_n = a_{n-1} + a_{n-2}$

Put $n = 17$, we get,

$a_{17} = a_{16} + a_{15}$.

40. Let (1, 1, 1), (1, –1, 1), (1, 1, –1) and (–1, –1, 1) be $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ vectors respectively and rest of the vectors are $-\vec{a}, -\vec{b}, -\vec{c}$ and $-\vec{d}$

Now, let us find the number of ways of selecting coplaner vectors.

Observe that out of any three coplaner vectors two are collinear (anti-parallel)

Number of ways selecting the anti-parallel pairs = 4

Number of ways of selecting the third vector = 6

Total = 24

Number of non-coplaner selections

$= {}^8C_3 - 24 = 56 - 24 = 32 = 2^5$

Therefore, $p = 5$.

41. Possible solutions are

1, 2, 3, 4, 10

1, 2, 3, 5, 9

1, 2, 3, 6, 8

1, 2, 4, 5, 8

1, 2, 4, 6, 7

1, 3, 4, 5, 7

2, 3, 4, 5, 6

Hence, 7 solutions are there.

42 Number of red lines = ${}^nC_2$

Number of blue lines = n

Hence, ${}^nC_2 - n = n$

$\Rightarrow \quad {}^nC_2 = 2n$

$\Rightarrow \quad \frac{n(n-1)}{2} = 2n$

$\Rightarrow \quad n^2 - n = 4n$

$\Rightarrow \quad n^2 = 5n$

$\Rightarrow \quad n = 0, 5$

Since $n \geq 2$, so $n = 5$.

CHAPTER

6

Binomial Theorem

CONCEPT BOOSTER

1. Definition

An algebraic expression which contains two and only two terms, is called a binomial expressions.

For examples, $a + b$, $2x + 3y$, $4p + 5q$, $x + \frac{1}{x}, \frac{3}{x} - \frac{x^2}{4}$, etc., are called binomial expressions.

In general, expressions containing more than two terms are known as multinomial expressions.

2. Factorial of a Natural Number, *n*.

It is a continued product of first n natural numbers. It is generally denoted as $n!$ or (n) and is defined as

$$\begin{aligned} n! &= 1.2.3 \ldots (n-1).n. \\ &= n(n-1)(n-2) \ldots 3.2.1 \\ &= n(n-1)! \\ &= n(n-1)(n-2)! \\ &= n(n-1)(n-2)(n-3)! \text{ and so on} \end{aligned}$$

(i) $(-n)!$ is not defined
(ii) $(0)! = 1$
(iii) $(1)! = 1$
(iv) $(2)! = 1$
(v) $(3)! = 6$
(vi) $(4)! = 24$
(vii) $(5)! = 120$
(viii) $(6)! = 720$
(ix) $(2n)! = 2^n \times (n!) \times \{1.3.5 \ldots (2n-1)\}$

3. Binomial Co-efficients

For $n \in N$, $r \in W$ and $r \leq n$, the expression ${}^nC_r$ is called a binomial co-efficient and it is defined as

$${}^nC_r = \frac{n!}{r! \times (n-r)!}.$$

(i) ${}^nC_r = {}^nC_{n-r}$
(ii) ${}^nC_0 = 1 = {}^nC_n$
(iii) ${}^nC_1 = n = {}^nC_{n-1}$
(iv) ${}^nC_2 = \frac{n(n-1)}{2} = {}^nC_{n-2}$
(v) ${}^nC_3 = \frac{n(n-1)(n-2)}{6} = {}^nC_{n-3}$

4. Binomial Theorem (for positive integral index)

For $a, b \in R$, $n \in I^+$, then

$$(a+b)^n = {}^nC_0 a^{n-0} b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + \cdots + {}^nC_r a^{n-r} b^r + \cdots + {}^nC_n a^{n-n} b^n,$$

where

${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ are called the binomial co-efficients.

This theorem can also be described in summarized form as

$$(a+b)^n = \sum_{r=0}^{n} {}^nC_r\, a^{n-r}\, b^r \qquad \ldots\text{(i)}$$

Some special cases:

1. Replacing b by $-b$ in Eq. (i), we get
$$(a-b)^n = \sum_{r=0}^{n} (-1)^r\, {}^nC_r\, a^{n-r}\, b^r \qquad \ldots\text{(ii)}$$
2. Replacing $a = 1$, $b = x$ in Eq. (ii), we get
$$(1+x)^n = \sum_{r=0}^{n} {}^nC_r\, x^r \qquad \ldots\text{(iii)}$$
3. Replacing x by $-x$ in Eq. (iii), we get
$$(1-x)^n = \sum_{r=0}^{n} (-1)^r\, {}^nC_r\, x^r$$
4. Adding Eqs (i) and (ii), we get
$$(a+b)^n + (a-b)^n = 2(a^n + {}^nC_2 a^{n-2} b^2 + {}^nC_4 a^{n-4} b^4 + \cdots)$$

5. Subtracting Eqs (i) and (ii), we get
$(a+b)^n - (a-b)^n$
$= 2({}^nC_1 a^{n-1} b^1 + {}^nC_3 a^{n-3} b^3 + {}^nC_5 a^{n-5} b^5 + \cdots)$

Notes

1. The number of terms in the expansion of $(a+b)^n$ is $(n+1)$.
2. The sum of the powers of a and b in each term in the expansion of $(a+b)^n$ is n.

4.1 Number of Terms in the Expansion of $(a+b)^n + (a-b)^n$

Case I: When n is even, the number of terms $= \left(\frac{n}{2}+1\right)$

Case II: When n is odd the number of terms $= \left(\frac{n+1}{2}\right)$

4.2 Number of Terms in the Expansion of $(a+b)^n + (a-b)^n$

Case I: When n is even, the number of terms $= \left(\frac{n}{2}\right)$

Case II: When n is odd, the number of terms $= \left(\frac{n+1}{2}\right)$

4.3 General Term in the Expansion of $(a+b)^n$

The $(r+1)$th term in the expansion of $(a+b)^n$ is known as the general term and is defined as

$$t_{r+1} = {}^nC_r\, a^{n-r}\, b^r$$

Notes

1. The general term in the expansion of $(1+x)^n$ is ${}^nC_r \cdot x^r$.
2. The general term in the expansion of $(1-x)^n$ is $(-1)^r\, {}^nC_r \cdot x^r$.
3. The general term in the expansion of $(a+x)^n$ is ${}^nC_r \cdot a^{n-r}\, x^r$.
4. The co-efficient of x^r in $(a+x)^n$ is ${}^nC_r \cdot a^{n-r}$.
5. The co-efficient of x^r in $(1+x)^n$ is ${}^nC_r$.
6. The co-efficient of x^r in $(1-x)^n$ is $(-1)^r\, {}^nC_r$.
7. The rth term from the end in the expansion of $(a+x)^n$ is
$= (n+1) - (r-1)$
$= (n-r+2)$th term from the end.

4.4 Middle Term in the Expansion of $(a+x)^n$.

Case I: When n is an even number

Let $n = 2m$.

Thus the middle term $= \left(\frac{2m}{2}+1\right)$th term
$= (m+1)$th term.
$= t_{m+1}$
$= {}^nC_m a^{n-m} x^m$
$= {}^nC_{n/2} a^{n/2} x^{n/2}$

Case II: When n is an odd number

Let $n = 2m+1$.

Thus, the middle terms

$= \left(\frac{2m+2}{2}\right)$th and $\left(\frac{2m+2}{2}+1\right)$th terms
$= (m+1)$th and $(m+2)$th terms.

Thus, $t_{m+1} = {}^nC_m\, a^{n-m} x^m = {}^nC_{\frac{n-1}{2}}\, a^{\frac{n+1}{2}}\, x^{\frac{n-1}{2}}$

and $t_{m+2} = {}^nC_{m+1}\, a^{n-m-1} x^{m+1} = {}^nC_{\frac{n+1}{2}}\, a^{\frac{n-1}{2}}\, x^{\frac{n+1}{2}}$

4.5 Greatest Co-efficient in the Expansion of $(a+x)^n$

The greatest co-efficient in the expansion of $(a+x)^n$ is the co-efficient of the middle term

If n is even, the greatest co-efficient $= {}^nC_{n/2} a^{n/2}$.

If n is odd, the greatest co-efficients are

$$= {}^nC_{(n-1)/2}\, a^{\left(\frac{n-1}{2}\right)} \text{ and } {}^nC_{(n+1)/2}\, a^{\left(\frac{n+1}{2}\right)}$$

4.6 Greatest Term in the Expansion of $(a+x)^n$, when n is a Positive Integer

Let t_r and t_{r+1} be the rth and $(r+1)$th terms respectively.

Thus, $t_r = {}^nC_{r-1} a^{n-r+1} x^{r-1}$

and $t_{r+1} = {}^nC_r a^{n-r} x^r$

Now, $$\left|\frac{t_{r+1}}{t_r}\right| = \left|\frac{{}^nC_r \cdot a^{n-r} \cdot x^r}{{}^nC_{r-1} \cdot a^{n-r+1} \cdot x^{r-1}}\right|$$

$$= \left|\frac{{}^nC_r}{{}^nC_{r-1}}\right| \times \left|\frac{x}{a}\right|$$

$$= \left|\left(\frac{n-r+1}{r}\right)\right| \times \left|\frac{x}{a}\right|$$

$$= \left|\left(\frac{n+1}{r}-1\right)\right| \times \left|\frac{x}{a}\right|$$

$$\Rightarrow \quad \left|\left(\frac{n+1}{r}-1\right)\right| \times \left|\frac{x}{a}\right| = \left|\frac{t_{r+1}}{t_r}\right| > 1$$

$$\Rightarrow \quad \left|\left(\frac{n+1}{r}-1\right)\right| \times \left|\frac{x}{a}\right| > 1$$

$$\Rightarrow \quad \left(\frac{n+1}{r}\right) > 1 + \left|\frac{a}{x}\right|$$

$$\Rightarrow \quad \left(\frac{r}{n+1}\right) < \frac{1}{\left(1+\left|\frac{a}{x}\right|\right)}$$

$$\Rightarrow \quad r < \frac{n+1}{\left(1+\left|\frac{a}{x}\right|\right)}$$

From this relation, the value of r is determined.

If r *is an integer*, say m, then m and $(m+1)$th terms are numerically the greatest terms of the given expansion and they are equal.

If *r is a fraction*, the next integer is numerically the greatest term.

Alternative method

To determine the greatest term in the expansion of $(a + x)^n$, we should use the following steps.

1. Calculate $m = \left|\dfrac{(n+1)|x|}{a+|x|}\right|$.
2. If $m \in I$, then t_m and t_{m+1} are numerically the greatest terms.
3. If $m \in I$, then $t_{[m]+1}$ is numerically the greatest term, where [,] = GIF.

4.7 Divisibility-related Problems in Binomial Theorem

Consider the expansion

$$(1+b)^n = (1 + {}^nC_1 b + {}^nC_2 b^2 + {}^nC_3 b^3 + \cdots + {}^nC_n b^n)$$

and so

$$(1+b)^n - 1 = ({}^nC_1 b + {}^nC_2 b^2 + {}^nC_3 b^3 + \cdots + {}^nC_n b^n)$$

$[(1 + b)^n - 1]$ is divisible by b

Also $[(1 + b)^n - bn - 1]$ is divisible by b^2 and so on.

Divisibility of $(a^n \pm b^n)$ by $(a \pm b)$, where $a, b, n \in N$

(i) *When n is even*, $(a^n - b^n)$ is divisible by $(a + b)$ and $(a - b)$

(ii) *When n is odd*, $(a^n - b^n)$ is divisible by $(a - b)$

(iii) *When n is odd*, $(a^n + b^n)$ is divisible by $(a + b)$.

4.8 Unit Digit of a Natural Number

The unit digit of a natural number depends on the period of that number.

We have, $2^1 = 2, 2^2 = 4, 2^3 = 8, 2^4 = 16$

Also, $2^5 = 32, 2^6 = 64, 2^7 = 128, 2^8 = 256$

Thus, the period of 2 is 4

Similarly, the period of 3 is 4 and also the period of 7 is 4.

For example,

(i) the unit digit of $2^{2013} = 2^{(4\times503+1)}$ is 2.

(ii) the unit digit of 2012^{2013} is same as 2^{2013}.

(iii) the unit digit of $3^{2014} = 3^{(4\times504+2)}$ is 9.

(iv) the unit digit of $(27)^{50} = 2^{150} = 3^{(4\times37+2)}$ is 9.

(v) The unit digit of 9^{9^9} is 9, since the period of 9 is 2.

4.9 Rational Terms in the Expansion of $(a^{1/p}, b^{1/q})^n$, where *a* and *b* are Prime Numbers, *p* and *q* are Integers and $n \in N$.

First of all, we find the $(r + 1)$th term in the expansion of $(a^{1/p} + b^{1/q})^n$.

Therefore, $t_{r+1} = {}^nC_r (a^{1/p})^{n-r} (b^{1/q})^r$

$$= {}^nC_r \times (a)^{\frac{n-r}{p}} (b)^{\frac{r}{q}}$$

Now, by inspection, putting the values of $0 \le r \le n$, when indices of a, b are integers.

4.10 Integral and Fractional part of a number

(i) If $(\sqrt{P} + Q)^n = I + f$, where I and n are natural numbers, n being odd and $0 \le f < 1$,
$(1 + f)f = k^n$, where $P - Q^2 = k$ and $\sqrt{Q} - P < 1$.

(ii) If $(\sqrt{P} + Q)^n = I + f$, where I and n are natural numbers, n being even and $0 \le f < 1$,
$(I + f)(1 - f) = k^n$, where $P - Q^2 = k$ and $\sqrt{Q} - P < 1$.

4.11 Properties of Binomial Co-efficients

Let ${}^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ are the binomial co-efficients in the expansion of $(1 + x)^n$.

P 1. ${}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n$

P 2. ${}^nC_0 + {}^nC_2 + {}^nC_4 + \cdots = 2^{n-1}$

P 3. ${}^nC_1 + {}^nC_3 + {}^nC_5 + \cdots = 2^{n-1}$

P 4. ${}^nC_r = {}^nC_{n-r}$

P 5. ${}^nC_x = {}^nC_y \Rightarrow x = y$ or $x + y = n$

P 6. $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \left(\dfrac{n-r+1}{r}\right)$

P 7. ${}^nC_r + {}^nC_{n-r} = {}^{n+1}C_r$

P 8. ${}^nC_r + {}^{n-1}C_r + {}^{n-2}C_r + {}^{n-3}C_r + \cdots + {}^rC_r = {}^{n+1}C_{r+1}$

P 9. Summation upto middle term

$$\begin{aligned}&{}^{2n}C_0 + {}^{2n}C_1 + {}^{2n}C_2 + {}^{2n}C_3 + \cdots + {}^{2n}C_n \\ &= 2^{2n-1} + \frac{1}{2}\,{}^{2n}C_n\end{aligned}$$

P 10. Summation of series

$$\begin{aligned}&{}^nC_3 + 2\,{}^{n+1}C_3 + 3\,{}^{n+2}C_3 + \ldots + n\,{}^{2n-1}C_3 \\ &= -\,{}^{2n}C_5 + {}^nC_5 + n\,{}^{2n}C_4\end{aligned}$$

P 11 ${}^nC_r = \dfrac{n}{r} \cdot {}^{n-1}C_{r-1} = \dfrac{n(n-1)}{r(r-1)} \cdot {}^{n-2}C_{r-2}$ and so on.

P 12. Sum of the series, when the sum of the lower suffices are the same

$$\begin{aligned}&{}^mC_r \cdot {}^nC_0 + {}^mC_{r-1} \cdot {}^nC_1 + {}^mC_{r-2} \cdot {}^nC_2 \\ &+ \cdots + {}^mC_m \cdot {}^nC_r = {}^{m+n}C_r\end{aligned}$$

P 13. Sum of the series, when the difference of the lower suffices are the same

$$\begin{aligned}&{}^nC_r \cdot {}^nC_n + {}^nC_{r-1} \cdot {}^nC_{n-1} + {}^nC_{r-2} \cdot {}^nC_{n-2} \\ &+ \cdots + {}^nC_0 \cdot {}^nC_{n-r} = {}^{2n}C_{n-r}\end{aligned}$$

5. Multinomial Theorem for Positive Integral Index

For $n \in I^+$,

$$\begin{aligned}&(a_1 + a_2 + a_3 + \cdots + a_m)^n \\ &= \sum_{k_1+k_2+k_3+\cdots+k_m=n} \binom{n}{k_1, k_2, k_3, \ldots, k_m} a_1^{k_1} a_2^{k_2} a_3^{k_3} \ldots a_m^{k_m}\end{aligned}$$

Proof

1. $(a_1+a_2)^2 = a_1^2 + 2a_1a_2 + a_2^2$

$$= {}^2C_0\, a_1^{2-0}\cdot a_2^0 + {}^2C_1\, a_1^{2-1}a_2^1 + {}^2C_2\, a_1^{2-2}a_2^2$$

$$= \sum_{k=0}^{2} {}^2C_k\, a_1^{2-k}\cdot a_2^k$$

$$= \sum_{k=0}^{2} \binom{2}{k} a_1^{2-k}\cdot a_2^k$$

$$= \sum_{k=0}^{2} \binom{2}{k_1, k_2} a_1^{k_1}\cdot a_2^{k_2},\; k_1+k_2=k$$

2. $(a_1+a_2)^3 = a^3 + 3a_1^2a_2 + 3a_1a_2^2 + a_2^3$

$$= {}^3C_0 a_1^{3-0}a_2^0 + {}^3C_1 a_1^{3-1}a_2^1 + {}^3C_2 a_1^{3-2}a_2^2 + {}^3C_3 a_1^{3-3}a_2^3$$

$$= \sum_{k=0}^{3} {}^3C_k\, a_1^{3-k}\cdot a_2^k$$

$$= \sum_{k=0}^{3} \binom{3}{k} a_1^{3-k}\cdot a_2^k$$

$$= \sum_{k=0}^{3} \binom{3}{k_1, k_2} \mathrm{a}_1^{k_1}\cdot a_2^{k_2},\; k_1+k_2=k$$

3. $(a_1+a_2+a_3)^2 = (a_1 + (a_2+a_3))^2$

$$= (a_1^2 + 2a_1(a_2+a_3) + (a_2+a_3)^2)$$

$$= \sum_{k=0}^{2} \binom{2}{k} a_1^{2-k}\cdot (a_2+a_3)^k$$

$$= \sum_{k_1+k_2=2} \binom{2}{k_1, k_2} a_1^{k_1}\cdot (a_2+a_3)^k$$

$$= \sum_{k_1+k_2=2} \binom{2}{k_1, k_2} a_1^{k_1}\cdot \sum_{r=0}^{k} a_2^{k-r}a_3^r$$

$$= \sum_{k_1+k_2+k_3=2} \binom{2}{k_1, k_2, k_3} a_1^{k_1}\cdot a_2^{k_2}a_3k^3$$

Thus,

$(a_1+a_2+a_3+\ldots+a_m)^n$

$$= \sum_{k_1+k_2+k_3+\cdots+k_m=n} \binom{n}{k_1, k_2, k_3, \ldots, k_m} a_1^{k_1}a_2^{k_2}a_3^{k_3}\ldots a_m^{k_m}.$$

(i) The co-efficient of $a_1^{k_1}a_2^{k_2}a_3^{k_3}\ldots a_m^{k_m}$ in the expansion of $(a_1+a_2+a_3+\ldots+a_m)^n$ is $\dfrac{n!}{k_1!k_2!\ldots k_m!}$.

(ii) The greatest co-efficient of $(a_1+a_2+a_3+\ldots+a_m)^n$ is $\dfrac{n!}{(q!)^{m-r}\times((q+1)!)^r}$,

where quotient = q, remainder = r, when n is divided by m.

(iii) The number of terms in the expansion of $(a_1+a_2+a_3+\ldots+a_m)^n$ is ${}^{n+m-1}C_{m-1}$.

(iv) If n is a positive integer and $a_1, a_2, \ldots, a_m \in C$, the co-efficient of x^r in the expansion of

$(a_1+a_2+a_3+\ldots+a_m)^n$ is

$$\sum\left(\frac{n!}{n_1!n_2!n_3!\ldots n_m!}\right)\times a_1^{n_1}a_2^{n_2}a_3^{n_3}\ldots a_m^{n_m}$$

where $n_1, n_2, \ldots, n_m$ are non-negative integers subject to the condition

$n_1 + n_2 + \ldots + n_m = n$ and $n_1 + 2n_2 + \ldots + mn_m = r$.

6. Binomial Theorem for any Index

Statement: Let $n \in R$ and x be a real number such that $|x| < 1$, then

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$
$$\cdots + \frac{n(n-1)(n-2)\ldots(n-r+1)}{n!}x^r + \cdots$$

Notes

1. If *n is a whole number* the condition $|x| < 1$ is unnecessary, while the same condition is essential *if n is a rational number*.
2. If *n is a negative integer* or a fraction, there are infinite number of terms.
3. If *n is a positive integer*, the expansion of $(1+x)^n$ contains $(n+1)$ terms and coincides with
$$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_nx^n.$$
4. If the *first term is not unity* and the index of the binomial is either a negative or a fraction, the expansion as follows.
$$(a+x)^n = \left\{a\left(1+\frac{x}{a}\right)\right\}^n$$
$$= a^n\left(1 + n\cdot\frac{x}{a} + \frac{n(n-1)}{2!}\left(\frac{x}{a}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{x}{a}\right)^3 + \cdots\right)$$
5. The general term in the expansion of $(1+x)^n$
$$= t_{r+1} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}\times x^r$$
6. The general term in the expansion of
$$(1-x)^n = t_{r+1} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}\times(-1)^r x^r$$
7. The general term in the expansion of $(1+x)^{-n}$
$$= t_{r+1} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}\times(-1)^r x^r.$$
8. The general term in the expansion of
$$(1-x)^{-n} = t_{r+1} = \frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}\times x^r$$

Important Expansions to Remember

(i) $(1+x)^{-1}=1-x+x^2-x^3+\cdots+(-1)^r x^r+\cdots$

(ii) $(1+x)^{-2}=1-2x+3x^2-4x^3+\cdots+(-1)^r(r+1)x^r+\cdots$

(iii) $(1+x)^{-3}=1-3x+6x^2-10x^3+\ldots+(-1)^r\frac{(r+1)(r+2)}{2}x^r+\cdots$

(iv) $(1-x)^{-1}=1+x+x^2+x^3+\ldots+x^r+\ldots$

(v) $(1-x)^{-2}=1+2x+3x^2+4x^3+\cdots+(r+1)x^r+\cdots$

(vi) $(1-x)^{-3}=1+3x+6x^2+10x^3+\ldots+\frac{(r+1)(r+2)}{2}x^r+\cdots$

3. If $n \in N$,

$$(1-x)^{-n}=1+{}^nC_1\cdot x+{}^{n+1}C_2\cdot x^2+{}^{n+2}C_3\cdot x^3+\cdots+{}^{n+r-1}C_r\cdot x^r+\cdots$$

Thus, the co-efficient of x^r in the expansion of $(1-x)^{-n}$ is ${}^{n+r-1}C_r$.

4. If $n \in Q$,

$$(1-x)^{-n}=1+nx+\frac{n(n+1)}{2!}x^2+\frac{n(n+1)(n+2)}{3!}x^3+\cdots$$

5. $(1-x)^3=1+3x+\frac{3.4}{1.2}x^2+\frac{3.4.5}{1.2.3}x^3+\frac{3.4.5.6}{1.2.3.4}x^4+\cdots$

6. $(1-x)^{-\frac{1}{2}}=1+\frac{1}{2}x+\frac{1.3}{2.4}x^2+\frac{1.3.5}{2.4.6}x^3+\frac{1.3.5.7}{2.4.6.8}x^4+\cdots$

7. Exponential Series

Leonhard Euler, the great Swiss Mathematician introduced and named the number e in his calculus text in 1748 AD.

Definition

The sum of the infinite series $1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\ldots$ to ∞ denoted by the number e is

1. $e=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$
2. e lies between 2 and 3.
3. $e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots$ to ∞
4. Let $a>0$, for all values of x,

$$a^x=1+x(\log_e a)+\frac{x^2}{2!}(\log_e a)^2+\frac{x^3}{3!}(\log_e a)+\cdots$$

Some Important Expansion to Remember

(i) $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots$ to ∞

(ii) $e^{-x}=1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\cdots$ to ∞

(iii) $\frac{e^x+e^{-x}}{2}=1+\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots$ to ∞

(iv) $\frac{e^x-e^{-x}}{2}=x+\frac{x^3}{3!}+\frac{x^5}{5!}+\frac{x^7}{7!}+\cdots$ to ∞

(v) $\frac{e+e^{-1}}{2}=1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots$

(vi) $\frac{e-e^{-1}}{2}=1+\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+\cdots$

(vii) $e=\sum_{n=0}^{\infty}\frac{1}{n!}=\sum_{n=0}^{\infty}\frac{1}{(n-1)!}=\sum_{n=0}^{\infty}\frac{1}{(n-2)!}$

$=\sum_{n=0}^{\infty}\frac{1}{(n-3)!}=\sum_{n=0}^{\infty}\frac{1}{(n-k)!}$

(viii) $\sum_{n=1}^{\infty}\frac{1}{n!}=1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$ to $\infty=e-1$

(ix) $\sum_{n=2}^{\infty}\frac{1}{n!}=\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$ to $\infty=e-2$

(x) $\sum_{n=0}^{\infty}\frac{1}{(n+1)!}=\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$ to $\infty=e-1$

(xi) $\sum_{n=0}^{\infty}\frac{1}{(n+2)!}=\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots$ to $\infty=e-2$

(xii) $\sum_{n=0}^{\infty}\frac{1}{2n!}=\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots$ to $\infty=\frac{e+e^{-1}}{2}$

(xiii) $\sum_{n=0}^{\infty}\frac{1}{(2n-1)!}=\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\cdots$ to $\infty=\frac{e-e^{-1}}{2}$

8. Logarithmic Series

We know that, if $ax=n \Leftrightarrow \log_a x=n$

Here, a is known as the base of the logarithms.

There are two types of logarithms.

(i) Naperian or Natural Logarithms, where the base is e.

(ii) Common Logarithms, where the base is 10.

Now, we shall obtain an expansion for $\log_e(1+x)$ as a series in powers of x which is valid only when $|x|<1$.

If $|x|<1$, then

$$\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$$

Some Important Expansion to Remember

(i) $\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$

(ii) $\log_e(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}+\cdots$

(iii) $\log\left(\frac{1+x}{1-x}\right)=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}+\cdots\right)$

(iv) $\log 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots$

(v) $\log(1-x^2)=-2\left(\frac{x^2}{2}+\frac{x^4}{4}+\frac{x^6}{6}+\cdots\right)$

EXERCISES

LEVEL I

(Questions based on Fundamentals)

ABC OF BINOMIAL EXPANSION

1. Write the expansion of $(x + 1)^6$.
2. Write the expansion of $\left(\frac{x}{5}+\frac{1}{x}\right)^5$.
3. Find the degree of the polynomial
$$(x+\sqrt{x^2-1})^6+(x-\sqrt{x^2-1})^6.$$
4. Find the number of terms in the expansion of
 (i) $(1 + x)^{2013}$
 (ii) $(1 + 2x + x^2)^{1007}$
 (iii) $(1 + 3x + 3x^2 + x^3)^{668}$
 (iv) $(1 + 4x + 6x^2 + 4x^3 + x^4)^{503}$
 (v) $(1 + x)(1 + x^2)^{10}$
 (vi) $(1 - x)^1(1 + x + x^2)^{10}$
5. Find the number of terms in the expansion of
$$(a + b)^{50} + (a - b)^{50}.$$
6. Find the number of terms in the expansion of $\left(x+\frac{1}{x}\right)^{99}+\left(x-\frac{1}{x}\right)^{99}$.
7. Find the number of terms in the expansion of $\left(x^2+\frac{1}{x^2}\right)^{20}-\left(x^2-\frac{1}{x^2}\right)^{20}$.
8. Find the number of terms in the expansion of $\left(\frac{2}{x}+x^3\right)^{2015}-\left(\frac{2}{x}-x^3\right)^{2015}$.

GENERAL TERM IN A BINOMIAL EXPANSION

9. Find the 13th term in the expansion of $\left(9x-\frac{1}{3\sqrt{x}}\right)^{18}$.
10. Find the 4th term from the end in the expansion of $\left(\frac{3}{x^2}-\frac{x^3}{6}\right)^7$.
11. Find the term independent of x in the expansion of $\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^6$.
12. Find the co-efficient of x^7 in the expansion of $\left(3x^2+\frac{1}{5x}\right)^{11}$.
13. Determine the value of x in the expansion of $(x+x^{\log_{10}x})^5$, where the 3rd term is 10,000.
14. Prove that the co-efficient of x^n in $(1 + x)^{2n}$ is twice the co-efficient of x^n in $(1 + x)^{2n-1}$.
15. If the co-efficients of 2nd, 3rd and 4th terms in the expansion of $(1 + x)^{2n}$ are in AP, prove that
$$2n^2 - 9n + 7 = 0$$
16. In the binomial expansion of $(1 + y)^n$, where n is a natural number, the co-efficients of the 5th, 6th and 7th terms are in AP. Find the value of n.
17. Let n be a positive integer. If the co-efficients of 2nd, 3rd and 4th terms in the expansion of $(1 + x)^n$ are in AP, find n.
18. Find the co-efficient of x^{10} in $(1 + x)^{50}$.
19. Find the co-efficient of x^{10} in $(1 + x^2)^{30}$.
20. Find the co-efficient of x^{50} in the expansion of $(1 + x)^{41}(1 - x + x^2)^{40}$.
21. Find the co-efficients of x, x^2, x^3 in the expansion of
$$(1 + x)^5 \times (1 - x)^6.$$
22. If in the expansion of $(1 + x)^m(1 - x)^n$, the co-efficient of x and x^2 are 3 and –6, respectively, find m and n.
23. Find the middle term in the expansion of $\left(\frac{2x^2}{3}-\frac{3}{2x}\right)^{12}$.
24. Find the middle term in the expansion of $\left(3x-\frac{x^3}{6}\right)^9$.
25. Prove that the middle term in the expansion of $(1 + x)^{2n}$ is $\frac{1.2.3\ldots(2n-1)}{(n)!}\times 2^n\times x^n$.
26. Prove that the co-efficients of the middle terms $(1 + x)^{2n}$ is equal to the sum of the co-efficients of the two middle terms in $(1 + x)^{2n-1}$.

GREATEST CO-EFFICIENTS/TERMS OF A BINOMIAL EXP

27. Find the greatest co-efficient of the polynomial $\left(\frac{1}{4}+\frac{2}{3}x\right)^{100}$.
28. If the sum of the co-efficients in the expansion of $(a + b)^n$ is 4096, then find the greatest co-efficient in the given expansion.
29. If the sum of the co-efficients in the expansion of $(1 + 2x)^n$ is 6561, find the greatest co-efficient in the given expansion.
30. Find the greatest term in the expansion of $(2 + 3x)^{10}$, when $x = 3/5$.
31. Find the greatest term in the expansion of $(3 - 5x)^{15}$, when $x = 1/5$.

DIVISIBILITY RELATED PROBLEMS

32. If n be a positive integer, prove that $(4^n - 3n - 1)$ is divisible by 9.
33. If n be a positive integer, prove that $3^{2n+2} - 8n - 9$ is divisible by 64.
34. Prove that $11^{n+2} + 12^{2n+1}$ is divisible by 133.

35. Prove that $6^{n+2} + 7^{2n+1}$ is divisible by 43 for all n.
36. Prove that $11^{10} - 1$ is divisible by 100.
37. Find the remainder when 7^{98} is divided by 5.
38. Prove that $1^{2013} + 2^{2013} + 3^{2013} + \ldots + 2011^{2013} + 2012^{2013}$ is divisible by 2013.
39. Prove that $1992^{1998} - 1955^{1998} - 1938^{1998} + 1901^{1998}$ is divisible by 1998.
40. Prove that $53^{53} - 33^3$ is divisible by 10.

UNIT DIGIT OF A NATURAL NUMBER

41. Find the unit digit of $(27)^{50} + (18)^{50}$.
42. Find the unit digit of $1! + 2! + 3! + 4! + 5! + \ldots + (33)!$.
43. Find the last two digits of $(27)^{27}$.
44. Find the last two digits of 3^{999}.
45. Find the last two digit of $(17)^{10}$.

RATIONAL TERM IN A BINOMIAL EXPANSION

46. Find the number of integral terms in the expansion of $(9^{1/4} + 8^{1/6})^{500}$.
47. Find the number of irrational terms in the expansion of $(3^{1/5} + 2^{1/3})^{1000}$.
48. Find the number of non-integral terms in the expansion of $(5^{1/8} + 2^{1/6})^{100}$.
49. Find the sum of rational terms in the expansion of $(2^{1/2} + 3^{1/5})^{10}$

INTEGRAL AND FRACTIONAL PART OF A NUMBER

50 If $(\sqrt{P} + Q)^n = I + f$, where I and n are natural numbers, n being odd and $0 \le f < 1$, show that
$$(I + f)f = k^n, \text{ where } P - Q^2 = k \text{ and } \sqrt{Q} - P < 1.$$
51 If $(\sqrt{P} + Q)^n = I + f$, where I and n are natural numbers, n being even and $0 \le f < 1$, show that
$$(I + f)(I - f)f = k^n, \text{ where } P - Q^2 = k \text{ and } \sqrt{Q} - P < 1.$$
52. Prove that the integral part of $(5\sqrt{5} + 11)^{2n+1}$ is even when $n \in N$ and hence if $R = (5\sqrt{5} + 11)^{2n+1}$ and $f = R - [R]$, where [,] = GIF, prove that $Rf = 4^{2n+1}$.
53. If $x = (8 + 3\sqrt{7})^n, n \in N$, prove that the integral part of x is an odd integer and show that $x - x^2 + x[x] = 1$, where [,] = GIF.
54. If $(8 + 3\sqrt{7})^n = \alpha + \beta$, where n and α are positive integers and β is a positive proper fraction, prove that $(1 - \beta)(\alpha + \beta) = 1$.
55. Find the integer just greater than the number $(3 + \sqrt{5})^5$.

PROPERTIES OF BINOMIAL CO-EFFICIENTS

56. Prove that the co-efficients of a^n and a^m in the expansion of $(1 + a)^{m+n}$ are equal.
57. If ${}^{10}C_r = {}^{10}C_{r+4}$, find the value of r.
58. Prove that
$$\frac{C_1}{C_0} + 2 \cdot \frac{C_2}{C_1} + 3 \cdot \frac{C_3}{C_2} + \cdots + n \cdot \frac{C_n}{C_{n-1}} = \frac{n(n+1)}{2}.$$
59. Find the value of
$$\frac{{}^{15}C_1}{{}^{15}C_0} + 2 \cdot \frac{{}^{15}C_2}{{}^{15}C_1} + 3 \cdot \frac{{}^{15}C_3}{{}^{15}C_2} + \cdots + 15 \cdot \frac{{}^{15}C_{15}}{{}^{15}C_{14}}.$$
60. Find the value of
$$\left(1 + \frac{C_1}{C_0}\right)\left(1 + \frac{C_2}{C_1}\right)\left(1 + \frac{C_3}{C_2}\right)\ldots\left(1 + \frac{C_n}{C_{n-1}}\right)$$
61. Find the value of
$$({}^nC_0 + {}^nC_1)\,({}^nC_1 + {}^nC_2)\,({}^nC_2 + {}^nC_3)\ldots({}^nC_n + {}^nC_{n-1}),$$
if ${}^nC_0 \cdot {}^nC_1 \cdot {}^nC_2 \ldots {}^nC_{n-1} = k$
62. Prove that ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$.
63. Evaluate $\binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2}$.
64. Evaluate: $\binom{n}{r} + 3\binom{n}{r-1} + 3\binom{n}{r-2} + \binom{n}{r-3}$.
65. Evaluate:
$$\binom{n}{r} + 4\binom{n}{r-1} + 6\binom{n}{r-2} + 4\binom{n}{r-3} + \binom{n}{r-4}.$$
66. Evaluate: ${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$.
67 Prove that
$${}^mC_m + {}^{m+1}C_m + {}^{m+2}C_m + \ldots + {}^nC_m = {}^{n+1}C_{m+1}.$$
68. Find the value of
$${}^{20}C_{20} + {}^{21}C_{20} + {}^{22}C_{20} + {}^{23}C_{20} + \ldots + {}^{2014}C_{20}.$$
69. Find the value of
$${}^{48}C_{47} + {}^{49}C_{47} + {}^{50}C_{47} + {}^{51}C_{47} + \ldots + {}^{1005}C_{47}.$$
70. Prove that
$${}^nC_r + {}^{n-1}C_r + {}^{n-2}C_r + {}^{n-3}C_r + \ldots + {}^rC_r = {}^{n+1}C_{r+1}.$$
71. Find the value of
$${}^{49}C_{10} + {}^{48}C_{10} + {}^{47}C_{10} + \ldots + {}^{10}C_{10}.$$
72. Find the value of
$${}^{2014}C_{100} + {}^{2013}C_{100} + {}^{2012}C_{100} + \ldots + {}^{100}C_{100}.$$
73. Find the value of
$${}^{70}C_{20} + {}^{69}C_{20} + {}^{68}C_{20} + {}^{67}C_{20} + \ldots + {}^{21}C_{20}.$$
74. Prove that
$${}^{2n}C_0 + {}^{2n}C_1 + {}^{2n}C_2 + {}^{2n}C_3 + \ldots + {}^{2n}C_n = 2^{2n-1} + \frac{1}{2}\,{}^{2n}C_n.$$
75. Find the sum of ${}^{30}C_0 + {}^{30}C_1 + {}^{30}C_2 + \ldots + {}^{30}C_{15}$.
76. Find the sum of ${}^{40}C_{20} + {}^{40}C_{21} + {}^{40}C_{22} + \ldots + {}^{40}C_{40}$.
77. Prove that
$${}^nC_3 + 2 \cdot {}^{n+1}C_3 + 3 \cdot {}^{n+2}C_3 + \ldots + n \cdot {}^{2n-2}C_3 = -{}^{2n}C_5 + {}^nC_5 + n \cdot {}^{2n}C_4.$$
78. Find the sum of
$${}^{100}C_{50} + 2 \cdot {}^{99}C_{49} + 3 \cdot {}^{98}C_{48} + \ldots + 51 \cdot {}^{50}C_0$$
79. Find the co-efficient of x^{50} in the expansion of $(1 + x)^{1000} + 2x(1 + x)^{999} + 3x^2(1 + x)^{998} + \ldots + 1001x^{1000}$.

80. Find the sum of $C_1 + 2\cdot C_2 + 3\cdot C_3 + \ldots + n\cdot C_n$.
81. Find the sum of $C_0 + 2\cdot C_1 + 3\cdot C_2 + \ldots + (n+1)\cdot C_n$.
82. Find the sum of $^2\cdot C_1 + 2^2\cdot C_2 + 3^2\cdot C_3 + \ldots + n^2\cdot C_n$.
83. Find the sum of

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \cdots + \frac{C_n}{n+1}.$$

84. Find the sum of

$$\frac{C_0}{1.2} + \frac{C_1}{2.3} + \frac{C_2}{3.4} + \cdots + \frac{C_n}{(n+1)(n+2)}.$$

85. Prove that

$$^mC_r\cdot{}^nC_0 + {}^mC_{r-1}\cdot{}^nC_1 + {}^mC_{r-2}\cdot{}^nC_2 + \ldots + {}^mC_m\cdot{}^nC_r = {}^{m+n}C_r$$

86. Find the sum of

$$C_0\cdot C_n + C_1\cdot C_{n-1} + C_2\cdot C_{n-2} + \ldots + C_n\cdot C_0.$$

87. Prove that

$$^nC_r\cdot{}^nC_n + {}^nC_{r-1}\cdot{}^nC_{n-1} + {}^nC_{r-2}\cdot{}^nC_{n-2} + \ldots + {}^nC_0\cdot{}^nC_{n-r} = {}^{2n}C_{n-r}$$

88. Find the sum of

$$({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 + \ldots + ({}^nC_n)^2 = {}^{2n}C_n.$$

MULTINOMIAL THEOREM

89. Find the sum of

$$C_0\cdot C_1 + C_1\cdot C_2 + C_2\cdot C_3 + \ldots + C_{n-1}\cdot C_n.$$

90 Find the co-efficient of $a^5b^4c^2$ in the expansion of $(a + b + c + d)^{15}$.
91. Find the co-efficient of $a^3b^4c^7$ in the expansion of $(ab + bc + ca)^8$.
92. Find the greatest co-efficient of $(a + b + c + d + e)^{53}$.
93. Find the number of terms in the expansion of
   (i) $(a + b + c)^n$
   (ii) $(a + b + c + d)^n$
   (iii) $(a + b + c + d + e)^n$.
94. Find the co-efficient of x^7 in the expansion of $(1 + 3x - 2x^3)^{10}$.

BINOMIAL THEOREM FOR ANY INDEX

95. Find the co-efficient of x^n in the expansion of $\dfrac{1}{1-3x}$.
96. Find the co-efficient of x^n in the expansion of $\dfrac{1}{1+4x}$.
97. Find the co-efficient of x^n in the expansion of $\dfrac{1}{1-9x+20x^2}$.
98. Find the co-efficient of x^n in the expansion of $\dfrac{1}{1-(a+b)x+abx^2}$.
99. Find the co-efficient of x^n in the expansion of $\left(\dfrac{1+x}{1-x}\right)$.
100. Find the co-efficient of x^n in the expansion of $\left(\dfrac{1+x}{1-x}\right)^2$.
101. Find the co-efficient of x^n in the expansion of $(1 - 2x)^{-1/2}$.
102. Find the first negative term in the expansion of $\left(1+\dfrac{3}{4}x\right)^{13/3}$, where $0 < x < 4/3$.
103. If $y = x - x^2 + x^3 - x^4 + \ldots$, where $|x| < 1$, prove that $x = \dfrac{y}{1-y}$.
104. If $y = 2x + 3x^2 + 4x^3 + \ldots$, where $|x| < 1$, prove that $x = 1 - \dfrac{1}{\sqrt{1+y}}$.
105 Find the sum of the series

$$1 + \frac{1}{3} + \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \cdots$$

106. If x be so small that its square and higher powers may be neglected, find the value of

$$\frac{(1+2x)^{1/2}(16+3x)^{1/4}}{(1-x)^2} \text{ at } x = \frac{1}{2}.$$

107 If for small values of x, $\dfrac{(1+2x)^{1/2}+(16+3x)^{1/4}}{(1-2x)^{1/4}}$ is very nearly equal to $a + bx$, find the values of a and b.
108. If x be a quantity so small that x^3 may be neglected in comparison of a^3, prove that

$$\sqrt{\frac{a}{a+x}} + \sqrt{\frac{a}{a-x}} = 2 + \frac{3x^2}{4a^2}.$$

109 If p be nearly equal to q, prove that

$$\left(\frac{5p+4q}{4p+5q}\right) = \left(\frac{p}{q}\right)^{1/9}.$$

110 If x be nearly equal to 1, prove that

$$\frac{mx^m - nx^n}{m-n} = x^{m+n}.$$

EXPONENTIAL SERIES

111. Find the co-efficient of x^n in the expansion of e^{5x}.
112. Find the co-efficient of x^n in the expansion of e^{5x+4}.
113. Find the co-efficient of x^n in $\left(\dfrac{1+3x+2x^2}{e^x}\right)$.
114. If $\dfrac{e^x}{1-x} = B_0 + B_1x + B_2x^2 + \cdots + B_nx^n$, prove that $B_n - B_{n-1} = \dfrac{1}{n!}$.
115. Find the value of

$$\frac{\left(1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots\right)}{\left(1+\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+\cdots\right)}.$$

116. Find the value of $(x + 1)^{x+1} + 10$, if
$$x=\left(1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots\right)^2-\left(1+\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+\cdots\right)^2.$$

117. If $a=\sum_{n=1}^{\infty}\left(\frac{2n}{(2n-1)!}\right), b=\sum_{n=1}^{\infty}\left(\frac{2n}{(2n+1)!}\right)$, find the value of ab.

118. Find the sum of
$$1+\frac{1+2}{1!}+\frac{1+2+3}{2!}+\frac{1+2+3+4}{3!}+\cdots.$$

119. Find the value of $\sum_{n=4}^{\infty}\left(\frac{C(n,4)}{P(n,n)}\right)=\frac{e}{24}$.

120. If $a=\sum_{n=0}^{\infty}\left(\frac{x^{3n}}{(3n)!}\right), b=\sum_{n=0}^{\infty}\left(\frac{x^{3n-2}}{(3n-2)!}\right)$,
and $c=\sum_{n=0}^{\infty}\left(\frac{x^{3n-1}}{(3n-1)!}\right)$,
prove that $a^3 + b^3 + c^3 - 3abc = 1$.

LOGARITHMIC SERIES

121. Prove that $\log_e 2 < 1 < \log_e 3$.

122. Find the value of
$$\frac{1}{4}-\frac{1}{2}\left(\frac{1}{4}\right)^2+\frac{1}{3}\left(\frac{1}{4}\right)^3-\frac{1}{4}\left(\frac{1}{4}\right)^4+\cdots.$$

123. Prove that
$$\log_e\left(\frac{x+1}{x}\right)=2\left(\frac{1}{(2x+1)}+\frac{1}{3(2x+1)^3}+\frac{1}{5(2x+1)^5}+\cdots\right).$$

124. Find the co-efficient of x^n in the expansion of $\log_e(1 + 6x + 8x^2)$.

125. If $y=x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots$,
prove that $x=y-\frac{y^2}{2!}+\frac{y^3}{3!}-\frac{y^4}{4!}+\cdots$.

126. Prove that
$$\log_e(1+x+x^2+x^3+x^4+\ldots)=\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}+\cdots\right).$$

127. Find the sum of
$$\frac{1}{2.3}+\frac{1}{4.5}+\frac{1}{6.7}+\cdots.$$

128. Find the sum of
$$\frac{1}{1.2.3}+\frac{1}{3.4.5}+\frac{1}{5.6.7}+\cdots.$$

129. Find the sum of
$$1+\left(\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4}+\left(\frac{1}{4}+\frac{1}{5}\right)\frac{1}{4^2}+\left(\frac{1}{6}+\frac{1}{7}\right)\frac{1}{4^3}+\cdots.$$

130. Find the sum of
$$1+\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)+\left(\frac{3-2\sqrt{2}}{12}\right)+\left(\frac{5\sqrt{2}-7}{24\sqrt{2}}\right)+\left(\frac{17-12\sqrt{2}}{80}\right)+\cdots$$

Level-II

(Mixed Problems)

1. The first three terms in the expansion of $(1 + ax)^n$, $(n \neq 0)$ are 1, $6x$, $16x^2$, respectively. The value of a and n are, respectively
(a) 2 and 9 (b) 3 and 2
(c) 2/3 and 9 (d) 3/2 and 6

2. The last term in the expansion of $\left(2^{1/3}-\frac{1}{2^{1/2}}\right)^n$ is $\left(\frac{1}{3\sqrt[3]{9}}\right)^{\log_3 8}$. The 5th term from the beginning is
(a) ${}^{10}C_6$ (b) $2 \cdot {}^{10}C_4$
(c) $\frac{1}{2} \cdot {}^{10}C_4$ (d) none

3. If the 6th term in the expansion of $\left(\frac{1}{x^{8/3}}+x^{2\log_{10}x}\right)^8$ is 5600, then $x =$
(a) 8 (b) 9 (c) 10 (d) none

4. In the expansion of $\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^9$, the term independent of x is
(a) ${}^9C_3\times\frac{1}{6^3}$ (b) ${}^9C_3\times\left(\frac{3}{2}\right)^3$
(c) ${}^9C_3$ (d) none

5. In the expansion of $\left(x+\frac{1}{x}\right)^{2n}$, where $n \in N$, the term independent of x is
(a) ${}^{2n}C_n$ (b) $\frac{(2n)!}{(n)!}$
(c) $\frac{(2n)!}{(n)!}\times 2^n$ (d) $\frac{(n)!}{(2n)!}$

6. The co-efficient of x^7 in the expansion of $\left(\frac{x^2}{2}-\frac{2}{x}\right)^8$ is
(a) –56 (b) 56 (c) –14 (d) 14

7. If the co-efficient of $(2r + 4)$ and $(r - 2)$th terms in the expansion of $(1 + x)^8$ are equal, then r is
(a) 12 (b) 10 (c) 8 (d) 6

8. If the co-efficients of t_rth, t_{r+1}th, t_{r+2}th terms of $(1 + x)^{14}$ are in AP, then r is
(a) 6 (b) 7 (c) 8 (d) 9

9. The co-efficient of x^3 in $\left(\sqrt{x^5}+\frac{3}{\sqrt{x^3}}\right)^6$ is
(a) 0 (b) 120 (c) 420 (d) 540

10. In the expansion of $(1 + ax)^n$, $n \in N$, the co-efficient of x and x^2 are 8 and 24, respectively, then
(a) $a = 2, n = 4$ (b) $a = 4, n = 2$
(c) $a = 2, n = 2$ (d) $a = -2, n = 4$.

11. The number of terms in the expansion of $(a + b + c)^n$ will be
(a) $n + 1$ (b) $n + 3$
(c) $\frac{(n+1)(n+2)}{2}$ (d) none

12. The expression
$$(x+\sqrt{x^3-1})^5+(x-\sqrt{x^3-1})^5$$
is a polynomial of degree
(a) 5 (b) 6 (c) 7 (d) 8

13. The number of terms whose values depend on x in the expansion of $\left(x^2-2+\frac{1}{x^2}\right)^n$ is
(a) $2n + 1$ (b) $2n$ (c) n (d) None

14. The number of distinct terms in the expansion of $(x + 2y - 3z + 5w - 7u)^n$ is
(a) $n + 1$ (b) ${}^{n+4}C_4$
(c) ${}^{n+4}C_n$ (d) $\frac{(n+1)(n+2)(n+3)(n+4)}{24}$

15. In how many terms in the expansion of $(x^{1/5} + y^{1/10})^{55}$ do not have fractional powers of the variable?
(a) 6 (b) 7 (c) 8 (d) 10

16. The middle term in the expansion of $\left(\frac{2x}{3}-\frac{3}{2x^2}\right)^{2n}$ is
(a) ${}^{2n}C_n$ (b) $(-1)^n \times {}^{2n}C_n \cdot x^n$
(c) ${}^{2n}C_n \cdot \frac{1}{x^n}$ (d) none

17. If the sum of the co-efficients in the expansion of $(x + y)^n$ is 1024, the value of the greatest co-efficient in the expansion is
(a) 356 (b) 252 (c) 210 (d) 120

18. The numerically greatest term of $(2 + 3x)^9$, when $x = 3/2$ is
(a) t_6 (b) t_7 (c) t_8 (d) none

19. $\binom{n}{0}+2\cdot\binom{n}{1}+2^2\cdot\binom{n}{2}+\cdots+2^n\cdot\binom{n}{n}=$
(a) 2^n (b) 0 (c) 3^n (d) none

20. $C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \ldots + C_{n-r}C_n =$
(a) $\frac{(2n)!}{(n-r)!\times(n+r)!}$ (b) $\frac{(n)!}{(n-r)!\times(n+r)!}$
(c) $\frac{(n)!}{(n-r)!}$ (d) None

21. The value of
$$\frac{C_1}{C_0}+2\cdot\frac{C_2}{C_1}+3\cdot\frac{C_3}{C_2}+\cdots+15\cdot\frac{C_{15}}{C_{14}} \text{ is}$$
(a) 100 (b) 120 (c) –120 (d) None

22. The sum of
$$C_0^2-C_1^2+C_2^2-C_3^2+\cdots+(-1)^n C_n^2 \text{ is}$$
(a) ${}^{2n}C_n$ (b) $(-1)^n\, {}^{2n}C_n$
(c) ${}^{2n}C_{n-1}$ (d) none

23. The sum of the co-efficients of all the integral powers of the expansion of $(1+2\sqrt{x})^{40}$ is
(a) $3^{40} + 1$ (b) $3^{40} - 1$
(c) $\frac{1}{2}(3^{40}-1)$ (d) $\frac{1}{2}(3^{40}+1)$

24. The sum of all the co-efficients in the binomial expansion of $(x^2 + x - 3)^{319}$ is
(a) 1 (b) 2 (c) –1 (d) 0

25. The sum of the co-efficients in the expansion of $(1 + x - 3x^2)^{2148}$ is
(a) 7 (b) 8 (c) –1 (d) 1

26. $\frac{1}{\sqrt{5+4x}}$ can be expanded by the binomial theorem, if
(a) $x < 1$ (b) $|x| < 1$
(c) $|x|<\frac{5}{4}$ (d) $|x|<\frac{4}{5}$

27. The co-efficient of x^4 in $\frac{1+2x+3x^2}{(1-x)^2}$ is
(a) 13 (b) 15 (c) 20 (d) 22

28. If n is a positive integer and three consecutive co-efficients in the expansion of $(1 + x)^n$ are in the ratio 6 : 33 : 110, then n is
(a) 4 (b) 6 (c) 12 (d) 16

29. The co-efficient of x^3 in the expansion of $(1 - x + x^2)^5$ is
(a) 10 (b) –20 (c) –50 (d) –30

30. The co-efficient of $a^8b^6c^4$ in the expansion of $(a + b + c)^{18}$ is
(a) ${}^{18}C_{14} \times {}^{14}C_{18}$ (b) ${}^{18}C_{10} \times {}^{10}C_6$
(c) ${}^{18}C_6 \times {}^{12}C_8$ (d) ${}^{18}C_4 \times {}^{14}C_6$

31. The co-efficient of x^3y^4z in the expansion of $(1 + x + y - z)^9$ is
(a) $2 \times {}^9C_7 \times {}^7C_4$ (b) $2 \times {}^9C_2 \times {}^7C_3$
(c) ${}^9C_7 \times {}^7C_4$ (d) none

32. The number of integral terms in the expansion of $(5^{1/2} + 7^{1/6})^{642}$ is
(a) 106 (b) 108 (c) 103 (d) 109

33. The sum of the rational terms in the expansion of $(2^{1/2} + 3^{1/5})^{10}$ is
(a) 42 (b) 9 (c) 41 (d) None

34. If $R=(\sqrt{2}+1)^{2n+1}$ and $f = R - [R]$, where [,] = GIF, then $[R]$ is
(a) $f+\frac{1}{f}$ (b) $f-\frac{1}{f}$
(c) $\frac{1}{f}-f$ (d) none

35. If $(5+2\sqrt{6})^n = I + f$, where $n \in N$ and $0 < f < 1$, then 1 equals
(a) $\frac{1}{f} - f$ (b) $\frac{1}{1+f} - f$
(c) $\frac{1}{1-f} - f$ (d) $\frac{1}{1-f} + f$
36. If $R = (7+4\sqrt{3})^n = I + f$, where $I \in N$ and $0 < f < 1$, then $R(1-f)$ equals
(a) $(7-4\sqrt{3})^n$ (b) $\frac{1}{(7+4\sqrt{3})}$
(c) 1 (d) none
37. The number of non-negative integral solutions of $x + y + 3z = 33$ is
(a) 120 (b) 135 (c) 210 (d) 520
38. The co-efficient of x^6 in the expansion of $(1 + x - x^2)^5$ is
(a) 80 (b) 84 (c) 88 (d) 92
39. If the 2nd term in the expansion of $(a^{1/13} + a^{3/2})^n$ is $14a^{5/2}$, the value of ${}^nC_3/{}^nC_2$ is
(a) 4 (b) 3 (c) 12 (d) 6
40. The co-efficient of $x^8y^6z^4$ in the expansion of $(x + y + z)$ is
(a) ${}^{18}C_{14} \times {}^{14}C_8$ (b) ${}^{18}C_{10} \times {}^{10}C_6$
(c) ${}^{18}C_6 \times {}^{12}C_8$ (d) none
41. The greatest value of the term independent of x in the expansion of $\left(x\sin\alpha + \frac{\cos\alpha}{x}\right)^{10}, \alpha \in R$ is
(a) 2^5 (b) $\frac{10!}{(5!)^2}$
(c) $\frac{10!}{2^5 \times (5!)^2}$ (d) none
42. If the 4th term in the expansion of $\left(\sqrt{x\left(\frac{1}{\log x + 1}\right)} + x^{1/12}\right)^6$ is equal to 200 and $x > 1$, then x is
(a) $10^{2^{1/2}}$ (b) 10
(c) 10^4 (d) none
43. Let $(1+x)^n = \sum_{r=0}^{n} C_r x^r$ such that
$$\frac{C_1}{C_0} + 2\cdot\frac{C_2}{C_1} + 3\cdot\frac{C_3}{C_2} + \cdots + n\cdot\frac{C_n}{C_{n-1}} = \frac{n(n+1)}{k},$$
the value of k is
(a) 1/2 (b) 2 (c) 1/3 (d) 3
44. Let $(1+x)^n = \sum_{r=0}^{n} C_r x^r$ and $k = \sum_{r=0}^{n} \frac{C_r}{r+1}$, the value of k is
(a) $\frac{2^{n+1}+1}{n+1}$ (b) $\frac{2^{n+1}-1}{n+1}$
(c) $\frac{2^n+1}{n+1}$ (d) $\frac{2^n-1}{n+1}$
45. The co-efficient of x^{50} in the expression $(1 + x)^{1000} + 2x(1 + x)^{999} + 3x^2(1 + x)^{998} + \ldots + 1001x^{1000}$ is
(a) ${}^{1000}C_{50}$ (b) ${}^{1001}C_{50}$ (c) ${}^{1002}C_{50}$ (d) ${}^{1000}C_{51}$
46. The co-efficient of x^{53} in the expansion of $\sum_{m=0}^{100} {}^{100}C_{50}(x-3)^{100-m}2^m$ is
(a) ${}^{1000}C_{47}$ (b) ${}^{1000}C_{53}$ (c) $-{}^{1000}C_{53}$ (d) $-{}^{1000}C_{100}$
47. The number of rational terms in the expansion of $(4^{1/3} + 6^{-1/4})^{20}$ is
(a) 3 (b) 4 (c) 4 (d) 16
48. The sum of ${}^rC_r + {}^{r+1}C_r + {}^{r+2}C_r + \ldots + {}^nC_r$ $(n \ge r)$ is
(a) ${}^nC_{r+1}$ (b) ${}^{n+1}C_{r+1}$ (c) ${}^{n+1}C_{r-1}$ (d) ${}^{n+1}C_{2r}$
49. The unit digit of $17^{1983} + 11^{1983} - 7^{1983}$ is
(a) 1 (b) 2 (c) 3 (d) 0
50. If $(1 + 2x + 3x^2)^{10} = a_0 + a_1x + a_2x^2 + \ldots + a_{20}x^{20}$, the value of a_1 is
(a) 10 (b) 20 (c) 210 (d) 420

Level III

(Problems for JEE-Advanced)

1. Find the co-efficient of x^4 in the expansion of $(1 + x + x^2 + x^3)^{10}$.
2. Find the co-efficient of x^4 in the expansion of $(1 + x - 2x^2)^7$.
3. Find the co-efficient of x^n in the expansion of $(1 - 2x + 3x^2 - 4x^3 + \ldots)^{-n}$.
4. Let $f(n) = \sum_{r=0}^{n} (r \cdot {}^{2n}C_{2r})$, find the value of $f(6)$.
5. Let $f(n) = \sum_{r=0}^{n}\sum_{k=r}^{n}\binom{k}{r}$, find the total number of divisors of $f(11)$.
6. Determine the value of x in the expansion of $(x + x^{\log_{10} x})^5$, where the 3rd term is 10,000.
7. Let n be a positive integer. If the co-efficient of 2nd, 3rd and 4th terms in the expansion of $(1 + x)^n$ are in AP, find n.
8. If in the expansion of $(1 + x)^m(1 - x)^n$, the co-efficients of x and x^2 are 3 and –6 respectively, find m and n.
9. Find the co-efficient of x^{10} in the expansion of $(1 + x)^{21} + (1 + x)^{22} + (1 + x)^{23} + \ldots + (1 + x)^{50}$.
10. Find the co-efficient of x^r in the expansion of
$$(x + 3)^{n-1} + (x + 3)^{n-1}(x + 2) + (x + 3)^{n-2}(x + 2)^2 + \ldots + (x + 2)^{n-1}$$

11. Find the co-efficient of t^{24} in
$$(1 + t^2)^{12} (1 + t^{12})(1 + t^{24}).$$

12 Find the co-efficient of x^{50} in the expansion of $(1+x)^{1000} + 2x(1+x)^{999} + 3x^2(1+x)^{998} + \ldots + 1001x^{1000}$.

13. Find the co-efficient of x^n in
$$(1 + x + 2x^2 + 3x^3 + \ldots + nx^n)^2.$$

14. Find the co-efficient of x^n in the expansion of
$$(1 + 2x + 3x^2 + 4x^3 + \ldots + nx^n)^2.$$

15. Find the co-efficient of
$$x^n \text{ in } \left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}\right)^2.$$

16. Find the number of different dissimilar terms in the sum of $(1 + x)^{2012} + (1 + x^2)^{2011} + (1 + x^3)^{2010}$.

17. Find the number of divisors of the number
$$N = {}^{2000}C_1 + 2 \cdot {}^{2000}C_2 + 3 \cdot {}^{2000}C_3 + \cdots + 2000 \cdot {}^{2000}C_{2000}.$$

18. Find the sum of the series
$$\sum_{r=0}^{n} (-1)^r \cdot {}^nC_r \left(\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\frac{15^r}{2^{4r}}+\cdots \text{ to } m \text{ terms}\right)$$

19. If n be a positive integer and $C_k = {}^nC_k$, find the value of
$$\sum_{k=1}^{n} k^3 \left(\frac{C_k}{C_{k-1}}\right)^2.$$

20. The co-efficients of three consecutive terms in the expansion of $(1 + x)^n$ are in the ratio 1 : 7 : 2, find the value of n.

21. Find the sum of the series
$$\left(\frac{{}^nC_1}{{}^nC_0}\right)^2+\left(\frac{2\cdot {}^nC_2}{{}^nC_1}\right)^2+\left(\frac{3\cdot {}^nC_3}{{}^nC_2}\right)^2+\cdots \text{ to } n \text{ terms.}$$

22. Prove that
$$1^{2017} + 2^{2017} + 3^{2017} + 4^{2017} + \ldots + 2016^{2017}$$
is divisible by 2017.

23. Let $f(x) = 1 - x + x^2 - x^3 + \ldots - x^{17}$
$$= a_0 + a_1(1 + x) + a_2(1 + x)^2 + \ldots + a_{17}(1 + x)^{17},$$
find the value of a_2.

24. Given $S_n = 1 + q + q^2 + \ldots + q^n$
and $D_n = 1+\left(\frac{q+1}{2}\right)+\left(\frac{q+1}{2}\right)^2+\cdots+\left(\frac{q+1}{2}\right)^n$,
prove that
$${}^{n+1}C_1 + {}^{n+1}C_2S_1 + {}^{n+1}C_3S_3 + \cdots + {}^{n+1}C_{n+1}S_n = 2^nD^n.$$

25. Let $a = (4^{1/401} - 1)$ and
$$b_n = {}^nC_1 + {}^nC_2 \cdot a + {}^nC_3 \cdot a^2 + \ldots + {}^nC_n \cdot a^{n-1}.$$
Find the value of $(b_{2006} - b_{2005})$.

26. Let $a = 3^{\frac{1}{323}} + 1$ and let
$$f(n) = {}^nC_0 \cdot a^{n-1} - {}^nC_1 \cdot a^{n-2} + {}^nC_2 \cdot a^{n-3} - {}^nC_4 a^{n-4} + \cdots + (-1)^{n-1}\, {}^nC_{n-1}$$
such that $f(2007) + f(2008) = 9\,m$, $m \in N$, find the value of m.

27. Prove that
$$\sum_{k=0}^{n} {}^nC_k \cdot \sin(kx) \cdot \cos(n-k)x = 2^{n-1}\sin(nx).$$

28. If $a_1, a_2, a_3, \ldots, a_{n+1}$ be in AP, prove that
$$\sum_{k=0}^{n} {}^nC_k \cdot a_{k+1} = 2^{n-1}(a_1 + a_{n+1}).$$

29. Prove that
$$C_0C_n - C_1C_{n-1} + C_2C_{n-2} - \cdots + (-1)^n C_nC_0 = \begin{cases} 0: & n \text{ is odd} \\ \dfrac{(-1)^{n/2}(n)!}{(n/2)! \times (n/2)!}: & n \text{ is even} \end{cases}$$

30. Find the sum of the series
$$\sum_{r=0}^{n}\left(\frac{n-3r+1}{n-r+1}\right)\left(\frac{{}^nC_r}{2^r}\right).$$

31. If $(1 + 2x + 3x^2)^{10} = a_0 + a_1x + a_2x^2 + \ldots + a_{20}x^{20}$, find the value of $(a_1 + a_2 + a_3 + a_4 + 5)$.

32. If the value of
$$\left(\frac{1}{1}+\frac{1}{2016}\right)\cdot\left(\frac{1}{2}+\frac{1}{2015}\right)\cdot\left(\frac{1}{3}+\frac{1}{2014}\right)\left(\frac{1}{4}+\frac{1}{2013}\right)\cdots\left(\frac{1}{1008}+\frac{1}{1009}\right) \text{ be } \frac{m^p}{(n)!},$$
where $m, n, p \in N$, find the value of $(m - n + p + 2)$.

33. If the co-efficient of x^n in the expansion of $\left(1+\sum_{k=0}^{n} k\cdot x^k\right)^2$ is 310, find the value of n.

34. Let $F(n) = {}^{n+1}C_2 + 2({}^nC_2 + {}^{n-1}C_2 + {}^{n-2}C_2 + \ldots + {}^2C_2)$, find the value of $[F(11) + 4]$.

35. Find the sum of the series
$$C_1^2+\left(\frac{1+2}{2}\right)C_2^2+\left(\frac{1+2+3}{3}\right)C_3^2+\cdots \text{ to } n \text{ terms.}$$

36. Let n be a positive integer and
$(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \ldots + a_{2n}x^{2n}$,
prove that
$$a_0^2 - a_1^2 + a_2^2 - a_3^2 + \cdots + a_{2n}^2 = a_n.$$

39. If $(1+x)^n = \sum_{r=0}^{n} {}^nC_r\, x^r$ and $\sum_{r=0}^{n}\left(\frac{1}{{}^nC_r}\right) = a$, find the value of $\sum_{0 \le i < j \le n}\sum\left(\frac{1}{{}^nC_i}+\frac{1}{{}^nC_j}\right)$ in terms of a and n.

39. If $(x + 1)(x + 2)(x + 3)\ldots(x + (n - 1))(x + n)$
$$= A_0 + A_1x + A_2x^2 + A_3x^3 + \ldots + A_nx^n,$$
prove that

(i) $A_1 + 2 \cdot A_2 + 3 \cdot A_3 + \ldots + n \cdot A_n$
$$= (n+1)! \times \left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n+1}\right)$$

(ii) $A_1 = (n)! \times \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right)$

40. If $(1 + x)^n = 1 + A_1x + A_2x^2 + \ldots + A_nx^n$ and $(1 + x)^{n+1} = 1 + B_1x + B_2x^2 + \ldots + B_nx^n$, prove that $B_r = A_r + A_{r-1}$.

41. If $a = \sum_{r=0}^{n}\left(\frac{1}{^nC_r}\right)$, find the sum of

$$\sum_{0 \le i < j \le n}\sum\left(\frac{1}{^nC_i} + \frac{1}{^nC_j}\right).$$

42. If $(1+x)^n = \sum_{r=0}^{n} {^nC_r}\, x^r$, find the value of $\sum_{i=0}^{n}\sum_{j=0}^{n}(C_i + C_j)$.

43. If $(1+x)^n = \sum_{r=0}^{n} {^nC_r}\, x^r$, find the value of $\sum_{0 \le i < j \le n}^{n}\sum(C_i + C_j)$.

44. If $(1+x)^n = \sum_{r=0}^{n} {^nC_r}\, x^r$, find the value of $\sum_{i=0}^{n}\sum_{j=0}^{n}(C_iC_j)$.

45. If $(1+x)^n = \sum_{r=0}^{n} {^nC_r}\, x^r$, find the value of $\sum_{0 \le i < j \le n}^{\text{n}}\sum(C_iC_j)$.

46. If $(1+x)^n = \sum_{r=0}^{n} {^nC_r}\, x^r$, find the value of $\sum_{0 \le i < j \le n}^{n}\sum((i \times j)C_iC_j)$.

47. Let $F(n) = \left(\sum_{i=0}^{n}\sum_{j=1}^{n}({^nC_j} \cdot {^jC_i})\right)$, where $i \le j$, find the value of $F(10)$.

48. Find the value of $\sum_{i=0}^{n}\sum_{j=0}^{n}\sum_{m=0}^{n}\sum_{p=0}^{n}(1)$.

49. If n is any positive integer, prove that

$$^nC_0^2 + {^nC_1^2} + {^nC_2^2} + \cdots + {^nC_n^2} = \frac{(2n)!}{(n)! \times (n)!}.$$

[Roorkee-JEE, 1983]

50. In the binomial expansion of $(1 + y)^n$, where n is a natural number, the co-efficients of the 5th, 6th and 7th terms are in AP. Find the value of n.

[Roorkee-JEE, 1984]

51. Find the terms in the expansion of $\left(\frac{x^{1/8}}{2} + x^{-1/8}\right)^8$ which does not contain x.

[Roorkee-JEE, 1985]

52. Find the co-efficient of x^n in the series

$$1 + \frac{(a+bx)}{1!} + \frac{(a+bx)^2}{2!} + \frac{(a+bx)^2}{3!} + \cdots$$

[Roorkee-JEE, 1985]

53. If A be the sum of the odd terms and B be the sum of even terms in the expansion of $(x + a)^n$, prove that $A^2 - B^2 = (x^2 - a^2)^n$ **[Roorkee-JEE, 1986]**

53. Find the sum of the co-efficients in the expansion of the binomial $(5p - 4q)^n$, where n is a positive integer.

[Roorkee-JEE, 1987]

54 Find the sum of

$$3 \cdot {^nC_0} - 8 \cdot {^nC_1} + 13 \cdot {^nC_2} - 18 \cdot {^nC_3} + \ldots \text{ upto } (n + 1) \text{ terms}$$

[Roorkee-JEE, 1988]

Note *No questions asked in 1989*

55. Find the co-efficient of x^{50} in the expansion of $(1 + x)^{1000} + 2x(1 + x)^{999} + 3x^2(1 + x)^{998} + \ldots + 1001x^{1000}$.

[Roorkee-JEE, 1990]

56. If n be a positive integer and $C_k = {^nC_k}$, find the value of $\sum_{k=1}^{n} k^3\left(\frac{C_k}{C_{k-1}}\right)^2$. **[Roorkee-JEE, 1991]**

57. Determine the value of x in the expansion of $(x + x^{\log_{10}x})^5$, where the 3rd term is 10,000.

[Roorkee-JEE, 1992]

58. Find the value of the expression

$$^{47}C_4 + \sum_{j=1}^{5} {^{47-j}C_3}.$$ **[Roorkee-JEE, 1993]**

59. Find the value of x for which the 6th term in the expansion of $(\sqrt{2^{\log(10-3^x)}} + \sqrt[5]{2^{(x-2)\log 3}})^m$ is equal to 21 and the co-efficients of 2nd, 3rd and 4th terms are the 1st, 3rd and 5th terms, respectively of an AP.

[Roorkee-JEE, 1993]

60. Given that the 4th term in the expansion of $\left(2 + \frac{3x}{8}\right)^{10}$ has the maximum numerical values, find the range of values of x for which this will be true.

[Roorkee-JEE, 1994]

Note *No questions asked in 1995.*

61. Let $(1 + x^2)^2(1 + x)^n = \sum_{k=0}^{n+4} a_k x^k$. If a_1, a_2 and a_3 be in AP, find n. **[Roorkee-JEE, 1996]**

62. In the expansion of $(x + a)^{15}$, if the 11th term is the GM of the 8th and 12th terms, which term in the expansion is the greatest. **[Roorkee-JEE, 1997]**

63. Find the largest co-efficient in the expansion of $(1 + x)^n$, given that the sum of the terms in the expansion is 4096. **[Roorkee-JEE, 2000]**

64. Find the co-efficient of x^{49} in the polynomial $\left(x-\frac{C_1}{C_0}\right)\left(x-2^2\frac{C_2}{C_1}\right)\left(x-3^2\frac{C_3}{C_2}\right)\cdots\left(x-50^2\frac{C_{50}}{C_{49}}\right)$, where $C_r = {}^{50}C_r$. **[Roorkee-JEE, 2001]**
65. Find the sum of
$$\frac{C_0}{1.2.3}+\frac{C_1}{2.3.4}+\frac{C_2}{3.4.5}+\cdots.$$
66. Find the sum of the series
$$1+\frac{1}{3}+\frac{1.3}{3.6}+\frac{1.3.5}{3.6.9}+\cdots.$$
67. If $\frac{e^x}{1-x}=B_0+B_1x+B_2x^2+\cdots+B_nx^n$, prove that $B_n-B_{n-1}=\frac{1}{n!}$
68. If $a=\sum_{n=1}^{\infty}\left(\frac{2n}{(2n-1)!}\right)$ and $b=\sum_{n=1}^{\infty}\left(\frac{2n}{(2n+1)!}\right)$, find the value of ab.
69. Find the sum of
$$1+\frac{1+2}{1!}+\frac{1+2+3}{2!}+\frac{1+2+3+4}{3!}+\cdots.$$
70. Find the value of $\sum_{n=4}^{\infty}\left(\frac{C(n,4)}{P(n,n)}\right)=\frac{e}{24}$.
71. If $a=\sum_{n=0}^{\infty}\left(\frac{x^{3n}}{(3n)!}\right), b=\sum_{n=0}^{\infty}\left(\frac{x^{3n-2}}{(3n-2)!}\right)$, and
$c=\sum_{n=0}^{\infty}\left(\frac{x^{3n-1}}{(3n-1)!}\right)$,
prove that $a^3+b^3+c^3-3abc=1$.
72. Find the sum of
$$\frac{1}{1.2.3}+\frac{1}{3.4.5}+\frac{1}{5.6.7}+\cdots.$$
73. Find the sum of
$$1+\left(\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4}+\left(\frac{1}{4}+\frac{1}{5}\right)\frac{1}{4^2}+\left(\frac{1}{6}+\frac{1}{7}\right)\frac{1}{4^3}+\cdots.$$
74. Find the sum of
$$1+\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)+\left(\frac{3-2\sqrt{2}}{12}\right)+\left(\frac{5\sqrt{2}-7}{24\sqrt{2}}\right)+\left(\frac{17-12\sqrt{2}}{80}\right)+\cdots.$$
75. Find the sum of
$$\frac{5}{1.2.3}+\frac{7}{2.3.4}+\frac{9}{3.4.5}+\cdots.$$
76. Find the sum of the series
$$1+\frac{2}{3}\cdot\frac{1}{2}+\frac{2}{3}\cdot\frac{5}{6}\cdot\frac{1}{2^2}+\frac{2}{3}\cdot\frac{5}{6}\cdot\frac{8}{9}\cdot\frac{1}{2^2}+\cdots.$$

Level IV

(Tougher Problems for JEE-Advanced)

1. If $(1+x+x^2)^n=\sum_{r=0}^{2n}a_rx^r$, prove that
$a_0+a_3+a_6+\ldots=a_1+a_4+a_7+\ldots$
$a_2+a_5+a_8+\ldots=3^{n-1}$
2. If $(1+x+x^2)^n=\sum_{r=0}^{2n}a_rx^r$, prove that
$$a_1+a_2+\cdots+a_{n-1}=\frac{1}{2}(3^n-a_n),$$
3. If $(1+x+x^2)^n=\sum_{r=0}^{2n}a_rx^r$, prove that
$a_0a_1-a_2a_3+a_4a_5-\ldots=0$.
4. Find the co-efficient of x^{50} in the expression
$(1+x)^{1000}+2x(1+x)^{999}+3x^2(1+x)^{998}+\ldots+1001x^{1000}$.
5. Let $(1+x^2)^2(1+x)^n=\sum_{k=0}^{n+4}a_kx^k$. If a_1, a_2, a_3 be in AP, find the value of n.
6. Find the co-efficient of x^{18} in the expansion of $(1+x)(1+x+x^2)\ldots(1+x+x^2+\ldots+x^{2n})$.
7. In the expansion of
$(1+x)(1+x+x^2)\ldots(1+x+x^2+\ldots+x^{2n})$,
prove that the sum of the co-efficients is $(2n+1)!$.
8. Prove that the value of
${}^nC_0+{}^{n+1}C_1+{}^{n+2}C_2+\ldots+{}^{n+k}C_k={}^{n+k+1}C_{n+1}$.
9. If $(1+x+2x^2)^{20}=a_0+a_1x+a_2x^2+\ldots+a_{40}x^{40}$, prove that $a_1+a_3+a_5+\ldots+a_{37}=2^{19}(2^{20}-21)$.
10. Prove that the co-efficient of x^n in the expansion of $\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}\right)^2$ is $\frac{2^n}{n!}$.
11. If $(1+x)^n=C_0+C_1x+C_2x^2+\ldots+C_nx^n$, prove that
$$\sum_{0\le i<j\le n}\sum(C_i-C_j)^2=(n+1)^{2n}C_n-2^{2n}.$$
12. If $(1+x)^n=C_0+C_1x+C_2x^2+\ldots+C_nx^n$, prove that
$$\sum_{0\le i<j\le n}\sum(i+j)C_iC_j=n\left(2^{n-1}-\frac{1}{2}{}^{2n}C_n\right).$$
13. If $(1+x)^n=C_0+C_1x+C_2x^2+\ldots+C_nx^n$, prove that
$$\sum_{0\le i<j\le n}\sum{}^nC_j\,{}^jC_i=3^n-1, i\le j$$
14. If $\sum_{r=0}^{2n}a_r(x-2)^r=\sum_{r=0}^{2n}b_r(x-3)^r$ and $a_k=1$, for all $k\ge n$, show that $b_n={}^{2n+1}C_{n+1}$.

15. Prove that the sum of $\sum_{r=0}^{m}\binom{10}{i}\binom{20}{m-i}$, where $\binom{p}{q}=0$, if $p>q$, is maximum when m is 15.
16. If ${}^nC_{r+1}=(k^2-3)^{n-1}C_r$, find k.
17. For $r = 0, 1, 2,..., 10$, let A_r, B_r and C_r denote respectively, the co-efficients of x^r in the expansion of $(1+x)^{10}$, $(1+x)^{20}$ and $(1+x)^{30}$. Prove that $\sum_{r=1}^{10}A_r(B_{10}B_r-C_{10}A_r)$ $=C_{10}-B_{10}$.
18. Find the remainder when $32^{32^{32}}$ is divided by 7.
19. Find the co-efficient of x^{49} in the polynomial $\left(x-\frac{C_1}{C_0}\right)\left(x-2^2\frac{C_2}{C_1}\right)\left(x-3^2\frac{C_3}{C_2}\right)\ldots\left(x-50^2\frac{C_{50}}{C_{49}}\right)$, where $C_r={}^{50}C_r$
20. Find the value of ${}^{100}C_{10}+5\cdot{}^{100}C_{11}+10\cdot{}^{100}C_{12}+10\cdot{}^{100}C_{13}+5\cdot{}^{100}C_{14}+{}^{100}C_{15}$
21. If $\sum_{r=0}^{n}(r^2+r+1)r!=2016\times 2016!$, find the value of n.
22. If the last digit of the number 9^{9^9} is m and the last digit of $2^{m^{100}}$ is n, find the value of $(n-m+2006)$.
23. Find the sum of $\sum_{r=0}^{n}\left(\frac{n-3r+1}{n-r+1}\right)\cdot\frac{{}^nC_r}{2}$.
24. If $\left(\sum_{r=0}^{2n}x^r\right)^n=\sum_{r=0}^{2n}a_rx^r$, prove that
$$\sum_{r=0}^{2n}\left(\frac{r}{a_r}\right)=n\sum_{r=0}^{2n}\left(\frac{1}{a_r}\right).$$
25. Find the sum of the roots (real or complex) of the equation $x^{2017}+\left(\frac{1}{2}-x\right)^{2017}=0$
26. If m be the number of terms in $(1-x)(1+x)^{20}$ and n the number of terms in $(1+x)^2(1-x)^{10}$, find the value of $m+n-5$.
27. Find the value of $\sum_{p=1}^{n}\left(\sum_{m=p}^{n}{}^nC_m\times{}^mC_p\right)$.
28. If $S_n={}^nC_0\cdot{}^nC_1+{}^nC_1\cdot{}^nC_2+{}^nC_2\cdot{}^nC_3+\ldots+{}^nC_{n-1}\cdot{}^nC_n$ such that $\frac{S_{n+1}}{S_n}=\frac{15}{4}$, find the value of n.
29. Find the sum of $3\cdot{}^nC_0-8\cdot{}^nC_1+13\cdot{}^nC_2$ $-18{}^nC_3+\ldots$up to$(n+1)$terms.
30. Find the value of $\sum_{r=0}^{n}\left(\frac{1}{(r+1)(r+2)}{}^nC_r\right)$.

EXPONENTIAL SERIES

31. Prove that $\sum_{n=2}^{\infty}\left(\frac{C(n,2)}{(n+1)!}\right)=\frac{e}{2}-1$.
32. Prove that
$$\sum_{n=1}^{\infty}\left(\frac{C(n,0)+C(n,0)+C(n,0)+\cdots+C(n,0)}{P(n,n)}\right)$$
$$=e^2-1$$
33. If $a=\sum_{n=0}^{\infty}\left(\frac{x^{3n}}{(3n)!}\right)$, $b=\sum_{n=0}^{\infty}\left(\frac{x^{3n-2}}{(3n-2)!}\right)$, and $c=\sum_{n=0}^{\infty}\left(\frac{x^{3n-1}}{(3n-1)!}\right)$, prove that $a^3+b^3+c^3-3abc=1$
34. Prove that
$$\frac{1}{1.2}+\frac{1.3}{1.2.3.4}+\frac{1.3.5}{1.2.3.4.5.6}+\cdots=\sqrt{e}-1.$$
35. If $a_n=\sum_{k=0}^{n}\left(\frac{2^k}{(k)!\times(n-k)!}\right)$, prove that $\sum_{r=0}^{\infty}a_r=e^3$.
36. Prove that $\sum_{n=4}^{\infty}\left(\frac{C(n,4)}{P(n,n)}\right)=\frac{e}{24}$.
37. Find the sum of
$$\frac{1}{3!}+\frac{2}{5!}+\frac{3}{7!}+\frac{4}{9!}+\cdots.$$
38. Find the sum of
$$\frac{1}{2!}+\frac{3}{4!}+\frac{5}{6!}+\frac{7}{8!}+\cdots.$$
39. Find the value of
$$C(n,2)\times\frac{3^{n-1}}{(n)!}.$$
40. Find the sum of
$$1^2+\frac{3^2}{1!}+\frac{5^2}{3!}+\frac{7^2}{5!}+\cdots.$$

LOGARITHMIC SERIES

41. Find the sum of
$$\frac{1}{2.3}+\frac{1}{4.5}+\frac{1}{6.7}+\cdots.$$
42. Find the sum of
$$\frac{1}{1.3}+\frac{1}{2.5}+\frac{1}{3.7}+\frac{1}{4.9}+\cdots.$$

43. Find the sum of
$$\frac{1}{1.2.3}+\frac{5}{3.4.5}+\frac{9}{5.6.7}+\cdots.$$
44. Find the sum of
$$\frac{5}{1.2.3}+\frac{7}{3.4.5}+\frac{9}{5.6.7}+\cdots.$$
45. Find the sum of
$$\frac{1}{1.2.3}+\frac{1}{3.4.5}+\frac{1}{5.6.7}+\cdots.$$
46. Find the sum of
$$1+\left(\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4}+\left(\frac{1}{4}+\frac{1}{5}\right)\frac{1}{4^2}+\left(\frac{1}{6}+\frac{1}{7}\right)\frac{1}{4^3}+\cdots.$$
47. Find the sum of
$$\frac{2}{1}\cdot\frac{1}{3}+\frac{3}{2}\cdot\frac{1}{9}+\frac{4}{3}\cdot\frac{1}{27}+\frac{5}{4}\cdot\frac{1}{81}+\cdots.$$
48. Find the sum of
$$2\left(1+\frac{(\log_e n)^2}{2!}+\frac{(\log_e n)^4}{4!}+\frac{(\log_e n)^6}{6!}+\cdots\right).$$
49. For $-1 < x < 1$, let
$$\log_e(1+x+x^2+x^3) = a_0 + a_1x + a_2x_2 + a_3x_3 + \ldots$$
Prove that
$$a_0 + a_1 + a_2 + a_3 + \ldots = \log_e 4.$$
50. Find the sum of
$$1+\left(\frac{\sqrt{2}-1}{2\sqrt{2}}\right)+\left(\frac{3-2\sqrt{2}}{12}\right)+\left(\frac{5\sqrt{2}-7}{24\sqrt{2}}\right)+\left(\frac{17-12\sqrt{2}}{80}\right)+\cdots$$

Linked Comprehension Type

Passage I

If n is a positive integer and if $(1+4x+4x^2)^n = \sum_{r=0}^{2n} a_r x^r$, where $a_i \in R$ and $i = 0, 1, 2, \ldots, 2n$

On the basis of the above information, answer the following questions.

1. The value of $2\sum_{r=0}^{n} a_{2r}$ is
   (a) $9^n - 1$ (b) $9^n + 1$ (c) $9^n - 2$ (d) $9^n + 2$
2. The value of $2\sum_{r=0}^{n} a_{2r-1}$ is
   (a) $9^n - 1$ (b) $9^n + 1$ (c) $9^n - 2$ (d) $9^n + 2$
3. The value of a_{2n-1} is
   (a) 2^{2n} (b) $(n-1)2^{2n}$
   (c) $n \cdot 2^{2n}$ (d) $(n+1)\cdot 2^{2n}$
4. The value of a_2 is
   (a) $8n$ (b) $8n^2 - 4$ (c) $8n^2 - 4n$ (d) $8n - 4$
5. The correct statement is
   (a) $a_r = a_{n-r}$, $0 \le r \le n$ (b) $a_{n+r} = a_{n-r}$, $0 \le r \le n$
   (c) $a_r = a_{2n-r}$, $0 \le r \le 2n$ (d) none

Passage II

Let $(1+x+x^2)^n = \sum_{r=0}^{n} a_r x^r$,

where $a_i \in R$,

$I = 0, 1, 2, \ldots, 2n$ and $n \in I^+$.

On the basis of the above information, answer the following questions.

1. The value of $\sum_{r=0}^{n-1} a_{2r}$ is
   (a) $\left(\frac{9^n - 2a_{2n} - 1}{4}\right)$ (b) $\left(\frac{9^n + 2a_{2n} + 1}{4}\right)$
   (c) $\left(\frac{9^n - 2a_{2n} + 1}{4}\right)$ (d) $\left(\frac{9^n + 2a_{2n} - 1}{4}\right)$
2. The value of $\sum_{r=1}^{n} a_{2r-1}$ is
   (a) $\left(\frac{9^n - 1}{2}\right)$ (b) $\left(\frac{3^{2n} - 1}{4}\right)$
   (c) $\left(\frac{3^{2n} + 1}{4}\right)$ (d) $\left(\frac{9^n + 1}{2}\right)$
3. The value of a_2 is
   (a) ${}^{4n+1}C_2$ (b) ${}^{3n+1}C_2$ (c) ${}^{2n+1}C_2$ (d) ${}^{n+1}C_2$
4. The value of a_{4n-1} is
   (a) $2n$ (b) $2n^2 + 4n$
   (c) $2n + 3$ (d) $2n^2 + 3n$
5. The correct statement is
   (a) $a_r = a_{n-r}$, $0 \le r \le n$ (b) $a_{n+r} = a_{n-r}$, $0 \le r \le n$
   (c) $a_r = a_{2n-r}$, $0 \le r \le 2n$ (d) $a_r = a_{4n-r}$, $0 \le r \le 4n$

Passage III

If m, n, r are positive integers and if $r < m$, $r < n$, then

$${}^mC_r \cdot {}^nC_0 + {}^mC_{r-1} \cdot {}^nC_1 + {}^mC_{r-2} \cdot {}^nC_2 + \ldots + {}^mC_0 \cdot {}^nC_r$$
$$= \text{Co-efficient of } x^r \text{ in } (1+x)^m \cdot (1+x)^n$$
$$= \text{Co-efficient of } x^r \text{ in } (1+x)^{m+n}$$
$$= {}^{m+n}C_r$$

1. The value of
   ${}^nC_0 \cdot {}^nC_n + {}^nC_1 \cdot {}^nC_{n-1} + {}^nC_2 \cdot {}^nC_{n-2} + \ldots + {}^nC_n \cdot {}^nC_0$ is
   (a) ${}^{2n}C_{n-1}$ (b) ${}^{2n}C_n$ (c) ${}^{2n}C_{n+1}$ (d) ${}^{2n}C_2$
2. The value of r, for which
   ${}^{30}C_r \cdot {}^{20}C_0 + {}^{30}C_{r-1} \cdot {}^{20}C_1 + \ldots + {}^{30}C_0 \cdot {}^{20}C_r$
   is maximum, is
   (a) 10 (b) 15 (c) 20 (d) 25
3. The value of $r (0 \le r \le 20)$ for which
   ${}^{20}C_r \cdot {}^{10}C_0 + {}^{20}C_{r-1} \cdot {}^{10}C_1 + \ldots + {}^{20}C_0 \cdot {}^{10}C_r$
   is minimum, is
   (a) 0 (b) 1 (c) 5 (d) 15

4. If $S_n = {}^nC_0 \cdot {}^nC_1 + {}^nC_1 \cdot {}^nC_2 + \dots + {}^nC_{n-1} \cdot {}^nC_n$
and if $\dfrac{S_{n+1}}{S_n} = 15$, then n is
(a) 2, 4 (b) 4, 6 (c) 6, 8 (d) 8, 10
5. If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx^n$ and n is odd, the value of
$C_0^2 - C_1^2 + C_2^2 - C_3^2 + \dots + (-1)^n C_n^2$ is
(a) 0 (b) ${}^{2n}C_n$
(c) $(-1)^n \times {}^{2n}C_{n-1}$ (d) ${}^{2n}C_{n-2}$

Passage IV

If a and b be prime numbers and n a natural number, free from radical terms or rational terms in the expansion of $(a^{1/p} + b^{1/q})$ are the terms in which the indices of a and b are integers.

On the basis of the above information, answer the following questions.

1. In the expansion of $(7^{1/3} + 11^{1/9})^{6561}$, the number of terms free from radicals, is
(a) 715 (b) 725 (c) 730 (d) 745
2. In the expansion of $(2^{1/2} + 3^{1/11})^{20}$, the number of irrational terms is
(a) 85 (b) 95 (c) 105 (d) 115
3. In the expansion of $(2^{1/5} + 3^{1/2})^{20}$, the sum of rational terms is
(a) 21 (b) 84 (c) 97 (d) None
4. The number of integral terms in the expansion of $(3^{1/2} + 5^{1/8})^{250}$ is
(a) 32 (b) 33 (c) 34 (d) 35
5. The number of terms with integral co-efficients in the expansion of $(9^{1/4} + 8^{1/6})^{500}$ is
(a) 501 (b) 250 (c) 253 (d) 251

Passage V

If n be a positive integer and $a_1, a_2, \dots a_n \in C$,
then $(a_1 + a_2 + a_3 + \dots + a_m)^n$

$$= \sum \left(\frac{n!}{n_1! n_2! n_3! n_4!} \times a_1^{n_1} a_2^{n_2} a_3^{n_3} \dots a_m^{n_m} \right)$$

where $n_1, n_2, n_3, \dots, n_m$ are all non-negative integers, subject to the condition

$$n_1 + n_2 + n_3 + \dots + n_m = n.$$

On the basis of the above information, answer the following questions.

1. The number of distinct terms in the expansion of $(x_1 + x_2 + \dots + x_n)^4$ is
(a) ${}^{n+1}C_4$ (b) ${}^{n+2}C_4$ (c) ${}^{n+3}C_4$ (d) ${}^{n+4}C_4$
2. The co-efficient of x^3y^4z in the expansion of $(1 + x - y + z)^9$ is
(a) 2320 (b) 2420 (c) 2520 (d) 2620
3. The co-efficient of $a^3b^4c^5$ in the expansion of $(ab + bc + ca)^6$ is
(a) 40 (b) 60 (c) 80 (d) 100.
4. The co-efficient of x^{39} in the expansion of $(1 + x + 2x^2)$ is
(a) 5.2^{19} (b) 5.2^{20} (c) 5.2^{21} (d) 5.2^{23}
5. If the co-efficients of x^{20} in $(1 - x + x^2)^{20}$ and in $(1 + x - x^2)$ are respectively a and b, then
(a) $a = b$ (b) $a > b$ (c) $a < b$ (d) $a + b = 0$

Passage VI

Suppose when m is divided by n, the quotient is q and the remainder is r. So we can say that $m = nq + r$, for every m, n, q, r are integers and n is non-zero.

On the basis of the above information, answer the following questions.

1. When 5^{97} is divided by 52, the remainder is
(a) 3 (b) 4 (c) 5 (d) 0
2. When $13^{99} - 19^{92}$ is divided by 162, the remainder is
(a) 3 (b) 4 (c) 5 (d) 0
3. When $27^{10} + 7^{51}$ is divided by 10, the remainder is
(a) 0 (b) 2 (c) 4 (d) 5
4. When $106^{85} - 85^{106}$ is divided by 7, the remainder is
(a) 0 (b) 1 (c) 2 (d) 3
5. When $53^{53} - 33^{3}$ is divided by 10, the remainder is
(a) 0 (b) 2 (c) 4 (d) 6

Matrix Match

1. Match the following columns:

Column I		Column II	
(A)	If λ be the number of terms in the expansion of $(5^{1/6} + 7^{1/9})^{1824}$, which are integers, then λ is divisible by	(P)	2
		(Q)	3
(B)	If λ be the number of terms in the expansion of $(5^{1/6} + 2^{1/8})^{100}$, which are rationals, then λ is divisible by	(R)	7
		(S)	13
(C)	If λ be the number of terms in the expansion of $(3^{1/4} + 4^{1/3})^{99}$, which are irrationals, then λ is divisible by	(T)	17

2. Match the following columns:

Column I		Column II	
(A)	If the last digit of the number 9^{9^9} be a and the last digit of $2^{a^{100}}$ be b, then	(P)	$a + b = 9$
		(Q)	$a + b = 11$
(B)	If the last digit of the number 2^{999} be a and the last digit of 3^{aaa} be b, then	(R)	$a + b = 7$
		(S)	$a + b = 4$
(C)	If the last digit of the number 2^{999} be a and the last digit of 3^{aaa} be b, then	(T)	$a^b + b^a = 9$

3. Match the following columns:

Column I		Column II	
(A)	The value of $^{50}C_{10} + {}^{49}C_{10} + {}^{48}C_{10} + \ldots + {}^{10}C_{10}$ is	(P)	$^{54}C_{10}$
		(Q)	$^{53}C_{10}$
(B)	The value of $^{50}C_1 + {}^{50}C_2 + {}^{50}C_3 + \ldots + {}^{50}C_{49}$ is	(R)	$^{51}C_{11}$
(C)	The value of $^{50}C_{10} + 3 \cdot {}^{50}C_9 + 3 \cdot {}^{50}C_8 + {}^{50}C_7$ is	(S)	$2^{50} - 2$
(D)	The value of $^{50}C_{10} + 4 \cdot {}^{50}C_9 + 6 \cdot {}^{50}C_8 + 4 \cdot {}^{50}C_7 + {}^{50}C_6$ is	(T)	$2^{50} - 1$

4. Match the following columns:

Column I		Column II	
(A)	The sum of $^{20}C_0 + {}^{20}C_1 + {}^{20}C_2 + \ldots + {}^{20}C_{10}$ is	(P)	1/2
(B)	The sum of $^{20}C_{10} + {}^{20}C_{11} + {}^{20}C_{12} + \ldots + {}^{20}C_{20}$ is	(Q)	$^{21}C_4 - {}^{10}C_4$
(C)	The sum of $\frac{1}{2}\,{}^{10}C_0 - {}^{10}C_1 + 2 \cdot {}^{10}C_2 - 2^2 \cdot {}^{10}C_3 + \ldots + 2^9 \cdot {}^{10}C_{10}$ is	(R)	$2^{19} + \frac{1}{2}\,{}^{20}C_{10}$
(D)	The sum of $^{10}C_3 + {}^{11}C_3 + {}^{12}C_3 + \ldots + {}^{20}C_3$ is	(T)	$2^{19} - \frac{1}{2}\,{}^{20}C_{10}$

Matching List Type
(Only One Option is correct)

This section contains four questions, each having two matching list. Choices for the correct combination of elements from List I and List II are given as options (A), (B), (C) and (D), out of which only ONE is correct.

Codes

	P	Q	R	S
(A)	2	3	1	4
(B)	3	2	1	4
(C)	2	3	4	1
(D)	3	2	4	2

Assertion and Reason

Codes

A Both A and R are individually true and R is the correct explanation of A.

B. Both A and R are individually true but R is not the correct explanation of A.

C. A is true but R is false.

D. A is false but R is true.

1. *Assertion (A):* The greatest co-efficient in the expansion of $(1 + 5x)^8$ is $^8C_4 \cdot 5^4$.
   Reason (R): The greatest co-efficient in the expansion of $(1 + x)^{2n}$ is the middle term.
2. *Assertion (A):* Number of dissimilar in the expansion of $(x + a)^{102} + (x - a)^{102}$ is 206.
   Reason (R): Number of terms in the expansion of $(a + b)^n$ is $n + 1$.
3. *Assertion (A):* The term independent of x in the expansion of $\left(x + \frac{1}{x} + 2\right)^{11}$ is $^{42}C_{21}$.
   Reason (R): In a binomial expansion, the middle term is independent of x.
4. *Assertion (A):* The number of terms in the expansion of $\left(x + \frac{1}{x} + 1\right)^n$ is $2n + 1$.
   Reason (R): The number of terms in the expansion of $(a_1 + a_2 + \ldots + a_m)^n$ is $^{n+m-1}C_{m-1}$.
5. *Assertion (A):* No three consecutive binomial co-efficients can be in GP and HP.
   Reason (R): Three consecutive binomial co-efficients are in AP.
6. *Assertion (A):* The co-efficient of $x^{3\lambda+2}$ in the expansion of $(a + x)^{\lambda} \cdot (b + x)^{\lambda+1} \cdot (c + x)^{\lambda+2}$ $\lambda \in N$ is $\lambda(a + b + c)$.
   Reason (R): The co-efficient of x^m in the expansion of $(a + x)^n$ is $^nC_m \cdot a^{n-m}$.
7. *Assertion (A):* $(\sqrt{2} - 1)^n$ can be expressed as $\sqrt{N} - \sqrt{N+1}$ for all $N > 1$ and n is a positive integer.
   Reason (R): $(\sqrt{2} - 1)^n$ can be expressed as $A + B\sqrt{2}$, where A and B are integers and n is a positive integer.
8. *Assertion (A):* If $\sum_{r=1}^{n} r^3 \left(\frac{^nC_r}{^nC_{r-1}}\right)^2 = 196$, the sum of the co-efficients of power of x in the expansion of $(x - 3x^2 + x^3)^n$ is –1.
   Reason (R): $\frac{^nC_r}{^nC_{r-1}} = \left(\frac{n - r + 1}{r}\right)$ for every $n \in N$ and $r \in W$.

Integer Type Questions

1. Find the co-efficient of x^{13} in the expansion of $(1 - x)^5 (1 + x + x^2 + x^3)^4$.
2. If the co-efficient of x^{20} in the expansion of $(1 + x^2)^{40}\left(x^2 + 2 + \frac{1}{x^2}\right)^{-5}$ is ^mC_n, find the value of $(m - n + 2)$.

3. Find the sum of the co-efficient of
$$(1 + 5x - 3x^2 + 4x^3 - 7x^4 + x^5)^{2017}.$$
4. Find the degree of the polynomial
$$(x+\sqrt{x^3-1})^5+(x-\sqrt{x^3-1})^5.$$
5. If the 9th term in the expansion of
$\{3^{\log_3\sqrt{(25^{x-1}+7)}}+3^{-1/8\log_3(5^{x-1}+1)}\}^{10}$ is 180, find x.
6. Let $F(n)=\sum_{r=0}^{n-1}\left(\frac{{}^nC_r}{{}^nC_r+{}^nC_{r+1}}\right)$, find the value of $F(16)$.
7. If $\sum_{r=0}^{10}\left(\frac{r}{{}^{10}C_r}\right)=20$, find the value of $\sum_{r=0}^{10}\left(\frac{1}{{}^{10}C_r}\right)$.
8. If m be unit digit of $(27^{50} + 18^{50})$ and n is the remainder when 7^{98} is divided by 5, find the value of $(m + n)$.
9. If the number of terms in the expansion of $\left(x+1+\frac{1}{x}\right)^n$ is 17, where $n \in I^+$, find n.
10. If $(r + 1)$th term in the expansion of $\left(\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right)^{10}$ is independent of x, find r.
11. If the co-efficients of $(2r + 4)$th and $(r - 2)$th terms in the expansion of $(1 + x)^{18}$ are equal, find r.
12. If m be the number of rational terms in the expansion of $(\sqrt{2}+3^{1/5})^{10}$ and n the number of integral terms in the expansion of $(7^{1/3}+\sqrt{5})^{12}$, find $(m + n + 2)$.
13. If $\sum_{r=0}^{n}\left(\frac{r+2}{r+1}\right){}^nC_r=\left(\frac{2^8-1}{6}\right)$, find the value of n.
14. If m be the value of $R\{1-R+[R]\}$, where $R=(2+\sqrt{3})$ and n the unit digit of 3^{100}, find the value of $(m + n + 4)$.
15. If the number $(5^{25} - 3^{25})$ be divisible by m, where m is the unit digit prime number and n the sum of the co-efficients of $(1 + 3x^{100} - 5x^{201})^{2018}$, find $(m + n)$.

Questions asked in Previous Years' JEE-Advanced Examinations

1. Prove that
$$({}^{2n}C_0)^2-({}^{2n}C_1)^2+({}^{2n}C_2)^2-({}^{2n}C_3)^2+\cdots+({}^{2n}C_n)^2 = (-1)^n({}^{2n}C_n)$$
[IIT-JEE, 1978]
2. Prove that
$$C_1^2-2C_2^2+3C_3^2-\cdots-2nC_{2n}^2=(-1)^{n-1}nC_n.$$
[IIT-JEE, 1979]
3. Given positive integers $r > 1$, $n > 2$ and the co-efficients of $(3r)$th and $(r + 2)$th terms in the binomial expansion of $(1 + x)^{2n}$ are equal, then
(a) $n = 2r$ (b) $n = 2r + 1$
(c) $n = 3r$ (d) None
[IIT-JEE, 1980]
4. The value of ${}^{47}C_4+\sum_{j=1}^{5}{}^{52-j}C_3$ is equal to
(a) ${}^{47}C_5$ (b) ${}^{52}C_5$ (c) ${}^{52}C_4$ (d) none
[IIT-JEE, 1980]
5. Let $u_1 = 1$, $u_2 = 1$, $u_{n+2} = u_{n+1} + u_n$ where $n > 1$. Use mathematical induction to show that
$u_n=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right]$, for all $n > 1$.
[IIT-JEE, 1981]
6. Prove that $7^{2n} + (2^{3n-3})(3^{n-1}) + 2$ is divisible by 25 for any natural number n. **[IIT-JEE, 1982]**
7. The sum of the co-efficients of the polynomials $(1 + x - 3x^2)^{2163}$ is... **[IIT-JEE, 1982]**
8. The co-efficient of x^{99} in the polynomial $(x - 1)(x - 2)(x - 3) \dots (x - 100)$ is...
9. The larger of $99^{50} + 100^{50}$ and 101^{50} is...
[IIT-JEE, 1982]
10. If $(1 + ax)^n = 1 + 8x + 24x^2 + \dots$, then a =..., and n =... **[IIT-JEE, 1983]**
11. The co-efficient of x^4 in $\left(\frac{x}{2}-\frac{3}{x^2}\right)^{10}$ is
(a) 405/256 (b) 504/259
(c) 450/263 (d) none
[IIT-JEE, 1983]
12. If $(1 + x)^n = C_0 + C_1x + C_2x^2 + \dots + C_nx$, show that the sum of the products of the C_is, taken two at a time, is represented by $\sum_{0\le i<j\le n}(C_iC_j)=2^{2n-1}-\frac{1}{2}\cdot\frac{(2n)!}{(n)!\times(n)!}$.
[IIT-JEE, 1983]
13. Prove that $n(n^2 - 1)$ is divisible by 24, by mathematical induction where n is any odd positive integer.
[IIT-JEE, 1983]
14. Given $S_n = 1 + q + q^2 + \dots + q^n$ and $D_n=1+\left(\frac{q+1}{2}\right)+\left(\frac{q+1}{2}\right)^2+\cdots+\left(\frac{q+1}{2}\right)^n$, prove that ${}^{n+1}C_1 + {}^{n+1}C_2S_1 + {}^{n+1}C_3S_3 + \dots + {}^{n+1}C_{n+1}S_n = 2^nD_n$
[IIT-JEE, 1984]
15. If p be a natural number, prove that $p^{n+1} + (p + 1)^{2n-1}$ is divisible by $p^2 + p + 1$ for every positive integer n.
[IIT-JEE, 1984]
16. Prove that $2.7^n + 3.5^n - 5$ is divisible by 24 for all natural number n. **[IIT-JEE, 1985]**
17. Find the sum of the series
$$\sum_{r=0}^{n}(-1)^r\,{}^nC_r\left[\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\frac{15^r}{2^{4r}}+\cdots m\text{ terms}\right].$$
18. If C_r stands for ${}^nC_r$, the sum of the series
$$\frac{2\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}{n!}\times[C_0^2+C_1^2+\cdots+(-1)^n(n+1)C_n^2],$$
where n is an even positive integer, is equal to

(a) 0 (b) $(-1)^{\frac{n+1}{2}}$
(c) $(-1)^n (n+2)$ (d) none

[IIT-JEE, 1986]

19. Prove that $\sum_{k=0}^{n} k^2\, {}^nC_k = n(n+1)2^{n-2}, n > 1$.

[IIT-JEE, 1986]

20. Prove by the mathematical induction that $\frac{(2n)!}{(2n)! \times (n)^2} \le \frac{1}{\sqrt{(3n+1)}}$ for every positive integer n.

[IIT-JEE, 1987]

21. Let $R = (5\sqrt{5}+11)^{2n+1}$ and $f = R - [R]$ where [,] denotes the greatest integer function, prove that
$Rf = 4^{2n+1}$ **[IIT-JEE, 1988]**

22. Prove that
${}^mC_0 \cdot {}^nC_k + {}^mC_1 \cdot {}^nC_{k-1} + {}^mC_2 \cdot {}^nC_{k-2} + \ldots + {}^mC_k \cdot {}^nC_0 = {}^{m+n}C_k$,
where m, n, k are positive integer and ${}^pC_q = 0, p < q$.

[IIT-JEE, 1989]

23. Prove that
$$C_0 - 2^2C_1 + 3^2C_2 - \ldots + (-1)^n(n+1)^2C_n = 0,$$
where $n > 1$ and $C_r = {}^nC_r$ **[IIT-JEE, 1989]**

24. Prove that $\left(\frac{n^7}{7} + \frac{n^5}{5} + \frac{2n^3}{3} - \frac{n}{105}\right)$ is an integer for every positive integer n. **[IIT-JEE, 1990]**

25. The product of n positive numbers is unity. Then their sum is
(a) *a* positive integer (b) divisible by n
(c) equal to $n + \frac{1}{n}$ (d) never less than n

[IIT-JEE, 1991]

26. Prove that for any non-negative integer s m, n, r and k
$$\sum_{m=0}^{k} (n-m)\left(\frac{(r+m)!}{k!}\right) = \frac{(r+k+1)!}{k!} \times \left[\frac{n}{r+1} - \frac{k}{r+2}\right]$$

[IIT-JEE, 1991]

27. If the sum of the co-efficients in the expansion of $(\alpha^2x^2 - 2\alpha x + 1)^{51}$ vanishes, the value of α is
(a) 2 (b) –1 (c) 1 (d) –2

[IIT-JEE, 1991]

28. If $\sum_{r=0}^{2n} a_r(x-2)^r = \sum_{r=0}^{2n} b_r(x-3)^r$ and $a_k = 1$ for all $k \ge n$, show that $b_n = {}^{2n+1}C_{n+1}$.

[IIT-JEE, 1992]

29. The co-efficient of x^{53} in the expansion of
$\sum_{m=0}^{100} {}^{100}C_m (x-3)^{100-m}\, 2^m$ is
(a) ${}^{100}C_{47}$ (b) ${}^{100}C_{53}$ (c) $-{}^{100}C_{53}$ (d) $-{}^{100}C_{100}$

[IIT-JEE, 1992]

30. The expansion $(x + \sqrt{x^3-1})^5 + (x - \sqrt{x^3-1})^5$ is a polynomial of degree
(a) 5 (b) 6 (c) 7 (d) 8

[IIT-JEE, 1992]

31. The value of
$$C_0 + 3 \cdot C_1 + 5 \cdot C_2 + 7 \cdot C_3 + \ldots + (2n+1) \cdot C_n$$
is equal to
(a) 2 (b) $2^n + n \cdot 2^{n-1}$
(c) $2^n(n+1)$ (d) none

[IIT-JEE, 1993]

32. Prove that $\sum_{r=1}^{k} (-3)^{r-1}\, {}^{3n}C_{2r-1} = 0$, where $k = \frac{3n}{2}$ and n is an even positive integer. **[IIT-JEE, 1993]**

33. The largest term in the expansion of $(3 + 2x)^{50}$, where $x = 1/5$ is
(a) 5th (b) 51th (C) 7th (d) 6th

[IIT-JEE, 1993]

34 Let n be a positive integer and
$(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \ldots + a_{2n}x^{2n}$,
prove that
$$a_0^2 - a_1^2 + a_2^2 - a_3^2 + \cdots + a_{2n}^2 = a_n$$

[IIT-JEE, 1994]

35. Let n be a positive integer. If the co-efficient of 2nd, 3rd and 4th terms in the expansion of $(1 + x)^n$ are in AP, find n.

[IIT-JEE, 1994]

No questions asked in 1995 and 1996.

36. Prove that $\frac{3!}{2(n+3)} = \sum_{r=0}^{n} (-1)^r \left(\frac{C_r}{{}^{r+3}C_r}\right)$.

[IIT-JEE, 1997]

37. The sum of the rational terms in the expansion of $\left(2^{\frac{1}{2}} + 3^{\frac{1}{5}}\right)^{10}$ is... **[IIT-JEE, 1997]**

38. If $a_n = \sum_{r=0}^{n} \left(\frac{1}{{}^nC_r}\right)$, then $\sum_{r=0}^{n} \left(\frac{r}{{}^nC_r}\right) =$
(a) $(n-1)a_n$ (b) na_n
(c) $\frac{na_n}{2}$ (d) none

[IIT-JEE, 1998]

39. If in the expansion of $(1 + x)^m(1 - x)^n$, the co-efficients of x and x^2 are 3 and –6, respectively, then m is
(a) 6 (b) 9 (c) 12 (d) 24

[IIT-JEE, 1999]

40. For *a* positive integer, let
$$a(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n-1}, \text{ then}$$
(a) $a(100) \le 100$ (b) $a(100) > 100$
(c) $a(200) \le 100$ (d) $a(200) > 100$

[IIT-JEE, 1999]

41. Let n be any positive integer. Prove that
$$\sum_{k=0}^{m} \left(\frac{\binom{2n-k}{k}}{\binom{2k-k}{k}}\right) \times \left(\frac{2n-4k+1}{2n-2k+1}\right) \times 2^{n-2k}$$

$$= \left(\frac{\binom{n}{m}}{\binom{2n-2m}{n-m}} \right) \times 2^{n-2m}$$

for each non-negative integer $m \le n$. $\left(\text{here} \binom{p}{q} = {}^pC_q \right)$

[IIT-JEE, 1999]

42. For $2 \le r \le n$, $\binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2} =$

(a) $\binom{n+1}{r-1}$ (b) $2\binom{n+1}{r-1}$

(c) $\binom{n+2}{r}$ (d) $2\binom{n+2}{2}$

[IIT-JEE, 2000]

43. For any positive integers m, n (with $n \ge m$),

let $\binom{m}{n} = {}^nC_m$. Prove that

$$\binom{n}{m} + \binom{n-1}{m} + \binom{n-2}{m} + \cdots + \binom{m}{m} = \binom{n+1}{m+1}$$

Hence or otherwise prove that

$$\binom{n}{m} + 2\binom{n-1}{m} + 3\binom{n-2}{m} + \cdots + (n-m+1)\binom{m}{m} = \binom{n+2}{m+2}$$

[IIT-JEE, 2001]

44. In the binomial expansion of $(a - b)^n$, $n \ge 5$, the sum of 5th and 6th terms is zero, then $\frac{a}{b}$ is

(a) $\left(\frac{n-5}{6}\right)$ (b) $\left(\frac{n-4}{5}\right)$

(c) $\left(\frac{5}{n-4}\right)$ (d) $\left(\frac{6}{n-5}\right)$

[IIT-JEE, 2002]

45. The sum of $\sum_{r=0}^{m} \binom{10}{i}\binom{20}{m-i}$, where $\binom{p}{q} = 0$, if $p > q$, is maximum, when m is

(a) 5 (b) 10 (c) 15 (d) 20

[IIT-JEE, 2002]

46. The co-efficient of t^{24} in $(1 + t^2)^{12}(1 + t^{12})(1 + t^{24})$ is

(a) ${}^{12}C_6 + 3$ (b) ${}^{12}C_6 + 1$ (c) ${}^{12}C_6$ (d) ${}^{12}C_6 + 2$

[IIT-JEE, 2003]

47. Prove that

$$2^k \binom{n}{0}\binom{n}{k} - 2^{k-1}\binom{n}{1}\binom{n-1}{k-1} - 2^{k-2}\binom{n}{2}\binom{n-2}{k-2} + \cdots + (-1)^k \binom{n}{k}\binom{n-k}{0} = \binom{n}{k}$$

[IIT-JEE, 2003]

48. If ${}^nC_{r+1} = (k^2 - 3)\, {}^{n-1}C_r$, then k lies in

(a) $(-\infty, 2]$ (b) $(2, \infty)$

(c) $[-\sqrt{3}, \sqrt{3}]$ (d) $(\sqrt{3}, 2]$

[IIT-JEE, 2004]

49. $\binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \binom{30}{2}\binom{30}{12} - \cdots + \binom{30}{20}\binom{30}{30}$ is equal to

(a) ${}^{65}C_{35}$ (b) ${}^{30}C_{10}$ (c) ${}^{60}C_{10}$ (d) ${}^{30}C_{11}$

[IIT-JEE, 2005]

No questions asked in between 2006 to 2009.

50. For $r = 0, 1, 2, \ldots, 10$, let A_r, B_r and C_r denote respectively, the co-efficients of x^r in the expansion of $(1 + x)^{10}$, $(1 + x)^{20}$ and $(1 + x)^{30}$.

Then $\sum_{r=1}^{10} A_r(B_{10}B_r - C_{10}A_r)$ is equal to

(a) $B_{10} - C_{10}$ (b) $A_{10}(B_{10}^2 - C_{10}A_{10})$

(c) 0 (d) $C_{10} - B_{10}$

[IIT-JEE, 2010]

No questions asked in 2011 and 2012.

51. The coefficients of three consecutive terms in the expansion of $(1 + x)^{n+5}$ are in the ratio 5 : 10 : 14. Then n = ... **[IIT-JEE, 2013]**

52. The coefficient of x^{11} in the expansion of $(1 + x^2)^4(1 + x^3)^7(1 + x^4)^{12}$ is

(a) 1051 (b) 1106 (c) 1113 (d) 1120

[IIT-JEE, 2014]

Answers

Level II

1. (a)	2. (a)	3. (c)	4. (a)	5. (a)
6. (c)	7. (d)	8. (d)	9. (a)	10. (a)
11. (c)	12. (c)	13. (b)	14. (b,c,d)	15. (a)
16. (b)	17. (b)	18. (b)	19. (c)	20. (a)
21. (b)	22. (d)	23. (d)	24. (c)	25. (d)
26. (c)	27. (d)	28. (c)	29. (d)	30. (a,b,d)
31. (b)	32. (b)	33. (c)	34. (c)	35. (c)
36. (c)	37. (c)	38. (b)	39. (a)	40. (d)
41. (c)	42. (b)	43. (b)	44. (b)	45. (a)
46. (c)	47. (a)	48. (b)	49. (a)	50. (b)

Level IV

COMPREHENSIVE LINK PASSAGES

Passage-I:	1. (b)	2. (a)	3. (c)	4. (c)	5. (d)
Passage-II:	1. (c)	2. (b)	3. (c)	4. (a)	5. (d)
Passage-III:	1. (b)	2. (d)	3. (a)	4. (a)	5. (a)
Passage-IV:	1. (c)	2. (c)	3. (d)	4. (b)	5. (d)
Passage-V:	1. (c)	2. (c)	3. (b)	4. (c)	5. (b)
Passage-VI:	1. (c)	2. (d)	3. (b)	4. (a)	5. (d)

MATCH MATRIX

1. A → (P, Q, T), B → (P), C → (R, S)
2. A → (Q, R), B → (P, R, T), C → (S)
3. A → (R), B → (S), C → (Q), D → (P)
4. A → (R), B → (R), C → (P), D → (Q)

ASSERTION AND REASON

1. (D)	2. (D)	3. (C)	4. (B)
5. (B)	6. (D)	7. (B)	8. (D)

INTEGER TYPE QUESTIONS

1. 4	2. 7	3. 1	4. 7	5. 1
6. 8	7. 4	8. 7	9. 8	10. 4
11. 6	12. 7	13. 5	14. 6	15. 3

Hints and Solutions

Level I

1. We have,

$$(x+1)^6$$
$$= {}^6C_0x^6 + {}^6C_1x^5 + {}^6C_2x^4 + {}^6C_3x^3 + {}^6C_2x^2 + {}^6C_1x + {}^6C_0x^0$$
$$= x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1$$

2. We have,

$$\left(\frac{x}{5}+\frac{1}{x}\right)^5$$
$$= {}^5C_0\left(\frac{x}{5}\right)^5\left(\frac{1}{x}\right)^0 + {}^5C_1\left(\frac{x}{5}\right)^4\left(\frac{1}{x}\right)^1 + {}^5C_2\left(\frac{x}{5}\right)^3\left(\frac{1}{x}\right)^2$$
$$+ {}^5C_3\left(\frac{x}{5}\right)^2\left(\frac{1}{x}\right)^3 + {}^5C_4\left(\frac{x}{5}\right)^1\left(\frac{1}{x}\right)^4 + {}^5C_5\left(\frac{x}{5}\right)^0\left(\frac{1}{x}\right)^5$$

3. We have,

$$(x+\sqrt{x^2-1})^6 + (x-\sqrt{x^2-1})^6$$
$$= (x+a)^6 + (x-a)^6,\ a = \sqrt{x^2-1}$$
$$= 2({}^6C_0x^6a^0 + {}^6C_2x^4a^2 + {}^6C_4x^2a^4 + {}^6C_6x^0a^6)$$
$$= 2({}^6C_0x^6 + {}^6C_2x^4a^2 + {}^6C_4x^2a^4 + {}^6C_6a^6)$$
$$= 2[{}^6C_0x^6 + {}^6C_2x^4(x^2-1) + {}^6C_4x^2(x^2-1)^2 + {}^6C_6(x^2-1)^3]$$

Hence, the degree of the polynomial is 6.

4. We have, the number of terms in
   (i) $(1+x)^{2013}$ is 2014.
   (ii) $(1+2x+x^2)^{1007}$
   $= ((1+x)^2)^{1007} = (1+x)^{2014}$ is 2015.
   (iii) $(1+3x+3x^2+x^3)^{668}$
   $= ((1+x)^3)^{668} = (1+x)^{2014}$ is 2005.
   (iv) $(1+4x+6x^2+4x^3+x^4)^{503}$
   $= ((1+x)^4)^{503} = (1+x)^{2012}$ is 2013.
   (v) $(1+x)(1+x^2)^{10}$ is 12.
   (vi) $(1-x)^{11}(1+x+x^2)^{10}$
   $= (1-x)\{(1-x)(1+x+x^2)\}^{10}$
   $= (1-x)(1-x^3)^{10}$ is 12.

5. The total number of terms in the expansion of $(a+b)^{50} + (a-b)^{50} = \left(\frac{50}{2}+1\right) = 26.$

6. The total number of terms in the expansion of $\left(x+\frac{1}{x}\right)^{99} + \left(x-\frac{1}{x}\right)^{99} = \left(\frac{99+1}{2}\right) = 50.$

7. The total number of terms in the expansion of $\left(x^2+\frac{1}{x^2}\right)^{20} - \left(x^2-\frac{1}{x^2}\right)^{20} = \left(\frac{20}{2}\right) = 10$

8. The total number of terms in the expansion of $\left(\frac{2}{x}+x^3\right)^{2015} - \left(\frac{2}{x}-x^3\right)^{2015} = \left(\frac{2015+1}{2}\right) = 1008$

9. We have

$$t_{13}$$
$$= t_{12+1}$$
$$= {}^{18}C_{12} \times (9x)^{18-12} \times \left(-\frac{1}{3\sqrt{x}}\right)^{12}$$
$$= {}^{18}C_{12} \times (9x)^6 \times \left(\frac{1}{3^{12}x^6}\right)$$
$$= {}^{18}C_{12}$$

10. We have, 4th term from the end = (7 + 1) – (4 – 1) = 5th term from the beginning.

Thus, $t_5 = t_{4+1} = {}^7C_4 \times \left(\frac{3}{x^2}\right)^{7-4} \times \left(-\frac{x^3}{6}\right)^4$

$$= {}^7C_4 \times \left(\frac{3}{x^2}\right)^3 \times \left(\frac{x^3}{6}\right)^4$$
$$= {}^7C_4 \times \left(\frac{x^6}{48}\right) = \frac{35}{48} \cdot x^6$$

11. Let t_{r+1}th term be the independent of x.

$$t_{r+1} = {}^6C_r \times \left(\frac{3x^2}{2}\right)^{6-r} \times \left(-\frac{1}{3x}\right)^r$$

$$= {}^6C_r \times \left(\frac{3}{2}\right)^{6-r} \times \left(-\frac{1}{3}\right)^r \times x^{12-2r-r}$$

Since the term is independent of x, so
$12 - 3r = 0$, i.e. $r = 4$
Thus, the 5th term is the independent of x.

12. Let t_{r+1} term has the co-efficient of x^7.

Now, $t_{r+1} = {}^{11}C_r \times (3x^2)^{11-r} \times \left(\frac{1}{5x}\right)^r$

$$= {}^{11}C_r \times (3)^{11-r} \times \left(\frac{1}{5}\right)^r \times x^{22-2r-r}$$

$\therefore\quad 22 - 3r = 7$
$\Rightarrow\quad 3r = 15$
$\Rightarrow\quad r = 5$

Thus, the co-efficient of x^7 is ${}^{11}C_5 \times 3^6 \times \left(\frac{1}{5}\right)^5$.

13. Given,
$t_3 = 10000$
$\Rightarrow\quad {}^5C_2\,(x)^{5-2}(x^{\log_{10}x})^2 = 10000$
$\Rightarrow\quad 10 \cdot (x)^3 (x^{2\log_{10}x}) = 10000$
$\Rightarrow\quad (x)^{3+2\log_{10}x} = 1000$
$\Rightarrow\quad (x)^{3+2\log_{10}x} = 10^3$
$\Rightarrow\quad 3 + 2\log_{10}x = \log_{10^3} x = \frac{1}{3}\log_{10}x$
$\Rightarrow\quad \left(2 - \frac{1}{3}\right)\log_{10}x = -3$
$\Rightarrow\quad \log_{10}x = -\frac{9}{5}$
$\Rightarrow\quad x = 10^{-9/5}$

14. We have, the co-efficient of x^n in
$(1 + x)^{2n}$
$= {}^{2n}C_n$
$= \dfrac{(2n)!}{(n)! \times (n)!}$
$= \dfrac{2n \times (2n-1)!}{(n)! \times (n)!}$
$= \dfrac{2 \times (2n-1)!}{(n)! \times (n-1)!}$
$= 2 \times \dfrac{(2n-1)!}{(n)! \times (n-1)!}$
$= 2 \times {}^{2n-1}C_n$
$= 2$ times the co-efficient of x^n in $(1 + x)^{2n-1}$.

15. Since,
${}^{2n}C_1,\ {}^{2n}C_2,\ {}^{2n}C_3 \in \text{AP}$
$\Rightarrow\quad 2\,{}^{2n}C_2 = {}^{2n}C_1 + {}^{2n}C_3$
$\Rightarrow\quad 2 \cdot \left(\frac{2n(2n-1)}{2}\right) = 2n + \left(\frac{2n(2n-1)(2n-2)}{6}\right)$
$\Rightarrow\quad 2n(2n-1) = 2n + \left(\frac{2n(2n-1)(2n-2)}{6}\right)$
$\Rightarrow\quad (2n-1) = 1 + \frac{(2n-1)(2n-2)}{6}$
$\Rightarrow\quad (12n - 6) = 6 + (2n - 1)(2n - 2)$
$\Rightarrow\quad (12n - 6) = 6 + (4n^2 - 6n + 2)$
$\Rightarrow\quad (4n^2 - 18n + 14) = 0$
$\Rightarrow\quad (2n^2 - 9n + 7) = 0$

16. Since,
${}^nC_4, {}^nC_5, {}^nC_6$ are in AP
$\Rightarrow\quad 2\,{}^nC_5 = {}^nC_4 + {}^nC_6$
$\Rightarrow\quad 2\frac{n!}{5! \times (n-5)!} = \frac{n!}{4! \times (n-4)!} + \frac{n!}{6! \times (n-6)!}$
$\Rightarrow\quad \frac{2}{5 \times (n-5)} = \frac{1}{(n-4) \times (n-5)} + \frac{1}{6 \times 5}$
$\Rightarrow\quad \frac{2}{5 \times (n-5)} - \frac{1}{(n-4) \times (n-5)} = \frac{1}{6 \times 5}$
$\Rightarrow\quad \frac{2(n-4) - 5}{5 \times (n-4) \times (n-5)} = \frac{1}{6 \times 5}$
$\Rightarrow\quad \frac{2n - 13}{(n-4) \times (n-5)} = \frac{1}{6}$
$\Rightarrow\quad 6(2n - 13) = (n - 4) \times (n - 5)$
$\Rightarrow\quad 12n - 78 = n^2 - 9n + 20$
$\Rightarrow\quad n^2 - 21n + 98 = 0$
$\Rightarrow\quad (n - 7)(n - 14) = 0$
$\Rightarrow\quad n = 14, 7$
Hence, the values of n are 7 and 14.

17 We have
$t_2 = {}^nC_1 x,\ t_3 = {}^nC_2 x^2,\ t_4 = {}^nC_3 x^3$
Since $\quad {}^nC_1,\ {}^nC_2,\ {}^nC_3 \in \text{AP}$
$\Rightarrow\quad 2\,{}^nC_2 = {}^nC_1 + {}^nC_3$
$\Rightarrow\quad 2 \cdot \frac{n(n-1)}{2} = n + \frac{n(n-1)(n-2)}{6}$
$\Rightarrow\quad n(n-1) = n + \frac{n(n-1)(n-2)}{6}$
$\Rightarrow\quad 6n(n - 1) = 6n + n(n - 1)(n - 2)$
$\Rightarrow\quad 6n^2 - 6n = 6n + n(n^2 - 3n + 2)$
$\Rightarrow\quad 6n^2 - 6n = 6n + n^3 - 3n^2 + 2n$
$\Rightarrow\quad n^3 - 9n^2 + 14n = 0$
$\Rightarrow\quad n(n^2 - 9n + 14) = 0$
$\Rightarrow\quad n(n - 2)(n - 7) = 0$
$\Rightarrow\quad n = 0, 2, 7$
$\Rightarrow\quad n = 7$

18. The co-efficient of x^{10} in $(1 + x)^{50}$ is ${}^{50}C_{10}$.

19. We have,
$(1 + x^2)^{30}$

$$= 1 + {}^{30}C_1 x^2 + {}^{30}C_2 (x^2)^2 + {}^{30}C_3 (x^2)^3 + \cdots + {}^{30}C_5 (x^2)^5 + \cdots + {}^{30}C_{30} (x^2)^{30}$$

Thus, the co-efficient of x^{10} is ${}^{30}C_5$.

20. We have,

$$(1+x)^{41}(1-x+x^2)^{40}$$
$$= (1+x) \times \{(1+x)(1-x+x^2)\}^{40}$$
$$= (1+x) \times (1+x^3)^{40}$$
$$= (1+x) \times [1 + {}^{40}C_1 x^3 + {}^{40}C_1 (x^3)^2 + \cdots + {}^{40}C_{16}(x^3)^{16} + {}^{40}C_{17}(x^3)^{17} + \cdots]$$

Thus, the co-efficient of x^{50} is 0.

21. We have,

$$(1+x)^5 \times (1-x)^6$$
$$= (1 + {}^5C_1 x + {}^5C_2 x^2 + {}^5C_3 x^3 + \cdots) \times (1 - {}^6C_1 x + {}^6C_2 x^2 - {}^6C_3 x^3 + \cdots)$$

Thus, co-efficient of $x = {}^5C_1 - {}^6C_1 = 5 - 6 = -1$

The co-efficient of $x^2 = {}^6C_2 + {}^5C_2 - {}^6C_1 \times {}^5C_1$
$= 15 + 10 - 30 = -5$

The co-efficient of x^3
$= {}^5C_3 - {}^6C_3 + {}^5C_1 \times {}^6C_2 - {}^6C_1 \times {}^5C_2$
$= 10 - 20 + 5.15 - 6.10$
$= 10 - 20 + 75 - 60$
$= 5$

22. We have,

$$(1+x)^m(1-x)^n$$
$$= (1 + {}^mC_1 x + {}^mC_2 x^2 + \cdots) \times (1 - {}^nC_1 x + {}^nC_2 x^2 - \cdots)$$

The co-efficient of $x = 3$

$\Rightarrow \quad {}^mC_1 - {}^nC_1 = 3$

$\Rightarrow \quad m - n = 3 \quad \ldots\text{(i)}$

The co-efficient of $x^2 = -6$

$\Rightarrow \quad {}^mC_2 + {}^nC_2 - {}^mC_1 \times {}^mC_1 = -6$

$\Rightarrow \quad \dfrac{m(m-1)}{2} + \dfrac{n(n-1)}{2} - mn = -6$

$\Rightarrow \quad m(m-1) + n(n-1) - 2mn = -12$

$\Rightarrow \quad m^2 + n^2 - 2mn - (m+n) = -12$

$\Rightarrow \quad (m-n)^2 - (m+n) = -12$

$\Rightarrow \quad (-3)^2 - (m+n) = -12$

$\Rightarrow \quad -(m+n) = -21$

$\Rightarrow \quad (m+n) = 21 \quad \ldots\text{(ii)}$

Solving Eqs (i) and (ii), we get

$m = 12$ and $n = 9$

Hence, the values of m and n are 12 and 9, respectively.

23. Since n is even, so the middle term $= \left(\dfrac{12}{2} + 1\right) = 7$th term.

$$\text{Thus, } t_7 = t_{6+1} = {}^{12}C_6 \times \left(\frac{2x^2}{3}\right)^{12-6} \times \left(-\frac{3}{2x}\right)^6$$
$$= {}^{12}C_6 \times x^6$$

24. Since n is odd, so, the middle terms are

$\left(\dfrac{9+1}{2}\right)$th and $\left(\dfrac{9+3}{2}\right)$th terms

= 5th and 6th terms.

$$\text{Thus, } t_5 = t_{4+1} = {}^9C_4 \times (3x)^{9-4} \times \left(-\frac{x^3}{6}\right)^4$$
$$= {}^9C_4 \times \left(\frac{3}{16}\right) \times x^{17}$$
$$\text{and } t_6 = t_{5+1} = {}^9C_5 \times (3x)^{9-5} \times \left(-\frac{x^3}{6}\right)^5$$
$$= {}^9C_5 \times \left(-\frac{x^{19}}{96}\right)$$

25. Since n is even, so the middle term is $(n+1)$th term.

Thus, $t_{n+1} = {}^{2n}C_n \times x^n$

$$= \frac{(2n)!}{(n)! \times (n)!} \times x^n$$
$$= \frac{2^n \times (n)! \times \{1.3.5..(2n-1)\}}{(n)! \times (n)!} \times x^n$$
$$= \frac{2^n \times \{1.3.5..(2n-1)\}}{(n)!} \times x^n$$

26. The sum of the co-efficients of two middle terms in $(1+x)^{2n-1}$

$= {}^{2n-1}C_{n-1} + {}^{2n-1}C_n$
$= {}^{2n-1}C_n + {}^{2n-1}C_{n-1}$
$= {}^{(2n-1)+1}C_n$
$= {}^{2n}C_n$
= the co-efficient of the middle term in $(1+x)^{2n}$.

27. Since n is even, so the greatest co-efficient

$$= {}^{100}C_{100/2} \times \left(\frac{1}{2}\right)^{100/2} \times \left(\frac{2}{3}\right)^{100/2}$$
$$= {}^{100}C_{50} \times \left(\frac{1}{2}\right)^{50} \times \left(\frac{2}{3}\right)^{50}$$
$$= {}^{100}C_{50} \times \left(\frac{1}{3}\right)^{50}$$

28. Put $a = 1 = b$, $\Rightarrow 2^n = 4096 = 2^{12}$

$\Rightarrow \quad n = 12$

Thus, the greatest co-efficient is ${}^{12}C_6$.

29. Put $x = 1$, $\Rightarrow 3^n = 6561 = 3^9$

$\Rightarrow \quad n = 9$

Thus, the greatest co-efficients are

$24 \times {}^9C_4$ and $2^5 \times {}^9C_5$.

30. We have,

$$m = \left|\frac{(n+1)|x|}{a + |x|}\right|$$
$$= \left|\frac{(10+1)\left(\frac{3}{2} \times \frac{3}{5}\right)}{1 + \left(\frac{3}{2} \times \frac{3}{5}\right)}\right| = \frac{99}{19} = 5\frac{4}{19}$$

$\Rightarrow \quad m \neq$ not an integer

Thus, the greatest term $= t_{\left[5\frac{4}{19}\right]+1}$

$= t_6 = 6$th term.

31. We have,

$$m = \left|\frac{(n+1)|x|}{a+|x|}\right|$$

$$= \left|\frac{(15+1)\left(-\frac{5}{3}\times\frac{1}{5}\right)}{1+\left(-\frac{5}{3}\times\frac{1}{5}\right)}\right| = \frac{\frac{16}{3}}{\frac{4}{3}} = 4$$

$\Rightarrow \quad m =$ an integer

Thus, the greatest term are t_4 and t_{4+1}, i.e 4th and 5th terms.

32. We have,

$$(4^n - 3n - 1)$$
$$= (1+3)^n - 3n - 1$$
$$= (1 + {}^nC_1 \cdot 3 + {}^nC_2 \cdot 3^2 + \cdots + {}^nC_n \cdot 3^n) - 3n - 1$$
$$= (1 + 3n + {}^nC_2 \cdot 3^2 + \cdots + {}^nC_n \cdot 3^n) - 3n - 1$$
$$= ({}^nC_2 \cdot 3^2 + {}^nC_3 \cdot 3^3 + \cdots + {}^nC_n \cdot 3^n)$$
$$= 3^2({}^nC_2 + {}^nC_3 \cdot 3 + \cdots + {}^nC_n \cdot 3^{n-2})$$

Thus, $(4^n - 3n - 1)$ is divisible by 9.

33. We have,

$$3^{2n+2} - 8n - 9$$
$$= 3^2.3^{2n} - 8n - 9$$
$$= 9.9^n - 8n - 9$$
$$= 9.(1+8)^n - 8n - 9$$
$$= 9(1 + {}^nC_1 \cdot 8 + {}^nC_2 \cdot 8^2 + {}^nC_3 \cdot 8^3 + \cdots + {}^nC_n \cdot 8^n) - 8n - 9$$
$$= 9(8n + {}^nC_2 \cdot 8^2 + {}^nC_3 \cdot 8^3 + \cdots + {}^nC_n \cdot 8^n) - 8n$$
$$= (8^2 n + {}^nC_2 \cdot 8^2 \cdot 9 + {}^nC_3 \cdot 8^3 \cdot 9 + \cdots + {}^nC_n \cdot 8^n \cdot 9)$$
$$= 8^2(n + {}^nC_2 \cdot 9 + {}^nC_3 \cdot 8.9 + \cdots + {}^nC_n \cdot 8^{n-2} \cdot 9)$$

Thus, it is divisible by 64.

34. We have,

$$11^{n+2} + 12^{2n+1}$$
$$= 11^2.11^n + 12.12^{2n}$$
$$= 121.11^n + 12.(144)^n$$
$$= 121.11^n + 12.(11 + 133)^n$$
$$= 121.11^n + 12[11^n + {}^nC_1 \cdot (11)^{n-1}133 + {}^nC_2 \cdot (11)^{n-2}133^2 + \cdots + {}^nC_n \cdot (11)^{n-n}133^n]$$
$$= 133[11^n + {}^nC_1 \cdot (11)^{n-1} + \cdots + {}^nC_n \cdot 133^{n-1}]$$

Thus, it is divisible by 133.

35. We have,

$$6^{n+2} + 7^{2n+1}$$
$$= 6^n.6^2 + 7^{2n}.7$$
$$= 36.6^n + 7(6+43)^n$$
$$= 36.6^n + 7[6^n + {}^nC_1 \cdot (6)^{n-1}43 + {}^nC_2 \cdot (6)^{n-2}43^2 + \cdots + {}^nC_n \cdot (6)^{n-n}43^n]$$
$$= 43[6^n + {}^nC_1 \cdot (6)^{n-1} + \cdots + {}^nC_n \cdot 43^{n-1}]$$

Thus, it is divisible by 43.

36. We have,

$$11^{10} - 1$$
$$= (1+10)^{10} - 1$$
$$= (1 + {}^{10}C_1 \cdot 10 + {}^{10}C_2 \cdot 10^2 + {}^{10}C_3 \cdot 10^3 + \cdots + {}^{10}C_{100} \cdot 10^{100}) - 1$$
$$= ({}^{10}C_1 \cdot 10 + {}^{10}C_2 \cdot 10^2 + {}^{10}C_3 \cdot 10^3 + \cdots + {}^{10}C_{100} \cdot 10^{100})$$
$$= 100({}^{10}C_2 + {}^{10}C_3 \cdot 10 + \cdots + {}^{10}C_{100} \cdot 10^{98})$$

Thus, it is divisible by 100.

37. We have,

$$7^{98}$$
$$= (7^2)^{49}$$
$$= (50-1)^{49}$$
$$= (50^{49} - {}^{49}C_1 \cdot 50^{48} + {}^{49}C_2 \cdot 50^{47} - {}^{49}C_3 \cdot 50^{46} + \cdots - {}^{49}C_{49} \cdot 1)$$
$$= (50^{49} - {}^{49}C_1 \cdot 50^{48} + {}^{49}C_2 \cdot 50^{47} - {}^{49}C_3 \cdot 50^{46} + \cdots - 5) + 4$$

Thus, the remainder is 4.

38. We have

$$1^{2013} + 2^{2013} + 3^{2013} + \ldots + 2011^{2013} + 2012^{2013}$$
$$= (1^{2013} + 2012^{2013}) + (2^{2013} + 2011^{2013}) + \ldots + (1006^{2013} + 1007^{2013})$$

Here, each bracket is of the form of $(a^{2n+1} + b^{2n+1})$ and so is divisible by $(a + b)$.

Hence, the given expression is divisible by 2013.

39. We have

$$1992^{1998} - 1955^{1998} - 1938^{1998} + 1901^{1998}$$
$$= (1992^{1998} - 1938^{1998}) - (1955^{1998} - 1901^{1998})$$

Clearly, it is divisible by 54.

Also, the given expression can be written as

$$(1992^{1998} - 1955^{1998}) - (1938^{1998} - 1901^{1998})$$

It is divisible by 37.

Thus, the given expression

$1992^{1998} - 1955^{1998} - 1938^{1998} + 1901^{1998}$

is divisible by $54 \times 37 = 1998$

40. We have,

$$53^{53} - 33^3$$
$$= ((53^{53} - 43^{53}) + (43^{53} - 33^3))$$
$$= ((53^{53} - 43^{53}) + (43^{53} - 33^{43}) + (33^{43} - 33^3))$$

Clearly, each bracket is divisible by 10.

Now, $(33^{43} - 33^3) = 33^3(33^{40} - 1)$
$= 33^3((1 + 32)^{40} - 1)$
$= 33^3((1 + {}^{40}C_1 \cdot 32 + {}^{40}C_2 \cdot 32^2 + \cdots + {}^{40}C_{40} \cdot 32^{40}) - 1)$
$= 33^3({}^{40}C_1 \cdot 32 + {}^{40}C_2 \cdot 32^2 + \cdots + {}^{40}C_{40} \cdot 32^{40})$

Here, it is divisible by 10.

41. We have $(27)^{50} + (18)^{50}$.
The unit digit if $(27)^{50}$ is 9.
The unit digit of $(18)^{50} = (2 \times 3^2)^{50}$
$= (2^{50} \times 3^{100})$
$= (2^{(4\times12+2)} \times 3^{4\times25})$ is 4×1
Thus, the unit digit of $(27)^{50} + (18)^{50}$ is unit digit of $(9 + 4)$ i.e. 3.

42. We know that $5! + 6! + 7! + \ldots + (33)!$ is divisible by 10.
Now, the unit digit of
$1! + 2! + 3! + 4! + 5! + \ldots + (33)!$ is the unit digit of $1! + 2! + 3! + 4!$.
Thus, the unit digit of $1! + 2! + 3! + 4!$ is the unit digit of $(1 + 2 + 6 + 24)$ (= 33) is 3.

43. We have,
$(27)^{27}$
$= (3)^{81} = 3(3)^{80} = 3(9)^{40} = 3(81)^{20}$
$= 3(80 + 1)^{20} = 3(1 + 80)^{20}$
$= 3[1 + {}^{20}C_1 \cdot 80 + {}^{20}C_2 \cdot 80^2 + {}^{20}C_3 \cdot 80^3 + \cdots + {}^{20}C_{19} \cdot 80^{19} + {}^{20}C_{20} \cdot 80^{20}]$
$= 3(1 + 1600)$ + a multiple of 100
$= 4803$ + a multiple of 100
Clearly, last two digits = 03.

44. We have,
3^{999}
$= 3(3^{998}) = 3(9^{499})$
$= 3 \times 9(9^{498}) = 27(81^{249})$
$= 27(1 + 80)^{249}$
$= 27[1 + {}^{249}C_1 \cdot 80 + {}^{249}C_2 \cdot 80^2 + {}^{249}C_3 \cdot 80^3 + \cdots + {}^{249}C_{249} \cdot 80^{249}]$
$= 27(1 + {}^{249}C_1 \cdot 80) + 27[{}^{249}C_2 \cdot 80^2 + {}^{249}C_3 \cdot 80^3 + \cdots + {}^{249}C_{249} \cdot 80^{249}]$
$= 27(1 + 249 \times 80)$ + a multiple of 100
$= 27 \times 19921$ + a multiple of 100
$= 537867$ + a multiple of 100
Thus, the last two digit = 67.

45. We have,
$(17)^{10}$
$= (17^2)^5$
$= (289)^5$
$= (290 - 1)^5$
$= (290^5 - {}^5C_1 \cdot 290^4 + {}^5C_2 \cdot 290^3 - {}^5C_3 \cdot 290^2 + {}^5C_4 \cdot 290 - 1)$
$= (290^5 - {}^5C_1 \cdot 290^4 + {}^5C_2 \cdot 290^3 - {}^5C_3 \cdot 290^2) + (5 \times 290 - 1)$
$= (5 \times 290 - 1)$ + a multiple of 100
$= 1449$ + a multiple of 100
Thus, the last two digit is 49.

46. We have,
$$t_{r+1} = {}^{500}C_r \times (9^{1/4})^{500-r} \times (8^{1/6})^r$$
$$= {}^{500}C_r \times (9)^{\frac{500-r}{4}} \times (8)^{\frac{r}{6}}$$
$$= {}^{500}C_r \times (3)^{\frac{500-r}{2}} \times (2)^{\frac{r}{2}}$$
Thus, $r = 0, 2, 4, 6, 8, \ldots, 500$
Hence, the number of integral terms = 250 + 1 = 251.

47 We have,
$$t_{r+1} = {}^{1000}C_r \times (3^{1/5})^{1000-r} \times (2^{1/3})^r$$
$$= {}^{1000}C_r \times (3)^{\frac{1000-r}{5}} \times (2)^{\frac{r}{3}}$$
where $r = 0, 1, 2, 3, \ldots, 1000$.
The total number of terms = 1001
The given term will be rational if the indices of 3 and 5 are integers. Thus, the given term will be rational, when the values of r are 0, 15, 30, 45, …, 990.
The total number of rational terms = 66 +1 = 67
Hence, the total number of irrational terms = (1001 – 67)
= 934

48. We have,
$$t_{r+1} = {}^{100}C_r \times (5^{1/8})^{100-r} \times (2^{1/6})^r$$
$$= {}^{100}C_r \times (5)^{\frac{100-r}{8}} \times (2)^{\frac{r}{6}}$$
The above expression is rational, if
$\left(\dfrac{100-r}{8}\right)$ and $\left(\dfrac{r}{6}\right)$ are integers.
Clearly, the values of r are 12, 36, 60, 84.
Thus, the number of non-integral terms = 101 – 4
= 97

49. We have,
$$t_{r+1} = {}^{10}C_r (2^{1/2})^{10-r} (3^{1/5})^r$$
$$= {}^{10}C_r (2)^{\frac{10-r}{2}} (3)^{\frac{r}{5}}$$
where $r = 0, 10$
When $r = 0$, $t_1 = {}^{10}C_0 (2)^5 (3)^0 = 32$
When $r = 10$, $t_{11} = {}^{10}C_{10} (2)^0 (3)^2 = 9$
Therefore the sum of the rational terms,
$t_1 + t_{11} = 32 + 9 = 41$

50. Given $\sqrt{Q} - P < 1$
$\Rightarrow \quad 0 < (\sqrt{Q} - P)^n < 1$
Let $(\sqrt{Q} - P)^n = f'$, where $0 < f' < 1$

Now, $(\sqrt{P}+Q)^n - (\sqrt{P}-Q)^n = 1 + f - f'$

Since n is odd, the LHS of the above expression contains an even powers of $\sqrt{P}$.

Hence, the terms in LHS and I are integers.

Therefore, $f - f'$ is an integer.

But $0 < f < 1$ and $-1 < f' < 0$

$\Rightarrow$ $0 < f < 1$ and $-1 < -f' < 0$

$\Rightarrow$ $-1 < f - f' < 1$

$\Rightarrow$ $f - f' = 0$

$\Rightarrow$ $f = f'$

Thus, $(I + f) f' = (\sqrt{P}+Q)^n (\sqrt{P}-Q)^n$

$= \{(\sqrt{P}+Q)(\sqrt{P}-Q)\}^n$

$= (P - Q^2)^n$

$= k^n$

51. Given $\sqrt{Q} - P < 1$

$\Rightarrow$ $0 < (\sqrt{Q} - P)^n < 1$

Let $(\sqrt{Q} - P)^n = f'$, where $0 < f' < 1$.

Now, $(\sqrt{P}+Q)^n + (\sqrt{P}-Q)^n = I + f + f'$

$=$ Even integer

But $0 < f < 1$ and $0 < f' < 1$

$\Rightarrow$ $0 < f + f' < 2$

$\Rightarrow$ $f + f' = 1$

$\Rightarrow$ $f' = 1 - f$

Hence,

$(I + f)(I - f) = (I + f)f'$

$= (\sqrt{P}+Q)^n \times (\sqrt{P}-Q)^n$

$= (P - Q^2)^n$

$= k^n$

52 Let I and f denote the integral and fractional part of R, respectively.

Given $f = R - [R]$

Let $R = I + f = (5\sqrt{5}+11)^{2n+1}$, $0 < f < 1$

Also, $f' = (5\sqrt{5}-11)^{2n+1}$, $0 < f' < 1$

Now,

$R - f' = (5\sqrt{5}+11)^{2n+1} - (5\sqrt{5}-11)^{2n+1}$

$=$ an even integer.

But $0 < f < 1$ and $0 < f' < 1$

$\Rightarrow$ $0 < f < 1$ and $-1 < -f' < 0$

$\Rightarrow$ $-1 < f - f' < 1$

$\Rightarrow$ $f - f' = 0$

$\Rightarrow$ $f = f'$

Therefore, $Rf = Rf'$

$= (5\sqrt{5}+11)^{2n+1} \cdot (5\sqrt{5}-11)^{2n+1}$

$= (125 - 121)^{2n+1}$

$= 4^{2n+1}$

53. Let $x = I + f = (8 + 3\sqrt{7})^n$, where $0 < f < 1$,

and $f' = (8 - 3\sqrt{7})^n$, $0 < f' < 1$.

Now,

$1 + f + f' = (8 + 3\sqrt{7})^n + (8 - 3\sqrt{7})^n$

$=$ an even integer.

But $0 < f < 1$ and $0 < f' < 1$

$\Rightarrow$ $0 < f + f' < 2$

$\Rightarrow$ $f + f' = 1$

Therefore, $I + f + f' = 2k$

$\Rightarrow$ $I + 1 = 2k$

$\Rightarrow$ $I = 2k - 1$

$\Rightarrow$ $[x] = 2k - 1$

Now,

$x - x^2 + x[x] = x - x(x - [x])$

$= x - xf$

$= x(1 - f)$

$= xf'$

$= (8 + 3\sqrt{7})^n \cdot (8 - 3\sqrt{7})^n$

$= (64 - 63)^n$

$= 1^n = 1$

54. Given $\alpha + \beta = (8 + 3\sqrt{7})^n$, where $0 < \beta < 1$.

Let $\beta' = (8 - 3\sqrt{7})^n$, where $0 < \beta' < 1$.

Now,

$\alpha + \beta + \beta' = (8 + 3\sqrt{7})^n + (8 - 3\sqrt{7})^n$

$=$ an even integer.

Also, $0 < \beta < 1$ and $0 < \beta' < 1$

$\Rightarrow$ $0 < \beta + \beta' < 2$

$\Rightarrow$ $\beta + \beta' = 1$

$\Rightarrow$ $\beta' = 1 - \beta$

Thus,

$(1 - \beta)(\alpha + \beta) = (8 - 3\sqrt{7})^n \cdot (8 + 3\sqrt{7})^n$

$= (64 - 63)^n$

$= 1^n = 1$

55. Let $I + f = (3 + \sqrt{5})^5$, where $0 < f < 1$,

and $f' = (3 - \sqrt{5})^5$, where $0 < f' < 1$.

Now,

$I + f + f' = (3 + \sqrt{5})^5 + (3 - \sqrt{5})^5$

$= 2({}^5C_0 \cdot 3^5 + {}^5C_2 \cdot 3^3 \cdot 5 + {}^5C_4 \cdot 3^1 \cdot 5^2)$

$= 2(243 + 10.135 + 5.75)$

$= 2(243 + 1350 + 275)$

$= 3936$

Thus, the number just grater than the number $(3 + \sqrt{5})^5$ is $I + f + f' = 3936$.

56. The co-efficient of a^n in $(1 + a)^{m+n}$

$= {}^{m+n}C_n$

$= {}^{m+n}C_{m+n-n}$

$= {}^{m+n}C_m$

$=$ the co-efficient of a^m in $(1 + a)^{m+n}$.

57. We have,

$$^{10}C_r = {}^{10}C_{r+4}$$

$\Rightarrow \quad r + r + 4 = 10$

$\Rightarrow \quad 2r = 10 - 4 = 6$

$\Rightarrow \quad r = 3$

58. We have,

$$\frac{C_1}{C_0} + 2 \cdot \frac{C_2}{C_1} + 3 \cdot \frac{C_3}{C_2} + \cdots + n \cdot \frac{C_n}{C_{n-1}}$$

$$= \frac{n}{1} + 2 \cdot \left(\frac{n-1}{2}\right) + 3\left(\frac{n-2}{3}\right) + \cdots + n\left(\frac{1}{n}\right)$$

$$= n + (n-1) + (n-2) + \ldots + 3 + 2 + 1$$

$$= 1 + 2 + 3 + \ldots + n$$

$$= \frac{n(n+1)}{2}$$

59. We have

$$\frac{^{15}C_1}{^{15}C_0} + 2 \cdot \frac{^{15}C_2}{^{15}C_1} + 3 \cdot \frac{^{15}C_3}{^{15}C_2} + \cdots + 15 \cdot \frac{^{15}C_{15}}{^{15}C_{14}}$$

$$= \frac{15 \times (15 + 1')}{2}$$

$$= \frac{15 \times 16}{2}$$

$$= 120$$

60. We have,

$$\left(1 + \frac{C_1}{C_0}\right)\left(1 + \frac{C_2}{C_1}\right)\left(1 + \frac{C_3}{C_2}\right)\ldots\left(1 + \frac{C_n}{C_{n-1}}\right)$$

$$= \left(1 + \frac{n}{1}\right)\left(1 + \frac{n-1}{2}\right)\left(1 + \frac{n-2}{3}\right)\ldots\left(1 + \frac{1}{n}\right)$$

$$= \left(\frac{1+n}{1}\right)\left(\frac{1+n}{2}\right)\left(\frac{1+n}{3}\right)\ldots\left(\frac{1+n}{n}\right)$$

$$= \frac{(n+1)^n}{\{1.2.3\ldots n\}}$$

$$= \frac{(n+1)^n}{(n)!}$$

61. We have,

$$({}^nC_0 + {}^nC_1)({}^nC_1 + {}^nC_2)({}^nC_2 + {}^nC_3)\ldots({}^nC_n + {}^nC_{n-1})$$

$$\left(1 + \frac{^nC_1}{^nC_0}\right)\left(1 + \frac{^nC_2}{^nC_1}\right)\left(1 + \frac{^nC_3}{^nC_2}\right)\ldots\left(1 + \frac{^nC_n}{^nC_{n-1}}\right) \times C_0 C_1 C_2 \ldots C_{n-1}$$

$$= \left(\frac{1+n}{1}\right)\left(\frac{1+n}{2}\right)\left(\frac{1+n}{3}\right)\cdots\left(\frac{1+n}{n}\right) \times k$$

$$= \frac{(n+1)^n}{(n)!} \times k$$

62. We have,

$$^nC_r + {}^nC_{r-1}$$

$$= \frac{n!}{r! \times (n-r)!} + \frac{n!}{(r-1)! \times (n-r+1)!}$$

$$= \frac{n!}{(r-1)! \times (n-r)!}\left(\frac{1}{r} + \frac{1}{n-r+1}\right)$$

$$= \frac{n!}{(r-1)! \times (n-r)!}\left(\frac{n-r+1+r}{r(n-r+1)}\right)$$

$$= \frac{(n+1) \times n!}{r \times (r-1)! \times (n-r+1) \times (n-r)!}$$

$$= \frac{(n+1)!}{r! \times (n-r+1)!}$$

$$= {}^{n+1}C_r$$

63. We have,

$$\binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2}$$

$$= {}^nC_r + 2\,{}^nC_{r-1} + {}^nC_{r-2}$$

$$= ({}^nC_r + {}^nC_{r-1}) + ({}^nC_{r-1} + {}^nC_{r-2})$$

$$= ({}^{n+1}C_r + {}^{n+1}C_{r-1})$$

$$= {}^{n+2}C_r$$

64. We have,

$$\binom{n}{r} + 3\binom{n}{r-1} + 3\binom{n}{r-2} + \binom{n}{r-3}$$

$$= \left(\binom{n}{r} + \binom{n}{r-1}\right) + 2\left(\binom{n}{r-1} + \binom{n}{r-2}\right) + \left(\binom{n}{r-2} + \binom{n}{r-3}\right)$$

$$= \binom{n+1}{r} + 2\binom{n+1}{r-1} + \binom{n+1}{r-2}$$

$$= \left(\binom{n+1}{r} + \binom{n+1}{r-1}\right) + \left(\binom{n+1}{r-1} + \binom{n+1}{r-2}\right)$$

$$= \binom{n+2}{r} + \binom{n+2}{r-1}$$

$$= \binom{n+3}{r}$$

65. We have,

$$\binom{n}{r} + 4\binom{n}{r-1} + 6\binom{n}{r-2} + 4\binom{n}{r-3} + \binom{n}{r-4}$$

$$= \left(\binom{n}{r} + \binom{n}{r-1}\right) + 3\left(\binom{n}{r-1} + \binom{n}{r-2}\right) + 3\left(\binom{n}{r-2} + \binom{n}{r-3}\right) + \left(\binom{n}{r-3} + \binom{n}{r-4}\right)$$

$$= \binom{n+1}{r} + 3\binom{n+1}{r-1} + 3\binom{n+1}{r-2} + \binom{n+1}{r-3}$$

$$=\binom{n+4}{r}$$

66. We have,

$$^{47}C_4+\sum_{j=1}^{5}{}^{52-j}C_3$$
$$= {}^{47}C_4+({}^{51}C_3+{}^{50}C_3+{}^{49}C_3+{}^{48}C_3+{}^{47}C_3)$$
$$=({}^{47}C_4+{}^{47}C_3)+({}^{51}C_3+{}^{50}C_3+{}^{49}C_3+{}^{48}C_3)$$
$$=({}^{48}C_4+{}^{48}C_3)+({}^{51}C_3+{}^{50}C_3+{}^{49}C_3)$$
$$=({}^{49}C_4+{}^{49}C_3)+({}^{51}C_3+{}^{50}C_3)$$
$$=({}^{50}C_4+{}^{50}C_3)+({}^{51}C_3)$$
$$=({}^{51}C_4+{}^{51}C_3)$$
$$={}^{52}C_4$$

67. We have,

$${}^{m}C_m+{}^{m+1}C_m+{}^{m+2}C_m+\cdots+{}^{n}C_m$$
$$={}^{m+1}C_{m+1}+{}^{m+1}C_m+{}^{m+2}C_m+\cdots+{}^{n}C_m$$
$$=({}^{m+1}C_{m+1}+{}^{m+1}C_m)+{}^{m+2}C_m+\cdots+{}^{n}C_m$$
$$=({}^{m+2}C_{m+1}+{}^{m+2}C_m)+{}^{m+3}C_m+\cdots+{}^{n}C_m$$
$$=({}^{m+3}C_{m+1}+{}^{m+3}C_m)+{}^{m+4}C_m+\cdots+{}^{n}C_m$$
$$={}^{m+4}C_{m+1}+\ldots+{}^{n}C_m$$
$$={}^{n+1}C_{m+1}$$

68. $${}^{20}C_{20}+{}^{21}C_{20}+{}^{22}C_{20}+{}^{23}C_{20}+\ldots+{}^{2014}C_{20}$$
$$={}^{2015}C_{21}$$

69. $${}^{48}C_{47}+{}^{49}C_{47}+{}^{50}C_{47}+{}^{51}C_{47}+\cdots+{}^{1005}C_{47}$$
$$=({}^{47}C_{47}+{}^{48}C_{47}+{}^{49}C_{47}+\cdots+{}^{1005}C_{47})-{}^{47}C_{47}$$
$$={}^{1006}C_{48}-1$$

70. We have,

$${}^{n}C_r+{}^{n-1}C_r+{}^{n-2}C_r+{}^{n-3}C_r+\ldots+{}^{r}C_r$$
= Co-efficient of x^r in the expansion of
$$(1+x)^n+(1+x)^{n-1}+(1+x)^{n-2}+\cdots+(1+x)^r$$
$$=(1+x)^r+(1+x)^{r+1}+(1+x)^{r+2}+\cdots+(1+x)^n$$
$$=(1+x)^r(1+(1+x)+(1+x)^2+\cdots+(1+x)^{n-r})$$
$$=(1+x)^r\left(\frac{(1+x)^{n-r+1}-1}{(1+x)-1}\right)$$
$$=\left(\frac{(1+x)^{n+1}-(1+x)^r}{x}\right)$$

$\therefore$ Co-efficient of x^r in $\left(\dfrac{(1+x)^{n+1}-(1+x)^r}{x}\right)$

= Co-efficient of x^{r+1} in $(1+x)^{n+1}-(1+x)^r$

$={}^{n+1}C_{r+1}$

71. $${}^{49}C_{10}+{}^{48}C_{10}+{}^{47}C_{10}+\ldots+{}^{10}C_{10}={}^{50}C_{11}$$

72. $${}^{2014}C_{100}+{}^{2013}C_{100}+{}^{2012}C_{100}+\ldots+{}^{100}C_{100}={}^{2015}C_{101}$$

73. $${}^{70}C_{20}+{}^{69}C_{20}+{}^{68}C_{20}+{}^{67}C_{20}+\ldots+{}^{21}C_{20}$$
$$={}^{70}C_{20}+{}^{69}C_{20}+{}^{68}C_{20}+\ldots+{}^{21}C_{20}+{}^{20}C_{20}-{}^{20}C_{20}$$
$$={}^{71}C_{21}+-1$$

74. Let $S={}^{2n}C_0+{}^{2n}C_1+{}^{2n}C_2+{}^{2n}C_3+\ldots+{}^{2n}C_n$ …(i)

$S={}^{2n}C_{2n}+{}^{2n}C_{2n-1}+{}^{2n}C_{2n-2}+{}^{2n}C_{2n-3}+\ldots+{}^{2n}C_{2n-n}$ …(ii)

and $S={}^{2n}C_{2n}+{}^{2n}C_{2n-1}+{}^{2n}C_{2n-2}+{}^{2n}C_{2n-3}+\ldots+C_n$

Adding Eqs (i) and (ii), we get

$$2S=({}^{2n}C_0+{}^{2n}C_1+{}^{2n}C_2+\ldots+{}^{2n}C_n+{}^{2n}C_{n+1}+\ldots+{}^{2n}C_{2n})={}^{2n}C_n$$
$$2S=2^{2n}+{}^{2n}C_n$$
$$S=2^{2n-1}+\frac{1}{2}\times{}^{2n}C_n$$

Hence, the result.

75. The required sum $=2^{29}+\dfrac{1}{2}\times{}^{30}C_{15}$

76. Required sum $=2^{39}+\dfrac{1}{2}\times{}^{40}C_{20}$

77. We have,

$${}^{n}C_3+2\cdot{}^{n+1}C_3+3\cdot{}^{n+2}C_3+\cdots+n\cdot{}^{2n-1}C_3$$
= Co-efficient of x^3 in the expansion of
$$(1+x)^n+2(1+x)^{n+1}+3(1+x)^{n+2}+\cdots+n\cdot(1+x)^{2n-1}$$

Let $S=(1+x)^n+2(1+x)^{n+1}+3(1+x)^{n+2}+\ldots+n.(1+x)^{2n-1}$ …(i)

$\therefore$ $(1+x)S=(1+x)^{n+1}+2(1+x)^{n+2}+3(1+x)^{n+3}+\ldots+n.(1+x)^{2n}$ …(ii)

Subtracting Eqs (ii) from (i), we get

$$-xS=(1+x)^n+(1+x)^{n+1}+\cdots+(1+x)^{2n-1}+(1+x)^{2n}-n(1+x)^{2n}$$
$$=(1+x)^n(1+(1+x)+\cdots+(1+x)^{n-1})-n(1+x)^{2n}$$
$$=(1+x)^n\left(\frac{(1+x)^n-1}{(1+x)-1}\right)-n(1+x)^{2n}$$
$$=\left(\frac{(1+x)^{2n}-(1+x)^n}{x}\right)-n(1+x)^{2n}$$
$$\Rightarrow\quad S=\left(\frac{(1+x)^n-(1+x)^{2n}}{x^2}\right)+\frac{n(1+x)^{2n}}{x}$$

Hence, the co-efficient of x^3 in

$$\left(\frac{(1+x)^n-(1+x)^{2n}}{x^2}\right)-\frac{n(1+x)^{2n}}{x}$$

= the co-efficient of x^5 in

$[(1+x)^n-(1+x)^{2n}-nx(1+x)^{2n}]$

$=-{}^{2n}C_5+{}^{n}C_5+n.{}^{2n}C_4$

78. We have,

$${}^{100}C_{50} + 2\cdot{}^{99}C_{49} + 3\cdot{}^{98}C_{48} + \cdots + 51\cdot{}^{50}C_0$$

$$= {}^{100}C_{50} + 2\cdot{}^{99}C_{50} + 3\cdot{}^{98}C_{50} + \cdots + 51\cdot{}^{50}C_{50}$$

= Co-efficient of x^{50} in the expansion of

$$(1+x)^{100} + 2\cdot(1+x)^{99} + 3\cdot(1+x)^{98} + \cdots + 51\cdot(1+x)^{50}$$

Let $S = (1+x)^{100} + 2\cdot(1+x)^{99} + 3\cdot(1+x)^{98} + \ldots + 51.(1+x)^{50}$ ...(i)

$$\therefore \quad \frac{S}{(1+x)} = (1+x)^{99} + 2\cdot(1+x)^{98} + 3\cdot(1+x)^{97} + \ldots + 51.(1+x)^{49} \quad \ldots\text{(ii)}$$

Subtracting Eqs (ii) from (i), we get

$$\left(1-\frac{1}{1+x}\right)S = (1+x)^{100} + (1+x)^{99} + \cdots + (1+x)^{50} - 51(1+x)^{49}$$

$$\left(\frac{x}{1+x}\right)S = (1+x)^{100} + (1+x)^{99} + \cdots + (1+x)^{50} - 51(1+x)^{49}$$

$$= (1+x)^{50}\left(\frac{(1+x)^{51}-1}{(1+x)-1}\right) - 51(1+x)^{49}$$

$$= \left(\frac{(1+x)^{101}-(1+x)^{50}}{x}\right) - 51(1+x)^{49}$$

$$\Rightarrow \quad S = \left(\frac{(1+x)^{102}-(1+x)^{51}}{x^2}\right) - 51\frac{(1+x)^{50}}{x}$$

Hence, the co-efficient of x^{50} in

$$\left(\frac{(1+x)^{102}-(1+x)^{51}}{x^2}\right) - 51\frac{(1+x)^{50}}{x}$$

= the co-efficient of x^{52} in

$$[(1+x)^{102} - (1+x)^{51} - 51x(1+x)^{50}]$$

$$= {}^{102}C_{52}$$

79. We have

$$(1+x)^{1000} + 2x(1+x)^{999} + 3x^2(1+x)^{998} + \ldots + 1001x^{1000}$$

$$= (1+x)^{1000}\left(1 + 2\cdot\left(\frac{x}{1+x}\right) + 3\left(\frac{x}{1+x}\right)^2 + \cdots + 1001\cdot\left(\frac{x}{1+x}\right)^{1000}\right) \quad \ldots\text{(i)}$$

Let

$$S = \left(1 + 2\cdot\left(\frac{x}{1+x}\right) + 3\left(\frac{x}{1+x}\right)^2 + \cdots + 1001\cdot\left(\frac{x}{1+x}\right)^{1000}\right)$$

$$r = \left(\frac{x}{1+x}\right)$$

$$S = (1 + 2\cdot r + 3\cdot r^2 + \cdots + 1001\cdot r^{1000}), \quad \ldots\text{(ii)}$$

$$\therefore \quad r.S = (r + 2.r^2 + 3.r^3 + \ldots + 1001.r^{1001}) \quad \ldots\text{(iii)}$$

Subtracting Eqs (iii) from (ii), we get

$$(1-r)\cdot S = (1 + r + r^2 + r^3 + \cdots + r^{1000}) - 1001r^{1001}$$

$$= \left(\frac{1-r^{1001}}{1-r}\right) - 1001r^{1001}$$

$$\Rightarrow \quad S = \left(\frac{1-r^{1001}}{(1-r)^2}\right) - 1001\frac{r^{1001}}{(1-r)}$$

$$= \left(\frac{1-\left(\frac{x}{1+x}\right)^{1001}}{\left(1-\frac{x}{1+x}\right)^2}\right) - 1001\frac{\left(\frac{x}{1+x}\right)^{1001}}{\left(1-\frac{x}{1+x}\right)}$$

$$= (1+x)^{1000}\left(\left(\frac{1-\left(\frac{x}{1+x}\right)^{1001}}{\left(1-\frac{x}{1+x}\right)^2}\right) - 1001\frac{\left(\frac{x}{1+x}\right)^{1001}}{\left(1-\frac{x}{1+x}\right)}\right)$$

$$= \left(\left(\frac{(1+x)^{1000} - \left(\frac{x^{1001}}{1+x}\right)}{\left(\frac{1}{1+x}\right)^2}\right) - 1001\frac{\left(\frac{x^{1001}}{1+x}\right)}{\left(\frac{1}{1+x}\right)}\right)$$

$$= (1+x)^{1002} - x^{1001}(1+x) - 1001x^{1001}$$

Thus, the co-efficient of x^{50} in the expn of

$$[(1+x)^{1002} - x^{1001}(1+x) - 1001x^{1001}]$$

$$= {}^{1002}C_{50}$$

80. Let $t_r = r.C_r$

$$= r.{}^nC_r$$

$$= r\times\frac{n}{r}\times{}^{n-1}C_{r-1} = n\cdot{}^{n-1}C_{r-1}$$

Thus, $S_n = \sum_{r=1}^{n} t_r = \sum_{r=1}^{n}(n\,{}^{n-1}C_{r-1})$

$$= n\sum_{r=1}^{n}({}^{n-1}C_{r-1}) = n2^{n-1}$$

81. Let $t_{r+1} = (r+1)\cdot C_r = (r+1)\cdot{}^nC_r$

$$= r\cdot{}^nC_r + {}^nC_r$$

$$= r\cdot\frac{n}{r}\cdot{}^{n-1}C_{r-1} + {}^nC_r$$

$$= n\cdot{}^{n-1}C_{r-1} + {}^nC_r$$

Thus, $S_n = \sum_{r=0}^{n} t_{r+1} = \sum_{r=0}^{n} (n \cdot {}^{n-1}C_{r-1} + {}^{n}C_r)$

$= n\sum_{r=0}^{n} {}^{n-1}C_{r-1} + \sum_{r=0}^{n} {}^{n}C_r$

$= n.2^{n-1} + 2^n$

$= (n + 2)2^{n-1}$

82. Let $t_r = r^2.C_r$

$= r^2.{}^nC_r$

$= \{r(r-1) + r\} \cdot {}^nC_r$

$= \{r(r-1)\,{}^nC_r + r\,{}^nC_r\}$

$= \left\{r(r-1) \times \left(\frac{n(n-1)}{r(r-1)}\right) {}^{n-2}C_{r-2} + r\left(\frac{n}{r}\right) {}^{n-1}C_{r-1}\right\}$

$= \{n(n-1)\,{}^{n-2}C_{r-2} + n\,{}^{n-1}C_{r-1}\}$

Thus, $S_n = \sum_{r=1}^{n} t_r$

$= \sum_{r=1}^{n} (n(n-1)\,{}^{n-2}C_{r-2} + n\,{}^{n-1}C_{r-1})$

$= n(n-1)\sum_{r=1}^{n} {}^{n-2}C_{r-2} + \sum_{r=1}^{n} n\,{}^{n-1}C_{r-1}$

$= n(n-1).2^{n-2} + n.2^{n-1}$

$= n[(n-1) \cdot 2^{n-2} + 2^{n-1}]$

$= n \cdot 2^{n-2}(n - 1 + 2)$

$= n(n-1)2^{n-2}$

83. Let $t_{r+1} = \frac{C_r}{r+1} = \frac{{}^nC_r}{r+1}$

$= \frac{1}{n+1}\left(\frac{n+1}{r+1} \times {}^nC_r\right)$

$= \frac{{}^{n+1}C_{r+1}}{n+1}$

Thus, $S_n = \sum_{r=0}^{n} t_{r+1} = \sum_{r=0}^{n}\left(\frac{{}^{n+1}C_{r+1}}{n+1}\right)$

$= \frac{1}{n+1} \times \sum_{r=0}^{n} ({}^{n+1}C_{r+1})$

$= \frac{1}{n+1} \times (2^{n+1} - {}^{n+1}C_0)$

$= \frac{2^{n+1} - 1}{n+1}$

84. Let $t_{r+1} = \frac{C_r}{(r+1)(r+2)} = \frac{{}^nC_r}{(r+1)(r+2)}$

$= \frac{1}{(n+1)(n+2)} \times \frac{(n+1)(n+2)}{(r+1)(r+2)} \times {}^nC_r$

$= \frac{{}^{n+2}C_{r+2}}{(n+1)(n+2)}$

Thus, $S_n = \sum_{r=0}^{n} t_{r+1} = \sum_{r=0}^{n} \frac{{}^{n+2}C_{r+2}}{(n+1)(n+2)}$

$= \frac{1}{(n+1)(n+2)} \sum_{r=0}^{n} ({}^{n+2}C_{r+2})$

$= \frac{2^{n+2} - {}^{n+2}C_0 - {}^{n+2}C_1}{(n+1)(n+2)}$

$= \frac{2^{n+2} - 1 - (n+2)}{(n+1)(n+2)}$

$= \frac{2^{n+2} - n - 3}{(n+1)(n+2)}$

85. We have,

$(1 + x)^m = {}^mC_0 + {}^mC_1 \cdot x + {}^mC_2 \cdot x^2 + \cdots + {}^mC_r \cdot x^r + {}^mC_r \cdot x^r + \cdots + {}^mC_m \cdot x^m$ …(i)

Also, $(1 + x)^n = {}^nC_0 + {}^nC_1 \cdot x + {}^nC_2 \cdot x^2 + \cdots + {}^nC_r \cdot x^r + {}^nC_{r+1} \cdot x^{r+1} + \cdots + {}^nC_n \cdot x^n$ …(ii)

Multiplying Eqs (i) and (ii), we get,

$(1+x)^{m+n} = ({}^mC_r \cdot {}^nC_0 + {}^mC_{r-1} \cdot {}^nC_1 + {}^mC_{r-2} \cdot {}^nC_2 + \cdots + {}^mC_0 \cdot {}^nC_r)x^r + (\ldots)x + (\ldots)x^2 + \ldots$

Comparing the co-efficients of x^r from both the sides, we get

${}^mC_r \cdot {}^nC_0 + {}^mC_{r-1} \cdot {}^nC_1 + {}^mC_{r-2} \cdot {}^nC_2 + \cdots + {}^mC_0 \cdot {}^nC_r) = {}^{m+n}C_r$

86. We have,

$(1 + x)^n = {}^nC_0 + {}^nC_1 \cdot x + {}^nC_2 \cdot x^2 + \cdots + {}^nC_r \cdot x^r + {}^nC_{r+1} \cdot x^{r+1} + \cdots + {}^nC_n \cdot x^n$ …(i)

Also, $(1 + x)^n = {}^nC_n x^n + {}^nC_1 \cdot x + {}^nC_2 \cdot x^2 + \cdots + {}^nC_r \cdot x^r + {}^nC_{r+1} \cdot x^{r+1} + \cdots + {}^nC_n \cdot x^n$ …(ii)

Multiplying Eqs (i) and (ii), we get

$(1 + x)^{2n} = ({}^nC_0 \cdot {}^nC_n + {}^nC_1 \cdot {}^nC_{n-1} + {}^nC_2 \cdot {}^nC_{n-2} + \cdots + {}^nC_n \cdot {}^nC_0)x^n + (\ldots)x + (\ldots)x^2 + \ldots$

Comparing the co-efficients of x^n from both the sides, we get

$({}^nC_0 \cdot {}^nC_n + {}^nC_1 \cdot {}^nC_{n-1} + {}^nC_2 \cdot {}^nC_{n-2} + \cdots + {}^nC_n \cdot {}^nC_0) = {}^{2n}C_n$

87. We have,

$$(1+x)^n = {}^nC_0 + {}^nC_1 \cdot x + {}^nC_2 \cdot x^2 + \cdots + {}^nC_r \cdot x^r + {}^nC_{r+1} \cdot x^{r+1} + \cdots + {}^nC_n \cdot x^n \quad \ldots\text{(i)}$$

Also, $$(x+1)^n = {}^nC_0 \cdot x^n + {}^nC_1 \cdot x^{n-1} + {}^nC_2 \cdot x^{n-2} + \cdots + {}^nC_r \cdot x^{n-r} + {}^nC_{r+1} \cdot x^{n-r-1} + \cdots + {}^nC_n \quad \ldots\text{(ii)}$$

Multiplying Eqs (i) and (ii), we get

$$(1+x)^{2n} = ({}^nC_r \cdot {}^nC_n + {}^nC_{r-1} \cdot {}^nC_{n-1} + {}^nC_{r-2} \cdot {}^nC_{n-2} + \cdots + {}^nC_0 \cdot {}^nC_{n-r})x^r + (\ldots)x + (\ldots)x^2$$

Comparing the co-efficients of x^{n+r} from both the sides we get,

$$({}^nC_r \cdot {}^nC_n + {}^nC_{r-1} \cdot {}^nC_{n-1} + \cdots + {}^nC_0 \cdot {}^nC_{n-r}) = {}^{2n}C_{n+r} = {}^{2n}C_{n-r}$$

88. We have,

$$(1+x)^n = {}^nC_0 + {}^nC_1 \cdot x + {}^nC_2 \cdot x^2 + \cdots + {}^nC_r \cdot x^r + {}^nC_{r+1} \cdot x^{r+1} + \cdots + {}^nC_n \cdot x^n \quad \ldots\text{(i)}$$

Also, $$(x+1)^n = {}^nC_0 \cdot x^n + {}^nC_1 \cdot x^{n-1} + {}^nC_2 \cdot x^{n-2} + \cdots + {}^nC_r \cdot x^{n-r} + {}^nC_{r+1} \cdot x^{n-r-1} + \cdots + {}^nC_n. \quad \ldots\text{(ii)}$$

Multiplying Eqs (i) and (ii), we get

$$(1+x)^{2n} = [({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 + \cdots + ({}^nC_n)^2]x^n + (\ldots)x + (\ldots)x^2 + \ldots$$

Comparing the co-efficients of x^n from both the sides, we get

$$(({}^nC_0)^2 + ({}^nC_1)^2 + ({}^nC_2)^2 + \cdots + ({}^nC_n)^2) = {}^{2n}C_n$$

89. We have,

$$(1+x)^n = {}^nC_0 + {}^nC_1 \cdot x + {}^nC_2 \cdot x^2 + \cdots + {}^nC_r \cdot x^r + {}^nC_{r+1} \cdot x^{r+1} + \cdots + {}^nC_n \cdot x^n \quad \ldots\text{(i)}$$

Also, $$(x+1)^n = {}^nC_0 \cdot x^n + {}^nC_1 \cdot x^{n-1} + {}^nC_2 \cdot x^{n-2} + \cdots + {}^nC_r \cdot x^{n-r} + {}^nC_{r+1} \cdot x^{n-r-1} + \cdots + {}^nC_n \quad \ldots\text{(ii)}$$

Multiplying Eqs (i) and (ii), we get

$$(1+x)^{2n} = ({}^nC_0 \cdot {}^nC_1 + {}^nC_1 \cdot {}^nC_2 + {}^nC_2 \cdot {}^nC_3 + \cdots + {}^nC_{n-1} \cdot {}^nC_n)x^{n+1} + (\ldots)x + (\ldots)x^2 + \ldots$$

Comparing the co-efficients of x^{n+1} from both the sides, we get

$$({}^nC_0 \cdot {}^nC_1 + {}^nC_1 \cdot {}^nC_2 + {}^nC_2 \cdot {}^nC_3 + \cdots + {}^nC_{n-1} \cdot {}^nC_n) = {}^{2n}C_{n+1} = {}^{2n}C_{n-1}$$

90. The co-efficient of $a^5b^4c^2d$ in the expansion of $(a+b+c+d)^{15}$ is

$$= \frac{15!}{5!4!2!1!}$$

91. Let $(ab)^x(bc)^y(ca)^z = a^3b^4c^7$

$\Rightarrow \quad a^{x+z} \cdot b^{x+y} \cdot c^{y+z} = a^3b^4c^7$

$\Rightarrow \quad x+z=3, x+y=4, y+z=7$

$x=0, y=4, z=3$

Thus,
the co-efficient of $a^3b^4c^7$ in the expansion of $(ab+bc+ca)^8$
= the co-efficient of $(ab)^{10}(bc)^4(ca)^3$ in the expansion of $(ab+bc+ca)^8$

$$= \frac{8!}{0!4!3!} = 280.$$

92. Here $n = 53$ and $m = 5$.
So, the quotient is 10 and the remainder is 3.
Thus, the greatest co-efficient

$$= \frac{53!}{(10!)^{5-3} \times ((10+1)!)^3}$$

$$= \frac{53!}{(10!)^2 \times (11!)^3}$$

93. The number of terms in the expansion of

(i) $(a+b+c)^n$ is ${}^{n+3-1}C_{3-1} = {}^{n+2}C_2$

(ii) $(a+b+c+d)^n$ is ${}^{n+4-1}C_{4-1} = {}^{n+3}C_3$

(iii) $(a+b+c+d+e)^n$ is ${}^{n+5-1}C_{5-1} = {}^{n+4}C_4$

94. The co-efficient of x^7 in the expansion of $(1+3x-2x^3)^{10}$

$$= \sum\left(\frac{10!}{n_1!n_2!n_3!}\right) \times (1)^{n_1}(3)^{n_2}(-2)^{n_3}$$

where $n_1 + n_2 + n_3 = 10$, $n_2 + 3n_3 = 7$.
The possible values of n_1, n_2, n_3 are shown in tabular form.

n_1	n_2	n_3
3	7	0
5	4	1
7	1	2

Hence,
the co-efficient of x^7

$$= \frac{10!}{3!7!0!} \times (1)^3(3)^7(-2)^0 + \frac{10!}{5!4!1!} \times (1)^5(3)^4(-2)^1 + \frac{10!}{7!1!2!} \times (1)^7(3)^1(-2)^2$$

$= 262440 - 204120 + 4320$

$= 62640$

95. We have,

$$\frac{1}{1-3x} = 1 + 3x + (3x)^2 + (3x)^3 + \cdots + (3x)^n + \cdots$$

Co-efficient of x^n in the expansion of $\frac{1}{1-3x} = 3^n$.

96. We have,

$$\frac{1}{1+4x}$$
$$= (1+4x)^{-1}$$
$$= 1 - 4x + (-4x)^2 + \cdots + (-4x)^n + \cdots$$

Co-efficient of x^n in the expansion of $\frac{1}{1+4x} = (-1)^n 4^n$.

97. We have,

$$\frac{1}{1-9x+20x^2}$$
$$= \frac{1}{1-(4+5)x+20x^2}$$
$$= \frac{1}{(1-4x)(1-5x)}$$
$$= \frac{5}{(1-5x)} - \frac{4}{(1-4x)}$$

Co-efficient of x^n in the expansion of

$$\frac{1}{1-9x+20x^2} = 5.5^n - 4.4^n$$
$$= 5^{n-1} - 4^{n+1}$$

98. We have,

$$\frac{1}{1-(a+b)x+abx^2}$$
$$= \frac{1}{(1-ax)(1-bx)}$$
$$= \frac{1}{a-b}\left(\frac{a}{1-ax} - \frac{b}{1-bx}\right)$$
$$= \frac{1}{a-b}[a(1-ax)^{-1} - b(1-bx)^{-1}]$$

$$\therefore \text{ Co-efficient of } x^n = \frac{1}{a-b}(a \cdot a^n - b \cdot b^n)$$
$$= \frac{a^{n+1} - b^{n+1}}{a-b}$$

99. We have,

$$\left(\frac{1+x}{1-x}\right)$$
$$= (1+x)(1-x)^{-1}$$
$$= (1+x)(1+x+x^2+x^3+\cdots+x^{n-1}+x^n+\cdots)$$

$\therefore$ Co-efficient of $x^n = 1 + 1 = 2$

100. We have,

$$\left(\frac{1+x}{1-x}\right)^2$$
$$= (1+x)^2(1-x)^{-2}$$
$$= (1+2x+x^2)[1+2x+3x^2+4x^3+\ldots + (n-1)x^{n-2} + nx^{n-1} + (n+1)x^n + \ldots]$$

$$\therefore \text{ Co-efficient of } x^n = n - 1 + 2n + n + 1$$
$$= 4n$$

101. As we know that the general term in the expansion of $(1-x)^{-n}$ is

$$t_{r+1} = \frac{n(n+1)(n+2)\ldots(n+r-1)}{r!} \times x^r$$

Thus, the co-efficient of x^n

$$= \frac{\frac{1}{2}\left(\frac{1}{2}+1\right)\left(\frac{1}{2}+2\right)\ldots\left(\frac{1}{2}+n-1\right)}{n!} \times (2)^n$$
$$= \frac{1.3.5\ldots(2n-1)}{2^n \times n!} \times (2)^n$$
$$= \frac{1.3.5\ldots(2n-1)}{n!}$$

102. Let t_{r+1}th term be the negative term.

We have

$$t_{r+1} = \frac{n(n-1)(n-2)\ldots(n-(r-1))}{r!} \times \left(\frac{3}{4}x\right)^r$$

Thus, t_{r+1} is negative when

$$(n - r + 1) < 0$$
$$\Rightarrow \frac{13}{3} - r + 1 < 0$$
$$\Rightarrow r > \frac{16}{3} = 5\frac{1}{3}$$
$$\Rightarrow r = 6$$

103. We have,

$$y = x - x^2 + x^3 - x^4 + \ldots$$
$$\Rightarrow -y = x - x^2 + x^3 - x^4 + \ldots$$
$$\Rightarrow 1 - y = 1 - x + x^2 - x^3 + x^4 - \ldots$$
$$\Rightarrow 1 - y = (1+x)^{-1}$$
$$\Rightarrow 1 - y = \frac{1}{(1+x)}$$
$$\Rightarrow y = 1 - \frac{1}{1+x} = \frac{x}{1+x}$$

104 We have,

$$y = 2x + 3x^2 + 4x^3 + \ldots$$
$$\Rightarrow 1 + y = 1 + 2x + 3x^2 + 4x^3 + \ldots$$
$$\Rightarrow 1 + y = (1-x)^{-2} = \frac{1}{(1-x)^2}$$
$$\Rightarrow (1-x)^2 = \frac{1}{1+y}$$
$$\Rightarrow 1 - x = \sqrt{\frac{1}{1+y}}$$
$$\Rightarrow x = 1 - \sqrt{\frac{1}{1+y}}$$

105. We have,

$$(1+x)^n$$
$$= 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

Comparing the co-efficient of x and x^2, we get

$$nx = \frac{1}{3}, \quad \frac{n(n-1)}{2}x^2 = \frac{1}{6}$$

$$\Rightarrow \quad nx = \frac{1}{3}, \quad \frac{nx(nx-x)}{2} = \frac{1}{6}$$

$$\Rightarrow \quad \frac{nx(nx-x)}{2} = \frac{1}{6}$$

$$\Rightarrow \quad nx(nx-x) = \frac{1}{3}$$

$$\Rightarrow \quad \frac{1}{3}\left(\frac{1}{3} - x\right) = \frac{1}{3}$$

$$\Rightarrow \quad \left|\frac{1}{3} - x\right| = 1$$

$$\Rightarrow \quad x = \frac{1}{3} - 1 = -\frac{2}{3}$$

Put $x = -\frac{2}{3}$, we get,

$$n \times -\frac{2}{3} = \frac{1}{3}$$

$$\Rightarrow \quad n = -\frac{1}{2}$$

Therefore,

$$1 + \frac{1}{3} + \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \cdots = (1+x)^n$$

$$= \left(1 - \frac{2}{3}\right)^{-1/2}$$

$$= \left(\frac{1}{3}\right)^{-1/2}$$

$$= \sqrt{3}$$

106. We have,

$$\frac{(1+2x)^{1/2}(16+3x)^{1/4}}{(1-x)^2}$$

$$= \frac{(1+2x)^{1/2}\left[16\left(1+\frac{3}{16}x\right)\right]^{1/4}}{(1-x)^2}$$

$$= \frac{2(1+2x)^{1/2}\left(1+\frac{3}{16}x\right)^{1/4}}{(1-x)^2}$$

$$= \frac{2\left(1+\frac{1}{2}\cdot 2x+\cdots\right)\left(1+\frac{1}{4}\cdot\frac{3}{16}x+\cdots\right)}{(1-x)^2}$$

$$= \frac{2(1+x)\left(1+\frac{3}{64}x\right)}{(1-x)^2}$$

$$= 2\left(1+x+\frac{3}{64}x\right) \times (1-x)^{-2}$$

$$= 2\left(1+x+\frac{3}{64}x\right) \times (1+2x+\cdots)$$

$$= 2\left(1+\frac{67}{64}x\right) \times (1+2x+\cdots)$$

$$= 2\left(1+\frac{67}{64}x+2x\right)$$

$$= 2\left(1+\frac{67}{64}\cdot\frac{1}{2}+2\cdot\frac{1}{2}\right)$$

$$= 2\left(2+\frac{67}{128}\right)$$

$$= \left|\frac{256+67}{64}\right| = \frac{323}{67}$$

107. We have,

$$\frac{(1+2x)^{1/2}+(16+3x)^{1/4}}{(1-2x)^{1/4}}$$

$$= \frac{(1+x+\cdots)+2\left(1+\frac{3}{16}x\right)^{1/4}}{(1-2x)^{1/4}}$$

$$= \frac{(1+x+\cdots)+2\left(1+\frac{1}{4}\cdot\frac{3}{16}x+\cdots\right)}{(1-2x)^{1/4}}$$

$$= \frac{(1+x)+\left(2+\frac{3}{32}x\right)}{(1-2x)^{1/4}}$$

$$= \left(3+\frac{35}{32}x\right) \times (1-2x)^{-1/4}$$

$$= \left(3+\frac{35}{32}x\right) \times \left(1+\frac{1}{4}\cdot 2x+\cdots\right)$$

$$= \left(3+\frac{35}{32}x\right) \times \left(1+\frac{x}{2}\right)$$

$$= 3+\frac{35x}{32}+\frac{3x}{2}$$

$$= \left|3+\frac{83x}{32}\right|$$

Thus, $a = 3$ and $b = \frac{83}{32}$

108. We have,

$$\sqrt{\frac{a}{a+x}} + \sqrt{\frac{a}{a-x}}$$

$$= \sqrt{\frac{1}{\left(1+\frac{x}{a}\right)}} + \sqrt{\frac{1}{\left(1-\frac{x}{a}\right)}}$$

$$= \left(1+\frac{x}{a}\right)^{-1/2} + \left(1-\frac{x}{a}\right)^{-1/2}$$

$$= \left(1-\frac{x}{2a}+\frac{3}{8}\frac{x^2}{a^2}+\cdots\right) + \left(1+\frac{x}{a}+\frac{3}{8}\frac{x^2}{a^2}+\cdots\right)$$

$$= 2+\frac{3}{4}\frac{x^2}{a^2}+\cdots$$

109. Let $p = q + h$

$\Rightarrow \quad \frac{p}{q} = 1 + \frac{h}{q}$

We have,

$$\frac{5p + 4q}{4p + 5q}$$

$$= \frac{5\left(\frac{p}{q}\right) + 4}{4\left(\frac{p}{q}\right) + 5}$$

$$= \frac{5\left(1 + \frac{h}{q}\right) + 4}{4\left(1 + \frac{h}{q}\right) + 5}$$

$$= \frac{\left(1 + \frac{5h}{9q}\right)}{\left(1 + \frac{4h}{9q}\right)}$$

$$= \left(1 + \frac{5h}{9q}\right) \times \left(1 + \frac{4h}{9q}\right)^{-1}$$

$$= \left(1 + \frac{5h}{9q}\right) \times \left(1 - \frac{4h}{9q}\right)$$

$$= \left|1 + \frac{5h}{9q} - \frac{4h}{9q}\right|$$

$$= \left|1 + \frac{h}{9q}\right| = \left(1 + \frac{h}{q}\right)^{1/9}$$

$$= \left(\frac{p}{q}\right)^{1/9}$$

110. Let $x = 1 + h$

We have,

$$\frac{mx^m - nx^n}{m - n}$$

$$= \frac{m(1+h)^m - n(1+h)^n}{m - n}$$

$$= \frac{m(1 + mh) - n(1 + nh)}{m - n}$$

$$= \frac{(m - n) - (m^2 - n^2)h}{m - n}$$

$$= 1 - (m + n)h$$

$$= (1 - h)^{(m+n)}$$

$$= x^{(m+n)}$$

111. We have,

$$e^{5x} = 1 + 5x + \frac{(5x)^2}{2!} + \frac{(5x)^3}{3!} + \frac{(5x)^4}{4!} + \cdots + \frac{(5x)^n}{n!} + \cdots$$

Thus,

the co-efficient of x^n in $e^{5x} = \frac{5^n}{n!}$

112. We have,

$$e^{5x+4}$$

$$= e^4.e^{5x}$$

$$= e^4\left(1 + \frac{5x}{1} + \frac{(5x)^2}{2!} + \frac{(5x)^3}{3!} + \cdots + \frac{(5x)^n}{n!} + \cdots\right)$$

Thus,

the co-efficient of x^n in $e^{5x+4} = e^4 \times \left(\frac{5^n}{n!}\right)$

113. We have,

$$\left(\frac{1 + 3x + 2x^2}{e^x}\right)$$

$$= (1 + 3x + 2x^2) \times e^{-x}$$

$$= (1 + 3x + 2x^2) \times \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + \frac{(-1)^n x^n}{n!} + \cdots\right)$$

Thus,

the co-efficient of x^n

$$= \frac{(-1)^n}{n!} + \frac{3 \cdot (-1)^{n-1}}{(n-1)!} + \frac{2 \cdot (-1)^{n-2}}{(n-2)!}$$

114. Given,

$$\frac{e^x}{1 - x} = B_0 + B_1 x + B_2 x^2 + \cdots + B_n x^n$$

Now,

$e^x \times (1 - x)^{-1}$

$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots\right) \times (1 + x + x^2 + x^3 + \cdots + x^n + \cdots)$$

$$= \left(\frac{1}{n!} + \frac{1}{(n-1)!} + \frac{1}{(n-2)!} + \cdots + \frac{1}{1!} + 1\right)x^n + \left(\frac{1}{(n-1)!} + \frac{1}{(n-2)!} + \frac{1}{(n-3)!} + \cdots + \frac{1}{1!} + 1\right)x^{n-1}$$

Thus, comparing the co-efficients of x^{n-1} and x^n, we get

$$B_n = \frac{1}{n!} + \frac{1}{(n-1)!} + \frac{1}{(n-2)!} + \cdots + \frac{1}{1!} + 1$$

and

$$B_{n-1} = \frac{1}{(n-1)!} + \frac{1}{(n-2)!} + \frac{1}{(n-3)!} + \cdots + \frac{1}{1!} + 1$$

Therefore, $B_n - B_{n-1} = \frac{1}{n!}$.

115 We have,

$$1 + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots = \frac{1}{2}(e + e^{-1})$$

and $1 + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \cdots = \frac{1}{2}(e - e^{-1})$

Now, $\dfrac{1+\dfrac{1}{2!}+\dfrac{1}{4!}+\dfrac{1}{6!}+\cdots}{1+\dfrac{1}{3!}+\dfrac{1}{5!}+\dfrac{1}{7!}+\cdots}$

$$=\frac{\frac{1}{2}(e+e^{-1})}{\frac{1}{2}(e-e^{-1})}$$

$$=\frac{e^2+1}{e^2-1}$$

116. We have,

$$x=\left(1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots\right)^2-\left(1+\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+\cdots\right)^2$$

$$=\frac{1}{4}(e+e^{-1})^2-\frac{1}{4}(e-e^{-1})^2$$

$$=\frac{1}{4}\left(e^2+\frac{1}{e^2}+2\right)-\frac{1}{4}\left(e^2+\frac{1}{e^2}-2\right)$$

$$=\frac{1}{4}\left(e^2+\frac{1}{e^2}+2-e^2-\frac{1}{e^2}+2\right)$$

$$\Rightarrow \quad x=\frac{1}{4}(4)=1$$

Therefore,

$$(x+1)^{x+1}+10=(2)^2+10 = 4+10 = 14$$

117. We have,

$$a=\sum_{n=1}^{\infty}\left(\frac{2n}{(2n-1)!}\right)$$

$$=\sum_{n=1}^{\infty}\left(\frac{(2n-1)+1}{(2n-1)!}\right)$$

$$=\sum_{n=1}^{\infty}\left(\frac{1}{(2n-2)!}+\frac{1}{(2n-1)!}\right)$$

$$=\sum_{n=1}^{\infty}\left(\frac{1}{(2n-2)!}\right)+\sum_{n=1}^{\infty}\left(\frac{1}{(2n-1)!}\right)$$

$$=\frac{1}{2}(e+e^{-1})-\frac{1}{2}(e-e^{-1})$$

$$=e$$

Also, $b=\displaystyle\sum_{n=1}^{\infty}\left(\frac{2n}{(2n+1)!}\right)$

$$=\sum_{n=1}^{\infty}\left(\frac{(2n+1)-1}{(2n+1)!}\right)$$

$$=\sum_{n=1}^{\infty}\left(\frac{1}{(2n)!}-\frac{1}{(2n+1)!}\right)$$

$$=\sum_{n=1}^{\infty}\left(\frac{1}{(2n)!}\right)-\sum_{n=1}^{\infty}\left(\frac{1}{(2n+1)!}\right)$$

$$=\frac{1}{2}(e+e^{-1}-1)-\frac{1}{2}\left(e-e^{-1}-\frac{1}{2}\right)$$

$$=\frac{1}{e}$$

Therefore, $ab=1$.

118. Let $t_n=\dfrac{1+2+3+\cdots+n}{n!}$

$$=\frac{n(n+1)}{2\times n!}$$

$$=\frac{1}{2}\frac{(n+1)}{(n-1)!}$$

$$=\frac{1}{2}\frac{[(n-1)+2]}{(n-1)!}$$

$$=\frac{1}{2}\left(\frac{1}{(n-2)!}+\frac{1}{(n-1)!}\right)$$

Therefore, $S=\displaystyle\sum_{n=1}^{\infty}t_n$

$$=\sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{1}{(n-2)!}+\frac{1}{(n-1)!}\right)$$

$$=\frac{1}{2}\left[\sum_{n=1}^{\infty}\left(\frac{1}{(n-2)!}\right)+\sum_{n=1}^{\infty}\left(\frac{1}{(n-1)!}\right)\right]$$

$$=\frac{1}{2}(e+e)$$

$$=e$$

119. We have,

$$\frac{C(n,4)}{P(n,n)}$$

$$=\frac{{}^nC_4}{n!}$$

$$=\frac{(n)!}{4!\times(n-4)!}\times\frac{1}{n!}$$

$$=\frac{1}{4!\times(n-4)!}$$

$$=\frac{1}{24}\times\frac{1}{(n-4)!}$$

Therefore,

$$\sum_{n=4}^{\infty}\left(\frac{C(n,4)}{P(n,n)}\right)=\sum_{n=4}^{\infty}\frac{1}{24}\times\frac{1}{(n-4)!}$$

$$=\frac{1}{24}\left(\sum_{n=4}^{\infty}\frac{1}{(n-4)!}\right)$$

$$=\frac{1}{24}\times e$$

$$=\frac{e}{24}$$

120. We have,

$$a = \sum_{n=0}^{\infty}\left(\frac{x^{3n}}{(3n)!}\right)$$
$$= \left(1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \frac{x^9}{9!} + \cdots\right)$$

Also,

$$b = \sum_{n=0}^{\infty}\left(\frac{x^{3n-2}}{(3n-2)!}\right)$$
$$= \left(\frac{x}{1} + \frac{x^4}{4!} + \frac{x^7}{7!} + \frac{x^{10}}{10!} + \cdots\right)$$

Further, $c = \sum_{n=0}^{\infty}\left(\frac{x^{3n-1}}{(3n-1)!}\right)$

$$= \left(\frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \frac{x^{11}}{11!} + \cdots\right)$$

Now,

$$a + b + c = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = e^x$$

$a + b\omega + c\omega^2 = e^{\omega x}$

and $a + b\omega^2 + c\omega = e^{\omega^2 x}$

Therefore, $a^3 + b^3 + c^3 - 3abc$

$= (a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

$= e^x . e^{\omega x} . e^{\omega^2 x}$

$= e^{x(1+\omega+\omega^2)}$

$= e^{x.0} = 1$

121. As we know that,

$2 < e < 3$

$\Rightarrow \log_e 2 < \log_e e < \log_e 3$

$\Rightarrow \log_e 2 < 1 < \log_e 3$

Hence, the result.

122. As we know that,

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

Put $x = \frac{1}{4}$, we get

$$\frac{1}{4} - \frac{1}{2}\left(\frac{1}{4}\right)^2 + \frac{1}{3}\left(\frac{1}{4}\right)^3 - \frac{1}{4}\left(\frac{1}{4}\right)^4 + \cdots$$
$$= \log_e\left(1 + \frac{1}{4}\right) = \log_e\left(\frac{5}{4}\right)$$

123. As we know that,

$$\log\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right)$$

Replacing x by $\frac{1}{2x+1}$, we get

$$2\left(\frac{1}{(2x+1)} + \frac{1}{3(2x+1)^3} + \frac{1}{5(2x+1)^5} + \cdots\right)$$
$$= \log\left(\frac{1 + \frac{1}{2x+1}}{1 - \frac{1}{2x+1}}\right)$$
$$= \log\left(\frac{\frac{2x+1+1}{2x+1}}{\frac{2x+1-1}{2x+1}}\right)$$
$$= \log\left(\frac{2x+2}{2x}\right)$$
$$= \log\left(\frac{x+1}{x}\right)$$

124. We have,

$\log_e(1 + 6x + 8x^2)$

$= \log_e(1 + 4x)(1 + 2x)$

$= \log_e(1 + 4x) + \log_e(1 + 2x)$

$$= 4x - \frac{(4x)^2}{2} + \frac{(4x)^3}{3} - \frac{(4x)^4}{4} + \cdots + \frac{(-1)^{n+1}(4x)^n}{n} + \cdots$$
$$+ 2x - \frac{(2x)^2}{2} + \frac{(2x)^3}{3} - \frac{(2x)^4}{4} + \cdots + \frac{(-1)^{n+1}(2x)^n}{n} + \cdots$$

Thus,

the co-efficient of x^n

$$= \frac{(-1)^{n+1}}{n}(4^n + 2^n)$$

125. Given,

$$y = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$
$$\Rightarrow -y = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right)$$

$\Rightarrow -y = \log_e(1 - x)$

$\Rightarrow (1 - x) = e^{-y}$

$\Rightarrow x = 1 - e^{-y}$

$$= 1 - \left(1 - y + \frac{y^2}{2!} - \frac{y^3}{3!} + \frac{y^4}{4!} - \frac{y^5}{5!} + \cdots\right)$$
$$= \left(y - \frac{y^2}{2!} + \frac{y^3}{3!} - \frac{y^4}{4!} + \frac{y^5}{5!} - \cdots\right)$$

126. We have,

$\log_e(1 + x + x^2 + x^3 + x^4 + \ldots)$

$$= \log_e\left(\frac{1}{1-x}\right)$$

$= \log_e 1 - \log_e(1 - x)$

$= -\log_e(1 - x)$

$$= \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \cdots\right)$$

Hence, the result.

127. We have,

$$\frac{1}{2.3}+\frac{1}{4.5}+\frac{1}{6.7}+\cdots$$
$$=\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{6}-\frac{1}{7}\right)+\cdots$$
$$=-\left(-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots\right)$$
$$=1-\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots\right)$$
$$= 1-\log_e 2$$
$$=\log_e e-\log_e 2$$
$$=\log_e\left(\frac{e}{2}\right)$$

128. Let $t_n=\dfrac{1}{(2n-1)2n(2n+1)}$

$$=\frac{1}{2}\left(\frac{(2n+1)-(2n-1)}{(2n-1)2n(2n+1)}\right)$$
$$=\frac{1}{2}\left(\frac{1}{(2n-1)2n}-\frac{1}{(2n-1)2n}\right)$$
$$=\frac{1}{2}\left(\frac{2n-(2n-1)}{(2n-1)2n}-\frac{(2n+1)-2n}{(2n+1)2n}\right)$$
$$=\frac{1}{2}\left(\frac{1}{(2n-1)}-\frac{1}{2n}-\frac{1}{2n}+\frac{1}{(2n+1)}\right)$$
$$=\frac{1}{2}\left(\frac{1}{|2n-1|}-\frac{1}{n}+\frac{1}{|2n+1|}\right)$$

Therefore,
the sum

$$=\frac{1}{2}\left[\left(1+\frac{1}{3}+\frac{1}{5}+\cdots\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\right)+\left(\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots\right)\right]$$
$$=\frac{1}{2}\left(-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\cdots\right)$$
$$=\frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\cdots\right)-\frac{1}{2}$$
$$=\frac{1}{2}\log_e 2-\frac{1}{2}$$
$$=\frac{1}{2}(\log_e 2-1)$$
$$=\frac{1}{2}\left(\log_e\left(\frac{2}{e}\right)\right)$$

129. We have

$$1+\left(\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4}+\left(\frac{1}{4}+\frac{1}{5}\right)\frac{1}{4^2}+\left(\frac{1}{6}+\frac{1}{7}\right)\frac{1}{4^3}+\cdots$$
$$=\left(\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{4}\cdot\left(\frac{1}{4}\right)^2+\left(\frac{1}{6}\right)\cdot\left(\frac{1}{4}\right)^3+\cdots\right)+\left(1+\frac{1}{3}\cdot\frac{1}{4}+\frac{1}{5}\cdot\left(\frac{1}{4}\right)^2+\left(\frac{1}{7}\right)\cdot\left(\frac{1}{4}\right)^3+\cdots\right)$$
$$=-\frac{\log\left(1-\frac{1}{16}\right)}{2}+\log\left(\frac{\left(1+\frac{1}{4}\right)}{\left(1-\frac{1}{4}\right)}\right)$$
$$=-\frac{\log\left(\frac{15}{16}\right)}{2}+\frac{1}{2}\log\left(\frac{5}{3}\right)$$
$$=-\frac{1}{2}\left[\log\left(\frac{15}{16}\right)-\log\left(\frac{5}{3}\right)\right]$$
$$=-\frac{1}{2}\log\left(\frac{15}{16}\times\frac{3}{5}\right)$$
$$=-\frac{1}{2}\log\left(\frac{9}{16}\right)$$
$$=-\frac{1}{2}\log\left(\frac{3}{4}\right)^2$$
$$=-\log\left(\frac{3}{4}\right)$$
$$=\log\left(\frac{4}{3}\right)$$

130. Let $\left(\dfrac{\sqrt{2}-1}{\sqrt{2}}\right)=x$

Then,
the sum

$$=1+\frac{x}{2}+\frac{x^2}{6}+\frac{x^3}{12}+\frac{x^4}{20}+\cdots$$
$$=1+\frac{x}{1.2}+\frac{x^2}{2.3}+\frac{x^3}{3.4}+\frac{x^4}{4.5}+\cdots$$
$$=\left[1+\left(1-\frac{1}{2}\right)x+\left(\frac{1}{2}-\frac{1}{3}\right)x^2+\left(\frac{1}{3}-\frac{1}{4}\right)x^3+\left(\frac{1}{4}-\frac{1}{5}\right)x^4+\cdots\right]$$
$$=\left(1+x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots\right)-\left(\frac{x}{2}+\frac{x^2}{3}+\frac{x^3}{4}+\cdots\right)$$

$$= 1 + \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right) - \frac{1}{x}\left(\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right)$$

$$= 1 - \log(1-x) - \frac{1}{x}(-\log(1-x) - x)$$

$$= 2 + \left(\frac{1}{x} - 1\right)\log(1-x)$$

Thus,
the sum

$$= 2 + \left(\frac{\sqrt{2}}{\sqrt{2}-1} - 1\right)\log\left(1 - \frac{\sqrt{2}-1}{\sqrt{2}}\right)$$

$$= 2 + \frac{1}{(\sqrt{2}-1)}\log\left(\frac{1}{\sqrt{2}}\right)$$

$$= 2 + (\sqrt{2}+1) \times -\frac{1}{2}\log(2)$$

$$= 2 - \frac{\sqrt{2}+1}{2} \times \log(2)$$

Level III

1. We have,

$$(1 + x + x^2 + x^3)^{10}$$
$$= [(1 + x) + x^2(1 + x)]^{10}$$
$$= [(1 + x)(1 + x^2)]^{10}$$
$$= (1 + x)^{10} \times (1 + x^2)^{10}$$
$$= (1 + {}^{10}C_1 x + {}^{10}C_2 x^2 + {}^{10}C_3 x^3 + {}^{10}C_4 x^4 + \cdots) \times (1 + {}^{10}C_1 x^2 + {}^{10}C_2 x^4 + \cdots)$$

$\therefore$ Co-efficient of x^4

$$= {}^{10}C_4 + {}^{10}C_2 + {}^{10}C_1 \times {}^{10}C_2$$
$$= \frac{10.9.8.7}{24} + \frac{10.9}{2} + 10 \cdot \frac{10.9}{2}$$
$$= 210 + 45 + 450$$
$$= 705$$

2. We have,

$$(1 + x - 2x^2)^7 = ((1 + x) - 2x^2)^7$$
$$= \sum_{r=0}^{7} [{}^7C_r (1+x)^{7-r} (-2x^2)^r]$$

$\therefore$ Co-efficient of x^4

$$= {}^7C_0 \cdot {}^7C_4 + {}^7C_1 \cdot (-2) \cdot {}^6C_4 + {}^7C_2 \cdot (-2)^2\, {}^5C_0$$
$$= 1.35 - 7.2.15 + 21.4.1$$
$$= 35 - 210 + 84$$
$$= -91$$

3. $(1 - 2x + 3x^2 - 4x^3 + \ldots)^{-n} = ((1 + x)^{-2})^{-n} = (1 + x)^{2n}$

Hence,
the co-efficient of x^n in $(1 + x)^{2n}$

$$= {}^{2n}C_n$$
$$= \frac{(2n)!}{(n!) \times (n!)} = \frac{(2n)!}{(n!)^2}$$

4. We have,

$$f(n) = \sum_{r=0}^{n} (r \cdot {}^{2n}C_{2r})$$
$$= \frac{1}{2}\sum_{r=0}^{n} (2r \cdot {}^{2n}C_{2r})$$
$$= \frac{1}{2}\sum_{r=0}^{n} \left(2r \times \frac{2n}{2r}\, {}^{2n-1}C_{2r-1}\right)$$
$$= \frac{1}{2}\sum_{r=0}^{n} (2n\, {}^{2n-1}C_{2r-1})$$
$$= n\sum_{r=0}^{n} ({}^{2n-1}C_{2r-1})$$
$$= n \cdot \left(\frac{2^{2n-1}}{2}\right)$$
$$= n \cdot 2^{2n-2}$$

Hence,
the value of $f(6)$

$$= 6 \times 2^{12-2} = 6 \times 2^{10}$$

5. We have,

$$f(n) = \sum_{r=0}^{n}\sum_{k=r}^{n}\binom{k}{r}$$
$$= \sum_{r=0}^{n} ({}^rC_r + {}^{r+1}C_r + {}^{r+2}C_r + {}^{r+3}C_r + \cdots + {}^nC_r)$$
$$= \sum_{r=0}^{n} {}^{n+1}C_{r+1}$$
$$= (2^{n+1} - 1)$$

Now, $f(1) = (2^{n+1} - 1)$
$= 4095$
$= 3^2 \times 5 \times 7 \times 13$

Hence, the total number of divisors

$$= (2 + 1) \times (1 + 1) \times (1 + 1) \times (1 + 1)$$
$$= 3 \times 2 \times 2 \times 2$$
$$= 24$$

6. Given $t_3 = 10000$

$$\Rightarrow \quad {}^5C_2\, (x)^{5-2} (x^{\log_{10} x})^2 = 10000$$
$$\Rightarrow \quad 10 \cdot (x)^3 (x^{2\log_{10} x}) = 10000$$
$$\Rightarrow \quad (x)^{3+2\log_{10} x} = 1000$$
$$\Rightarrow \quad (x)^{3+2\log_{10} x} = 10^3$$
$$\Rightarrow \quad 3 + 2\log_{10} x = \log_{10^3} x = \frac{1}{3}\log_{10} x$$
$$\Rightarrow \quad \left(2 - \frac{1}{3}\right)\log_{10} x = -3$$
$$\Rightarrow \quad \log_{10} x = -\frac{9}{5}$$
$$\Rightarrow \quad x = 10^{-9/5}$$

7. We have $t_2 = {}^nC_1x, t_3 = {}^nC_2x^2, t_4 = {}^nC_3x^3$
and ${}^nC_1, {}^nC_2, {}^nC_3 \in$ AP
$\Rightarrow \quad 2\,{}^nC_2 = {}^nC_1 + {}^nC_3$
$\Rightarrow \quad 2 \cdot \dfrac{n(n-1)}{2} = n + \dfrac{n(n-1)(n-2)}{6}$
$\Rightarrow \quad n(n-1) = n + \dfrac{n(n-1)(n-2)}{6}$
$\Rightarrow \quad 6n(n-1) = 6n + n(n-1)(n-2)$
$\Rightarrow \quad 6n^2 - 6n = 6n + n(n^2 - 3n + 2)$
$\Rightarrow \quad 6n^2 - 6n = 6n + n^3 - 3n^2 + 2n$
$\Rightarrow \quad n^3 - 9n^2 + 14n = 0$
$\Rightarrow \quad n(n^2 - 9n + 14) = 0$
$\Rightarrow \quad n(n-2)(n-7) = 0$
$\Rightarrow \quad n = 0, 2, 7$
$\Rightarrow \quad n = 7$

8. We have,
$(1+x)^m(1-x)^n$
$= (1 + {}^mC_1x + {}^mC_2x^2 + \ldots) \times (1 + {}^nC_1x + {}^nC_2x^2 + \ldots)$
$\therefore$ Co-efficient of $x = 3$
$\Rightarrow \quad {}^mC_1 - {}^nC_1 = 3$
$\Rightarrow \quad m - n = 3$ …(i)
Also, Co-efficient of $x^2 = -6$
$\Rightarrow \quad {}^mC_2 + {}^nC_2 - {}^mC_1 \times {}^mC_1 = -6$
$\Rightarrow \quad \dfrac{m(m-1)}{2} + \dfrac{n(n-1)}{2} - mn = -6$
$\Rightarrow \quad m(m-1) + n(n-1) - 2mn = -12$
$\Rightarrow \quad m^2 + n^2 - 2mn - (m+n) = -12$
$\Rightarrow \quad (m-n)^2 - (m+n) = -12$
$\Rightarrow \quad (-3)^2 - (m+n) = -12$
$\Rightarrow \quad -(m+n) = -21$
$\Rightarrow \quad |m+n| = 21$ …(ii)
Solving Eqs (i) and (ii), we get,
$m = 12$ and $n = 9$.

9. We have,
$$(1+x)^{21} + (1+x)^{22} + (1+x)^{23} + \cdots + (1+x)^{50}$$
$$= (1+x)^{21}[1 + (1+x)^1 + (1+x)^2 + \cdots + (1+x)^{29}]$$
$$= (1+x)^{21}\left(\frac{(1+x)^{30} - 1}{(1+x) - 1}\right)$$
$$= \left(\frac{(1+x)^{51} - (1+x)^{21}}{x}\right)$$
$\therefore$ Co-efficient of x^{10} in $\left(\dfrac{(1+x)^{51} - (1+x)^{21}}{x}\right)$
= Co-efficient of x^{11} in $(1+x)^{51} - (1+x)^{21}$
$= {}^{51}C_{10} - {}^{21}C_{10}$

10. We have,
$(x+3)^{n-1} + (x+3)^{n-1}(x+2)$
$+ (x+3)^{n-2}(x+2)^2 + \ldots + (x+2)^{n-1}$
$$= (x+3)^{n-1}\left(1 + \left(\frac{x+2}{x+3}\right) + \left(\frac{x+2}{x+3}\right)^2 + \cdots + \left(\frac{x+2}{x+3}\right)^{n-1}\right)$$
$$= (x+3)^{n-1}\left(\frac{\left(\frac{x+2}{x+3}\right)^n - 1}{\left(\frac{x+2}{x+3}\right) - 1}\right)$$
$$= (x+3)^n\left(1 - \left(\frac{x+2}{x+3}\right)^n\right)$$
$= (x+3)^n - (x+2)^n$
Thus,
the co-efficient of x^r in $((3+x)^n - (2+x)^n)$
= the co-efficient of x^r in $((3+x)^n - (2+x)^n)$
$= {}^nC_r(3^{n-r} - 2^{n-r})$

11. $(1+t^2)^{12}(1+t^{12})(1+t^{24})$
$= (1+t^2)^{12}(1 + t^{12} + t^{24} + t^{36})$
$= [1 + {}^{12}C_1t^2 + {}^{12}C_2(t^2)^2 + \cdots + {}^{12}C_6(t^2)^6 + \cdots + {}^{12}C_{12}(t^2)^{12} + \cdots] \times (1 + t^{12} + t^{24} + t^{36})$
Thus,
the co-efficients of $t^{24} = {}^{12}C_{12} + 1 + {}^{12}C_6$
$= {}^{12}C_6 + 2$

12. Let the sum
$$(1+x)^{1000} \times \left(1 + 2\left(\frac{x}{1+x}\right) + 3\left(\frac{x}{1+x}\right)^2 + \cdots + 1001\left(\frac{x}{1+x}\right)^{1000}\right)$$
$= (1+x)^{1000} \times S,$
Let
$$\left[S = \left(1 + 2\left(\frac{x}{1+x}\right) + 3\left(\frac{x}{1+x}\right)^2 + \cdots + 1001\left(\frac{x}{1+x}\right)^{1000}\right) \quad \text{(i)}\right.$$
$$\left(\frac{x}{1+x}\right)S = \left(\left(\frac{x}{1+x}\right) + 2\left(\frac{x}{1+x}\right)^2 + \cdots + 1000\left(\frac{x}{1+x}\right)^{1000} + 1001\left(\frac{x}{1+x}\right)^{1001}\right) \text{(ii)}$$
Subtracting Eqs (i) and (ii), we get,
$$\left(1 - \frac{x}{x+1}\right)S = 1 + \left(\frac{x}{x+1}\right) + \left(\frac{x}{x+1}\right)^2 + \cdots + \left(\frac{x}{x+1}\right)^{1000} - \left(\frac{x}{x+1}\right)^{1001}$$

$$= \frac{1 - \left(\frac{x}{x+1}\right)^{1001}}{1 - \left(\frac{x}{1+x}\right)} - \left(\frac{x}{x+1}\right)^{1001}$$

$$= \frac{(x+1)^{1001} - x^{1001}}{(x+1)^{1000}} - \left(\frac{x}{x+1}\right)^{1001}$$

Therefore, the sum

$$= (1 + x^{1000})\left[\frac{(x+1)^{1001} - x^{1001}}{(x+1)^{1000}} - \left(\frac{x}{x+1}\right)^{1001}\right]$$

$$= \left[(x+1)^{1001} - x^{1001} - \left(\frac{x^{1001}}{x+1}\right)\right]$$

Thus, the co-efficient of x^{50} is ${}^{1001}C_{50}$.

13. We have,

$(1 + x + 2x^2 + 3x^3 + \ldots + nx^n)^2$
$= (1 + x + 2x^2 + 3x^3 + \ldots + nx^n)$
$\times (1 + x + 2x^2 + 3x^3 + 4x^4 + \ldots + nx^n)^2$
$= [1.n + 1(n-1) + 2(n-2)$
$+ 3(n-3) + \ldots + (n-1).1 + n.1]$

Let $t_r = r(n - r) = nr - r^2$

Thus,

the sum of co-efficients of x^n $S_n = (\Sigma t_r) + 2n$

$$= [\Sigma(nr - r^2)] + 2n$$

$$= [n\Sigma r - \Sigma r^2] + 2n$$

$$= \left[n\left(\frac{n(n+1)}{2}\right) - \frac{n(n+1)(2n+1)}{6}\right] + 2n$$

$$= \left(\frac{n(n+1)}{2}\right)\left(n - \frac{(2n+1)}{3}\right) + 2n$$

$$= \left(\frac{n(n+1)}{2}\right)\left(\frac{n-1}{3}\right) + 2n$$

$$= \frac{n(n^2-1)}{6} + 2n$$

$$= \frac{n(n^2+11)}{6}$$

14. We have

$(1 + 2x + 3x^2 + 4x^3 + \ldots + nx^n)^2$
$= (1 + 2x + 3x^2 + 4x^3 + \ldots + nx^n)^2$
$\times (1 + 2x + 3x^2 + 4x^3 + \ldots + nx^n)^2$

Thus, the co-efficient of x^n

$= [1.n + 2(n-1) + 3(n-2) + \ldots + (n-1).2 + n.1]$

Let $t_r = r[n - (r - 1)]$
$= (n + 1)r - r^2$

Thus, $S_n = \Sigma\,(n+1)r - r^2$

$$= (n+1)\Sigma r - \Sigma r^2$$

$$= (n+1)\left(\frac{n(n+1)}{2}\right) - \frac{n(n+1)(2n+1)}{6}$$

$$= \left(\frac{n(n+1)}{2}\right)\left((n+1) - \frac{(2n+1)}{3}\right)$$

$$= \left(\frac{n(n+1)}{2}\right)\left(\frac{(3n+3)-(2n+1)}{3}\right)$$

$$= \left(\frac{n(n+1)}{2}\right)\left(\frac{n+2}{3}\right)$$

$$= \left|\frac{n(n+1)(n+2)}{6}\right|$$

15. We have,

$$\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right)^2$$

$$= \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right) \times \left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\right)$$

Thus, the co-efficient of x^n

$$= 1 \cdot \frac{1}{n!} + \frac{1}{1!} \cdot \frac{1}{(n-1)!} + \frac{1}{2!} \cdot \frac{1}{(n-2)!} + \frac{1}{3!} \cdot \frac{1}{(n-3)!} + \cdots + \frac{1}{n!} \cdot 1$$

$$= \frac{1}{n!}\left(\frac{n!}{n!} + \frac{n!}{1!(n-1)!} + \frac{n!}{2!(n-2)!} + \cdots + \frac{n!}{n!0!}\right)$$

$$= \frac{1}{n!}(1 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n)$$

$$= \frac{2^n}{n!}$$

16. Hence the number of different dissimilar terms

$= (1 + 2012) + (2011 - 1006) + (2010 - 1005)$
$= 2013 + 1005 + 1005$
$= 4023$

17. We have,

$N = {}^{2000}C_1 + 2.{}^{2000}C_2 + 3.{}^{2000}C_3 + \ldots + 2000.{}^{2000}C_{2000}$
$= 2000.2^{2000-1}$
$= 2000.2^{1999}$
$= 2^4 \times 5^3 \times 2^{1999}$
$= 2^{2003} \times 5^3$

Hence,

the total number of divisors $= (2003 + 1) \times (3 + 1)$
$= 2004 \times 4$
$= 8016$

18 We have,

$$\sum_{r=0}^{n} (-1)^r \cdot {}^nC_r \cdot \left(\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \frac{15^r}{2^{4r}} + \cdots \text{ to } m \text{ terms}\right)$$

$$= \sum_{r=0}^{n} (-1)^r \cdot {}^nC_r \cdot \left[\left(\frac{1}{2}\right)^r + \left(\frac{3}{4}\right)^r + \left(\frac{7}{8}\right)^r + \left(\frac{15}{16}\right)^r + \cdots\right]$$

$$= \sum_{r=0}^{n} (-1)^r \cdot {}^nC_r \cdot \left(\frac{1}{2}\right)^r + \sum_{r=0}^{n} (-1)^r \cdot {}^nC_r \cdot \left(\frac{3}{4}\right)^r$$

$$+ \sum_{r=0}^{n} (-1)^r \cdot {}^nC_r \cdot \left(\frac{7}{8}\right)^r + \sum_{r=0}^{n} (-1)^r \cdot {}^nC_r \cdot \left(\frac{15}{16}\right)^r$$

+ … upto m terms

$$= \left(1-\frac{1}{2}\right)^n + \left(1-\frac{3}{4}\right)^n + \left(1-\frac{7}{8}\right)^n + \left(1-\frac{15}{16}\right)^n + \cdots$$

$$= \left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n + \left(\frac{1}{8}\right)^n + \left(\frac{1}{16}\right)^n + \cdots$$

$$= \left(\frac{1}{2}\right)^n + \left(\frac{1}{2^2}\right)^n + \left(\frac{1}{2^3}\right)^n + \left(\frac{1}{2^4}\right)^n + \cdots$$

$$= \left(\frac{1}{2}\right)^n \left(\frac{1-\left(\frac{1}{2}\right)^m}{1-\frac{1}{2^n}}\right) = \frac{2^{mn}-1}{2^{mn}(2^n-1)}$$

19. We have,

$$\sum_{k=1}^{n} k^3 \left(\frac{C_k}{C_{k-1}}\right)^2$$

$$= \sum_{k=1}^{n} k^3 \left(\frac{n-k+1}{k}\right)^2$$

$$= \sum_{k=1}^{n} k(n-k+1)^2$$

$$= \sum_{k=1}^{n} k(n^2+k^2+1-2kn+2n-2k)$$

$$= \sum_{k=1}^{n} k((n^2+2n+1)-2(n+1)k+k^2)$$

$$= (n^2+2n+1)\sum_{k=1}^{n} k - 2(n+1)\sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k^3$$

$$= (n+1)^2\left(\frac{n(n+1)}{2}\right)$$

$$= -2(n+1)\left(\frac{n(n+1)(2n+1)}{6}\right) + \left(\frac{n(n+1)}{2}\right)^2$$

$$= \frac{n(n+1)^2(n+2)}{12}$$

20. Let t_{r-1}, t_r, t_{r+1} be three consecutive terms in the given expansion.

Clearly, the co-efficients of t_{r-1}, t_r, t_{r+1} will be ${}^nC_{r-2}$, ${}^nC_{r-1}$, ${}^nC_r$, respectively.

Thus, $\frac{{}^nC_{r-2}}{{}^nC_{r-1}} = \frac{1}{7}$ and $\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{1}{6}$

$$\Rightarrow \quad \frac{{}^nC_{r-1}}{{}^nC_{r-2}} = 7 \text{ and } \frac{{}^nC_r}{{}^nC_{r-1}} = 6$$

$$\left(\frac{n-(r-1)+1}{r-1}\right) = 7 \text{ and } \left(\frac{n-r+1}{r}\right) = 6$$

$$\left(\frac{n-r+2}{r-1}\right) = 7 \text{ and } \left(\frac{n-r+1}{r}\right) = 6$$

$\Rightarrow \quad n - 8r + 9 = 0$ and $n - 7r + 1 = 0$

Solving, we get,

$r = 8$ and $n = 55$.

21. We have

$$r \cdot \frac{{}^nC_r}{{}^nC_{r-1}} = |n-r+1|$$

Hence, the sum,

$$S_n = \sum_{r=1}^{n} (n-r+1)^2$$

$$= \sum_{r=1}^{n} (n-(r-1))^2$$

$$= \sum_{r=1}^{n} (n^2+(r-1)^2-2n(r-1))$$

$$= \sum_{r=1}^{n} n^2 + \sum_{r=1}^{n} (r-1)^2 - \sum_{r=1}^{n} 2n(r-1)$$

$$= n^2\sum_{r=1}^{n} 1 + \sum_{r=1}^{n} (r-1)^2 - 2n\sum_{r=1}^{n} (r-1)$$

$$= n^2 \cdot n + \frac{(n-1)\cdot n\cdot(2n-1)}{6} - 2n\cdot\frac{n(n-1)}{2}$$

$$= n^3 + \frac{n(n-1)(2n-1)}{6} - n^2(n-1)$$

$$= \frac{n(14n^2-9n+1)}{6}$$

22. As we know that, $x^n + y^n$ is divisible by $(x+y)$, when n is odd.

Now, the given expression can be expressed as

$$(1^{2017} + 2016^{2017}) + (2^{2017} + 2015^{2017})$$
$$+ (3^{2017} + 2014^{2017}) + (4^{2017} + 2013^{2017})$$
$$+ \ldots + (1008^{2017} + 1009^{2017})$$

Clearly, the given expression is divisible by 2017.

23. Put $y = 1 + x$, then

$$f(x) = 1 - x + x^2 - x^3 + \ldots - x^{17}$$
$$= 1 - (y-1) + (y-1)^2 - (y-1)^3 + \ldots - (y-1)^{17}$$
$$= a_0 + a_1y + a_2y^2 + a_3y^3 + \ldots + a_{17}y^{17}$$

Comparing the co-efficients of y^2, we get

$$a_2 = 1 + 3 + 6 + 10 + \ldots$$

$$a_2 = \frac{n(n+1)(n+2)}{6},$$

When $n = 16$

$$a_2 = \frac{16\times17\times18}{6} = 816$$

24. We have,

$$S_n = 1 + q + q^2 + \ldots + q^n$$

$$= \frac{1-q^{n+1}}{1-q}$$

and $D_n = 1 + \left(\frac{q+1}{2}\right) + \left(\frac{q+1}{2}\right)^2 + \cdots + \left(\frac{q+1}{2}\right)^n$

$$= \frac{1-\left(\frac{1+q}{2}\right)^{n+1}}{1-\left(\frac{1+q}{2}\right)} = \frac{2^{n+1}-(1+q)^{n+1}}{2^n(1-q)}$$

Now, ${}^{n+1}C_1 + {}^{n+1}C_2S_1 + {}^{n+1}C_3S_3 + \ldots + {}^{n+1}C_{n+1}S_n$

$$= \frac{1}{1-q}[{}^{n+1}C_1(1-q) + {}^{n+1}C_2(1-q^2) + {}^{n+1}C_3(1-q^3) + \cdots + {}^{n+1}C_{n+1}(1-q^{n+1})]$$

$$= \frac{1}{1-q}\left[\sum_{k=1}^{n+1} {}^{n+1}C_k - \sum_{k=1}^{n+1} {}^{n+1}C_k q^k\right]$$

$$= \frac{1}{1-q}((2^{n+1}-1) - \{(1+q)^{n+1}-1\})$$

$$= \left|\frac{2^{n+1}-(1+q)^{n+1}}{(1-q)}\right|$$

$$= 2^n D_n$$

25. Given, $a = (4^{1/401} - 1)$

$\Rightarrow \quad (a+1)^{401} = 4$

Now, $b_n = {}^nC_1 + {}^nC_2 \cdot a + {}^nC_3 \cdot a^2 + \ldots + {}^nC_n \cdot a^{n-1}$

$\therefore \quad 1 + ab_n = {}^nC_1 \cdot a + {}^nC_2 \cdot a^2 + {}^nC_3 \cdot a^3 + \ldots + {}^nC_n \cdot a^n$

$\Rightarrow \quad 1 + ab_n = (1+a)^n$

$\Rightarrow \quad b_n = \frac{(1+a)^n - 1}{a}$

Hence,
the value of $(b_{2006} - b_{2005})$

$$= \frac{(1+a)^{2006}-1}{a} - \frac{(1+a)^{2005}-1}{a}$$

$$= \frac{(1+a)^{2006}-(1+a)^{2005}}{a}$$

$$= \frac{(1+a)^{2005}(1+a-1)}{a}$$

$$= \frac{(1+a)^{2005}\times a}{a}$$

$$= (1+a)^{2005}$$

$$= 4^{\frac{2005}{401}}$$

$$= 4^5 = 2^{10}$$

26. We have,

$$f(n) = {}^nC_0 \cdot a^{n-1} - {}^nC_1 \cdot a^{n-2} + {}^nC_2 \cdot a^{n-3} - {}^nC_4 a^{n-4} + \ldots + (-1)^{n-1}\,{}^nC_{n-1}$$

$\Rightarrow \quad af(n) + (-1)^n = {}^nC_0 \cdot a^n - {}^nC_1 \cdot a^{n-1} + {}^nC_2 \cdot a^{n-2} - {}^nC_4 a^{n-3} + \cdots + (-1)^{n-1}\,{}^nC_{n-1} \cdot a - 1$

$\Rightarrow \quad af(n) + (-1)^n = (a-1)^n$

$\Rightarrow \quad f(n) = \frac{(a-1)^n - (-1)^n}{a}$

Now,

$f(2007) + f(2008)$

$$= \frac{(a-1)^{2007}-(-1)^{2007}}{a} + \frac{(a-1)^{2008}-(-1)^{2008}}{a}$$

$$= \frac{(a-1)^{2007}+1}{a} + \frac{(a-1)^{2008}-1}{a}$$

$$= \frac{(a-1)^{2007}+(a-1)^{2008}}{a}$$

$$= \frac{(a-1)^{2007}(1+a-1)}{a}$$

$$= \frac{(a-1)^{2007}\times a}{a}$$

$$= (a-1)^{2007}$$

$$= 3^{\frac{2007}{323}} = 3^9$$

Now, $9m = 3^9$

$\Rightarrow \quad m = \frac{3^9}{9} = 3^7 = 2187$

27. Let $\quad S = \sum_{k=0}^{n} {}^nC_k \cdot \sin(kx) \cdot \cos(n-k)x$

$\Rightarrow \quad S = \sum_{k=0}^{n} {}^nC_{n-k} \cdot \sin[(n-k)x] \cdot \cos(k)x$

Adding, we get

$$2S = \sum_{k=0}^{n} {}^nC_k \cdot \sin(kx) \cdot \cos(n-k)x + \sum_{k=0}^{n} {}^nC_k \cdot \sin[(n-k)x] \cdot \cos(k)x$$

$$= \sum_{k=0}^{n} {}^nC_k \cdot \sin[kx + (n-k)x]$$

$$= \sum_{k=0}^{n} {}^nC_k \cdot \sin(n)x$$

$$= \sin(n)x \sum_{k=0}^{n} {}^nC_k$$

$$= \sin(n)x \times 2^n$$

$\Rightarrow \quad S = 2^{n-1} \times \sin(n)x$

28. Let d be the common difference.

So, $a_{n-1} = a + nd$

We have,

$$\sum_{k=0}^{n} {}^nC_k \cdot a_{k+1}$$

$$= \sum_{k=0}^{n} {}^nC_k \cdot (a + kd)$$
$$= a\sum_{k=0}^{n} {}^nC_k \cdot + d\sum_{k=0}^{n} k\,{}^nC_k$$
$$= a \cdot 2^n + d \cdot n\,2^{n-1}$$
$$= 2^{n-1}(2a + nd)$$
$$= 2^{n-1}[a + (a + nd)]$$
$$= 2^{n-1}(a_1 + a_{n+1})$$

Hence, the result.

29. We have

$$(1 + x)^n = C_0 + C_1x + \dots + C_{n-2}x^{n-2} + C_{n-1}x^{n-1} + C_nx^n \quad \dots\text{(i)}$$

Replacing x by $-x$, we get

$$(1 - x)^n = C_0 - C_1x + \dots + (-1)^{n-1}C_{n-1}x^{n-1} + (-1)^n C_nx^n \quad \dots\text{(ii)}$$

Multiplying Eqs (i) and (ii), we get

$$(1 - x^2)^n$$
$$= [C_0 + C_1x + C_2x^2 + \cdots + C_{n-1}x^{n-1} + C_nx^n]$$
$$\times [C_0 - C_1x + \cdots + (-1)^{n-1}C_{n-1}x^{n-1} + (-1)^n C_nx^n]$$
$$= [C_0C_n - C_1C_{n-1} + C_2C_{n-2} - \cdots + (-1)^n C_nC_0]x^n + (\dots)x^{n-1} + (\dots)x^{n-2} + \dots$$

Comparing the co-efficients of x^n from both the sides, we get

$$(C_0C_n - C_1C_{n-1} + C_2C_{n-2} - \cdots + (-1)^n C_nC_0)$$
$$= (-1)^{n/2}\ {}^nC_{n/2}$$
$$= \frac{(-1)^{n/2} \times (n)!}{(n/2)!(n/2)!}$$
$$= \begin{cases} 0: & n \text{ is odd} \\ \dfrac{(-1)^{n/2}(n)!}{(n/2)! \times (n/2)!}: & n \text{ is even} \end{cases}$$

Hence, the result.

30. We have,

$$\sum_{r=0}^{n}\left(\frac{n-3r+1}{n-r+1}\right)\left(\frac{{}^nC_r}{2^r}\right)$$
$$= \sum_{r=0}^{n}\left(1 - \frac{2r}{n-r+1}\right)\left(\frac{{}^nC_r}{2^r}\right)$$
$$= \sum_{r=0}^{n}\left(\frac{{}^nC_r}{2^r}\right) - \sum_{r=0}^{n}\left(\frac{2r}{n-r+1} \times \frac{{}^nC_r}{2^r}\right)$$
$$= \sum_{r=0}^{n}\left(\frac{{}^nC_r}{2^r}\right) - \sum_{r=0}^{n}\left(\frac{r}{n-r+1} \times \frac{{}^nC_r}{2^{r-1}}\right)$$
$$= \sum_{r=0}^{n}\left(\frac{{}^nC_r}{2^r}\right) - \sum_{r=0}^{n}\left(\frac{{}^nC_{r-1}}{2^{r-1}}\right)$$
$$= \sum_{r=0}^{n}\left(\frac{{}^nC_r}{2^r}\right) - \sum_{r=0}^{n}\left(\frac{{}^nC_r}{2^r}\right) + \left(\frac{{}^nC_n}{2^n}\right)$$
$$= \frac{{}^nC_n}{2^n}$$
$$= \frac{1}{2^n}$$

31. We have,

$$(1 + 2x + 3x^2)^{10}$$
$$= [1 + (2x + 3x^2)]^{10}$$
$$= [1 + {}^{10}C_1(2x + 3x^2) + {}^{10}C_2(2x + 3x^2)^2 + {}^{10}C_3(2x + 3x^2)^3 + {}^{10}C_4(2x + 3x^2)^4 + \dots]$$
$$= 1 + (2 \cdot {}^{10}C_1)x + (3 \cdot {}^{10}C_1 + 4 \cdot {}^{10}C_2)x^2 + (6 \cdot {}^{10}C_2 + 8 \cdot {}^{10}C_3)x^3 + (9 \cdot {}^{10}C_2 + 36 \cdot {}^{10}C_3 + 16 \cdot {}^{10}C_4)x^4 + \dots$$

Comparing the co-efficients of x, x^2, x^3 and x^4, we get

$$a_1 = 2 \cdot {}^{10}C_1 = 2 \cdot 10 = 20$$
$$a_2 = 3 \cdot {}^{10}C_1 + 4 \cdot {}^{10}C_2 = 30 + 180 = 210$$
$$a_3 = (6 \cdot {}^{10}C_2 + 8 \cdot {}^{10}C_3) = 1230$$

and $a_4 = 9 \cdot {}^{10}C_2 + 36 \cdot {}^{10}C_3 + 16 \cdot {}^{10}C_4$
$= 405 + 4320 + 3360 = 8085$

Hence, the value of

$$a_1 + a_2 + a_3 + a_4 + 5$$
$$= 20 + 210 + 1230 + 8085 + 5$$
$$= 9550$$

32. We have,

$$\left(\frac{1}{1} + \frac{1}{2016}\right)\cdot\left(\frac{1}{2} + \frac{1}{2015}\right)\cdot\left(\frac{1}{3} + \frac{1}{2014}\right)\left(\frac{1}{4} + \frac{1}{2013}\right)\dots\left(\frac{1}{1008} + \frac{1}{1009}\right)$$
$$= \left(\frac{2017}{1.2016} \times \frac{2017}{2.2015} \times \frac{2017}{3.2014} \times \cdots \times \frac{2017}{1008.1009}\right)$$
$$= \frac{(2017)^{1008}}{(2016)!}$$

Clearly, $m = 2017$, $n = 2016$ and $p = 1008$.

Hence, the value of

$$m - n + p + 2 = 2017 - 2016 + 1008 + 2$$
$$= 1011$$

33. We have,

$$\left(1 + \sum_{k=0}^{n} k \cdot x^k\right)^2$$
$$= (1 + x + 2x^2 + 3x^3 +_ \dots + nx^n)^2$$

$= (1 + x + 2x^2 + 3x^3 + \ldots + nx^n)$
$\times (1 + x + 2x^2 + 3x^3 + \ldots + nx^n)$

Co-efficient of $x^n = 1 \cdot n + 1 \cdot (n-1) + 2 \cdot (n-2) + \ldots + (n-1) \cdot 1 + n \cdot 1$

$$= \frac{n^2(n+1)}{2} - \frac{n(n+1)(2n+1)}{6} + 2n$$

where $t_r = nr - r^2$

$$= \frac{n(n+1)}{2}\left(n - \frac{2n+1}{3}\right) + 2n$$

$$= \frac{n(n+1)}{2} \cdot \left(\frac{n-1}{3}\right) + 2n$$

$$= \frac{n(n^2-1)}{6} + 2n$$

$$= \frac{n(n^2+11)}{6}$$

It is given that

$$\frac{n(n^2+11)}{6} = 310$$

$\Rightarrow \quad n(n^2 + 1) = 1860$
$\Rightarrow \quad n = 12$

34. We have

$F(n) = {}^{n+1}C_2 + 2({}^nC_2 + {}^{n-1}C_2 + {}^{n-2}C_2 + \ \ldots + {}^2C_2)$
$= {}^{n+1}C_2 = 2\ {}^{n+1}C_3$
$= ({}^{n+1}C_2 + {}^{n+1}C_3) + {}^{n+1}C_3$
$= {}^{n+2}C_3 + {}^{n+1}C_3$

$$= \frac{(n+2)(n+1)n}{6} + \frac{(n+1)n(n-1)}{6}$$

$$= \frac{n(n+1)}{6}(n+2+n-1)$$

$$= \frac{n(n+1)(2n+1)}{6}$$

Hence, the value of

$F(11) + 4$

$$= \frac{11.12.23}{6} + 4$$

$= 506 + 4 = 510$

35. Let

$$t_r = \left(\frac{1+2+3+\cdots+r}{r}\right)C_r^2$$

$$= \left(\frac{r(r+1)}{2r}\right)C_r^2$$

$$= \left(\frac{(r+1)}{2}\right){}^nC_r^2$$

$$= \frac{1}{2}(r\,{}^nC_r^2 + {}^nC_r^2)$$

Thus,

$$S_n = \frac{1}{2}\left(\sum_{r=0}^{n} r\,{}^nC_r^2 + \sum_{r=0}^{n} {}^nC_r^2\right)$$

$$= \frac{1}{2}[(1 \cdot {}^nC_1^2 + 2 \cdot {}^nC_2^2 + 3 \cdot {}^nC_2^2 + \cdots + n \cdot {}^nC_n^2)$$
$$+ ({}^nC_1^2 + {}^nC_2^2 + {}^nC_2^2 + {}^nC_3^2 \cdots + {}^nC_n^2)]$$

$$= \frac{1}{2}(n \cdot {}^{2n-1}C_{n-1} + {}^{2n}C_n)$$

38. We have,

$(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + \ldots + a_{2n}x^{2n}$ …(i)

Replace x by $-x$, we get,

$(1 - x + x^2)^n = a_0 - a_1x + a_2x^2 + \ldots + a_{2n}x^{2n}$ …(ii)

Replace x by $\frac{1}{x}$ in Eq. (i), we get

$$\left(1 + \frac{1}{x} + \frac{1}{x^2}\right)^n = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_{2n}}{x^{2n}} \quad \ldots\text{(iii)}$$

Multiplying Eqs (ii) and (i), we get,

$$(1 - x + x^2)^n \left(1 + \frac{1}{x} + \frac{1}{x^2}\right)^n$$
$$= (a_0^2 - a_1^2 + a_2^2 - a_3^2 + \cdots + a_n^2) + (\ldots)x + \cdots$$

Comparing the co-efficients of the constant terms from both the sides, we get

$(a_0^2 - a_1^2 + a_2^2 - a_3^2 + \cdots + a_n^2)$
= Co-efficient of constant term in

$$(1 - x + x^2)^n (1 + x + x^2)^n \times \frac{1}{x^{2n}}$$

= Co-efficient of x^{2n} in
$(1 - x + x^2)^n(1 + x + x^2)^n$
$= [(1 + x^2)^2 - x^2]^n = (1 + x^2 + x^2)^n$
= Co-efficient of t^n in $(1 + t + t^2)^n$
$= a_n$

39. We have,

$(x + 1)(x + 2)(x + 3)\ldots(x + (n - 1))(x + n)$
$= A_0 + A_1x + A_2x^2 + A_3x^3 + \ldots + A_nx^n$
$\log\{(x + 1)(x + 2)\ldots(x + (n - 1))(x + n)\}$
$= \log(A_0 + A_1x + A_2x^2 + \ldots + A_nx^n)$

(i) Differentiating both sides w.r.t x and put $x = 1$, we get

$$\left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{(n+1)}\right)$$
$$= \frac{A_1 + 2A_2 + 3A_3 + \cdots + nA_n}{A_0 + A_1 + A_2 + A_3 + \cdots + A_n}$$
$$= \frac{A_1 + 2A_2 + 3A_3 + \cdots + nA_n}{(n+1)!}$$

Thus, $A_1 + 2A_2 + 3A_3 + \ldots + nA_n$

$$= (n+1)! \times \left(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n+1}\right)$$

(ii) Differentiating both sides w.r.t x and put $x = 0$, we get

$$\frac{A_1}{A_0} = \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n+1}\right)$$

$$A_1 = A_0\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n+1}\right)$$

$$A_1 = n! \times \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n+1}\right)$$

40. Clearly,

$$A_r = C_r$$

$$\Rightarrow \quad {}^nA_r = {}^nC_r$$

Also,

$$B_r = C_r$$

$$\Rightarrow \quad {}^{n+1}B_r = {}^{n+1}C_r$$

Now, $A_r + A_{r-1} = {}^nA_r + {}^nA_{r+1}$

$$= {}^nC_r + {}^nC_{r+1}$$

$$= {}^{n+1}C_r$$

$$= {}^{n+1}B_r$$

$$= B_r$$

Hence, the result.

41. Let $S = \sum_{0 \le i < j \le n}\sum \left(\frac{1}{{}^nC_i} + \frac{1}{{}^nC_j}\right)$

$$= \left(\sum_{0 \le i < j \le n}\sum \left(\frac{n-i}{{}^nC_{n-\text{i}}} + \frac{n-j}{{}^nC_{n-j}}\right)\right)$$

$$= \left[n \sum_{0 \le i < j \le n}\sum \left(\frac{1}{{}^nC_i} + \frac{1}{{}^nC_j}\right) - \sum_{0 \le i < j \le n}\sum \left(\frac{i}{{}^nC_{n-i}} + \frac{\text{j}}{{}^nC_{n-j}}\right)\right]$$

$$= n \sum_{0 \le i < j \le n}\sum \left(\frac{1}{{}^nC_i} + \frac{1}{{}^nC_j}\right) - S$$

$$\Rightarrow \quad 2S = n \sum_{0 \le i < j \le n}\sum \left(\frac{1}{{}^nC_i} + \frac{1}{{}^nC_j}\right)$$

$$\Rightarrow \quad 2S = n\left(\sum_{r=0}^{n}\left(\frac{n-r}{{}^nC_r}\right) + \sum_{r=0}^{n}\left(\frac{r}{{}^nC_r}\right)\right)$$

$$\Rightarrow \quad 2S = n\left(\sum_{r=0}^{n}\left(\frac{n}{{}^nC_r}\right)\right)$$

$$\Rightarrow \quad 2S = n^2\left(\sum_{r=0}^{n}\left(\frac{1}{{}^nC_r}\right)\right) = n^2 a$$

$$\Rightarrow \quad S = \frac{n^2 a}{2}$$

42. We have,

$$\sum_{i=0}^{n}\sum_{j=0}^{n}(C_i + C_j)$$

$$= \sum_{i=0}^{n}\sum_{j=0}^{\text{n}}(C_i) + \sum_{i=0}^{n}\sum_{j=0}^{n}(C_j)$$

$$= \sum_{j=0}^{n}\left(\sum_{i=0}^{n}(C_i)\right) + \sum_{i=0}^{n}\left(\sum_{j=0}^{n}(C_j)\right)$$

$$= \sum_{j=0}^{n}(2^n) + \sum_{i=0}^{n}(2^n)$$

$$= 2^n\sum_{j=0}^{n}(1) + 2^n\sum_{i=0}^{n}(1)$$

$$= 2^n(n+1) + 2^n(n+1)$$

$$= (n+1)2^{n-1}$$

43. We know that

$$\sum_{i=0}^{n}\sum_{j=0}^{n}(C_i + C_j) = \sum_{i=0}^{n}(C_i + C_j) + 2\sum_{0 \le i < j \le n}^{\text{n}}\sum (C_i + C_j)$$

$$\Rightarrow \quad (n+1)2^{n+1} = 2^n + 2^n + 2\sum_{0 \le i < j \le n}^{n}\sum (C_i + C_j)$$

$$\Rightarrow \quad (n+1)2^{n+1} = 2.2^n + 2\sum_{0 \le i < j \le n}^{n}\sum (C_i + C_j)`$$

$$\Rightarrow \quad (n+1)2^n = 2^n + \sum_{0 \le i < j \le n}^{n}\sum (C_i + C_j)$$

$$\Rightarrow \quad \sum_{0 \le i < j \le n}^{n}\sum (C_i + C_j) = n \cdot 2^n$$

44. We have,

$$\sum_{i=0}^{n}\sum_{j=0}^{n}(C_iC_j) = \sum_{i=0}^{n}(C_i)\sum_{j=0}^{n}(C_j)$$

$$= (2^n) \times (2^n)$$

$$= (2^{2n})$$

45. We know that

$$\sum_{i=0}^{n}\sum_{j=0}^{n}(C_i\,C_j) = \left(\sum_{i=0}^{n}C_i^2\right) + 2\sum_{0 \le i < j \le n}^{n}\sum (C_iC_j)$$

$$\Rightarrow \quad \sum_{i=0}^{n}(C_i)\sum_{j=0}^{n}(C_j) = \left(\sum_{i=0}^{n}C_i^2\right) + 2\sum_{0 \le i < j \le n}^{n}\sum (C_iC_j)$$

$$\Rightarrow \quad 2^n \cdot 2^n = {}^{2n}C_n + 2 \cdot \sum_{0 \le i < j \le n}^{n}\sum (C_iC_j)$$

$$\Rightarrow \quad 2^{2n} = {}^{2n}C_n + 2 \cdot \sum_{0 \le i < j \le n}^{n}\sum (C_iC_j)$$

$$\Rightarrow \quad \sum_{0 \le i < j \le n}^{n}\sum (C_iC_j) = \frac{1}{2}(2^{2n} - {}^{2n}C_n)$$

46. We have,

$$\sum_{0 \le i < j \le n}^{n}\sum ((i \times j)C_iC_j)$$

$$= \sum_{0 \le i < j \le n}^{n} \sum [(i \times C_i)(j \times C_j)]$$

$$= \sum_{0 \le i < j \le n}^{n} \sum [(i \times {}^nC_i)(j \times {}^nC_j)]$$

$$= \sum_{0 \le i < j \le n}^{n} \sum (n \cdot {}^{n-1}C_{i-1})(n \cdot {}^{n-1}C_{j-1})$$

$$= n^2 \cdot \frac{1}{2}(2^{2(n-1)} - {}^{2(n-1)}C_{n-1})$$

47. We have,

$$F(n) = \left(\sum_{i=0}^{n} \sum_{j=1}^{n} ({}^nC_j \cdot {}^jC_i) \right)$$

$$= \left(\sum_{j=1}^{n} ({}^nC_j \cdot ({}^jC_0 + {}^jC_1 + {}^jC_2 + \cdots + {}^jC_j)) \right)$$

$$= \left(\sum_{j=1}^{n} ({}^nC_j \cdot 2^j) \right)$$

$$= \left(\sum_{j=0}^{n} ({}^nC_j \cdot 2^j) \right) - {}^nC_0$$

$= (1 + 2)^n - 1$

$= 3^n - 1$

Hence, the value of $F(10)$

$= (3^{10} - 1)$

48. We have, $\sum_{i=0}^{n} \sum_{j=0}^{n} \sum_{m=0}^{n} \sum_{p=0}^{n} (1)$

$$= \sum_{i=0}^{n} (1) \sum_{j=0}^{n} (1) \sum_{m=0}^{n} (1) \sum_{p=0}^{n} (1)$$

$= (n + 1).(n + 1).(n + 1).(n + 1)$

$= (n + 1)^4$

49. We have,

$(1 + x)^n$

$= {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_n x^n$ ...(i)

Also, $(x + 1)^n$

$= {}^nC_0 x^n + {}^nC_1 x^{n-1} + {}^nC_2 x^{n-2} + \ldots + {}^nC_n$ ...(ii)

Multiplying Eqs (i) and (ii), we get

$$(1 + x)^{2n} = ({}^nC_0^2 + {}^nC_0^2 + {}^nC_0^2 + \cdots + {}^nC_0^2)x^n + (\ldots)x^{n-1} + (\ldots)x^{n-2} + \cdots$$

Comparing the co-efficients of x^n from both the sides we get,

$$({}^nC_0^2 + {}^nC_0^2 + {}^nC_0^2 + \cdots + {}^nC_0^2) = {}^{2n}C_n = \frac{(2n)!}{(n)! \times (n)!}$$

50. Since ${}^nC_4, {}^nC_5, {}^nC_6$ are in AP.

$\Rightarrow \quad 2\,{}^nC_5 = {}^nC_4 + {}^nC_6$

$$\Rightarrow \quad 2\frac{n!}{5! \times (n-5)!} = \frac{n!}{4! \times (n-4)!} + \frac{n!}{6! \times (n-6)!}$$

$$\Rightarrow \quad \frac{2}{5 \times (n-5)} = \frac{1}{(n-4) \times (n-5)} + \frac{1}{6 \times 5}$$

$$\Rightarrow \quad \frac{2}{5 \times (n-5)} - \frac{1}{(n-4) \times (n-5)} = \frac{1}{6 \times 5}$$

$$\Rightarrow \quad \frac{2(n-4) - 5}{5 \times (n-4) \times (n-5)} = \frac{1}{6 \times 5}$$

$$\Rightarrow \quad \frac{2n - 13}{(n-4) \times (n-5)} = \frac{1}{6}$$

$\Rightarrow \quad 6(2n - 13) = (n - 4) \times (n - 5)$

$\Rightarrow \quad 12n - 78 = n^2 - 9n + 20$

$\Rightarrow \quad n^2 - 21n + 98 = 0$

$\Rightarrow \quad (n - 7)(n - 14) = 0$

$\Rightarrow \quad n = 14, 7$

51. Let $t_{r+1} = {}^8C_r \left(\frac{x^{1/8}}{2}\right)^{r-8} (x^{-1/8})^r$

$$= {}^8C_r \left(\frac{1}{2}\right)^{\frac{8-r}{8}} (x)^{\frac{8-r-r}{8}}$$

Now, $\frac{8 - 2r}{8} = 0$

$\Rightarrow \quad 2r = 8$

$\Rightarrow \quad r = 4$

Thus, the 5th term does not contain x.

52. We have

$$1 + \frac{(a + bx)}{1!} + \frac{(a + bx)^2}{2!} + \frac{(a + bx)^2}{3!} + \cdots$$

$= e^{a+bx}$

$= e^a.e^{bx}$

$$= e^a \cdot \left(1 + \frac{bx}{1!} + \frac{(bx)^2}{2!} + \cdots + \frac{(bx)^n}{n!} + \cdots\right)$$

Thus,

the co-efficient of $x^n = e^a \times \frac{b^n}{n!}$

53. We have,

$(x + a)^n$

$$= {}^nC_0 x^n + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + {}^nC_3 x^{n-3} a^3 + {}^nC_4 x^{n-4} a^4 + {}^nC_5 x^{n-5} a^5 + {}^nC_6 x^{n-6} a^6 + \cdots$$

$= t_1 + t_2 + t_3 + t_4 + t_5 + t_6 + \ldots$

$= (t_1 + t_2 + t_5 + \ldots) + (t_2 + t_4 + t_6 + \ldots)$

$= A + B$ ...(i)

Also, $(x - a)^n$

$$= {}^nC_0 x^n - {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 - {}^nC_3 x^{n-3} a^3 + {}^nC_4 x^{n-4} a^4 - {}^nC_5 x^{n-5} a^5 + {}^nC_6 x^{n-6} a^6 + \cdots$$

$= (t_1 + t_3 + t_5 + \ldots) - (t_2 + t_4 + t_6 + \ldots)$

$= A - B$ ...(ii)

Multiplying Eqs (i) and (ii), we get

Thus, $A^2 - B^2 = (x + a)^n(x - a)^n$

$= (x^2 - a^2)^n$

53. Given expression = $(5p - 4q)^n$

Put $p = 1$, $q = 1$

Thus,

the sum of the co-efficients = $(5 - 4)^n$

$= 1$

54. Let $t_{r+1} = (-1)^r(5r - 2)C_r$

$= (-1)^r(5r - 2)^nC_r$

$= (-1)^r(5r\,{}^nC_r - 2\,{}^nC_r)$

$= (-1)^r(5n\,{}^{n-1}C_{r-1} - 2\,{}^nC_r)$

$= (-1)^r5n\,{}^{n-1}C_{r-1} - (-1)^r2\,{}^nC_r$

$\therefore \quad S_n = \sum_{r=0}^{n} t_{r+1}$

$$= 5n\sum_{r=0}^{n}(-1)^r\,{}^{n-1}C_{r-1} - 2\sum_{r=0}^{n}(-1)^r\,{}^nC_r$$

$= 5n.0 - 2.0$

$= 0$

Thus, the sum of $S_{n+1} = 0$.

55. Let the sum

where

$$\text{Let} \quad S = \left(1 + 2\left(\frac{x}{1+x}\right) + 3\left(\frac{x}{1+x}\right)^2 + \cdots + 1001\left(\frac{x}{1+x}\right)^{1000}\right) \quad \text{(i)}$$

$$\therefore \quad \left(\frac{x}{1+x}\right)S = \left(\left(\frac{x}{1+x}\right) + 2\left(\frac{x}{1+x}\right)^2 + \cdots + 1000\left(\frac{x}{1+x}\right)^{1000} + 1001\left(\frac{x}{1+x}\right)^{1001}\right) \quad \text{(ii)}$$

On subtracting Eq. (ii) from Eq. (i) we get

$$\left(1 - \frac{x}{x+1}\right)S$$

$$= 1 + \left(\frac{x}{x+1}\right) + \left(\frac{x}{x+1}\right)^2 + \cdots + \left(\frac{x}{x+1}\right)^{1000} - \left(\frac{x}{x+1}\right)^{1001}$$

$$= \frac{1 - \left(\frac{x}{x+1}\right)^{1001}}{1 - \left(\frac{x}{1+x}\right)} - \left(\frac{x}{x+1}\right)^{1001}$$

$$= \frac{(x+1)^{1001} - x^{1001}}{(x+1)^{1000}} - \left(\frac{x}{x+1}\right)^{1001}\Bigg]$$

Therefore the sum,

$$(1 + x)^{1000} \times \left(1 + 2\left(\frac{x}{1+x}\right) + 3\left(\frac{x}{1+x}\right)^2 + \cdots + 1001\left(\frac{x}{1+x}\right)^{1000}\right)$$

$= (1 + x)^{1000} \times S,$

$$(1 + x^{1000})\left(\frac{(x+1)^{1001} - x^{1001}}{(x+1)^{1000}} - \left(\frac{x}{x+1}\right)^{1001}\right)$$

$$= \left((x+1)^{1001} - x^{1001} - \left(\frac{x^{1001}}{x+1}\right)\right)$$

Thus, the co-efficient of x^{50} is ${}^{1001}C_{50}$.

56. We have,

$$\frac{C_k}{C_{k-1}} = \frac{{}^nC_k}{{}^nC_{k-1}} = \frac{(n-k+1)}{k}$$

Now,

$$k^3 \times \frac{(n-k+1)^2}{k^2} = k \times (n - k + 1)^2$$

$= k \times [(n + 1) - k]^2$

$= k \times [(n + 1)^2 - 2(n + 1)k + k^2]$

$= (n + 1)^2\,k - 2(n + 1)k^2 + k^3$

Therefore, the sum

$$= \sum_{k=1}^{n} k^3\left(\frac{C_k}{C_{k-1}}\right)^2$$

$$= \sum_{k=1}^{n}((n+1)^2k - 2(n+1)k^2 + k^3)$$

$$= (n+1)^2\sum_{k=1}^{n}k - 2(n+1)\sum_{k=1}^{n}k^2 + \sum_{k=1}^{n}k^3$$

$$= (n+1)^2 \times \frac{n(n+1)}{2} - 2(n+1)\left(\frac{n(n+1)(2n+1)}{6}\right) + \left(\frac{n(n+1)}{2}\right)^2$$

$$= \frac{n(n+1)^2}{12}(6n + 6 - 8n - 4 + 3n)$$

$$= \frac{n(n+1)^2(n+2)}{12}$$

57. Given,

$t_3 = 10000$

$\Rightarrow \quad {}^5C_2\,(x)^{5-2}(x^{\log_{10}x})^2 = 10{,}000$

$\Rightarrow \quad 10 \cdot (x)^3(x^{2\log_{10}x}) = 10{,}000$

$\Rightarrow \quad (x)^{3+2\log_{10}x} = 10{,}00$

$\Rightarrow \quad (x)^{3+2\log_{10}x} = 10^3$

$\Rightarrow \quad 3 + 2\log_{10}x = \log_{10^3} x = \frac{1}{3}\log_{10}x$

$\Rightarrow \quad \left(2 - \frac{1}{3}\right)\log_{10}x = -3$

$\Rightarrow \quad \log_{10}x = -\frac{9}{5}$

$\Rightarrow \quad x = 10^{-9/5}$

58. We have,

$$^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$$

$= {}^{47}C_4 + {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{47}C_3$
$= ({}^{47}C_4 + {}^{47}C_3) + {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3$
$= ({}^{48}C_4 + {}^{48}C_3) + {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3$
$= ({}^{49}C_4 + {}^{49}C_3) + {}^{51}C_3 + {}^{50}C_3$
$= ({}^{50}C_4 + {}^{50}C_3) + {}^{51}C_3$
$= |{}^{51}C_4 + {}^{51}C_3|$
$= {}^{52}C_4$

59. Let t_2, t_3, t_5 be the first, third and 5th terms, respectively of the given AP.

Clearly,

$2 \cdot {}^mC_2 = {}^mC_1 + {}^mC_3$

$\Rightarrow \quad 2 \cdot \left(\frac{m(m-1)}{2}\right) = m + \left(\frac{m(m-1)(m-2)}{6}\right)$

$\Rightarrow \quad 6m(m-1) = (6m + m(m-1)(m-2))$

$\Rightarrow \quad 6(m-1) = [6 + (m-1)(m-2)]$

$\Rightarrow \quad m^2 - 9m + 14 = 0$

$\Rightarrow \quad m = 2, 7$

Since the 6th term is 21, so the value of $m = 7$.

It is also given that

$${}^7C_5\left(\sqrt{2^{\log(10-3^x)}}\right)^{7-5}\left(\sqrt[5]{2^{(x-2)\log 3}}\right)^5 = 21$$

$\Rightarrow \quad 21\left(\sqrt{2^{\log(10-3^x)}}\right)^2\left(\sqrt[5]{2^{(x-2)\log 3}}\right)^5 = 21$

$\Rightarrow \quad \left(\sqrt{2^{\log(10-3^x)}}\right)^2\left(\sqrt[5]{2^{(x-2)\log 3}}\right)^5 = 1$

$\Rightarrow \quad (2^{\log(10-3^x)})(2^{(x-2)\log 3}) = 1$

$\Rightarrow \quad |2^{\log(10-3^x)+(x-2)\log 3}| = 2^0$

$\Rightarrow \quad \log(10-3^x) + (x-2)\log 3 = 0$

$\Rightarrow \quad \log(10-3^x) + \log 3^{(x-2)} = 0$

$\Rightarrow \quad \log\{(10-3^x)\cdot 3^{(x-2)}\} = \log(1)$

$\Rightarrow \quad |(10-3^x)\cdot 3^{(x-2)}| = 1$

$\Rightarrow \quad \left|\frac{(10-3^x)\cdot 3^x}{9}\right| = 1$

$\Rightarrow \quad (3^x)^2 - 10(3^x) + 9 = 0$

$\Rightarrow \quad (3^x - 9)(3^x - 1) = 0$

$\Rightarrow \quad 3x = 9, 1$

$\Rightarrow \quad x = 2, 0$

Hence, the values of x are 0 and 2.

60. We have,

$$\left(2 + \frac{3x}{8}\right)^{10} = 2^{10}\left(1 + \frac{3}{16}x\right)^{10}$$

Now, $\frac{t_4}{t_5} = \left(\frac{10-4+1}{4-1}\right)\left(\frac{3x}{16}\right) \geq 1$

$\Rightarrow \quad \frac{7}{3}\left(\frac{3x}{16}\right) \geq 1$

$\Rightarrow \quad \left(\frac{7x}{16}\right) \geq 1$

$\Rightarrow \quad x \geq \frac{16}{7}$

61. Given,

$$(1 + x^2)^2(1 + x)^n = \sum_{k=0}^{n+4} a_k x^k$$

$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$
$= (1 + 2x^2 + x^4)(1 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \cdots)$

Comparing the co-efficients of x, x^2 and x^3, we get

$a_1 = {}^nC_1,\ a_2 = {}^nC_2 + 2,\ a_3 = {}^nC_3 + 2\,{}^nC_1$

Since $a_1, a_2, a_3 \in$ AP

$\Rightarrow \quad 2a_2 = a_1 + a_3$

$\Rightarrow \quad 2({}^nC_2 + 2) = {}^nC_1 + ({}^nC_3 + 2 \cdot {}^nC_1)$

$\Rightarrow \quad 2\left(\frac{n(n-1)}{2} + 2\right) = n + \left(\frac{n(n-1)(n-2)}{6} + 2n\right)$

$\Rightarrow \quad n(n-1) + 4 = 3n + \frac{n(n-1)(n-2)}{6}$

$\Rightarrow \quad 6n(n-1) + 24 = 18n + n(n-1)(n-2)$

$\Rightarrow \quad 6n^2 - 6n + 24 = 18n + n(n^2 - 3n + 2)$

$\Rightarrow \quad 6n^2 - 6n + 24 = 18n + (n^3 - 3n^2 + 2n)$

$\Rightarrow \quad n^3 - 9n^2 + 26n - 24 = 0$

$\Rightarrow \quad (n-2)(n-3)(n-4) = 0$

$\Rightarrow \quad n = 2, 3, 4$

62 Given expansion is $(x + a)^{15}$.

$\Rightarrow \quad t_{11} = {}^{15}C_{10}x^5a^{10},\ t_8 = {}^{15}C_7x^7a^8,\ t_{12} = {}^{15}C_{11}x^4a^{11}$

Now it is given that, $(t_{11})^2 = (t_{12}.t_8)$

$\Rightarrow \quad ({}^{15}C_{10}x^5a^{10})^2 = ({}^{15}C_7x^7a^8 \times {}^{15}C_{11}x^4a^{11})$

$\Rightarrow \quad ({}^{15}C_{10} \times {}^{15}C_{10} \times x^{10}a^{20}) = ({}^{15}C_7 \times {}^{15}C_{11} \times x^{11}a^{19})$

$\Rightarrow \quad \frac{x}{a} = \left(\frac{{}^{15}C_{10} \times {}^{15}C_{10}}{{}^{15}C_7 \times {}^{15}C_{11}}\right) = 0.986$

For the greatest term,

$$m = \frac{16 \times (a/x)}{1 + (a/x)} = \frac{16 \times 0.986}{1.986} = 7.943$$

Hence, the greatest term is 8th term.

63. Given $\quad 2^n = 4096 = 2^{12}$

$\Rightarrow \quad n = 12$

The largest co-efficient is the middle term of the given expansion.

Middle term = $\left(\frac{12}{2}+1\right)$th = 7th term

Thus, $t_7 = t_{6-1} = {}^{12}C_6 x^6$

Therefore, the co-efficient of the middle term = ${}^{12}C_6$

64. We have

$$\left(x-\frac{C_1}{C_0}\right)\left(x-2^2\frac{C_2}{C_1}\right)\left(x-3^2\frac{C_3}{C_2}\right)\cdots\left(x-50^2\frac{C_{50}}{C_{49}}\right)$$

$= (x-1)(x-2^2\cdot 2)(x-3^2\cdot 3)\ldots(x-50^2\cdot 50)$

$= (x-1)(x-2^3)(x-3^3)\ldots(x-50^3)$

Thus,

the co-efficient of x^{49}

$= -(1+2^3+3^3+\ldots+50^3)$

$= -\left(\frac{50\times 51}{2}\right)^2$

$= -(25\times 51)^2$

$= -(1275)^2$

65. Let $t_r = \frac{C_r}{(r+1)(r+2)(r+3)}$

$$= \frac{{}^nC_r}{(r+1)(r+2)(r+3)}$$

$$= \frac{{}^{n+3}C_{r+3}}{(n+1)(n+2)(n+3)}$$

Thus, $S_n = \sum_{r=0}^{n} t_r$

$$= \sum_{r=0}^{n}\left(\frac{{}^{n+3}C_{r+3}}{(n+1)(n+2)(n+3)}\right)$$

$$= \left(\frac{2^{n+3} - {}^{n+3}C_0 - {}^{n+3}C_1 - {}^{n+3}C_2}{(n+1)(n+2)(n+3)}\right)$$

$$= \left(\frac{2^{n+3} - 1 - (n+3) - (n+2)(n+3)/2}{(n+1)(n+2)(n+3)}\right)$$

$$= \left(\frac{2^{n+4} - 2 - (2n+6) - (n+2)(n+3)}{2(n+1)(n+2)(n+3)}\right)$$

$$= \frac{2^{n+4} - n^2 - 7n - 14}{2(n+1)(n+2)(n+3)}$$

66. We have,

$(1+x)^n$

$$= 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

Comparing the co-efficient of x and x^2, we get

$$nx = \frac{1}{3}, \frac{n(n-1)}{2}x^2 = \frac{1}{6}$$

$$\Rightarrow \quad nx = \frac{1}{3}, \frac{nx(nx-x)}{2} = \frac{1}{6}$$

$$\Rightarrow \quad \frac{nx(nx-x)}{2} = \frac{1}{6}$$

$$\Rightarrow \quad nx(nx-x) = \frac{1}{3}$$

$$\Rightarrow \quad \frac{1}{3}\left(\frac{1}{3}-x\right) = \frac{1}{3}$$

$$\Rightarrow \quad \left|\frac{1}{3}-x\right| = 1$$

$$\Rightarrow \quad x = \frac{1}{3} - 1 = -\frac{2}{3}$$

Putting $x = -\frac{2}{3}$, we get,

$$n \times -\frac{2}{3} = \frac{1}{3}$$

$$\Rightarrow \quad n = -\frac{1}{2}$$

Therefore,

$$1 + \frac{1}{3} + \frac{1.3}{3.6} + \frac{1.3.5}{3.6.9} + \cdots = (1+x)^n$$

$$= \left(1-\frac{2}{3}\right)^{-1/2}$$

$$= \left(\frac{1}{3}\right)^{-1/2}$$

$$= \sqrt{3}$$

67. Given

$$\frac{e^x}{1-x} = B_0 + B_1 x + B_2 x^2 + \cdots + B_n x^n$$

Now,

$e^x \times (1-x)^{-1}$

$$= \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots\right)$$

$$\times (1 + x + x^2 + x^3 + \ldots + x^n + \ldots)$$

$$= \left(\frac{1}{n!} + \frac{1}{(n-1)!} + \frac{1}{(n-2)!} + \cdots + \frac{1}{1!} + 1\right)x^n$$

$$+ \left(\frac{1}{(n-1)!} + \frac{1}{(n-2)!} + \frac{1}{(n-3)!} + \cdots + \frac{1}{1!} + 1\right)x^{n-1}$$

Thus, comparing the co-efficients of x^{n-1} and x^n, we get

$$B_n = \frac{1}{n!} + \frac{1}{(n-1)!} + \frac{1}{(n-2)!} + \cdots + \frac{1}{1!} + 1$$

and

$$B_{n-1} = \frac{1}{(n-1)!} + \frac{1}{(n-2)!} + \frac{1}{(n-3)!} + \cdots + \frac{1}{1!} + 1$$

Therefore, $B_n - B_{n-1} = \frac{1}{n!}$.

68. We have,

$$a = \sum_{n=1}^{\infty}\left(\frac{2n}{(2n-1)!}\right)$$

$$= \sum_{n=1}^{\infty}\left(\frac{(2n-1)+1}{(2n-1)!}\right)$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{(2n-2)!}+\frac{1}{(2n-1)!}\right)$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{(2n-2)!}\right)+\sum_{n=1}^{\infty}\left(\frac{1}{(2n-1)!}\right)$$

$$= \frac{1}{2}(e+e^{-1})-\frac{1}{2}(e-e^{-1})$$

$$= e$$

Also,

$$b = \sum_{n=1}^{\infty}\left(\frac{2n}{(2n+1)!}\right)$$

$$= \sum_{n=1}^{\infty}\left(\frac{(2n+1)-1}{(2n+1)!}\right)$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{(2n)!}-\frac{1}{(2n+1)!}\right)$$

$$= \sum_{n=1}^{\infty}\left(\frac{1}{(2n)!}\right)-\sum_{n=1}^{\infty}\left(\frac{1}{(2n+1)!}\right)$$

$$= \frac{1}{2}(e+e^{-1}-1)-\frac{1}{2}\left(e-e^{-1}-\frac{1}{2}\right)$$

$$= \frac{1}{e}$$

Therefore, $ab = 1$.

69. Let $t_n = \dfrac{1+2+3+\cdots+n}{n!}$

$$= \frac{n(n+1)}{2\times n!}$$

$$= \frac{1}{2}\frac{(n+1)}{(n-1)!}$$

$$= \frac{1}{2}\frac{[(n-1)+2]}{(n-1)!}$$

$$= \frac{1}{2}\left(\frac{1}{(n-2)!}+\frac{1}{(n-1)!}\right)$$

Therefore, $S = \sum_{n=1}^{\infty} t_n$

$$= \sum_{n=1}^{\infty}\frac{1}{2}\left(\frac{1}{(n-2)!}+\frac{1}{(n-1)!}\right)$$

$$= \frac{1}{2}\left(\sum_{n=1}^{\infty}\left(\frac{1}{(n-2)!}\right)+\sum_{n=1}^{\infty}\left(\frac{1}{(n-1)!}\right)\right)$$

$$= \frac{1}{2}(e+e)$$

$$= e$$

70. We have,

$$\frac{C(n,4)}{P(n,n)} = \frac{{}^nC_4}{n!}$$

$$= \frac{(n)!}{4!\times(n-4)!}\times\frac{1}{n!}$$

$$= \frac{1}{4!\times(n-4)!}$$

$$= \frac{1}{24}\times\frac{1}{(n-4)!}$$

Therefore,

$$\sum_{n=4}^{\infty}\left(\frac{C(n,4)}{P(n,n)}\right) = \sum_{n=4}^{\infty}\frac{1}{24}\times\frac{1}{(n-4)!}$$

$$= \frac{1}{24}\left(\sum_{n=4}^{\infty}\frac{1}{(n-4)!}\right)$$

$$= \frac{e}{24}$$

71. We have,

$$a = \sum_{n=0}^{\infty}\left(\frac{x^{3n}}{(3n)!}\right)$$

$$= \left(1+\frac{x^3}{3!}+\frac{x^6}{6!}+\frac{x^9}{9!}+\cdots\right)$$

Now,

$$b = \sum_{n=0}^{\infty}\left(\frac{x^{3n-2}}{(3n-2)!}\right)$$

$$= \left(\frac{x}{1}+\frac{x^4}{4!}+\frac{x^7}{7!}+\frac{x^{10}}{10!}+\cdots\right)$$

Also,

$$c = \sum_{n=0}^{\infty}\left(\frac{x^{3n-1}}{(3n-1)!}\right)$$

$$= \left(\frac{x^2}{2!}+\frac{x^5}{5!}+\frac{x^8}{8!}+\frac{x^{11}}{11!}+\cdots\right)$$

Now,

$$a+b+c = 1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\cdots = e^x$$

$$a+b\omega+c\omega^2 = e^{\omega x}$$

and $a+b\omega^2+c\omega = e^{\omega^2 x}$

Therefore,

$$a^3+b^3+c^3-3abc$$

$$= (a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$$

$$= e^x\cdot e^{\omega x}\cdot e^{\omega^2 x}$$

$$= e^{x(1+\omega+\omega^2)}$$

$$= e^{x.0} = 1$$

72. Let $t_n = \dfrac{1}{(2n-1)2n(2n+1)}$

$$= \frac{1}{2}\left(\frac{(2n+1)-(2n-1)}{(2n-1)2n(2n+1)}\right)$$

$$= \frac{1}{2}\left(\frac{1}{(2n-1)2n} - \frac{1}{(2n-1)2n}\right)$$

$$= \frac{1}{2}\left(\frac{2n-(2n-1)}{(2n-1)2n} - \frac{(2n+1)-2n}{(2n+1)2n}\right)$$

$$= \frac{1}{2}\left(\frac{1}{(2n-1)} - \frac{1}{2n} - \frac{1}{2n} + \frac{1}{(2n+1)}\right)$$

$$= \frac{1}{2}\left(\frac{1}{(2n-1)} - \frac{1}{n} + \frac{1}{(2n+1)}\right)$$

Therefore, the sum

$$= \frac{1}{2}\left[\left(\left(1+\frac{1}{3}+\frac{1}{5}+\cdots\right) - \left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots\right)\right) + \left(\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\cdots\right)\right]$$

$$= \frac{1}{2}\left(-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\cdots\right)$$

$$= \frac{1}{2}\left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\cdots\right) - \frac{1}{2}$$

$$= \frac{1}{2}\log_e 2 - \frac{1}{2}$$

$$= \frac{1}{2}(\log_e 2 - 1)$$

$$= \frac{1}{2}\left(\log_e\left(\frac{2}{e}\right)\right)$$

73. We have

$$1+\left(\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4}+\left(\frac{1}{4}+\frac{1}{5}\right)\frac{1}{4^2}+\left(\frac{1}{6}+\frac{1}{7}\right)\frac{1}{4^3}+\cdots$$

$$= \left(\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{4}\cdot\left(\frac{1}{4}\right)^2+\left(\frac{1}{6}\right)\cdot\left(\frac{1}{4}\right)^3+\cdots\right) + \left(1+\frac{1}{3}\cdot\frac{1}{4}+\frac{1}{5}\cdot\left(\frac{1}{4}\right)^2+\left(\frac{1}{7}\right)\cdot\left(\frac{1}{4}\right)^3+\cdots\right)$$

$$= -\frac{\log\left(1-\frac{1}{16}\right)}{2} + \log\left(\frac{\left(1+\frac{1}{4}\right)}{\left(1-\frac{1}{4}\right)}\right)$$

$$= -\frac{\log\left(\frac{15}{16}\right)}{2} + \frac{1}{2}\log\left(\frac{5}{3}\right)$$

$$= -\frac{1}{2}\left(\log\left(\frac{15}{16}\right) - \log\left(\frac{5}{3}\right)\right)$$

$$= -\frac{1}{2}\left(\log\left(\frac{15}{16}\times\frac{3}{5}\right)\right)$$

$$= -\frac{1}{2}\left(\log\left(\frac{9}{16}\right)\right)$$

$$= -\frac{1}{2}\left(\log\left(\frac{3}{4}\right)^2\right)$$

$$= -\left(\log\left(\frac{3}{4}\right)\right)$$

$$= \left|\log\left(\frac{4}{3}\right)\right|$$

74. Let $\left(\dfrac{\sqrt{2}-1}{\sqrt{2}}\right) = x$

Thus, the sum

$$= 1+\frac{x}{2}+\frac{x^2}{6}+\frac{x^3}{12}+\frac{x^4}{20}+\cdots$$

$$= 1+\frac{x}{1.2}+\frac{x^2}{2.3}+\frac{x^3}{3.4}+\frac{x^4}{4.5}+\cdots$$

$$= \left[1+\left(1-\frac{1}{2}\right)x+\left(\frac{1}{2}-\frac{1}{3}\right)x^2 + \left(\frac{1}{3}-\frac{1}{4}\right)x^3+\left(\frac{1}{4}-\frac{1}{5}\right)x^4+\cdots\right]$$

$$= \left(1+x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots\right) - \left(\frac{x}{2}+\frac{x^2}{3}+\frac{x^3}{4}+\cdots\right)$$

$$= 1+\left(x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots\right) - \frac{1}{x}\left(\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\cdots\right)$$

$$= 1-\log(1-x)-\frac{1}{x}(-\log(1-x)-x)$$

$$= 2+\left(\frac{1}{x}-1\right)\log(1-x)$$

Thus, the sum

$$= 2+\left(\frac{\sqrt{2}}{\sqrt{2}-1}-1\right)\log\left(1-\frac{\sqrt{2}-1}{\sqrt{2}}\right)$$

$$= 2+\frac{1}{(\sqrt{2}-1)}\log\left(\frac{1}{\sqrt{2}}\right)$$

$$= 2+(\sqrt{2}+1)\times -\frac{1}{2}\log(2)$$

$$= 2-\frac{|\sqrt{2}+1|}{2}\times\log(2)$$

75. Let $t_n = \frac{4n+1}{n(n+1)(n+2)}$

$$= \frac{4}{(n+1)(n+2)} + \frac{1}{n(n+1)(n+2)}$$

$$= \frac{4}{(n+1)(n+2)} + \frac{1}{2}\left(\frac{(n+2)-n}{n(n+1)(n+2)}\right)$$

$$= \frac{4}{(n+1)(n+2)} + \frac{1}{2}\left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right)$$

$$= \frac{7/2}{(n+1)(n+2)} + \frac{1}{2}\left(\frac{1}{n(n+1)}\right)$$

$$= \frac{7}{2}\left(\frac{1}{(n+1)} - \frac{1}{(n+2)}\right) + \frac{1}{2}\left(\frac{1}{n} - \frac{1}{(n+1)}\right)$$

Thus, $S = t_1 + t_2 + t_3 + t_4 + \ldots$

$$= -\frac{7}{2}(\log 2 - 1) + \frac{1}{2}\log 2$$

$$= \frac{7}{2} - 3\log 2$$

76. We have,

$$(1+x)^n$$

$$= 1 + nx + \frac{n(n-1)}{2}x^2 + \frac{n(n-1)(n-2)}{6}x^3 + \cdots$$

Thus, $nx = \frac{2}{3}\cdot\frac{1}{2} = \frac{1}{3}$ …(i)

and $\frac{n(n-1)}{2}x^2 = \frac{2}{3}\cdot\frac{5}{6}\cdot\frac{1}{2^2}$

$\Rightarrow$ $nx(nx - x) = \frac{5}{18}$

$\Rightarrow$ $\frac{1}{3}\left(\frac{1}{3} - x\right) = \frac{5}{18}$ from (i)

$\Rightarrow$ $\left|\frac{1}{3} - x\right| = \frac{5}{6}$

$\Rightarrow$ $x = \left|\frac{1}{3} - \frac{5}{6}\right| = -\frac{3}{6} = -\frac{1}{2}$

and $n\cdot\left(-\frac{1}{2}\right) = \frac{1}{3}$

$\Rightarrow$ $n = -\frac{2}{3}$

Hence, the sum

$$= (1+x)^n$$

$$= \left(1 - \frac{1}{2}\right)^{-2/3}$$

$$= 2^{2/3} = 4^{1/3}$$

Level IV

(Tougher Problems for JEE-Advanced)

1. We have,

$$(1 + x + x^2)^n = \sum_{r=0}^{2n} a_r x^r \quad \ldots\text{(i)}$$

$\Rightarrow a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_{2n}x^{2n} = (1 + x + x^2)^n$

Put $x = 1$, we get

$$a_0 + a_1 + a_2 + a_3 + \cdots + a_{2n} = 3^n \quad \ldots\text{(ii)}$$

Put $x = \omega$ in Eq. (i), we get

$$(a_0 + a_3 + a_6 + \ldots) + (a_1 + a_4 + a_7 + \ldots)\omega$$
$$(a_2 + a_5 + a_8 + \ldots)\omega^2 = 0$$

$\Rightarrow$ $A + B\omega + C\omega^2 = 0$

$\Rightarrow$ $A + B\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) + C\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = 0$

$\Rightarrow$ $\left(A - \frac{B}{2} - \frac{C}{2}\right) + i\frac{\sqrt{3}}{2}(B - C) = 0$

Comparing the real and imaginary parts, we get

$\Rightarrow$ $\left(A - \frac{B}{2} - \frac{C}{2}\right) = 0, (B - C) = 0$

$\Rightarrow$ $A = \frac{B+C}{2}, B = C$

$\Rightarrow$ $A = B = C$

From Eq. (ii), we get

$$(a_0 + a_3 + a_6 + \ldots) + (a_1 + a_4 + a_7 + \ldots)$$
$$+ (a_2 + a_5 + a_8 + \ldots) = 3^n$$

Thus,

$$(a_0 + a_3 + a_6 + \ldots) = |a_1 + a_4 + a_7 + \ldots|$$
$$= |a_2 + a_5 + a_8 + \ldots| = 3^{n-1}$$

2. Given,

$$\left(\sum_{r=0}^{2} x^r\right)^n = \sum_{r=0}^{2n} a_r x^r$$

$\Rightarrow (1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_{2n}x^{2n}$ …(i)

Replacing x by $1/x$, we get

$$\left(1 + \frac{1}{x} + \frac{1}{x^2}\right)^n = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \cdots + \frac{a_{2n}}{x^{2n}}$$

$\Rightarrow (1 + x + x^2)^n = |a_0x^{2n} + a_1x^{2n-1} + \cdots + a_{2n-1}x + a_{2n}|$ …(ii)

From Eqs (i) and (ii), we get

$$a_0 = a_{2n}$$
$$a_1 = a_{2n-1}$$
$$a_2 = a_{2n-2}$$
$$\vdots$$
$$a_n = a_n$$
$$\vdots$$
$$a_{2n} = a_0$$

Putting $x = 1$, Eq. in (i), we get

$$a_0 + a_1 + a_2 + \ldots + a_{2n-1} + a_{2n} = 3^n$$

$$\Rightarrow \quad 2(a_0 + a_1 + a_2 + \ldots + a_{n-1}) + a_n = 3^n$$

$$\Rightarrow \quad (a_0 + a_1 + a_2 + \cdots + a_{n-1}) + \frac{a_n}{2} = \frac{3^n}{2}$$

$$\Rightarrow \quad (a_0 + a_1 + a_2 + \cdots + a_{n-1}) = \frac{1}{2}(3^n - a_n)$$

3. Given

$$\left(\sum_{r=0}^{2} x^r\right)^n = \sum_{r=0}^{2n} a_r x^r$$

$$\Rightarrow (1 + x + x^2)^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_{2n} x^{2n} \quad \ldots\text{(i)}$$

Replacing x by $1/x$, we get

$$\left(1 + \frac{1}{x} + \frac{1}{x^2}\right)^n = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \cdots + \frac{a_{2n}}{x^{2n}}$$

$$\Rightarrow (1 + x + x^2)^n = (a_0 x^{2n} + a_1 x^{2n-1} + \cdots + a_{2n-1} x + a_{2n}) \quad \ldots\text{(ii)}$$

From Eq. (i) and (ii), we get,

$$a_0 = a_{2n}$$
$$a_1 = a_{2n-1}$$
$$a_2 = a_{2n-2}$$
$$\vdots$$
$$a_n = a_n$$
$$\vdots$$
$$a_{2n-2} = a_2$$
$$a_{2n-1} = a_1$$
$$a_{2n} = a_0$$

Hence, the value of

$$a_0 a_1 - a_2 a_3 + a_4 a_5 - \ldots - a_{2n-5} a_{2n-4} + a_{2n-3} a_{2n-2} - a_{2n-1} a_{2n}$$

$$= a_0 a_1 - a_2 a_3 + a_4 a_5 - \ldots - a_5 a_4 + a_3 a_2 - a_1 a_0$$

$$= 0$$

4. Let

$$S = \left(1 + 2\left(\frac{x}{1+x}\right) + 3\left(\frac{x}{1+x}\right)^2 + \cdots + 1001\left(\frac{x}{1+x}\right)^{1000}\right) \quad \text{(i)}$$

$$\left(\frac{x}{1+x}\right)S = \left(\left(\frac{x}{1+x}\right) + 2\left(\frac{x}{1+x}\right)^2 + \cdots + 1000\left(\frac{x}{1+x}\right)^{1000} + 1001\left(\frac{x}{1+x}\right)^{1001}\right) \quad \text{(ii)}$$

On subtracting Eq. (ii) from Eq. (i), we get

$$\left(1 - \frac{x}{x+1}\right)S$$

$$= 1 + \left(\frac{x}{x+1}\right) + \left(\frac{x}{x+1}\right)^2 + \cdots + \left(\frac{x}{x+1}\right)^{1000} - \left(\frac{x}{x+1}\right)^{1001}$$

$$= \frac{1 - \left(\frac{x}{x+1}\right)^{1001}}{1 - \left(\frac{x}{1+x}\right)} - \left(\frac{x}{x+1}\right)^{1001}$$

$$= \frac{(x+1)^{1001} - x^{1001}}{(x+1)^{1000}} - \left(\frac{x}{x+1}\right)^{1001}\Bigg]$$

Therefore the sum

$$(1 + x)^{1000} \times \left(1 + 2\left(\frac{x}{1+x}\right) + 3\left(\frac{x}{1+x}\right)^2 + \cdots + 1001\left(\frac{x}{1+x}\right)^{1000}\right)$$

$$= (1 + x)^{1000} \times S,$$

$$= (1 + x^{1000})\left(\frac{(x+1)^{1001} - x^{1001}}{(x+1)^{1000}} - \left(\frac{x}{x+1}\right)^{1001}\right)$$

$$= \left((x+1)^{1001} - x^{1001} - \left(\frac{x^{1001}}{x+1}\right)\right)$$

Thus, the co-efficient of x^{50} is ${}^{1001}C_{50}$.

5. We have,

$$(1 + x^2)^2 (1 + x)^n = \sum_{k=0}^{n+4} a_k x^k$$

$$\Rightarrow \quad (1 + 2x^2 + x^4) \times (1 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \cdots)$$
$$= a_0 + a_1 x + a_2 x^2 + a_3 x_3 + \cdots$$

Comparing the co-efficient of x, x^2 and x^3, we get $a_1 = {}^nC_1$, $a_2 = 2 + {}^nC_2$, $a_3 = 2{}^nC_1 + {}^nC_3$

Since a_1, a_2 and a_3 are in AP, so

$$a_1 + a_3 = 2a_2$$

$$\Rightarrow \quad {}^nC_1 + 2{}^nC_1 + {}^nC_3 = 2(2 + {}^nC_2)$$

$$\Rightarrow \quad 3 \cdot {}^nC_1 + {}^nC_3 = 2(2 + {}^nC_2)$$

$$\Rightarrow \quad 3n + \frac{n(n-1)(n-2)}{6} = 4 + n(n-1)$$

$$\Rightarrow \quad \frac{18n + n^3 - 3n^2 + 2n}{6} = |n^2 - n + 4|$$

$$\Rightarrow \quad n^3 - 3n^2 + 20n = 6n^2 - 6n + 24$$

$$\Rightarrow \quad n^3 - 6n^2 + 26n - 24 = 0$$

$$\Rightarrow \quad (n - 2)(n - 3)(n - 4) = 0$$

$$\Rightarrow \quad n = 2, 3, 4$$

6. We have

$$(x+4)^{20} - C_1(x+4)^{19}.3 + C_2(x+4)^{18}.3^2 - \ldots + C_n 3^{20}$$
$$= (x+4-3)^{20}$$
$$= (1+x)^{20}$$

Thus,

the co-efficient of x^{18} in $(1+x)^{20} = {}^{20}C_{18} = {}^{20}C_2$

$$= \frac{20 \times 19}{2} = 190$$

7. Given expansion is

$$(1+x)(1+x+x^2) \ldots (1+x+x^2+\ldots+x^{2n})$$

putting $x = 1$, we get

$$2.3.4.5.6 \ldots (2n+1) = 1.2.3.4.5.6 \ldots (2n+1)$$
$$= (2n+1)!$$

8. We have

$${}^nC_0 + {}^{n+1}C_1 + {}^{n+2}C_2 + \cdots + {}^{n+k}C_k$$
$$= {}^nC_n + {}^{n+1}C_n + {}^{n+2}C_n + {}^{n+3}C_n + \cdots + {}^{n+k}C_n$$

= Co-efficient of x^n in the expansion of

$$(1+x)^n + (1+x)^{n+1} + (1+x)^{n+2} + \ldots + (1+x)^{n+k}$$
$$= (1+x)^n(1+(1+x)^1+(1+x)^2+\ldots+(1+x)^k)$$
$$= (1+x)^n\left(\frac{(1+x)^{k+1}-1}{(1+x)-1}\right)$$
$$= \frac{(1+x)^{n+k+1}-(1+x)^n}{x}$$

Co-efficient of x^{n+1} in the expansion of

$$(1+x)^{n+k+1} - (1+x)^n$$
$$= {}^{n+k+1}C_n$$
$$= {}^{n+k+1}C_{k+1}$$

9. Putting $x = 1$ and $x = -1$, we get

$$a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + \cdots = 4^{20} = 2^{40}$$
$$a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + \cdots = 2^{20}$$

subtracting, we get

$$2(a_1 + a_3 + a_5 + \cdots + a_{37} + a_{39}) = 2^{20}(2^{20}-1)$$
$$(a_1 + a_3 + a_5 + \cdots + a_{37} + a_{39}) = 2^{19}(2^{20}-1)$$
$$(a_1 + a_3 + a_5 + \cdots + a_{37}) = 2^{19}(2^{20}-1) - a_{39}$$
$$a_1 + a_3 + a_5 + \cdots + a_{37} = 2^{19}(2^{20}-21)$$

Since a_{39} = co-efficient of x^{39} in $(1+x+2x^2)^{20}$
$= 2^{19}.20$

10. We have,

$$\left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}\right)^2$$
$$= \left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}\right) \times \left(1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}\right)$$

Thus,

the co-efficient of x^n

$$= 1 \cdot \frac{1}{n!} + \frac{1}{1!} \cdot \frac{1}{(n-1)!} + \frac{1}{2!} \cdot \frac{1}{(n-2)!} + \frac{1}{3!} \cdot \frac{1}{(n-3)!} + \frac{1}{4!} \cdot \frac{1}{(n-4)!} + \cdots + \frac{1}{n!} \cdot 1$$
$$= \frac{1}{n!}\left(\frac{n!}{0! \cdot n!} + \frac{n!}{1!(n-1)!} + \frac{n!}{2!(n-2)!} + \cdots + \frac{n!}{n!(0)!}\right)$$
$$= \frac{1}{n!}({}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n)$$
$$= \frac{2^n}{n!}$$

11. Let

$$I = \sum_{0 \le i < j \le n} \sum ({}^nC_i)^2$$
$$= \sum_{r=0}^{n-1} ({}^nC_r)^2 (1+1+\cdots+1)(n-r) \text{ times}$$
$$= \sum_{r=0}^{n-1} (n-r)({}^nC_r)^2 \quad \ldots(i)$$
$$= \sum_{r=0}^{n} r({}^nC_r)^2 \quad \ldots(ii)$$

Adding Eqs (i) and (ii), we get

$$2I = \sum_{r=0}^{n} n({}^nC_r)^2 = n\sum_{r=0}^{n} ({}^nC_r)^2$$
$$\Rightarrow \quad I = \frac{n}{2}({}^{2n}C_n)$$

Similarly,

$$\sum_{0 \le i < j \le n} \sum ({}^nC_j)^2 = \frac{n}{2}({}^{2n}C_n)$$

Thus, $\displaystyle\sum_{0 \le i < j \le n} \sum ({}^nC_i - {}^nC_j)^2$

$$= \sum_{0 \le i < j \le n} \sum ({}^nC_i)^2 + ({}^nC_j)^2 - 2({}^nC_i \cdot {}^nC_j)$$
$$= \frac{n}{2}\,{}^{2n}C_n + \frac{n}{2}\,{}^{2n}C_n - (2^{2n} - {}^{2n}C_n)$$
$$= n\,{}^{2n}C_n - (2^{2n} - {}^{2n}C_n)$$
$$= (n+1)\,{}^{2n}C_n - 2^{2n}$$

12. Let $I = \displaystyle\sum_{0 \le i < j \le n} \sum (i+j)C_iC_j$

$$= \sum_{0 \le i < j \le n} \sum (i+j)\,{}^nC_i\,{}^nC_j$$
$$= \sum_{0 \le i < j \le n} \sum (n-i+n-j)\,{}^nC_i\,{}^nC_j$$
$$= \sum_{0 \le i < j \le n} \sum (2n-i-j)\,{}^nC_i\,{}^nC_j$$

Thus, $2I = \sum_{0 \le i < j \le n} \sum 2n \, {}^nC_i \, {}^nC_j$

$= 2n \sum_{0 \le i < j \le n} \sum {}^nC_i \, {}^nC_j$

$\Rightarrow \quad I\, I = n \sum_{0 \le i < j \le n} \sum {}^nC_i \, {}^nC_j$

$= n\left(\dfrac{\left(\sum_{i=0}^{n} {}^nC_i\right)^2 - \sum_{i=0}^{n} ({}^nC_i)^2}{2}\right)$

$= \dfrac{n}{2}(2^{2n} - {}^{2n}C_n)$

13. Let $I = \sum_{i=0}^{n} \sum_{j=0}^{n} {}^nC_j \, {}^jC_i$

$= {}^nC_1({}^1C_0 + {}^1C_1) + {}^nC_2({}^2C_0 + {}^2C_1 + {}^2C_2)$
$+ {}^nC_3({}^3C_0 + {}^3C_1 + {}^3C_2 + {}^3C_3) + \ldots$
$+ {}^nC_n({}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_n)$
$= {}^nC_1(2) + {}^nC_2(2)^2 + {}^nC_3(2)^3 + \ldots + {}^nC_n(2)^n$
$= (1 + 2)^n - 1$
$= (3^n - 1)$

14. Put $y = x - 3$, then

$x - 2 = 1 + y$

Thus, $\sum_{r=0}^{2n} a_r (1+y)^r = \sum_{r=0}^{2n} b_r y^r$

$a_0 + a_1(1 + y) + a_2(1 + y)^2 + a_3(1 + y)^3$
$+ a_4(1 + y)^4 + \ldots + a_{n-1}(1 + y)^{n-1}$
$+ a_n(1 + y)^n + a_{n+1}(1 + y)^{n+1} + \ldots + a_{2n}(1 + y)^{2n}$
$= b_0 + b_1y + b_2y^2 + b_3y^3 + \ldots + b_{2n}y^{2n}$

It is given that $a_k = 1$, for $k \ge n$.

Thus, $a_0 + a_1(1 + y) + a_2(1 + y)^2 + a_3(1 + y)^3 + a_4(1 + y)^4 + \ldots + a_{n-1}(1 + y)^{n-1}$
$+ (1 + y)^n + (1 + y)^{n+1} + \ldots + (1 + y)^{2n}$
$= b_0 + b_1y + b_2y^2 + b_3y^3 + \ldots + b_{2n}y^{2n}$

Comparing the co-efficients of y^n from both sides, we get

$b_n = {}^nC_n + {}^{n+1}C_n + {}^{n+2}C_n + \ldots + {}^{n+n}C_n$
$b_n = {}^nC_n + {}^{n+1}C_n + {}^{n+2}C_n + \ldots + {}^{2n}C_n$
$b_n = {}^{2n+1}C_{n+1}$

15. We have

$\sum_{r=0}^{m} \binom{10}{i}\binom{20}{m-i}$ = the number of ways of selecting m items out of 10 items of one kind and 20 items of another kind

= the number of ways of selecting m out of 30 items

$= {}^{30}C_m$

But ${}^{30}C_m$ is maximum when $m = 15$.

16. We have

${}^nC_{r-1} = (k^2 - 3)\, {}^{n-1}C_r$

$\Rightarrow \quad (k^2 - 3) = \dfrac{{}^nC_{r+1}}{{}^{n-1}C_r} = \dfrac{n}{r+1}$

Here, $1 \le r \le n + 1$

$\Rightarrow \quad 0 < \dfrac{n}{n+1} \le \dfrac{r}{n+1} \le 1$

Thus, $0 < k^2 - 3 \le 1$

$\Rightarrow \quad 3 \le k^2 \le 4$

$\Rightarrow \quad \sqrt{3} < k \le 2$

17. We have

$A_r = {}^{10}C_r,\ B_r = {}^{10}C_r,\ C_r = {}^{30}C_r$

Now,

$\sum_{r=1}^{10} A_r B_r = A_1B_1 + A_2B_2 + \ldots + A_{10}B_{10}$
$= {}^{10}C_1 \, {}^{20}C_1 + {}^{10}C_2 \, {}^{20}C_2 + {}^{10}C_3 \, {}^{20}C_3$
$+ \ldots + {}^{10}C_{10} \, {}^{20}C_{10}$

Co-efficient of x^{20} in the expansion of

$(1 + x)^{10}(1 + x)^{20} - 1$
$= {}^{30}C_{20} - 1 = {}^{30}C_{10} - 1 = C_{10} - 1$

Also,

$\sum_{r=1}^{10} (A_r)^2 = \sum_{r=1}^{10} ({}^{10}C_r)^2$
$= ({}^{10}C_1)^2 + ({}^{10}C_2)^2 + ({}^{10}C_3)^2 + \ldots + ({}^{10}C_{10})^2$

Co-efficient of x^{10} in the expansion of

$(1 + x)^{10}\,(1 + x)^{10} - 1$
$= {}^{20}B_{10} - 1$
$= B_{10} - 1$

Therefore,

$\sum_{r=1}^{10} A_r (B_{10}B_r - C_{10}A_r) = \sum_{r=1}^{10} B_{10}(A_rB_r) - C_{10}(A_r)^2$
$= B_{10}(C_{10} - 1) - C_{10}(B_{10} - 1)$
$= (C_{10} - B_{10})$

18. We have,

$32^{32} = (2^5)^{32} = (2)^{160}$
$= (3 - 1)^{160}$
$= 3^{160} - {}^{160}C_1 . 3^{159} + {}^{160}C_2 \cdot 3^{158}$
$- {}^{160}C_3 \cdot 3^{157} + \cdots + {}^{160}C_{160} \cdot 1$
$= 3(3^{159} - {}^{160}C_1 \cdot 3^{158} + {}^{160}C_2 \cdot 3^{157}$
$- \cdots + {}^{160}C_{159}) + 1$
$= 3m + 1$, where m is integer.

Now,

$32^{32^{32}} = 32^{m+1}$
$= 2^{5(3m+1)}$
$= 2^{15m+5}$
$= 2^{3(5m+1)}.2^2$
$= 4.8^{(5m+1)}$
$= 4.(7 + 1)^{(5m+1)}$

$= 4 \cdot [7^{5m+1} + {}^{5m+1}C_1 \cdot 7^{5m} + {}^{5m+1}C_2 \cdot 7^{5m-1}$
$+ \ldots + {}^{5m+1}C_{5m-1} \cdot 7^2 + 5^{m+1}C_{5m} \cdot 7 + {}^{5m+1}C_{5m+1}]$
$= 4[7(7^{5m} + 5^{m+1}C_1 \cdot 7^{5m-1} + \ldots + {}^{5m+1}C_{5m}) + 1]$
$= 4(7n + 1)$
$= 28n + 4$

Hence, the remainder is 4.

19. We have

$$\left(x - \frac{C_1}{C_0}\right)\left(x - 2^2\frac{C_2}{C_1}\right)\left(x - 3^2\frac{C_3}{C_2}\right)\ldots\left(x - 50^2\frac{C_{50}}{C_{49}}\right)$$
$= (x - 1)(x - 2^2.2)(x - 3^2.3) \ldots (x - 50^2.50)$
$= (x - 1)(x - 2^3)(x - 3^3) \ldots (x - 50^3)$

Thus,

the co-efficient of $x^{49} = -(1 + 2^3 + 3^3 + \ldots + 50^3)$
$$= -\left(\frac{50 \times 51}{2}\right)^2$$
$= -(25 \times 51)^2$
$= -(1275)^2$

20. We have

${}^{100}C_{10} + 5 \cdot {}^{100}C_{11} + 10 \cdot {}^{100}C_{12}$
$+ 10 \cdot {}^{100}C_{13} + 5 \cdot {}^{100}C_{14} + {}^{100}C_{15}$
$= ({}^{100}C_{10} + {}^{100}C_{11})$
$+ 4({}^{100}C_{11} + {}^{100}C_{12}) + 6({}^{100}C_{12} + {}^{100}C_{13})$
$4({}^{102}C_{13} + {}^{102}C_{14}) + ({}^{102}C_{14} + {}^{102}C_{15})$
$= {}^{101}C_{11} + 4^{101}C_{12} + 6^{101}C_{13} + 4 \cdot {}^{101}C_{14} + {}^{101}C_{15}$
$= ({}^{101}C_{11} + {}^{101}C_{12}) + 3({}^{101}C_{12} + 2^{101}C_{13} + {}^{101}C_{14})$
$({}^{101}C_{14} + {}^{101}C_{15})$
$= {}^{102}C_{12} + 3 \cdot {}^{102}C_{13} + 3 \cdot {}^{102}C_{14} + {}^{102}C_{15}$
$= ({}^{102}C_{12} + {}^{102}C_{13}) + 2({}^{102}C_{13} + {}^{102}C_{14}) + {}^{103}C_{15}$
$= {}^{103}C_{13} + 2^{103}C_{14} + {}^{103}C_{15}$
$= ({}^{103}C_{13} + {}^{103}C_{14}) + ({}^{103}C_{14} + {}^{103}C_{15})$
$= |{}^{104}C_{14} + {}^{104}C_{15}|$
$= {}^{105}C_{15}$

21. We have

$(r^2 + r + 1)r! = \{(r + 2)(r + 1) - 2(r + 1) + 1\}\ r!$
$= (r + 2)! - 2(r + 1)! + r!$

Thus,

$S = (1 + 2! + 3! + 4! + \ldots + n! + (n + 1)! + (n + 2)!)$
$- 2(1 + 2! + 3! + 4! + \ldots + n! + (n + 1)!)$
$+ (1 + 2! + 3! + 4! + 5! + \ldots + n!)$
$= (n + 2)! - (n + 1)!$
$= (n + 1)!(n + 2 - 1)$
$= (n + 1)!(n + 1)$
$= (n + 1)\ (n + 1)!$

Clearly, $n + 1 = 2016$

$\Rightarrow \quad n = 2015$

22. Clearly $m = 9$, since the period of 9 is 2.

Also, $n = 2$, since the period is 4.

Thus, the value of

$(n - m + 2006)$
$= 9 - 2 + 2006$
$= 2013$

23. We have,

$$\sum_{r=0}^{n}\left(\frac{n-3r+1}{n-r+1}\right)\cdot\frac{{}^nC_r}{2}$$
$$= \sum_{r=0}^{n}\left(1 - \frac{2r}{n-r+1}\right)\frac{{}^nC_r}{2^r}$$
$$= \sum_{r=0}^{n}\frac{{}^nC_r}{2^r} - \sum_{r=0}^{n}\frac{2r}{n-r+1}\cdot\frac{{}^nC_r}{2^r}$$
$$= \sum_{r=0}^{n}\frac{{}^nC_r}{2^r} - \sum_{r=0}^{n}\frac{r}{n-r+1}\cdot\frac{1}{2^{r-1}}\times\frac{n!}{r!\times(n-r)!}$$
$$= \sum_{r=0}^{n}\frac{{}^nC_r}{2^r} - \sum_{r=0}^{n}\frac{1}{2^{r-1}}\times\frac{n!}{r-1!\times(n-r+1)!}$$
$$= \sum_{r=0}^{n}\frac{{}^nC_r}{2^r} - \sum_{r=0}^{n}\frac{{}^nC_{r-1}}{2^{r-1}}$$
$$= \left(1+\frac{1}{2}\right)^n - \left(\left(1+\frac{1}{2}\right)^n - \frac{{}^nC_n}{2^n}\right)$$
$$= \left(\frac{3}{2}\right)^n - \left[\left(\frac{3}{2}\right)^n - \frac{1}{2^n}\right] = \frac{1}{2^n}$$

24. Given,

$$\left(\sum_{r=0}^{2} x^r\right)^n = \sum_{r=0}^{2n} a_r x^r$$

$(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_{2n}x^{2n}$ …(i)

Replacing x by $1/x$, we get

$$\left(1 + \frac{1}{x} + \frac{1}{x^2}\right)^n = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \frac{a_3}{x^3} + \cdots + \frac{a_{2n}}{x^{2n}}$$
$$\Rightarrow (1 + x + x^2)^n = (a_0x^{2n} + a_1x^{2n-1} + \cdots + a_{2n-1}x + a_{2n}) \quad \ldots\text{(ii)}$$

From Eqs (i) and (ii), we get

$a_0 = a_{2n}$
$a_1 = a_{2n-1}$
$\vdots$
$a_r = a_{2n-r}$

Now,

$$\sum_{r=0}^{2n}\left(\frac{r}{a_r}\right)$$
$$= \sum_{r=0}^{2n}\left(\frac{2n-(2n-r)}{a_r}\right)$$
$$= \sum_{r=0}^{2n}\left(\frac{2n}{a_r}\right) - \sum_{r=0}^{2n}\left(\frac{(2n-r)}{a_r}\right)$$
$$= \sum_{r=0}^{2n}\left(\frac{2n}{a_r}\right) - \sum_{r=0}^{2n}\left(\frac{(2n-r)}{a_{2n-r}}\right)$$
$$= 2n\sum_{r=0}^{2n}\left(\frac{1}{a_r}\right) - \sum_{r=0}^{2n}\left(\frac{r}{a_r}\right)$$

$$\Rightarrow \quad 2\sum_{r=0}^{2n}\left(\frac{r}{a_r}\right) = 2n\sum_{r=0}^{2n}\left(\frac{1}{a_r}\right)$$

$$\Rightarrow \quad \sum_{r=0}^{2n}\left(\frac{r}{a_r}\right) = n\sum_{r=0}^{2n}\left(\frac{1}{a_r}\right)$$

Hence, the result.

25. Given equation is

$$x^{2017} + \left(\frac{1}{2} - x\right)^{2017} = 0$$

$$\Rightarrow \quad \left(x - \frac{1}{2}\right)^{2017} - x^{2017} = 0$$

$$\Rightarrow \quad x^{2017} - {}^{2017}C_1\, x^{2016}\left(\frac{1}{2}\right) + {}^{2017}C_2 \cdot x^{2015}\left(\frac{1}{2}\right)^2 - {}^{2017}C_3 \cdot x^{2014}\left(\frac{1}{2}\right)^3 + \cdots - x^{2017} = 0$$

$$\Rightarrow \quad -{}^{2017}C_1\, x^{2016}\left(\frac{1}{2}\right) + {}^{2017}C_2 \cdot x^{2015}\left(\frac{1}{2}\right)^2 - {}^{2017}C_3 \cdot x^{2014}\left(\frac{1}{2}\right)^3 + \cdots = 0$$

$$\Rightarrow \quad {}^{2017}C_1\, x^{2016}\left(\frac{1}{2}\right) - {}^{2017}C_2 \cdot x^{2015}\left(\frac{1}{2}\right)^2 + {}^{2017}C_3 \cdot x^{2014}\left(\frac{1}{2}\right)^3 - \cdots = 0$$

Hence,

$$\text{the sum of the roots} = {}^{2017}C_2 \cdot \left(\frac{1}{2}\right)^2 / {}^{2017}C_1 \cdot \left(\frac{1}{2}\right)$$

$$= {}^{2017}C_2 \cdot \left(\frac{1}{2}\right) / {}^{2017}C_1$$

$$= \frac{2017 \times 2016}{2017 \times 4}$$

$$= 504$$

26. Clearly $m = 21 + 1 = 22$
and $n = 11 + 2 = 13$
Hence, the value of
$m + n - 5 = 22 + 13 - 5$
$= 30$

27. We have

$$\sum_{p=1}^{n}\left(\sum_{m=p}^{n} {}^nC_m \times {}^mC_p\right) = \sum_{m=1}^{n}\left\{{}^nC_m \sum_{p=1}^{m} {}^mC_p\right\}$$

$$= \sum_{m=1}^{n}\{{}^nC_m(2^m - 1)\}$$

$$= \sum_{m=1}^{n}\{{}^nC_m \cdot 2^m\} - \sum_{m=1}^{n}({}^nC_m)$$

$$= (3^n - 1) - (2^n - 1)$$

$$= |3^n - 2^n|$$

28. Clearly, $S_n = {}^{2n}C_{n-1}$

Now, $\dfrac{S_{n+1}}{S_n} = \dfrac{15}{4}$

$$\Rightarrow \quad \frac{{}^{2n+2}C_n}{{}^{2n}C_{n-1}} = \frac{15}{4}$$

$$\Rightarrow \quad \frac{(2n+2)(2n+1)}{n(n+2)} = \frac{15}{4}$$

$\Rightarrow \quad n^2 - 6n + 8$

$\Rightarrow \quad (n-2)(n-4) = 0$

$\Rightarrow \quad n = 2, 4$

Hence, the values of n are 2 and 4.

29. Let $t_{r+1} = (-1)^r(5r - 2)C_r$

$= (-1)^r(5r - 2)^nC_r$

$= (-1)^r(5r\,{}^nC_r - 2\,{}^nC_r)$

$= (-1)^r(5n\,{}^{n-1}C_{r-1} - 2\,{}^nC_r)$

$= (-1)^r\, 5n\,{}^{n-1}C_{r-1} - (-1)^r\, 2\,{}^nC_r$

$$S_n = \sum_{r=0}^{n} t_{r+1}$$

$$= 5n\sum_{r=0}^{n}(-1)^r\,{}^{n-1}C_{r-1} - 2\sum_{r=0}^{n}(-1)^r\,{}^nC_r$$

$= 5n.0 - 2.0$

$= 0$

Thus, the sum of $S_{n+1} = 0$.

30. Let $t_{r+1} = \left|\dfrac{C_r}{(r+1)(r+2)}\right|$

$$= \left|\frac{{}^nC_r}{(r+1)(r+2)}\right|$$

$$= \frac{1}{(n+1)(n+2)}\left(\frac{(n+1)(n+2)}{(r+1)(r+2)} \times {}^nC_r\right)$$

$$= \frac{{}^{n+2}C_{r+2}}{(n+1)(n+2)}$$

Thus, $S = \displaystyle\sum_{r=0}^{n} t_{r+1}$

$$= \sum_{r=0}^{n}\frac{{}^{n+2}C_{r+2}}{(n+1)(n+2)}$$

$$= \frac{1}{(n+1)(n+2)}\sum_{r=0}^{n}({}^{n+2}C_{r+2})$$

$$= \frac{(2^{n+2} - {}^{n+2}C_1 - {}^{n+2}C_0)}{(n+1)(n+2)}$$

$$= \frac{[2^{n+2} - (n+2) - 1]}{(n+1)(n+2)}$$

$$= \frac{|2^{n+2} - n - 3|}{(n+1)(n+2)}$$

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31. Let $t_n = \left|\dfrac{C(n,2)}{(n+1)!}\right|$

$$= \frac{\frac{n(n-1)}{2}}{(n+1)!}$$

$$= \frac{1}{2} \times \frac{n(n-1)}{(n+1)!}$$

$$= \frac{1}{2} \times \left(\frac{(n+1)(n-2)+2}{(n+1)!}\right)$$

$$= \frac{1}{2} \times \left(\frac{(n-2)}{n!} + \frac{2}{(n+1)!}\right)$$

$$= \frac{1}{2} \times \left(\frac{1}{(n-1)!} - \frac{2}{n!} + \frac{2}{(n+1)!}\right)$$

Thus,

$$S = t_1 + t_2 + t_3 + t_4 + \dots$$

$$= \frac{1}{2}\left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\right) - \left(\frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots\right) + \left(\frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots\right)$$

$$= \frac{1}{2}(e-1) - (e-2) + \left(e - 1 - 1 - \frac{1}{2}\right)$$

$$= \left|\frac{e}{2} - 1\right|$$

32. Let $t_n = \dfrac{{}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \cdots + {}^nC_n}{{}^nP_n}$

$$= \frac{2^n}{n!}$$

Thus, $S = t_1 + t_2 + t_3 + t_4 + \dots$

$$= \left(\frac{2}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{2^4}{4!} + \cdots\right)$$

$$= |e^2 - 1|$$

33. We have

$$a = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \frac{x^9}{9!} + \cdots$$

$$b = \frac{x}{1!} + \frac{x^4}{4!} + \frac{x^7}{7!} + \frac{x^{10}}{10!} + \cdots$$

and $c = \dfrac{x^2}{2!} + \dfrac{x^5}{5!} + \dfrac{x^8}{8!} + \dfrac{x^{11}}{11!} + \cdots$

Now, $a^3 + b^3 + c^3 - 3abc$

$$= (a+b+c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$$

$$= e^x \cdot e^{x\omega} \cdot \omega^{x\omega^2}$$

$$= e^{x + x\omega + x\omega^2}$$

$$= e^{x(1+\omega+\omega^2)}$$

$$= e^{x \cdot 0}$$

$$= 1$$

34. We have

$$\frac{1}{1.2} + \frac{1.3}{1.2.3.4} + \frac{1.3.5}{1.2.3.4.5.6} + \cdots$$

$$= \frac{1}{2!} + \frac{1.3}{4!} + \frac{1.3.5}{6!} + \frac{1.3.5.7}{8!}$$

Let $t_n = \dfrac{1.3.5.7\dots(2n-1)}{(2n)!}$

$$= \frac{1.2.3.4.5.6\dots(2n-1)2n}{(2n)!(2.4.6.8\dots 2n)}$$

$$= \frac{(2n)!}{(2n)!(2.4.6.8\dots 2n)}$$

$$= \frac{1}{2^n(1.2.3\dots n)}$$

$$= \frac{(1/2)^n}{n!}$$

Thus, $S = t_1 + t_2 + t_3 + t_4 + \dots$

$$= \frac{(1/2)}{1!} + \frac{(1/2)^2}{2!} + \frac{(1/2)^3}{3!} + \frac{(1/2)^4}{4!} + \cdots$$

$$= e^{1/2} - 1$$

$$= (\sqrt{e} - 1)$$

35. We have

$$a_n = \sum_{k=0}^{n}\left(\frac{2^k}{(k)! \times (n-k)!}\right)$$

$$= \frac{1}{1! \times (n)!} + \frac{2}{1! \times (n-1)!} + \frac{2^2}{2! \times (n-2)!} + \frac{2^3}{3! \times (n-3)!} + \cdots$$

$$= \frac{1}{n!}\left(\frac{n!}{1! \times (n)!} + \frac{2(n!)}{1! \times (n-1)!} + \frac{2^2(n!)}{2! \times (n-2)!} + \cdots\right)$$

$$= \frac{1}{n!}({}^nC_n + 2 \cdot {}^nC_{n-1} + 2^2 \cdot {}^nC_{n-2} + \cdots + 2^n \cdot {}^nC_0)$$

$$= \frac{1}{n!}({}^nC_0 + 2 \cdot {}^nC_1 + 2^2 \cdot {}^nC_2 + \cdots + 2^n \cdot {}^nC_n)$$

$$= \frac{1}{n!}(1+2)^n$$

$$= \frac{3^n}{n!}$$

Now,

$$\sum_{r=0}^{\infty} a_r = a_0 + a_1 + a_2 + a_3 + a_4 + \dots$$

$$= \left(1 + \frac{3}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!} + \cdots\right)$$

$$= e^3$$

36. Let $t_n = \dfrac{C(n, 4)}{P(n, n)}$

$$= \frac{n(n-1)(n-2)(n-3)}{24(n!)}$$

$$= \frac{n(n-1)(n-2)(n-3)}{24[n(n-1)(n-2)(n-3)](n-4)!}$$

$$= \frac{1}{24(n-4)!}$$

Thus,

$$S = t_4 + t_5 + t_6 + t_7 + t_8 + \ldots$$

$$= \frac{1}{24}\left(1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots\right)$$

$$= \frac{e}{24}$$

37. Let $t_n = \dfrac{n}{(2n+1)!}$

$$= \frac{1}{2}\left(\frac{(2n+1)-1}{(2n+1)!}\right)$$

$$= \frac{1}{2}\left(\frac{1}{(2n)!} - \frac{1}{(2n+1)!}\right)$$

Thus, $S = \dfrac{1}{2}\displaystyle\sum_{n=1}^{\infty}\left(\frac{1}{(2n)!} - \frac{1}{(2n+1)!}\right)$

$$= \frac{1}{2}\left(\left(\frac{1}{2!} - \frac{1}{3!}\right) + \left(\frac{1}{4!} - \frac{1}{5!}\right) + \left(\frac{1}{6!} - \frac{1}{7!}\right) + \left(\frac{1}{8!} - \frac{1}{9!}\right) + \cdots\right)$$

$$= \frac{1}{2}\left(\left(1 - \frac{1}{1!}\right) + \left(\frac{1}{2!} - \frac{1}{3!}\right) + \left(\frac{1}{4!} - \frac{1}{5!}\right) + \left(\frac{1}{6!} - \frac{1}{7!}\right) + \cdots\right)$$

$$= \frac{1}{2} \times e^{-1} = \frac{1}{2e}$$

38. Let $t_n = \dfrac{2n-1}{2n!}$

$$= \frac{2n}{2n!} - \frac{1}{2n!}$$

$$= \frac{1}{(2n-1)!} - \frac{1}{2n!}$$

Thus,

$$S = \sum_{n=1}^{\infty}\left(\frac{1}{(2n-1)!} - \frac{1}{2n!}\right)$$

$$= \left(\left(\frac{1}{1!} - \frac{1}{2!}\right) + \left(\frac{1}{3!} - \frac{1}{4!}\right) + \left(\frac{1}{5!} - \frac{1}{6!}\right) + \left(\frac{1}{7!} - \frac{1}{8!}\right) + \cdots\right)$$

$$= -\left(-\frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \frac{1}{8!} + \cdots\right)$$

$$= \left(1 - \frac{1}{e}\right)$$

39. We have,

$$C(n, 2) \times \frac{3^{n-1}}{(n)!}$$

$$= \frac{n(n-1)}{2} \times \frac{3^{n-1}}{(n)!}$$

$$= \frac{1}{2} \times \frac{3^{n-1}}{(n-2)!}$$

Thus $S = \displaystyle\sum_{n=0}^{\infty} C(n, 2) \times \frac{3^{n-1}}{(n)!}$

$$= \frac{1}{2} \times \left(\frac{3^1}{1!} + \frac{3^2}{1!} + \frac{3^3}{2!} + \frac{3^4}{3!} + \frac{3^5}{4!} + \cdots\right)$$

$$= \frac{3}{2} \times \left(\frac{1}{1!} + \frac{3}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} + \cdots\right)$$

$$= \frac{3e^3}{2}$$

40. Consider $\dfrac{3^2}{1!} + \dfrac{5^2}{3!} + \dfrac{7^2}{5!} + \cdots$

Let $t_n = \dfrac{(2n+1)^2}{(2n-1)!}$

$$= \frac{(2n+1)(2n+1)}{(2n-1)!}$$

$$= \frac{4n^2 + 4n + 1}{(2n-1)!}$$

$$= \frac{(4n^2 - 1) + (4n + 2)}{(2n-1)!}$$

$$= \frac{(2n-1)(2n+1) + 2(2n+1)}{(2n-1)!}$$

$$= \frac{(2n+1)}{(2n-2)!} + \frac{2(2n-1) + 4}{(2n-1)!}$$

$$= \frac{(2n-2) + 3}{(2n-2)!} + \frac{2}{(2n-2)!} + \frac{4}{(2n-1)!}$$

$$= \frac{1}{(2n-3)!} + \frac{5}{(2n-2)!} + \frac{4}{(2n-1)!}$$

Thus,

$$S = t_1 + t_2 + t_3 + t_4 + t_5 + \ldots$$

$$= \left(\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \cdots\right) + 5\left(\frac{1}{1!} + \frac{1}{2!} + \frac{1}{4!} + \frac{1}{6!} + \cdots\right) + 4\left(\frac{1}{1!} + \frac{1}{3!} + \frac{1}{5!} + \frac{1}{7!} + \cdots\right)$$

$$= \left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\frac{1}{7!}+\cdots\right)$$
$$+4\left(\frac{1}{1!}+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\cdots\right)$$
$$+4\left(\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+\cdots\right)$$
$$= \left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\frac{1}{7!}+\cdots\right)$$
$$+4\left(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\frac{1}{5!}+\frac{1}{6!}+\frac{1}{7!}+\cdots\right)$$
$$= e+4$$
$$= 5e$$

Hence, the required sum is $1 + 5e$.

LOGARITHMIC SERIES

41. We have

$$\frac{1}{2.3}+\frac{1}{4.5}+\frac{1}{6.7}+\cdots$$
$$= \left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{6}-\frac{1}{7}\right)+\cdots$$
$$= -\left(-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\cdots\right)$$
$$= \log 2 - 1$$
$$= \log_e\left(\frac{2}{e}\right)$$

42. We have $\frac{1}{1.3}+\frac{1}{2.5}+\frac{1}{3.7}+\frac{1}{4.9}+\cdots$

Let $t_n = \frac{1}{n(2n+1)}$

$$= \frac{2}{2n(2n+1)}$$
$$= 2\left(\frac{1}{2n}-\frac{1}{2n+1}\right)$$

Thus,

$S = t_1 + t_2 + t_3 + t_4 + \ldots$

$$= 2\left[\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{6}-\frac{1}{7}\right)+\left(\frac{1}{8}-\frac{1}{9}\right)+\cdots\right]$$
$$= -2\left[-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}+\cdots\right]$$
$$= -2(\log 2 - 1)$$

43. Let $t_n = \frac{1+(n-1)4}{(2n-1)2n(2n+1)}$

$$= \frac{4n-3}{(2n-1)2n(2n+1)}$$
$$= \frac{2(2n-1)-1}{(2n-1)2n(2n+1)}$$
$$= \frac{2}{2n(2n+1)} - \frac{1}{(2n-1)2n(2n+1)}$$
$$= \frac{2}{2n(2n+1)} - \frac{1}{2}\frac{(2n+1)-(2n-1)}{(2n-1)2n(2n+1)}$$
$$= 2\left(\frac{1}{2n}-\frac{1}{(2n+1)}\right)$$
$$-\frac{1}{2}\left(\frac{1}{2n(2n-1)}-\frac{1}{2n(2n+1)}\right)$$
$$= \frac{5}{2}\left(\frac{1}{2n}-\frac{1}{(2n+1)}\right)-\frac{1}{2}\left(\frac{1}{2n(2n-1)}\right)$$
$$= \frac{5}{2}\left(\frac{1}{2n}-\frac{1}{(2n+1)}\right)-\frac{1}{2}\left(\frac{1}{(2n-1)}-\frac{1}{2n}\right)$$
$$= -\frac{5}{2}(\log 2 - 1) - \frac{1}{2}\log 2$$
$$= \frac{5}{2} - 3\log_e 2$$

44. Let $t_n = \frac{4n+1}{(2n-1)2n(2n+1)}$

$$= \frac{2(2n+1)-1}{(2n-1)2n(2n+1)}$$
$$= \frac{2}{(2n-1)2n} - \frac{1}{(2n-1)2n(2n+1)}$$
$$= \frac{2}{(2n-1)2n} - \frac{1}{2}\left(\frac{(2n+1)-(2n-1)}{(2n-1)2n(2n+1)}\right)$$
$$= \frac{2}{(2n-1)2n} - \frac{1}{2}\left(\frac{1}{(2n-1)2n}-\frac{1}{2n(2n+1)}\right)$$
$$= \frac{3}{2}\left(\frac{1}{(2n-1)2n}\right)+\frac{1}{2}\left(\frac{1}{2n(2n+1)}\right)$$
$$= \frac{3}{2}\left(\frac{1}{(2n-1)}-\frac{1}{2n}\right)+\frac{1}{2}\left(\frac{1}{2n}-\frac{1}{(2n+1)}\right)$$
$$= \frac{3}{2}\left(\frac{1}{(2n-1)}-\frac{1}{2n}\right)-\frac{1}{2}\left(-\frac{1}{2n}+\frac{1}{(2n+1)}\right)$$

Thus, $S = t_1 + t_2 + t_3 + t_4 + \ldots$

$$= \frac{3}{2}\log 2 - \frac{1}{2}(\log 2 - 1)$$
$$= \log 2 + \frac{1}{2}$$

45. Let $t_n = \frac{1}{(2n-1)2n(2n+1)}$

$$= \frac{1}{2}\left(\frac{(2n+1)-(2n-1)}{(2n-1)2n(2n+1)}\right)$$
$$= \frac{1}{2}\left(\frac{1}{(2n-1)2n}-\frac{1}{2n(2n+1)}\right)$$
$$= \frac{1}{4}\left(\frac{1}{(2n-1)}-\frac{1}{2n}\right)-\frac{1}{4}\left(\frac{1}{2n}-\frac{1}{(2n+1)}\right)$$

Thus, $S = t_1 + t_2 + t_3 + t_4 + \ldots$

$$= \frac{1}{4}\log 2 + \frac{1}{4}(\log 2 - 1)$$
$$= \frac{1}{2}\log 2 - \frac{1}{4}$$
$$= \frac{1}{2}\left(\log 2 - \frac{1}{2}\right)$$

46. We have,

$$1 + \left(\frac{1}{2}+\frac{1}{3}\right)\frac{1}{4} + \left(\frac{1}{4}+\frac{1}{5}\right)\frac{1}{4^2} + \left(\frac{1}{6}+\frac{1}{7}\right)\frac{1}{4^3} + \cdots$$
$$= \left(\frac{1}{2}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{1}{4^2} + \frac{1}{6}\cdot\frac{1}{4^3} + \cdots\right)$$
$$+ \left(1 + \frac{1}{3}\cdot\frac{1}{4} + \frac{1}{5}\cdot\frac{1}{4^2} + \frac{1}{7}\cdot\frac{1}{4^3} + \cdots\right)$$
$$= \frac{1}{2}\left(\frac{1}{4} + \frac{1}{2}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{3}\cdot\left(\frac{1}{4}\right)^3 + \cdots\right)$$
$$+ 2\left(\frac{1}{2} + \frac{1}{3}\cdot\left(\frac{1}{2}\right)^3 + \frac{1}{5}\cdot\left(\frac{1}{2}\right)^4 + \frac{1}{7}\cdot\left(\frac{1}{2}\right)^5 + \cdots\right)$$
$$= \frac{1}{2}\left\{-\log\left(1-\frac{1}{4}\right)\right\} + \log\left\{\frac{\left(1+\frac{1}{2}\right)}{\left(1-\frac{1}{2}\right)}\right\}$$
$$= \frac{1}{2}\left\{\log\left(\frac{4}{3}\right)\right\} + \log 3$$
$$= \log 2 + \frac{1}{2}\log 3$$
$$= \log(2\sqrt{3})$$
$$= \log_e(\sqrt{12})$$

47. We have

$$\frac{2}{1}\cdot\frac{1}{3} + \frac{3}{2}\cdot\frac{1}{9} + \frac{4}{3}\cdot\frac{1}{27} + \frac{5}{4}\cdot\frac{1}{81} + \cdots$$
$$= \frac{2}{1}\cdot\left(\frac{1}{3}\right) + \frac{3}{2}\cdot\left(\frac{1}{3}\right)^2 + \frac{4}{3}\cdot\left(\frac{1}{3}\right)^3 + \frac{5}{4}\cdot\left(\frac{1}{3}\right)^4 + \cdots$$
$$= (1+1)\left(\frac{1}{3}\right) + \left(1+\frac{1}{2}\right)\left(\frac{1}{3}\right)^2 + \left(1+\frac{1}{3}\right)\left(\frac{1}{3}\right)^3 + \cdots$$
$$= \left(\frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4} + \frac{1}{3^5} + \cdots\right)$$
$$+ \left[\frac{1}{3} + \frac{1}{2}\cdot\left(\frac{1}{3}\right)^2 + \frac{1}{3}\left(\frac{1}{3}\right)^3 + \frac{1}{4}\left(\frac{1}{3}\right)^4 + \cdots\right]$$
$$= \frac{(1/3)}{\left|1-\frac{1}{3}\right|} + \log\left(1-\frac{1}{3}\right)$$
$$= \frac{1}{2} + \log\left(\frac{2}{3}\right)$$

48. We know that

$$e^x + e^{-x} = 2\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots\right)$$

Putting $x = \log_e n$, we get

$$2\left(1 + \frac{(\log_e n)^2}{2!} + \frac{(\log_e n)^4}{4!} + \frac{(\log_e n)^6}{6!} + \cdots\right)$$
$$= e^{\log_e n} + e^{-\log_e n}$$
$$= e^{\log_e n} + e^{\log_e\left(\frac{1}{n}\right)}$$
$$= \left(n + \frac{1}{n}\right)$$

49. We have

$$\log_e(1 + x + x^2 + x^3)$$
$$= a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$$

Putting $x = 1$, we get

$$a_0 + a_1 + a_2 + a_3 + a_4 + \ldots = \log_e 4$$

50 Let $\left(\frac{\sqrt{2}-1}{\sqrt{2}}\right) = x$.

Thus, the sum

$$= 1 + \frac{x}{2} + \frac{x^2}{6} + \frac{x^3}{12} + \frac{x^4}{20} + \cdots$$
$$= 1 + \frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \frac{x^4}{4.5} + \cdots$$
$$= \left[1 + \left(1-\frac{1}{2}\right)x + \left(\frac{1}{2}-\frac{1}{3}\right)x^2 + \left(\frac{1}{3}-\frac{1}{4}\right)x^3 + \left(\frac{1}{4}-\frac{1}{5}\right)x^4 + \cdots\right]$$
$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right) - \left(\frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \cdots\right)$$
$$= 1 + \left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right) - \frac{1}{x}\left(\frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right)$$
$$= 1 - \log(1-x) - \frac{1}{x}(-\log(1-x) - x)$$
$$= 2 + \left(\frac{1}{x} - 1\right)\log(1-x)$$

Thus, the sum

$$= 2 + \left(\frac{\sqrt{2}}{\sqrt{2}-1} - 1\right)\log\left(1 - \frac{\sqrt{2}-1}{\sqrt{2}}\right)$$

$$= 2 + \frac{1}{(\sqrt{2}-1)}\log\left(\frac{1}{\sqrt{2}}\right) \quad \left(\because \frac{1}{\sqrt{2}-1} = \sqrt{2}+1\right)$$

$$= 2 + (\sqrt{2}+1) \times -\frac{1}{2}\log(2)$$

$$= 2 - \frac{(\sqrt{2}+1)}{2} \times \log(2)$$

Integer Type Questions

1. We have,

$$(1-x)^5(1+x+x^2+x^3)^4$$
$$= (1-x)^5(1+x)^4(1+x^2)^4$$
$$= (1-x)^4(1+x)^4(1+x^2)^4(1-x)$$
$$= (1-x^2)^4(1+x^2)^4(1-x)$$
$$= (1-x^4)^4 \times (1-x)$$
$$= (1 - {}^4C_1 \cdot x^4 + {}^4C_2 x^8 - {}^4C_3 x^{12} + \ldots) \times (1-x)$$

Hence, the co-efficients of x^{13} is ${}^4C_3 = 4$.

2. We have,

$$(1+x^2)^{40}\left(x^2 + 2 + \frac{1}{x^2}\right)^{-5}$$
$$= (1+x^2)^{40}\left(\left(x + \frac{1}{x}\right)^2\right)^{-5}$$
$$= (1+x^2)^{40}\left(x + \frac{1}{x}\right)^{-10}$$
$$= (1+x^2)^{40}\left(\frac{x^2+1}{x}\right)^{-10}$$
$$= (1+x^2)^{30} \times x^{10}$$

Co-efficient of x^{20} in $(1+x^2)^{30} \times x^{10}$

= Co-efficient of x^{10} in $(1+x^2)^{30}$

$= {}^{30}C_5 = {}^{30}C_{25}$

Thus, $m = 30$ and $n = 25$.

Hence, the value of

$$m - n + 2 = 30 - 25 + 2$$
$$= 7$$

3. We have,

$$(1 + 5x - 3x^2 + 4x^3 - 7x^4 + x^5)^{2017}$$

Putting $x = 1$ in the given expression, we get

Sum of the co-efficients $= (1 + 5 - 3 + 4\ 0\ 7 + 1)^{2017}$

$= (11 - 10)^{2017}$

$= 1$

4. We have,

$$(x + \sqrt{x^3-1})^5 + (x - \sqrt{x^3-1})^5$$
$$= (x+a)^5 + (x-a)^5,\ a = \sqrt{x^3-1}$$
$$= 2({}^5C_0 x^5 + {}^5C_2 x^3(x^3-1) + {}^5C_4 x(x^3-1)^2)$$

Thus, the degree of the polynomial is 7.

5. We have $t_9 = t_{8+1}$

$$= {}^{10}C_8 \times ((3^{\log_3\sqrt{25^{x-1}+7}})^2 (3^{-\frac{1}{8}\log(5^{x-1}+1)})^8)$$
$$= {}^{10}C_8 \times ((3^{\log_3(25^{x-1}+7)})(3^{-\log_3(5^{x-1}+1)}))$$
$$= {}^{10}C_8 \times ((3^{\log_3(25^{x-1}+7)-\log_3(5^{x-1}+1)}))$$
$$= {}^{10}C_8 \times \left(\left(3^{\log_3\left(\frac{25^{x-1}+7}{(5^{x-1}+1)}\right)}\right)\right)$$
$$= {}^{10}C_8 \times \left(\frac{25^{x-1}+7}{(5^{x-1}+1)}\right)$$
$$= 45 \times \left(\frac{5^{2(x-1)}+7}{(5^{x-1}+1)}\right)$$

Thus,

$$45 \times \left(\frac{5^{2(x-1)}+7}{(5^{x-1}+1)}\right) = 180$$

$\Rightarrow \left(\frac{5^{2(x-1)}+7}{(5^{x-1}+1)}\right) = 4$

$\Rightarrow 5^{2(x-1)} + 7 = 4(5^{x-1} + 1)$

$\Rightarrow a^2 - 4a + 3 = 0,\ a = 5^{x-1}$

$\Rightarrow (a-1)(a-3) = 0$

$\Rightarrow a = 1, 3$

$\Rightarrow 5^{x-1} = 1, 3$

$\Rightarrow 5^{x-1} = 1 = 5^0$

$\Rightarrow x - 1 = 0$

$\Rightarrow x = 1$

Hence, the value of x is 1.

6. We have,

$$F(n) = \sum_{r=0}^{n-1}\left(\frac{{}^nC_r}{{}^nC_r + {}^nC_{r+1}}\right)$$
$$= \sum_{r=0}^{n-1}\left(\frac{{}^nC_r}{{}^{n+1}C_{r+1}}\right)$$
$$= \sum_{r=0}^{n-1}\left(\frac{{}^nC_r}{\left(\frac{n+1}{r+1}\right){}^nC_r}\right)$$
$$= \sum_{r=0}^{n-1}\left(\frac{r+1}{n+1}\right)$$
$$= \frac{1}{n+1}\sum_{r=0}^{n-1}(r+1)$$
$$= \frac{(1+2+3+\cdots+n)}{n+1}$$
$$= \frac{n(n+1)/2}{n+1}$$
$$= \frac{n}{2}$$

$$\therefore \quad F(16) = \frac{16}{2} = 8$$

7. We have

$$\sum_{r=0}^{10}\left(\frac{r}{^{10}C_r}\right)=20$$

$$\Rightarrow \sum_{r=0}^{10}\left(\frac{10-r}{^{10}C_r}\right)=20$$

$$\Rightarrow \sum_{r=0}^{10}\left(\frac{10}{^{10}C_r}\right)-\sum_{r=0}^{10}\left(\frac{r}{^{10}C_r}\right)=20$$

$$\Rightarrow 10\left(\sum_{r=0}^{10}\left(\frac{1}{^{10}C_r}\right)\right)-20=20$$

$$\Rightarrow 10\left(\sum_{r=0}^{10}\left(\frac{1}{^{10}C_r}\right)\right)=40$$

$$\Rightarrow \left(\sum_{r=0}^{10}\left(\frac{1}{^{10}C_r}\right)\right)=4$$

Hence, the value of $\sum_{r=0}^{10}\left(\frac{1}{^{10}C_r}\right)$ is 4.

8. We have,

$$27^{50}+18^{50}=3^{150}+2^{50}\cdot 3^{100}$$
$$=3^{100}(3^{50}+2^{50})$$
$$=3^{100}(3^{(4\times 12+2)}+2^{(4\times 12+2)})$$
$$=3^{4\times 25}(3^{(4\times 12+2)}+2^{(4\times 12+2)})$$

Since the periods of 2 and 3 are 4, so the value of m is 3.

And $7^{98}=7^{2\times 49}$

$$=(50-1)^{49}$$
$$={}^{49}C_0(50)^{49}-{}^{49}C_1(50)^{48}+{}^{49}C_2(50)^{47}-\cdots+{}^{49}C_{48}(50)-1$$
$$=({}^{49}C_0(50)^{49}-{}^{49}C_1(50)^{48}+\cdots+{}^{49}C_{48}(50)-5)+4$$

Clearly, $n=4$.

Hence, the value of

$$m+n=3+4=7$$

9. We have

$$\left(x+1+\frac{1}{x}\right)^n$$
$$=\left(\frac{x^2+x+1}{x}\right)^n$$
$$=\frac{(1+x+x^2)^n}{x^n}$$
$$=a_0+a_1x+a_2x^2+a_3x^3+\cdots+a_{2n}x^{2n}$$

Clearly number of terms $=|2n+1|$

Thus, $|2n+1|=17$

$\Rightarrow \quad n=8$

Hence, the value of n is 8.

10. We have,

$$\left(\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right)^{10}$$
$$=\left(\frac{(x^{1/3}+1)(x^{2/3}-x^{1/3}+1)}{|x^{2/3}-x^{1/3}+1|}-\frac{(\sqrt{x}+1)(\sqrt{x}-1)}{\sqrt{x}(\sqrt{x}-1)}\right)^{10}$$
$$=\left((x^{1/3}+1)-\frac{(\sqrt{x}+1)}{\sqrt{x}}\right)^{10}$$
$$=\left((x^{1/3}+1)-\left(1+\frac{1}{\sqrt{x}}\right)\right)^{10}$$
$$=(x^{1/3}-x^{-1/2})^{10}$$

Now, $t_{r+1}={}^{10}C_r\times x^{\frac{10-r}{3}}\times x^{-\frac{r}{2}}$

$$={}^{10}C_r\times x^{\frac{10-r}{3}-\frac{r}{2}}$$
$$={}^{10}C_r\times x^{\frac{20-2r-3r}{6}}$$

Clearly, $20-5r=0$

$\Rightarrow \quad r=\frac{20}{5}=4$

Hence, the value of r is 4.

11. Clearly,

$${}^{18}C_{2r+3}={}^{18}C_{r-3}$$

$\Rightarrow \quad |2r+3+r-3|=18$

$\Rightarrow \quad 3r=18$

$\Rightarrow \quad r=6$

12. Let $t_{r+1}=\left({}^{10}C_r\times 2^{\frac{10-r}{2}}\times 3^{\frac{r}{5}}\right)$, $r=0, 1, 2, \ldots, 10$

The number of rational terms possible only when $r=0, 10$

Thus, $m=2$

Again, let $T_{r+1}=\left({}^{12}C_r\times 7^{\frac{12-r}{3}}\times 5^{\frac{r}{2}}\right)$

where $r=0, 1, 2, \ldots, 12$

The number of integral terms possible only when $r=0, 6, 12$

So, the value of n is 3.

Hence, the value of

$$m+n+2=2+3+2$$
$$=7$$

13. We have

$$\sum_{r=0}^{n}\left(\frac{r+2}{r+1}\right){}^nC_r$$
$$=\sum_{r=0}^{n}\left(1+\frac{1}{r+1}\right){}^nC_r$$
$$=\sum_{r=0}^{n}\left({}^nC_r+\frac{{}^nC_r}{r+1}\right)$$

$$= \sum_{r=0}^{n} ({}^nC_r) + \sum_{r=0}^{n} \left(\frac{{}^nC_r}{r+1}\right)$$

$$= \sum_{r=0}^{n} ({}^nC_r) + \frac{1}{n+1}\sum_{r=0}^{n} ({}^{n+1}C_{r+1})$$

$$= 2^n + \left(\frac{2^{n+1}-1}{n+1}\right)$$

$$= \frac{1}{n+1}((n+1)2^n + 2^{n+1} - 1)$$

$$= \left|\frac{2^n(n+1+2)-1}{n+1}\right|$$

Thus, $\left|\frac{2^n(n+3)-1}{n+1}\right| = \left|\frac{2^8-1}{6}\right|$

Solving we get

$n = 5$

Hence, the value of n is 5.

14. Now, $m = R\{1 - R + [R]\}$

$= (2+\sqrt{3})(2-\sqrt{3}) = 1$

and $n = 1$, where $3^{100} = 3^{4(25)} = 3^{4(4\times 6+1)}$ and period of 3 is 4.

Hence, the value of

$m + n + 4 = 1 + 1 + 4$

$= 6$

15. Clearly, $(5^{25} - 3^{25})$ is divisible by 2.

Thus, $m = 2$

and n = sum of the co-efficients of $(1 + 3x^{100} - 5x^{201})^{2018}$

$= (1 + 3 - 5)^{2018}$, where $x = 1$

$= 1$

Hence, $m + n = 2 + 1 = 3$

Previous Years' JEE-Advanced Examinations

1. We have,

$(1 + x)^{2n}$

$= {}^{2n}C_0 + {}^{2n}C_1x + {}^{2n}C_2x^2 + \ldots + {}^{2n}C_nx^n$ …(i)

Replacing x by $-\frac{1}{x}$, we get

$$\left(1-\frac{1}{x}\right)^{2n} = {}^{2n}C_0 - {}^{2n}C_1\left(\frac{1}{x}\right) + {}^{2n}C_2\left(\frac{1}{x}\right)^2 - \cdots + (-1)^n\,{}^{2n}C_n\left(\frac{1}{x}\right)^n \quad \ldots\text{(ii)}$$

Multiplying Eqs (i) and (ii), we get

$$(1+x)^{2n}\left(1-\frac{1}{x}\right)^{2n}$$

$= ({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \ldots + (-1)^n ({}^{2n}C_n)^2 + (\ldots)x + (\ldots)x^2 + (\ldots)x^3$

Comparing the co-efficients of the constant term from both the sides, we get

$({}^{2n}C_0)^2 - ({}^{2n}C_1)^2 + ({}^{2n}C_2)^2 - \ldots + (-1)^n ({}^{2n}C_n)^2$

$= \text{co-efficients of constant term in } (1-x^2)^{2n} \times \left(\frac{1}{x}\right)^{2n}$

$= (-1)^n \times {}^{2n}C_n$

2. We have,

$(1 + x)^{2n}$

$= {}^{2n}C_0 + {}^{2n}C_1x + {}^{2n}C_2x^2 + \ldots + {}^{2n}C_nx^n$ …(i)

Differentiating w.r.t x, we get

$2n(1 + x)^{2n-1}$

$= {}^{2n}C_1 + 2\cdot{}^{2n}C_2x + 3\cdot{}^{2n}C_3x^2 + \ldots + n\cdot{}^{2n}C_nx^{n-1}$

Replacing x by $-1/x$, we get

$$2n\left(1-\frac{1}{x}\right)^{2n-1}$$

$= {}^{2n}C_0 + {}^{2n}C_1x + {}^{2n}C_2x^2 + \ldots + {}^{2n}C_nx^n$

$$= {}^{2n}C_1 - 2\cdot{}^{2n}C_2\left(\frac{1}{x}\right) + 3\cdot{}^{2n}C_3\left(\frac{1}{x^2}\right) - \cdots + (-1)^{n-1}n\cdot{}^{2n}C_n\left(\frac{1}{x^{n-1}}\right) \quad \ldots\text{(ii)}$$

Multiplying Eqs (i) and (ii), we get

$$2n(1+x)^{2n}\left(1-\frac{1}{x}\right)^{2n-1}$$

$= (({}^{2n}C_1)^2 - 2({}^{2n}C_2)^2 + 3\cdot({}^{2n}C_3)^2 - \ldots + (-1)^n n({}^{2n}C_n)^2)x + (\ldots)x^2 + (\ldots)x^3 + \ldots$

Comparing the co-efficients of x from both the sides, we get

$({}^{2n}C_1)^2 - 2({}^{2n}C_2)^2 + 3\cdot({}^{2n}C_3)^2 - \ldots + (-1)^n n({}^{2n}C_n)^2$

$= \text{Co-efficient of } x \text{ in } 2n(1+x)^{2n}\left(1-\frac{1}{x}\right)^{2n-1}$

$=$ Co-efficient of x^{2n} in $2n(1 + x)^{2n}(x - 1)^{2n-1}$

$=$ Co-efficient of x^{2n} in $2n(1 + x)(x^2 - 1)^{2n-1}$

$=$ Co-efficient of t^n in $(-1)^{2n-1} \times 2n(1 - t)^{2n-1}$

$= (-1)^{2n-1} \times 2n \times (-1)^n \times {}^{2n-1}C_n$

$= (-1)^{n-1} \times 2n \times {}^{2n-1}C_n$

$= (-1)^{n-1} \times n \times {}^{2n}C_n$

3. Given,

co-efficients of $(3r)$th = co-efficients of $(r + 2)$th

$\Rightarrow$ ${}^{2n}C_{3r-1} = {}^{2n}C_{r+1}$

$\Rightarrow$ $3r - 1 = 2n - (r + 1)$

$\Rightarrow$ $4r = 2n$

$\Rightarrow$ $n = 2r$

4. We have,

$${}^{47}C_4 + \sum_{j=1}^{5} {}^{52-j}C_3$$

$= {}^{47}C_4 + {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3 + {}^{47}C_3$

$= ({}^{47}C_4 + {}^{47}C_3) + {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3 + {}^{48}C_3$

$= ({}^{48}C_4 + {}^{48}C_3) + {}^{51}C_3 + {}^{50}C_3 + {}^{49}C_3$

$= ({}^{49}C_4 + {}^{49}C_3) + {}^{51}C_3 + {}^{50}C_3$
$= ({}^{50}C_4 + {}^{50}C_3) + {}^{51}C_3$
$= |{}^{51}C_4 + {}^{51}C_3|$
$= {}^{52}C_4$

5. Do yourself.
6. Given,

$7^{2n} + (2^{3n-3})(3^{n-1}) + 2$
$= 7^{2n} + (2^3)^{n-3}(3^{n-1}) + 2$
$= 7^{2n} + (2^3 \times 3)^{n-1} + 2$
$= (49)^n + (8 \times 3)^{n-1} + 2$

$$= (50^n - {}^nC_1 \cdot 50^{n-1} + {}^nC_2 \cdot 50^{n-2} - \cdots + {}^nC_{n-1} \cdot 50 - 1) + (25^{n-1} - {}^{n-1}C_1 \cdot 25^{n-2} + {}^{n-1}C_2 \cdot 25^{n-3} - \cdots + {}^{n-1}C_{n-2} \cdot 25 - 1) + 2$$

Clearly it is divisible by 25.

7. Given polynomial is $(1 + x - 3x^2)^{2163}$.
Putting $x = 1$, we get
$(1 + 1 - 3)^{2163} = (-1)^{2163} = -1$
Hence, the sum of the co-efficients is -1.
8. Given $(x - 1)(x - 2)(x - 3) \ldots (x - 100)$
$= x^{100} - (1 + 2 + 3 + \ldots + 100)x^9 + \ldots$
Co-efficient of $x^{99} = -(1 + 2 + 3 + \ldots + 100)$

$$= -\frac{100}{2}(1 + 100)$$
$$= -50 \times 101$$
$$= -5050$$

9. $99^{50} + 100^{50}$
$= (100 - 1)^{50} + 100^{50}$
$= 10050 - {}^{50}C_1 100^{49} + {}^{50}C_2 100^{48} - \ldots + {}^{50}C_{50} 100^{50}$
and 101^{50}
$= (100 + 1)^{50}$
$= (100)^{50} + {}^{50}C_1 100^{49} + {}^{50}C_2 100^{48} + \ldots + {}^{50}C_{50}$
$= a$ positive number
Hence, $101^{50} > 99^{50} + 100^{50}$
10. Given,
$(1 + ax)^n = 1 + 8x + 24x^2 + \ldots$
$+ (1 + {}^nC_1(ax) + {}^nC_2(ax)^2 + \ldots)$
$= (1 + 8x + 24x^2 + \ldots)$
Comparing the co-efficients of x and x^2, we get,
${}^nC_1 \cdot a = 8$ and ${}^nC_2 \cdot a^2 = 24$
$\Rightarrow$ $na = 8$ and $n(n - 1)a^2 = 48$
Now, $((na)^2 - na \cdot a) = 48$
$\Rightarrow$ $(64 - 8a) = 48$
$\Rightarrow$ $8a = 64 - 48 = 16$
$\Rightarrow$ $a = 2$
Therefore, $a = 2$ and $n = 4$.
11. We have,

$$t_{r+1} = {}^{10}C_r \left(\frac{x}{2}\right)^{10-r} \left(-\frac{3}{x^2}\right)^r$$
$$= {}^{10}C_r \left(\frac{1}{2}\right)^{10-r} (-3)^r (x)^{10-r-2r}$$

Thus, $10 - 3r = 4$
$\Rightarrow$ $3r = 6$
$\Rightarrow$ $r = 2$
Hence, the co-efficient of x^4

$$= {}^{10}C_2 \times \left(\frac{1}{2}\right)^8 \times (-3)^2 = \frac{45 \times 9}{2^8} = \frac{405}{256}$$

12. We know that

$$2 \sum_{0 \le i < j < n} C_i C_j$$
$$= (C_0 + C_1 + C_2 + \ldots + C_n)^2 - (C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2)$$
$$= (2n)^2 - (2^n C_n)$$
$$= 2^{2n} - \frac{(2n)!}{(n)! \times (n)!}$$

Thus, $\displaystyle\sum_{0 \le i < j \le n} (C_i C_j) = 2^{2n-1} - \frac{1}{2} \cdot \frac{(2n)!}{(n)! \times (n)!}$

13. Do yourself.
14. We have $S_n = 1 + q + q^2 + \ldots + q^n$

$$= \frac{1 - q^{n+1}}{1 - q}$$

and $D_n = 1 + \left(\dfrac{q+1}{2}\right) + \left(\dfrac{q+1}{2}\right)^2 + \cdots + \left(\dfrac{q+1}{2}\right)^n$

$$= \frac{1 - \left(\frac{1+q}{2}\right)^{n+1}}{1 - \left(\frac{1+q}{2}\right)} = \frac{2^{n+1} - (1+q)^{n+1}}{2^n(1-q)}$$

Now, ${}^{n+1}C_1 + {}^{n+1}C_2 S_1 + {}^{n+1}C_3 S_3 + \ldots + {}^{n+1}C_{n+1} S_n$

$$= \frac{1}{1-q}[{}^{n+1}C_1(1-q) + {}^{n+1}C_2(1-q^2) + {}^{n+1}C_3(1-q^3) + \cdots + {}^{n+1}C_{n+1}(1-q^{n+1})]$$
$$= \frac{1}{1-q}\left[\sum_{k=1}^{n+1} {}^{n+1}C_k - \sum_{k=1}^{n+1} {}^{n+1}C_k q^k\right]$$
$$= \frac{1}{1-q}[(2^{n+1} - 1) - \{(1+q)^{n+1} - 1\}]$$
$$= \left|\frac{2^{n+1} - (1+q)^{n+1}}{(1-q)}\right|$$

$= 2^n D_n$

15. Do yourself by mathematical induction.
16. We have,
$2.7^n + 3.5^n - n$
$= 2(1 + 6)^n + 3(1 + 4)^n - 5$
$= 2(1 + {}^nC_1 \cdot 6 + {}^nC_2 \cdot 6^2 + {}^nC_3 \cdot 6^3 + \ldots + {}^nC_n \cdot 6^n)$
$+ 3(1 + {}^nC_1 \cdot 4 + {}^nC_2 \cdot 4^2 + \ldots {}^nC_n \cdot 4^n) - 5$

$= 2({}^nC_1 \cdot 6 + {}^nC_2 \cdot 6^2 + {}^nC_3 \cdot 6^3 + \dots + {}^nC_n \cdot 6^n)$
$+ 3({}^nC_1 \cdot 4 + {}^nC_2 \cdot 4^2 + \dots {}^nC_n \cdot 4^n)$

Clearly, it is divisible by 24.

17. We have,

$$\sum_{r=0}^{n} (-1)^r\, {}^nC_r \left[\frac{1}{2^r} + \frac{3^r}{2^{2r}} + \frac{7^r}{2^{3r}} + \frac{15^r}{2^{4r}} + \cdots m \text{ terms}\right]$$

$$= \sum_{k=1}^{m}\sum_{r=0}^{n} (-1)^n ({}^nC_r)\left(\frac{2^k-1}{2^r}\right)^n$$

$$= \sum_{k=1}^{m}\left(1 - \frac{2^k-1}{2^k}\right)^n$$

$$= \sum_{k=1}^{m}\left(\frac{1}{2^k}\right)^n$$

$$= \sum_{k=1}^{m}\left(\frac{1}{2^n}\right)^k$$

$$= \left|\frac{1}{2^n} + \left(\frac{1}{2^n}\right)^2 + \left(\frac{1}{2^n}\right)^3 + \cdots + \left(\frac{1}{2^n}\right)^m\right|$$

$$= \frac{\frac{1}{2^n}\left(1 - \frac{1}{2^{mn}}\right)}{\left|1 - \frac{1}{2^n}\right|} = \left|\frac{2^{mn}-1}{2^{mn}(2^n-1)}\right|$$

18. If C_r stands for ${}^nC_r$, then sum of the series

$$\frac{2\left(\frac{n}{2}\right)!\left(\frac{n}{2}\right)!}{n!} \times [C_0^2 + C_1^2 + \cdots + (-1)^n (n+1) C_n^2]$$

where n is an even +ve integer, is equal to

(a) 0 (b) $(-1)^{\frac{n+1}{2}}$
(c) $(-1)^n(n+2)$ (d) None

[IIT-JEE, 1986]

19. We have

$t_{k+1} = k^2 C_k$
$= [k(k-1) + k]\, {}^nC_k$
$= [k(k-1)]\, {}^nC_k + k\, {}^nC_k$

$$= k(k-1) \times \frac{n(n-1)}{k(k-1)}\, {}^{n-2}C_{k-2} + k \times \frac{n}{k}\, {}^{n-1}C_{k-1}$$

$= n(n-1)\, {}^{n-2}C_{k-2} + n\, {}^{n-1}C_{k-1}$

Therefore,

$$\sum_{k=0}^{n} k^2\, {}^nC_k = \sum_{k=0}^{n} t_{k+1}$$

$$= \sum_{k=0}^{n} (n(n-1)\, {}^{n-2}C_{k-2} + n\, {}^{n-1}C_{k-1})$$

$$= n(n-1)\sum_{k=0}^{n} ({}^{n-2}C_{k-2}) + n\sum_{k=0}^{n} ({}^{n-1}C_{k-1})$$

$= n(n-1)2^{n-2} + n2^{n-1}$

$= n \cdot 2^{n-2}(n - 1 + 2)$
$= n \cdot (n + 1)2^{n-2}$

20. Prove by the mathematical induction that $\frac{(2n)!}{(2n)! \times (n)^2} \le \frac{1}{\sqrt{(3n+1)}}$ for every +ve integer n.

[IIT-JEE, 1987]

20. Do yourself.

21. Let $F = (5\sqrt{5} - 11)^{2n+1}$

Then $0 < F < 1$

Also, $R - F$

$= (5\sqrt{5} + 11)^{2n+1} - (5\sqrt{5} - 11)^{2n+1}$
$=$ Even integer
$= 2m$ (say)

Thus, $R - F = 2m$

$\Rightarrow [R] + f - F = 2m$
$\Rightarrow f - F = 2m - [R]$

As, $0 < f < 1, 0 < F < 1$

$\Rightarrow 0 < f < 1, -1 < -F < 0$
$\Rightarrow -1 < f - F < 1$
$\Rightarrow f - F = 0$
$\Rightarrow f = F$

Now,

$Rf = (5\sqrt{5} + 11)^{2n+1}(5\sqrt{5} - 11)^{2n+1}$
$= ((5\sqrt{5} + 11) \times (5\sqrt{5} - 11))^{2n+1}$
$= (125 - 121)^{2n+1}$
$= 4^{2n+1}$

22. We have,

$(1 + x)^m$
$= {}^mC_0 + {}^mC_1 x + {}^mC_2 x^2 + \dots + {}^mC_k x^k + \dots$ …(i)

Also,

$(1 + x)^n$
$= {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 xx + \dots + {}^nC_k x^k + {}^nC_{k-1} x^{k-1} + {}^nC_{k-2} x^{k-2} + \dots$ …(ii)

Multiplying Eqs (i) and (ii), we get

$(1 + x)^{m+n} = ({}^mC_0 \cdot {}^nC_k + {}^mC_1 \cdot {}^nC_{k-1} + {}^mC_2 \cdot {}^nC_{k-2} + \dots + {}^mC_k \cdot {}^nC_0)x^k + (\dots)x^{k-1} + (\dots)x^{k-2} + \dots$

Comparing the co-efficients of x^k from both the sides, we get

${}^mC_0 \cdot {}^nC_k + {}^mC_1 \cdot {}^nC_{k-1} + {}^mC_2 \cdot {}^nC_{k-2} + \dots + {}^mC_k \cdot {}^nC_0 = {}^{m+n}C_k$

23. We have,

$C_0 + C_1 x + C_2 x^2 + \dots + C_n x^n = (1 + x)^n$

Multiplying by x, we get

$C_0 x + C_1 x^2 + C_2 x^3 + \dots + C_n x^{n+1} = x(1 + x)^n$

Differentiating w.r.t x, we get

$C_0 + 2^2 C_1 x + 3^2 C_2 x^2 + \dots + (n + 1) C_n X^n = (1 + x)^n + nx(1 + x)^{n-1}$

Again multiplying by x, we get

$C_0 x + 2C_1 x^2 + 3C_2 x^3 + \dots + (n + 1) C_n x^{n+1} = x(1 + x)^n + nx^2(1 + x)^{n-1}$

Differentiating w.r.t x, we get

$$C_0 + 2^2C_1x + 3^2C_2x^2 + \ldots + (n+1)^2 C_nx^n$$
$$= nx(1+x)^{n-1} + n(n-1)x^2(1+x)^{n-2} + 2nx(1+x)^{n-1}$$

Putting $x = -1$, we get

$$C_0 - 2^2C_1 + 3^2C_2 + \ldots + (-1)^n(n+1)^2C_n = 0$$

24. Do yourself, by mathematical induction.
25. The product of n +ve numbers is unity.
Then their sum is
(a) a +ve integer (b) divisible by n
(c) equal to $n + \frac{1}{n}$ (d) never less than n.
[IIT-JEE, 1991]
26. We have,

$$\sum_{m=0}^{k}(n-m)\left(\frac{(r+m)!}{k!}\right)$$
$$= \sum_{m=0}^{k}\left(n\left(\frac{(r+m)!}{k!}\right) - m\left(\frac{(r+m)!}{k!}\right)\right)$$
$$= (r!)\sum_{m=0}^{k}\left(n\left(\frac{(r+m)!}{r!\times k!}\right) - m\left(\frac{(r+m)!}{r!\times k!}\right)\right)$$
$$= (r!)\left(n\sum_{m=0}^{k}{}^{r+m}C_m - m\sum_{m=0}^{k}{}^{r+m}C_m\right)$$
$$= (r!)\Big[n[{}^{r}C_0 + {}^{r+1}C_1 + {}^{r+2}C_2 + \cdots + {}^{r+k}C_k]$$
$$- \sum_{r=0}^{k}[(m+r+1)-(r+1)^{r+m}C_m]\Big]$$
$$= r![n({}^{r+k+1}C_k) - (r+1)({}^{r+k+2}C_k) - ({}^{r+k+1}C_k)]$$
$$= r![n({}^{r+k+1}C_k) - (r+1)({}^{r+k+1}C_{k-1})]$$
$$= r!\left[n\left(\frac{(r+k+1)!}{(r+1)!\times k!}\right) - (r+1)\left(\frac{(r+k+1)!}{(r+2)!\times(k-1)!}\right)\right]$$
$$= \left[\frac{(r+k+1)!}{k!}\right]\left[\frac{n}{r+1} - \frac{k}{r+2}\right]$$

Hence, the result.

27. Given, $(\alpha^2x^2 - 2\alpha x + 1)^{51} = 0$
Putting $x = 1$, we get
$(\alpha^2 - 2\alpha + 1)^{51} = 0$
$\Rightarrow (\alpha^2 - 2\alpha + 1) = 0$
$\Rightarrow (\alpha - 1)^2 = 0$
$\Rightarrow (\alpha - 1) = 0$
$\Rightarrow \alpha = 1$
28. We have,

$$\sum_{r=0}^{2n} a_r(x-2)^r = \sum_{r=0}^{2n} b_r(x-3)^r$$
$$\Rightarrow \sum_{r=0}^{2n} a_r((x-3)+1)^r = \sum_{r=0}^{2n} b_r(x-3)^r$$
$$\Rightarrow \sum_{r=0}^{2n} a_r(y+1)^r = \sum_{r=0}^{2n} b_r y^r,\ y = (x-3)$$

Comparing the co-efficients from both the sides, we get

$$b_n = \sum_{r=0}^{2n} a^r({}^rC_n)$$
$$= \sum_{r=n}^{2n}({}^rC_n)$$
$$= |{}^nC_n + {}^{n+1}C_n + {}^{n+2}C_n + \cdots + {}^{2n}C_n|$$
$$= |{}^nC_0 + {}^{n+1}C_1 + {}^{n+2}C_2 + \cdots + {}^{2n}C_n|$$
$$= {}^{2n+1}C_n$$
$$= {}^{2n+1}C_{n+1}$$

29. We have,

$$\sum_{m=0}^{100}{}^{100}C_m(x-3)^{100-m}\,2^m = ((x-3)+2)^{100}$$
$$= (x-1)^{100}$$
$$= (1-x)^{100}$$

$\therefore$ Co-efficient of $x^{53} = (-1)^{53}\ {}^{100}C_{53} = -{}^{100}C_{53}$.

30. We have,

$$(x+\sqrt{x^3-1})^5 + (x-\sqrt{x^3-1})^5$$
$$= (x+a)^5 + (x-a)^5, \text{ where } a = \sqrt{x^3-1}$$
$$= 2(x^5 + {}^5C_2x^3a^2 + {}^5C_4xa^4)$$
$$= 2[x^5 + {}^5C_2x^3(x^3-1) + {}^5C_4x(x^3-1)^2]$$

Thus, the degree of the polynomial is 7.

31. Let $t_{r+1} = (2r+1)C_r$

$$= (2r+1)\,{}^nC_r$$
$$= 2r\,{}^nC_r + {}^nC_r$$
$$= 2r \times \frac{n}{r} \times {}^{n-1}C_{r-1} + {}^nC_r$$
$$= 2n \times {}^{n-1}C_{r-1} + {}^nC_r$$

Thus, $S_n = \sum_{r=0}^{n} t_{r+1}$

$$= \sum_{r=0}^{n}(2n\,{}^{n-1}C_{r-1} + {}^nC_r)$$
$$= 2n\sum_{r=0}^{n}{}^{n-1}C_{r-1} + \sum_{r=0}^{n}{}^nC_r$$
$$= 2n \cdot 2^{n-1} + 2^n$$
$$= n.2^n + 2^n$$
$$= (n+1)2^n$$

32. Let $n = 2\,m$, so that $k = 3\,m$
We have

$$\sum_{r=1}^{k}(-3)^{r-1}\ {}^{3n}C_{2r-1}$$
$$= \sum_{r=1}^{3m}(-3)^{r-1}\ {}^{6m}C_{2r-1}$$

$$= {}^{6m}C_1 - 3\cdot{}^{6m}C_3 + 3^2\cdot{}^{6m}C_5 - 3^3\,{}^{6m}C_7 + \ldots \quad \ldots\text{(i)}$$

Now,

$$(e^{-i\theta})^{6m} = (\cos\theta - i\sin\theta)^{6m}$$
$$= \cos^{6m}\theta - {}^{6m}C_1\cos^{6m-1}\theta\sin\theta + {}^{6m}C_2\cos^{6m-2}\theta\sin^2\theta$$

Thus,

$\cos(6m)\theta - i\sin(6m)\theta$

$= \cos^{6m}\theta - i\,{}^{6m}C_1\cos^{6m-1}\theta\sin\theta - {}^{6m}C_2\cos^{6m-2}\theta\sin^2\theta + \ldots$

Equating imaginary parts, we get

$$-\sin(6m)\theta = -\,{}^{6m}C_1\cos^{6m-1}\theta\sin\theta + {}^{6m}C_3\cos^{6m-3}\theta\sin^3\theta - \ldots$$

Put $\theta = \dfrac{\pi}{3} \Rightarrow 6\theta = 2\pi$

$$ {}^{6m}C_1\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}\right)^{6m-1} - 3\,{}^{6m}C_3\left(\frac{\sqrt{3}}{2}\right)^3\left(\frac{1}{2}\right)^{6m-3} + \cdots = 0$$

$${}^{6m}C_1 - 3\cdot{}^{6m}C_3 + 3^2\cdot{}^{6m}C_5 \ldots = 0$$

Hence, the result.

33. We have,

$$m = \left|\frac{(n+1)\,|x|}{a+|x|}\right| = \left|\frac{(50+1)\left|\left(\frac{2}{3}\times\frac{1}{5}\right)\right|}{1+\left|\left(\frac{2}{3}\times\frac{1}{5}\right)\right|}\right| = \left|\frac{51\times\frac{2}{15}}{\frac{17}{15}}\right| = 6$$

Thus, the greatest terms are 6th and 7th respectively.

34 We have

$$(1 - x + x^2)^n = a_0 + a_1x + a_2x^2 + \ldots + a_{2n}x^{2n} \quad \ldots\text{(i)}$$

Replacing x by $-x$ in Eq. we get

$$(1 - x + x^2)^n = a_0 - a_1x + a_2x^2 + \ldots + a_{2n}x^{2n} \quad \ldots\text{(ii)}$$

Replacing x by $\dfrac{1}{x}$ in Eq. (i), we get

$$\left(1+\frac{1}{x}+\frac{1}{x^2}\right)^n = a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_{2n}}{x^{2n}} \quad \ldots\text{(iii)}$$

Multiplying Eqs (ii) and (i), we get

$$(1-x+x^2)^n\left(1+\frac{1}{x}+\frac{1}{x^2}\right)^n = (a_0^2 - a_1^2 + a_2^2 - a_3^2 + \ldots + a_n^2) + (\ldots)x + \ldots$$

Comparing the co-efficients of the constant terms from both the sides, we get

$(a_0^2 - a_1^2 + a_2^2 - a_3^2 + \ldots + a_n^2)$

= Co-efficient of constant term in

$$(1-x+x^2)^n(1+x+x^2)^n \times \frac{1}{x^{2n}}$$

= Co-efficient of x^{2n} in

$(1 - x + x^2)^n(1 + x + x^2)^n$

$= [(1 + x^2)^2 - x^2]^n = (1 + x^2 + x^4)^n$

= Co-efficient of t^n in $(1 + t + t^2)^n$

$= a_n$

35. We have,

$t_2 = {}^nC_1x,\ t_3 = {}^nC_2x^2,\ t_4 = {}^nC_3x^3$

Since ${}^nC_1, {}^nC_2, {}^nC_3 \in$ AP

$\Rightarrow\ 2\,{}^nC_2 = {}^nC_1 + {}^nC_3$

$\Rightarrow\ 2\cdot\dfrac{n(n-1)}{2} = n + \dfrac{n(n-1)(n-2)}{6}$

$\Rightarrow\ n(n-1) = n + \dfrac{n(n-1)(n-2)}{6}$

$\Rightarrow\ 6n(n - 1) = 6n + n(n - 1)(n - 2)$

$\Rightarrow\ 6n^2 - 6n = 6n + n(n^2 - 3n + 2)$

$\Rightarrow\ 6n^2 - 6n = 6n + n^3 - 3n^2 + 2n$

$\Rightarrow\ n^3 - 9n^2 + 14n = 0$

$\Rightarrow\ n(n^2 - 9n + 14) = 0$

$\Rightarrow\ n(n - 2)(n - 7) = 0$

$\Rightarrow\ n = 0, 2, 7$

$\Rightarrow\ n = 7$

36. For $r \ge 0$, let

$$t_r = (-1)^r\left(\frac{C_r}{{}^{r+3}C_r}\right)$$
$$= (-1)^r\left(\frac{n!}{r!\times(n-r)!}\right)\times\left(\frac{r!\times 3!}{(r+3)!}\right)$$
$$= \frac{(-1)^r\,3!}{(n+1)(n+2)(n+3)}\times\left(\frac{(n+3)!}{(r+3)!\times(n-r)!}\right)$$
$$= \frac{3!}{(n+1)(n+2)(n+3)}\times|{}^{n+3}C_{r+3}\times(-1)^r|$$

Thus,

$$\sum_{r=0}^{n} t_r$$
$$= \sum_{r=0}^{n}\left(\frac{3!}{(n+1)(n+2)(n+3)}\times({}^{n+3}C_{r+3}\times(-1)^r)\right)$$
$$= \frac{3!(-1)^3}{(n+1)(n+2)(n+3)}\sum_{r=0}^{n}(({}^{n+3}C_{r+3}\times(-1)^{r+3}))$$
$$= \frac{3!(-1)^3}{(n+1)(n+2)(n+3)} \times[(1-1)^{n+3} - {}^{n+3}C_0 + {}^{n+3}C_1 - {}^{n+3}C_2]$$

$$= \frac{3!(-1)^3}{(n+1)(n+2)(n+3)} \times \left[-1 + (n+3) - \frac{(n+3)(n+2)}{2}\right]$$

$$= \frac{3!(-1)^3}{(n+1)(n+2)(n+3)} \times \left[(n+2) - \frac{(n+3)(n+2)}{2}\right]$$

$$= \frac{3!(-1)^3}{(n+1)(n+2)(n+3)} \times -\frac{1}{2}(n+2)(n+3)$$

$$= \frac{3!}{2(n+3)}$$

37. We have,

$$t_{r+1} = {}^{10}C_r (2^{1/2})^{10-r} (3^{1/5})^r$$
$$= {}^{10}C_r (2)^{\frac{10-r}{2}} (3)^{\frac{r}{5}}$$

where $r = 0, 10$

When $r = 0$, $t_1 = {}^{10}C_0 (2)^5 (3)^0 = 32$

When $r = 10$, $t_{11} = {}^{10}C_{10} (2)^0 (3)^2 = 9$

Therefore the sum of the rational terms,

$t_1 + t_{11} = 32 + 9 = 41$

38. We have,

$$\sum_{r=0}^{n}\left(\frac{r}{{}^nC_r}\right) = \sum_{r=0}^{n}\left(\frac{n-r}{{}^nC_{n-r}}\right)$$
$$= \sum_{r=0}^{n}\left(\frac{n}{{}^nC_{n-r}}\right) - \sum_{r=0}^{n}\left(\frac{r}{{}^nC_{n-r}}\right)$$
$$= n\sum_{r=0}^{n}\left(\frac{1}{{}^nC_r}\right) - \sum_{r=0}^{n}\left(\frac{r}{{}^nC_r}\right)$$

$$\Rightarrow \quad 2\sum_{r=0}^{n}\left(\frac{r}{{}^nC_r}\right) = n\sum_{r=0}^{n}\left(\frac{1}{{}^nC_r}\right) = na_n$$

$$\Rightarrow \quad \sum_{r=0}^{n}\left(\frac{r}{{}^nC_r}\right) = \frac{na_n}{2}$$

39. We have,

$(1 + x)^m (1 - x)^n$

$= (1 + {}^mC_1 x + {}^mC_2 x^2 + \ldots) \times (1 - {}^nC_1 x + {}^nC_2 x^2 - \ldots)$

Now, co-efficient of $x = 3$

$\Rightarrow \quad {}^mC_1 - {}^nC_1 = 3$

$\Rightarrow \quad m - n = 3$ …(i)

Also, co-efficient of $x^2 = -6$

$\Rightarrow \quad {}^mC_2 + {}^nC_2 - {}^mC_1 \times {}^mC_1 = -6$

$$\Rightarrow \quad \frac{m(m-1)}{2} + \frac{n(n-1)}{2} - mn = -6$$

$\Rightarrow \quad m(m - 1) + n(n - 1) - 2mn = -12$

$\Rightarrow \quad m^2 + n^2 - 2mn - (m + n) = -12$

$\Rightarrow \quad (m - n)^2 - (m + n) = -12$

$\Rightarrow \quad (-3)^2 - (m + n) = -12$

$\Rightarrow \quad -(m + n) = -21$

$\Rightarrow \quad |m + n| = 21$ …(ii)

Solving Eqs (i) and (ii), we get

$m = 12$ and $n = 9$.

Hence, the value of m is 12.

40. For a +ve integer n.

Let $a(n) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2n-1}$, then

(a) $a(100) \le 100$ (b) $a(100) > 100$
(c) $a(200) \le 100$ (d) $a(200) > 100$

[IIT-JEE, 1999]

41. Let n be any +ve integer. Prove that

$$\sum_{k=0}^{m}\left(\frac{\binom{2n-k}{k}}{\binom{2k-k}{k}}\right) \times \left(\frac{2n-4k+1}{2n-2k+1}\right) \times 2^{n-2k} = \left(\frac{\binom{n}{m}}{\binom{2n-2m}{n-m}}\right) \times 2^{n-2m}$$

for each non –ve integer $m \le n$

$\left(\text{Here} \binom{p}{q} = {}^pC_q\right)$ [IIT-JEE, 1999]

Solution:

42. $\binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2}$

$= {}^nC_r + 2{}^nC_{r-1} + {}^nC_{r-2}$

$= ({}^nC_r + {}^nC_{r-1}) + ({}^nC_{r-1} + {}^nC_{r-2})$

$= ({}^{n+1}C_r + {}^{n+1}C_{r-1})$

$= {}^{n+2}C_r$

43. We have,

$$\binom{n}{m} + \binom{n-1}{m} + \binom{n-2}{m} + \cdots + \binom{m}{m}$$

= Co-efficient of x^m in the expansion of

$(1 + x)^n + (1 + x)^{n-1} + (1 + x)^{n-2} + \ldots + (1 + x)^m$

$= (1 + x)^m ((1 + x)^{n-m} + (1 + x)^{n-m-1} + (1 + x)^{n-m-2} + \ldots + 1)$

$$= (1+x)^m \left(\frac{(1+x)^{n-m+1} - 1}{(1+x) - 1}\right)$$

$$= (1+x)^m \left(\frac{(1+x)^{n-m+1} - 1}{x}\right)$$

$$= \left(\frac{(1+x)^{n+1} - (1+x)^m}{x}\right)$$

$\Rightarrow$ Co-efficient of x^{m+1} in the expansion of

$(1 + x)^{n+1} - (1 + x)^m$

$$= {}^{n+1}C_{m+1} = \binom{n+1}{m+1}$$

44. Given,

$t_5 + t_6 = 0$

$\Rightarrow \quad {}^nC_4 a^{n-4}(-b)^4 + {}^nC_5 a^{n-5}(-b)^5 = 0$

$\Rightarrow \quad {}^nC_4 a^{n-4} b^4 - {}^nC_5 a^{n-5} b^5 = 0$

$\Rightarrow \quad {}^nC_4 a^{n-4} b^4 = {}^nC_5 a^{n-5} b^5$

$\Rightarrow \quad {}^nC_4 a = {}^nC_5 b$

$$\Rightarrow \quad \frac{a}{b} = \frac{{}^nC_5}{{}^nC_4} = \frac{(n-4)}{5}$$

45. We have

$\sum_{r=0}^{m}\binom{10}{i}\binom{20}{m-i}$ = the number of ways of selecting m persons out of 10 men and 20 women = the number of ways of selecting m out of 30 persons = ${}^{30}C_m$.

But ${}^{30}C_m$ is maximum when $m = 15$.

46. $(1 + t^2)^{12}(1 + t^{12})(1 + t^{24})$

$= (1 + t^2)^{12}(1 + t^{12} + t^{24} + t^{36})$

$= [1 + {}^{12}C_1 t^2 + {}^{12}C_2 (t^2)^2 + \ldots + {}^{12}C_6 (t^2)^6$

$+ \ldots + {}^{12}C_{12}(t^2)^{12} + \ldots] \times (1 + t^{12} + t^{24} + t^{36})$

Thus the co-efficients of $t^{24} = {}^{12}C_{12} + 1 + {}^{12}C_6$

$= {}^{12}C_6 + 2$

47. For $r \geq 0$, let

$$t_r = (-1)^r\, 2^{k-r}\binom{n}{r}\binom{n-r}{k-r}$$

$$= (-1)^r\, 2^{k-r} \times \left(\frac{n!}{r! \times (n-r)!}\right) \times \left(\frac{(n-r)!}{(k-r)! \times (n-k)!}\right)$$

$$= (-1)^r\, 2^{k-r} \times \left(\frac{n!}{k! \times (n-k)!}\right) \times \left(\frac{k!}{(k-r)! \times r!}\right)$$

$$= (-1)^r\, 2^{k-r} \times \binom{n}{k} \times \binom{k}{r}$$

Thus,

$$S = \sum_{r=0}^{k} (-1)^r\, 2^{k-r} \times \binom{n}{k} \times \binom{k}{r}$$

$$= \binom{n}{k} \sum_{r=0}^{k} (-1)^r\, 2^{k-r} \times \binom{k}{r}$$

$$= \binom{n}{k} \times (2-1)^k = \binom{n}{k}$$

48. We have,

${}^nC_{r+1} = (k^2 - 3)^{n-1}C_r$

$$\Rightarrow \quad (k^2 - 3) = \frac{{}^nC_{r+1}}{{}^{n-1}C_r} = \frac{n}{r+1}$$

Here, $1 \leq r \leq n + 1$

$$\Rightarrow \quad 0 < \frac{n}{n+1} \leq \frac{r}{n+1} \leq 1$$

Thus, $0 < k^2 - 3 \leq 1$

$\Rightarrow \quad 3 < k^2 \leq 4$

$\Rightarrow \quad \sqrt{3} < k \leq 2$

49. We have,

$(1 + x)^{30}$

$= {}^{30}C_0 + {}^{30}C_1 x + {}^{30}C_2 x^2 + {}^{30}C_3 x^3 + \ldots + {}^{30}C_{20} x^{30} + \ldots$

and $(1 - x)^{30}$

$= {}^{30}C_0 x^{30} + \ldots + {}^{30}C_{10} x^{20} - {}^{30}C_{11} x^{19} + \ldots + {}^{30}C_{30}$

Now,

$(1 + x)^{30}(1 - x)^{30}$

$= ({}^{30}C_0 \cdot {}^{30}C_{10} - {}^{30}C_1 \cdot {}^{30}C_{11}$

$+ {}^{30}C_2 \cdot {}^{30}C_{12} - \cdots + {}^{30}C_{10} \cdot {}^{30}C_{30}) x^{20}$

$+ (\ldots)x^{19} + \cdots + (\ldots)x^{30} + \cdots$

Comparing the co-efficients of x^{20} from both the sides, we get

${}^{30}C_0 \cdot {}^{30}C_{10} - {}^{30}C_1 \cdot {}^{30}C_{11}$

$+ {}^{30}C_2 \cdot {}^{30}C_{12} - \cdots + {}^{30}C_{10} \cdot {}^{30}C_{30} = {}^{30}C_{10}$

50. We have,

$A_r = {}^{10}C_r,\ B_r = {}^{10}C_r,\ C_r = {}^{30}C_r$

Now,

$$\sum_{r=1}^{10} A_r B_r = A_1B_1 + A_2B_2 + \cdots + A_{10}B_{10}$$

$$= {}^{10}C_1\, {}^{20}C_1 + {}^{10}C_2\, {}^{20}C_2 + {}^{10}C_3\, {}^{20}C_3 + \ldots + {}^{10}C_{10}\, {}^{20}C_{10}$$

Co-efficient of x^{20} in the expansion of

$(1 + x)^{10}(1 + x)^{20} - 1$

$= {}^{30}C_{20} - 1 = {}^{30}C_{10} - 1 = C_{10} - 1$

Also,

$$\sum_{r=1}^{10} (A_r)^2$$

$$= \sum_{r=1}^{10} ({}^{10}C_r)^2$$

$$= ({}^{10}C_1)^2 + ({}^{10}C_2)^2 + ({}^{10}C_3)^2 + \cdots + ({}^{10}C_{10})^2$$

Co-efficient of x^{10} in the expansion of

$(1 + x)^{10}(1 + x)^{10} - 1$

$= {}^{20}B_{10} - 1$

$= B_{10} - 1$

Therefore,

$$\sum_{r=1}^{10} A_r (B_{10}B_r - C_{10}A_r)$$

$$= \sum_{r=1}^{10} B_{10}(A_rB_r) - C_{10}(A_r)^2$$

$= B_{10}(C_{10} - 1) - C_{10}(B_{10} - 1)$

$= (C_{10} - B_{10})$

51. Let the three consecutive terms are t_r, t_{r-1}, t_{r-2} respectively.

Given,

$$^{n+5}C_{r-1} : {}^{n+5}C_r : {}^{n+5}C_{r+1} = 5 : 10 : 14$$

$$\Rightarrow \quad \frac{^{n+5}C_r}{^{n+5}C_{r-1}} = \frac{10}{5} \text{ and } \frac{^{n+5}C_{r+1}}{^{n+5}C_r} = \frac{14}{10}$$

$$\Rightarrow \quad \frac{(n+5)-r+1}{r} = 2 \text{ and } \frac{(n+5)-(r+1)+1}{r+1} = \frac{7}{5}$$

$$\Rightarrow \quad \frac{n+6}{r} = 3 \text{ and } \frac{(n+6)}{r+1} = \frac{12}{5}$$

$$\Rightarrow \quad 3r = \frac{12}{5}(r+1)$$

$\Rightarrow \quad r = 4$ and $n + 6 = 12$

Thus, $r = 4$ and $n = 6$.

52. We have,

$$(1 + x^2)^4(1 + x^3)^7(1 + x^4)^{12}$$

$$= (1 + {}^4C_1x^2 + {}^4C_2x^4 + {}^4C_3x^6 + {}^4C_4x^8)$$

$$\times (1 + {}^7C_1x^3 + {}^7C_2x^6 + {}^7C_3x^9 + {}^7C_4x^{12} + \ldots)$$

$$\times (1 + {}^{12}C_1x^4 + {}^{12}C_2x^8 + \ldots)$$

$$= (1 + {}^4C_1x^2 + {}^4C_2x^4 + {}^4C_3x^6 + {}^4C_4x^8)$$

$$\times [1 + {}^{12}C_1x^4 + {}^{12}C_2x^8 + {}^7C_1x^3 + ({}^7C_1 \times {}^{12}C_1)x^7$$

$$+ ({}^7C_1 \times {}^{12}C_2)x^{11} + {}^7C_3x^9 + \ldots]$$

Thus,

the co-efficient of x^{11}

$$= ({}^7C_1 \times {}^{12}C_2) + ({}^4C_1 \times {}^7C_3) + ({}^7C_1 \times {}^{12}C_1 \times {}^4C_2)$$

$$+ ({}^4C_4 \times {}^7C_1)$$

$$= 7 \times 66 + 4 \times 35 + 7 \times 12 \times 6 + 7$$

$$= 462 + 140 + 504 + 7$$

$$= 1113$$

CHAPTER

7

Matrices and Determinants

CONCEPT BOOSTER

Introduction

1. Matrix

A set of *mn* numbers (real or complex) arranged in the form of a rectangular array having *m* rows and *n* columns is called an $m \times n$ matrix. We read as *m* by *n* matrix.

An $m \times n$ matrix is usually written as

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

It is also denoted as $[a_{ij}]_{m \times n}$.

> ***Note*** A matrix is not a number. It just an ordered collection of numbers arranged in the form of a rectangular array.

1.1 Order

If a matrix has *m* rows and *n* columns, the order of the matrix is *m* by *n* or $m \times n$.

(i) The order of the matrix $[1 \quad 2 \quad 3]$ is 1×3.

(ii) The order of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 3 & 2 \\ 5 & 6 & 4 \end{bmatrix}$ is 3×3.

(iii) The order of the matrix $\begin{bmatrix} 1 & 4 & 8 \\ 2 & 3 & 5 \end{bmatrix}$ is 2×3.

2. Types of Matrices

(i) Row matrix

A matrix having only one row is called a row matrix.

For example, Let $A = [1 \quad 2 \quad 3]$.

(ii) Column matrix

A matrix having only one column is called a column matrix.

For example, $B = \begin{bmatrix} 1 \\ 2 \\ 5 \\ 9 \end{bmatrix}$.

(iii) Rectangular matrix

A matrix in which number of rows and number of columns are not equal is called a rectangular matrix.

For example, $C = \begin{bmatrix} 1 & 4 & 8 \\ 2 & 3 & 5 \end{bmatrix}$.

(iv) Square matrix

In a matrix, in which the number of rows is equal to the number of columns, it is called a square matrix.

For example, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 8 & 7 & 6 \end{pmatrix}$

are square matrices of order 2 and 3 respectively.

(v) Diagonal matrix

In a square matrix, if all the diagonal elements are non-zero and rest are zero is called a diagonal matrix.

For example, $A = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix}$, etc.

(vi) Scalar matrix

In a square matrix, if all the diagonal elements are the same and rest of the elements are zero, it is called a scalar matrix.

For example, $A = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}, B = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

(vii) Identity matrix
In a scalar matrix, if all the diagonal elements are 1, it is called an identity matrix.

For example, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, etc.

(viii) Non-zero matrix
In a matrix, if at-least one element is non-zero, it is called a non-zero matrix.

For example, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$

are non-zero matrices.

(ix) Zero matrix
In a matrix, if every elements are zero, it is known as zero matrix. It is denoted as **O**.

For example, $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, etc.

(x) Upper triangular matrix
In a square matrix, if all the elements below the leading elements are zero, it is called a upper triangular matrix.

For example, $A = \begin{bmatrix} 1 & 2 \\ 0 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 7 \end{bmatrix}$, etc.

(xi) Lower triangular matrix
In a square matrix, if all the elements above the leading elements are zero, it is called a lower triangular matrix.

For example, $A = \begin{bmatrix} 1 & 0 \\ 2 & 7 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 7 & 0 \\ 2 & 3 & 4 \end{bmatrix}$, etc.

(xii) Strictly triangular matrix
In a square matrix, if all the diagonal matrices are zero, it is called a strictly triangular matrix.

For example, $A = \begin{bmatrix} 0 & 3 \\ 2 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 2 & 3 \\ 5 & 0 & 7 \\ 3 & 6 & 0 \end{bmatrix}$, etc.

(xiii) Comparable matrices
Two matrices are said to be comparable matrices, if their orders are the same.

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are two comparable matrices.

(xiv) Trace of a matrix
The sum of the diagonal elements of a matrix is known as the trace of a matrix.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, then $\text{Tr}(A) = a_{11} + a_{22} + a_{33}$

(xv) Equality of two matrices
Two comparable matrices are said to be equal if their corresponding elements are the same.

If $\begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $a = 2, b = 3, c = 4$ and $d = 6$.

(xvi) Sub-matrix
Any matrix is obtained by eliminating some rows and some columns from a given matrix A, it is called a sub-matrix of A.

Let $A = \begin{bmatrix} 1 & 3 & 4 & 6 \\ 7 & 0 & 5 & 2 \\ 2 & 5 & 9 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 3 & 4 \\ 7 & 0 & 5 \end{bmatrix}$,

B is a sub-matrix of A.

3. Addition of Matrices

We can find the addition of two or more matrices if they are comparable matrices otherwise addition is not defined.

Let $A = \begin{bmatrix} 1 & 3 \\ 6 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$,

then $A + B = \begin{bmatrix} 3 & 6 \\ 10 & 6 \end{bmatrix}$

Properties of Addition of Matrices
(i) Matrix addition is commutative.
(ii) Matrix addition is associative.
(iii) Additive identity of a matrix exists.
(iv) Additive inverse of a matrix exists.

3.1 Scalar Multiplication

If $A = \begin{bmatrix} 2 & 4 \\ 6 & 7 \end{bmatrix}$, then $kA = \begin{bmatrix} 2k & 4k \\ 6k & 7k \end{bmatrix}$,

where k is the scalar multiple of A.

3.2 Negative of a matrix

If $A = \begin{bmatrix} 2 & 4 \\ 6 & 7 \end{bmatrix}$, then $-A = \begin{bmatrix} -2 & -4 \\ -5 & -7 \end{bmatrix}$

4. Multiplication of Matrices

If the number of columns of a first matrix is equal to the number of rows of a second matrix, we can find out the product, otherwise product is not defined.

If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ji}]_{n \times p}$,

then $AB = [c_{ij}]_{m \times p}$

Thus, if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$,

then $AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$

Properties of matrix multiplication

(i) In general, matrix multiplication is not commutative.
(ii) Matrix multiplication is associative.
(iii) Matrix multiplication is distributive over addition
(iv) Multiplicative identity of a matrix exists.

5. Transpose of a Matrix

Transpose of a matrix is obtained by interchanging rows into columns and columns into rows.

If A be any matrix, its transpose is denoted by A^T or A'.

For example, $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, then $A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

5.1 Properties of Transpose of Matrices

(i) $(A + B)^T = A^T + B^T$
(ii) $(A^T)^T = A$
(iii) $(kA)^T = k(A)^T$
(iv) $(AB)^T = B^T A^T$

5.2 Symmetric Matrix

A square matrix A is said to be symmetric $A^T = A$.

For example, $A = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}, B = \begin{bmatrix} 2 & 5 \\ 5 & 7 \end{bmatrix}$,

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 7 \end{bmatrix} \text{ and } D = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$$

etc., are symmetric matrices.

Example: If $A = \begin{pmatrix} x & x+3 \\ 6x-2 & 10x \end{pmatrix}$ be a symmetric matrix, find x.

5.3 Skew-symmetric Matrix

A square matrix is said to be skew-symmetric, if $A^T = -A$.

For example, $A = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 & 3 \\ -2 & 0 & 5 \\ -3 & -5 & 0 \end{bmatrix}$ are skew-symmetric matrices.

Example: If $A = \begin{pmatrix} x & x+2 \\ 5x-2 & 9x \end{pmatrix}$ is a skew-symmetric matrix, find x.

5.4 Properties of Symmetric and Skew-symmetric Matrices

(i) In skew-symmetric matrix, all the diagonal elements are zero.

Let $A = (a_{ij})$
Given $A^T = -A$
$\Rightarrow$ $(a_{ji}) = -(a_{ij})$
Put $i = j$
$\Rightarrow$ $(a_{ii}) = -(a_{ii})$
$\Rightarrow$ $2(a_{ii}) = \mathbf{O}$
$\Rightarrow$ $(a_{ii}) = \mathbf{O}$
$\Rightarrow$ $a_{11} = 0 = a_{22} = a_{33} = \ldots = a_{nn}$

(ii) For any square matrix A, $A + A^T$ is symmetric and $A - A^T$ is skew-symmetric.

(iii) Any square matrix can be expressed uniquely as a sum of a symmetric and a skew-symmetric matrices.

Let A be a square matrix.

Consider $A = \frac{1}{2}(2A)$

$$= \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$
$$= P + Q, \text{ (say)}$$

To prove $P' = P$ and $Q' = -Q$

Now, $P^T = \left[\frac{1}{2}(A + A^T)\right]^T$

$$= \frac{1}{2}(A + A^T)^T$$
$$= \frac{1}{2}[A^T + (A^T)^T]$$
$$= \frac{1}{2}(A^T + A)$$
$$= \frac{1}{2}(A + A^T) = P$$

Thus P is a symmetric matrix.

Also, $Q^T = \left(\frac{1}{2}(A - A^T)\right)^T$

$$= \frac{1}{2}(A - A^T)^T$$
$$= \frac{1}{2}[A^T - (A^T)^T]$$
$$= \frac{1}{2}(A^T - \mathrm{A})$$
$$= -\frac{1}{2}(A - A^T)$$
$$= -Q$$

Thus Q is skew-symmetric matrix

Hence, the result.

(iv) The sum of skew-symmetric matrices is again a skew-symmetric matrix.
(v) The square of a skew-symmetric matrix is not a skew symmetric.
(vi) The cube of a skew symmetric matrices is again skew-symmetric matrix.
(vii) If A is symmetric matrix, then kA is also symmetric.
(viii) If A is skew-symmetric matrix, then kA is also skew-symmetric matrix.
(ix) If A is symmetric, then AA^T and A^TA are symmetric matrices.

(xii) If A is symmetric (skew-symmetric) matrix, then $B^T AB$ is symmetric (skew-symmetric) matrix.

6. Determinant

For every square matrix of order n, there is associated number (real or complex) is called a determinant of the same order.

A determinant is a polynomial of the elements of a square matrix. It is scalar.

It has some finite values. Determinants are defined only for square matrices. Determinants of a non-square matrix is not defined.

Determinant of a square matrix A is denoted by det A or $|A|$.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Then $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

6.1 Minor of an Element of a Matrix

The minor of an element of a matrix is obtained after deleting the corresponding rows and corresponding columns.

Let $A = \begin{bmatrix} 1 & 3 \\ 5 & 8 \end{bmatrix}$.

Then
minor of 1 = 8
minor of 3 = 5
minor of 5 = 3, etc.

Also, let $B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

Then

minor of 1 = $\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = 45 - 48 = -3$

minor of 2 = $\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = 36 - 42. = -6$ and so on.

6.2 Co-factor of an Element of a Matrix

The co-factor of an element of a matrix is obtained after deleting the corresponding rows and corresponding columns with a proper sign. The sign scheme can be used for 2nd order matrix $\begin{pmatrix} + & - \\ - & + \end{pmatrix}$ and for 3rd order matrix $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$.

Let $A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$.

Then

co-factor of 2 = 5
co-factor of 3 = –4
co-factor of 4 = –3
co-factor of 5 = 2

Also, if $B = \begin{pmatrix} 1 & 0 & -2 \\ 4 & -5 & 6 \\ -7 & -3 & 6 \end{pmatrix}$,

the co-factor of

$1 = \begin{vmatrix} -5 & 6 \\ -3 & 6 \end{vmatrix} = -30 + 18 = -12.$

co-factor of $0 = \begin{vmatrix} 4 & 6 \\ -7 & 6 \end{vmatrix} = 24 + 42 = -66$ and so on.

6.3 Expansion of a Determinant

Expansion of a determinant is the sum of the product of the elements of any one row or column with their corresponding co-factors.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$,

then $|A| = a.d + b.(-c) = ad - bc$

Also, if $B = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$

then $|B| = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

$$a\begin{vmatrix} b & f \\ f & c \end{vmatrix} - h\begin{vmatrix} h & f \\ g & c \end{vmatrix} + g\begin{vmatrix} h & b \\ g & f \end{vmatrix}$$

$= a(bc - f^2) - h(ch - fg) + g(hf - bg)$
$= abc - af^2 - ch^2 + fgh + fgh - bg^2$
$= abc + 2fgh - af^2 - bg^2 - ch^2$

6.5 Properties of Determinants

1. The value of a determinant remains unchanged, if the rows and columns are interchanged.
   If A is any square matrix, then $|A| = |A^T|$.
2. If any two rows or columns of a determinant are interchanged, the value of determinant remains same but in opposite sign.

   Let $D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

   and $D_2 = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$.

   Then $D_1 = -D_2$
3. If every element of a row or column of a determinant is zero, the value of a determinant is zero.

   For example, $D_1 = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0$

4. If any two rows (or columns) of a determinant are identical, the value of determinant is zero.

Let $D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$,

then $D_1 = 0$

5. If any two rows or columns of a determinant are proportional, the value of a determinant is 0.

Let $D_1 = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 4 & 5 & 7 \end{vmatrix}$,

then $D_1 = 0$

6. If every elements of any one row or column is multiplied by a non-zero constant, the value of a determinant is multiplied by that number.

Let $D_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

and $D_2 = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, then $D_1 = kD_2$

7. If every element of a row or a column as a sum of two or more terms, the given determinant is equal to the sum of two or more determinants.
For example,

$$\begin{vmatrix} a_1 + x & b_1 + y & c_1 + z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & y & z \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

8. The value of a determinant is unchanged by adding to the elements of any row or column with the same multiples of the corresponding elements of any other row or column.

Let $D_1 = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ and $D_2 = \begin{vmatrix} a + cm & b + dm \\ c & d \end{vmatrix}$,

then $D_1 = D_2$

9. **Special determinants**

(i) **Symmetric determinant**

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$$

(ii) **Skew-symmetric determinant of odd order is zero, i.e.**

$$\begin{vmatrix} 0 & b & c \\ -b & 0 & a \\ -c & -a & 0 \end{vmatrix} = 0$$

(iii) **Circulant determinant**

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3 + b^3 + c^3 - 3abc)$$

(iv) **Some important determinants to remember**

1. $\begin{vmatrix} 1 & a & b + c \\ 1 & b & c + a \\ 1 & c & a + b \end{vmatrix} = 0$

2. $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a - b)(b - c)(c - a) = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

3. $\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a - b)(b - c)(c - a)(a + b + c)$

4. $\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$

$= (a - b)(b - c)(c - a)(ab + bc + ca)$

5. $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & 1 & \omega^2 \\ \omega & \omega^2 & 1 \end{vmatrix} = 0$,

where ω is the complex cube root of unity.

6. $\begin{vmatrix} 1 & bc & ca + ab \\ 1 & ca & ab + bc \\ 1 & ab & bc + ca \end{vmatrix} = 0$

7. $\begin{vmatrix} a - b & b - c & c - a \\ p - q & q - r & r - p \\ x - y & y - z & z - x \end{vmatrix} = 0$

8. $\begin{vmatrix} \sin A & \cos A & \sin(A + \theta) \\ \sin B & \cos B & \sin(B + \theta) \\ \sin C & \cos C & \sin(C + \theta) \end{vmatrix} = 0$

7. Cramers Rule Statement

The solutions of the system of equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

are given by

$$x = \frac{D_1}{D}, y = \frac{D_2}{D}, z = \frac{D_3}{D},$$

where

$$D=\begin{vmatrix}a_1 & b_1 & c_1\\a_2 & b_2 & c_2\\a_3 & b_3 & c_3\end{vmatrix}, D_1=\begin{vmatrix}d_1 & b_1 & c_1\\d_2 & b_2 & c_2\\d_3 & b_3 & c_3\end{vmatrix},$$

$$D_2=\begin{vmatrix}a_1 & d_1 & c_1\\a_2 & d_2 & c_2\\a_3 & d_3 & c_3\end{vmatrix} \text{ and } D_3=\begin{vmatrix}a_1 & b_1 & d_1\\a_2 & b_2 & d_2\\a_3 & b_3 & d_3\end{vmatrix}$$

Proof: Given $D=\begin{vmatrix}a_1 & b_1 & c_1\\a_2 & b_2 & c_2\\a_3 & b_3 & c_3\end{vmatrix}$

$$\Rightarrow \quad xD=x\begin{vmatrix}a_1 & b_1 & c_1\\a_2 & b_2 & c_2\\a_3 & b_3 & c_3\end{vmatrix}$$

$$=\begin{vmatrix}a_1x & b_1 & c_1\\a_2x & b_2 & c_2\\a_3x & b_3 & c_3\end{vmatrix}$$

$$\Rightarrow \quad =\begin{vmatrix}a_1x+b_1y+c_1z & b_1 & c_1\\a_2x+b_2y+c_2z & b_2 & c_2\\a_3x+b_3y+c_3z & b_3 & c_3\end{vmatrix}$$

$(C_1 \to C_1 + C_2 + C_3)$

$$\Rightarrow \quad xD=\begin{vmatrix}d_1 & b_1 & c_1\\d_2 & b_2 & c_2\\d_3 & b_3 & c_3\end{vmatrix}=D_1$$

$$\Rightarrow \quad x=\frac{D_1}{D}$$

Similarly, we can proved that

$$y=\frac{D_2}{D}, z=\frac{D_3}{D}$$

Hence, the result.

Nature of solutions of the system of equaions by Camers Rule

(i) If $D \neq 0$, the system of equations has a unique solution and is said to be consistent.

(ii) If $D = 0$ as well as $D_1 = 0 = D_2 = D_3$, the system of equations has infinitely many solutions and is said to be consistent.

(iii) If $D = 0$ and at least one of D_1, D_2, D_3 is non-zero, the system of equations has no solution and is said to be inconsistent.

8. Homogeneous System of Equations

The given homogenous system of equations are

$$a_1x + b_1y + c_1z = 0$$
$$a_2x + b_2y + c_2z = 0$$
$$a_3x + b_3y + c_3z = 0.$$

Let $$D=\begin{vmatrix}a_1 & b_1 & c_1\\a_2 & b_2 & c_2\\a_3 & b_3 & c_3\end{vmatrix}, D_1=\begin{vmatrix}0 & b_1 & c_1\\0 & b_2 & c_2\\0 & b_3 & c_3\end{vmatrix},$$

$$D_2=\begin{vmatrix}a_1 & 0 & c_1\\a_2 & 0 & c_2\\a_3 & 0 & c_3\end{vmatrix} \text{ and } D_3=\begin{vmatrix}a_1 & b_1 & 0\\a_2 & b_2 & 0\\a_3 & b_3 & 0\end{vmatrix}$$

Nature of solutions by homogeneous system of equations

(i) If $D \neq 0$, the system of equations has only trivial solution, say $x = 0 = y = z$, and the system of equations is said to be consistent.

(ii) If $D = 0$, the system of equations has non-trivial solution, i.e. infinite solutions and the system of equations is also said to be consistent.

9. Multiplication of Two Determinants

Two determinants can be multiplied by a variety of ways. row-by-column, row-by-row, column-by-column and column-by-row multiplication rule.

Let $A=\begin{vmatrix}a_1 & b_1\\a_2 & b_2\end{vmatrix}$ and $B=\begin{vmatrix}m_1 & n_1\\m_2 & n_2\end{vmatrix}$.

Then $AB=\begin{vmatrix}a_1 & b_1\\a_2 & b_2\end{vmatrix}\times\begin{vmatrix}m_1 & n_1\\m_2 & n_2\end{vmatrix}$

10. Differentiation of Determinant

Let $$F(x)=\begin{vmatrix}f(x) & g(x) & h(x)\\p(x) & q(x) & r(x)\\u(x) & v(x) & w(x)\end{vmatrix}$$

Then $$F'(x)=\begin{vmatrix}f'(x) & g'(x) & h'(x)\\p(x) & q(x) & r(x)\\u(x) & v(x) & w(x)\end{vmatrix}+\begin{vmatrix}f(x) & g(x) & h(x)\\p'(x) & q'(x) & r'(x)\\u(x) & v(x) & w(x)\end{vmatrix}+\begin{vmatrix}f(x) & g(x) & h(x)\\p(x) & q(x) & r(x)\\u'(x) & v'(x) & w'(x)\end{vmatrix}$$

or $$F'(x)=\begin{vmatrix}f'(x) & g(x) & h(x)\\p'(x) & q(x) & r(x)\\u'(x) & v(x) & w(x)\end{vmatrix}+\begin{vmatrix}f(x) & g'(x) & h(x)\\p(x) & q'(x) & r(x)\\u(x) & v'(x) & w(x)\end{vmatrix}+\begin{vmatrix}f(x) & g(x) & h'(x)\\p(x) & q(x) & r'(x)\\u(x) & v(x) & w'(x)\end{vmatrix}$$

11. Integration of Determinant

If $$F(x)=\begin{vmatrix} f(x) & g(x) & h(x) \\ a & b & c \\ 1 & m & n \end{vmatrix},$$

then $$\int_a^b F(x)\,dx=\begin{vmatrix} \int_a^b f(x)\,dx & \int_a^b g(x)\,dx & \int_a^b h(x)\,dx \\ a & b & c \\ 1 & m & n \end{vmatrix}$$

12. Summation of Determinants

Let $$\Delta_r=\begin{vmatrix} f(r) & g(r) & h(r) \\ a & b & c \\ p & q & r \end{vmatrix}$$

Then $$\sum_{r=1}^{n}\Delta_r=\begin{vmatrix} \sum_{r=1}^{n} f(r) & \sum_{r=1}^{n} g(r) & \sum_{r=1}^{n} h(r) \\ a & b & c \\ p & q & r \end{vmatrix}$$

13. Adjoint of a Matrix

The adjoint of a matrix is the transpose of the co-factors of the corresponding elements of a given matrix.

If A be any square matrix, then

$$\text{adj } A = (C_{ij})^T$$

Example 1: Let $A=\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$,

then $$\text{adj}(A)=\begin{pmatrix} 5 & -4 \\ -3 & 2 \end{pmatrix}^T=\begin{pmatrix} 5 & -3 \\ -4 & 2 \end{pmatrix}.$$

Example 2: Let $B=\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 4 \end{pmatrix}$,
then

$$\text{adj}(B)=\begin{pmatrix} \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \end{pmatrix}^T$$

$$=\begin{pmatrix} 0 & -2 & 1 \\ -2 & -5 & 4 \\ 1 & 4 & -3 \end{pmatrix}^T$$

$$=\begin{pmatrix} 0 & -2 & 1 \\ -2 & -5 & 4 \\ 1 & 4 & -3 \end{pmatrix}$$

Theorem: If A be any square matrix, then

$$A\cdot[\text{adj}(A)]=|A|\cdot I_n=[\text{adj}(A)]\cdot A$$

13.1 Properties of Adjoint of Matrix or Matrices

(i) If A be any square matrix of order n, then

$$|\text{adj}(A)|=|A|^{n-1}$$

Proof: We know that $A\cdot\text{adj}(A)=|A|\cdot I_n$

$\Rightarrow \quad |A\cdot\text{adj}(A)|=\|A|\cdot I_n|=|A|^n$

$\Rightarrow \quad |A||\text{adj}(A)|=\|A|\cdot I_n|=|A|^n$ $\quad (\because |AB|=|A||B|)$

$\Rightarrow \quad |\text{adj}(A)|=\dfrac{|A|^n}{|A|}=|A|^{n-1}$

Hence, the result.

(ii) If $|A| = 0$, then $|\text{adj}(A)| = 0$
i.e. if A is singular, then adj(A) is also singular

(iii) $\text{adj}(kA)=k^{n-1}(\text{adj}(A))$

(iv) $\text{adj}(A^T)=(\text{adj } A)^T$

(v) $\text{adj}(AB)=(\text{adj } B)(\text{adj } A)$

Proof: We have

$(AB)\,\text{adj}(AB)=|AB|\,I_n$

$\Rightarrow \quad (\text{adj } B)(\text{adj } A)(AB)\,\text{adj}(AB)$
$\quad =|AB|\,(\text{adj } B)(\text{adj } A)I_n$

$\Rightarrow \quad (\text{adj } B)\,|A|\,I_n B(\text{adj } AB)=|AB|(\text{adj } B)(\text{adj } A)$

$\Rightarrow \quad |A|\,(\text{adj } B)B(\text{adj } AB)=|AB|\,(\text{adj } B)(\text{adj } A)$

$\Rightarrow \quad |A||B|\,I_n(\text{adj } AB)=|AB|(\text{adj } B)(\text{adj } A)$

$\Rightarrow \quad |A|\,|B|\,(\text{adj } AB)=|AB|(\text{adj } B)(\text{adj } A)$

$\Rightarrow \quad (\text{adj } AB) = (\text{adj } B)(\text{adj } A)$

(vi) If A is a non-singular matrix, then

$$\text{adj}[\text{adj}(A)]=|A|^{n-2}\cdot A$$

Proof: We know that
$A.\text{adj}(A) = |A|.I_n$
Replace A by adj$A$, we get

$\text{adj}(A)\cdot\text{adj}[\text{adj}(A)]=|\text{adj}(A)|\cdot I_n$

$\Rightarrow \quad \text{adj}(A)\cdot\text{adj}[\text{adj}(A)]=|A|^{n-1}\,I_n$

$\Rightarrow \quad [A\,\text{adj}(A)]\cdot\text{adj}(\text{adj}(A))=|A|^{n-1}(AI_n)$

$\Rightarrow \quad |A|\cdot I_n\,\text{adj}[\text{adj}(A)]=|A|^{n-1}\cdot A$

$\Rightarrow \quad |A|\,\text{adj}[\text{adj}(A)]=|A|^{n-1}\cdot A$

$\Rightarrow \quad \text{adj}[\text{adj}(A)]=\dfrac{|A|^{n-1}\cdot A}{|A|}$

$\Rightarrow \quad \text{adj}[\text{adj}(A)]=|A|^{n-2}\cdot A$

(vii) $|\text{adj}[\text{adj}(A)]|=|A|^{(n-1)^2}$

Proof: We have $|[\text{adj}(A)]|=|A|^{n-1}$

Replace A by adj(A), we get,

$$|[\text{adj}\,\{\text{adj}\,(A)\}]| = |\text{adj}\,(A)|^{n-1}$$

$$\Rightarrow \quad |[\text{adj}\,\{\text{adj}\,(A)\}]| = ||A|^{n-1}|^{n-1}$$

$$\Rightarrow \quad |[\text{adj}\,\{\text{adj}\,(A)\}]| = |A|^{(n-1)^2}$$

(viii) If A be a square matrix of order n and B be its adjoint, then det $(AB + kI_n) = (\det A + k)^n$.

Proof: We have,

$$AB = A[\text{adj}(A)]$$

$$\Rightarrow \quad AB = |A|I_n$$

$$\Rightarrow \quad AB + kI_n = |A|I_n + kI_n$$

$$\Rightarrow \quad (AB + kI_n) = (|A| + k)\, I_n$$

$$\Rightarrow \quad \det\,(AB + kI_n) = \det\,[(|A| + k)\, I_n]$$

$$\Rightarrow \quad \det(AB + kI_n) = \det[(|A| + k)^n] \qquad (\because \det|kI_n| = k^n)$$

14. Singular and Non-singular Matrices Singular Matrix

If the determinant of a matrix is zero, it is called a singular matrix.

Let $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, then |A| = 4 – 4 = 0.

Thus A is a singular matrix.

Non-singular matrix

If the determinant of a matrix is non-zero, it is called non-singular matrix.

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$,

then $|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0$

Thus A is non-singular.

15. Inverse of a Matrix

For every non-singular square matrix of order n, there exists another square matrix of the same order such that $AB = I_N$, then B is called the inverse of A.

As we know that, $A.[\text{adj}(A)] = |A|.I_n$

$$\Rightarrow \quad A \cdot \left(\frac{[\text{adj}\,(A)]}{|A|}\right) = I_n$$

$$\Rightarrow \quad A^{-1} = \left(\frac{[\text{adj}\,(A)]}{|A|}\right)$$

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 - 6 = -2 \neq 0$$

Thus, its inverse exists.

Now, $\text{adj}\,(A) = \begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$

Hence, $A^{-1} = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix}$.

15.1 Properties of Inverse of a Matrix or Matrices

(i) $(A^{-1})^{-1} = A$

(ii) $|A^{-1}| = \frac{1}{|A|}$

(iii) $(A^T)^{-1} = (A^{-1})^T$

(iv) If k is non-zero and A is non-singular, then $(kA)^{-1} = \frac{1}{k}(A^{-1})$

(v) $(I^{-1}) = I$

(vi) $(\text{adj}\, A^{-1}) = (\text{adj}\, A)^{-1} = \frac{A}{|A|}$

(vii) If A and B be two non singular matrices, then $(AB)^{-1} = (B^{-1} A^{-1})$.

(viii) Inverse of a non-singular diagonal matrix is again a diagonal matrix, i.e.

if $A = \text{diag}\,(a_{11}, a_{22}, \ldots, a_{nn})$ where $a_{ii} \neq 0$

then $A^{-1} = \text{diag}\left(\frac{1}{a_{11}}, \frac{1}{a_{22}}, \ldots, \frac{1}{a_{nn}}\right)$.

For example,

If $A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$,

then $A^{-1} = \begin{pmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{pmatrix}$

(ix) If $A = \begin{pmatrix} \cos x & \sin x & 0 \\ -\sin x & \cos x & 0 \\ 0 & 0 & 1 \end{pmatrix} = F(x)$

then $A^{-1} = F(-x)$.

16. Solutions of the System of Equations by Matrix (Inverse) Method

Consider the system of equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

The given system of equations can be written in matrix form as

$$\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \times \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

$\Rightarrow \quad AX = B$

where

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$$

Nature of solutions by the matrix method

(i) If $|A| \neq 0$, the system of equations has a unique solution and the system of equation is said to be consistent.

(ii) If $|A| = 0$ and (adj $A)B = O$, the system of equations has infinity many solutions and the system of equations is said to be consistent.

(iii) If $|A| = 0$ and (adj $A)B \neq O$, the system of equations has no solution and the system of equations is said to be inconsistent.

17. Elementary Transformations of a Matrix

There are six operations (transformations) on a matrix, three of which are due to rows and three due to columns, which are known as elementary transformations.

(i) The interchange of any two rows (or columns). Symbolically, $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$

(ii) The multiplications of the elements of any one row (or column) by a non-zero constant, say k, i.e. symbolically,

$R_i \to kR_j$ or $C_i \to kC_j$

(iii) The addition to the elements of any one row (or column) with the corresponding elements of any other row (or column) multiplied by a non-zero number, i.e. symbolically,

$R_i \to R_i + kR_j$ or $C_i \to C_i + kC_j$

Equivalent matrices

Two matrices are said to be equivalent, if one is obtained from other by elementary transformations.

We generally write it as

$A \sim B.$

18. Advance Types of Matrices

1. Idempotent matrix

A square matrix A is said to be an idempotent matrix if $A^2 = A$.

For example, $A = \begin{bmatrix} 2 & -2 & 4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is an idempotent matrix.

2. Periodic matrix

A square matrix A is said to periodic with period k (where k is a least positive integer such that $A^{k+1} = A$, i.e. if $A^3 = A, A^5 = A, A^7 = A$, it is a periodic matrix and $A^{2+1} = A$, so its period = 2.

3. Nilpotent matrix

A square matrix A is called a nilpotent matrix if there exists $k \in N$ such that $A^k = 0$, where k is called the index of the nilpotent of matrix A.

For example, $A = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ is a nilpotent matrix.

4. Involutory matrix

A square matrix A is called an involutory matrix, if $A^2 = I$, i.e. $A^{-1} = A$.

For example, $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an involutory matrix.

5. Orthogonal matrix

A square matrix A is said to be orthogonal matrix, if $AA^T = I$, where A^T is the transpose of matrix A and I is an identity matrix.

For example, $A = \frac{1}{3}\begin{bmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ -2 & -2 & -1 \end{bmatrix}$ is an orthogonal matrix.

> ***Note***
>
> (i) If $AA^T = I$, then $A^{-1} = A^{T.}$
>
> (ii) Every orthogonal matrix is non-singular.
>
> (iii) I is an orthogonal matrix.
>
> (iv) If A and B are orthogonal, then AB is also orthogonal
>
> (v) If A is orthogonal, then A^{-1} and A^T are also orthogonal.

6. Conjugate of a matrix

Let A be any matrix containing complex number as its elements, then a matrix is obtained from A on replacing its elements by the corresponding conjugate complex numbers, is called the conjugate of the matrix A and it is denoted by $\bar{A}$.

For example, If $A = \begin{bmatrix} 1+2i & 2+3i \\ 1-i & 3+4i \end{bmatrix}$,

then $\bar{A} = \begin{bmatrix} 1-2i & 2-3i \\ 1+i & 3-4i \end{bmatrix}$.

7. Complex conjugate transpose of a matrix

The conjugate of the transpose of the matrix A is called the conjugate transpose of A and is denoted by A^θ.

If $A = \begin{pmatrix} 2+4i & 3 \\ 4 & \alpha + i\beta \end{pmatrix}$, then $A^\theta = \begin{pmatrix} 2-4i & 3 \\ 4 & \alpha - i\beta \end{pmatrix}$.

8. Hermitian matrix

A square matrix A is called a hermitian matrix, if $A^\theta = A$.

For example, $A = \begin{pmatrix} 2 & \alpha + i\beta \\ \alpha - i\beta & 3 \end{pmatrix}$.

9. Skew hermitian matrix

A square matrix A is called a skew-hermitian matrix, if $A^\theta = -A$.

For example, $A = \begin{pmatrix} 2i & -\alpha - i\beta \\ \alpha + i\beta & -i \end{pmatrix}$

10. Unitary matrix

A square matrix A is called unitary, if $AA^{\theta} = I$.

For example, $A = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1+i \\ 1-i & -1 \end{pmatrix}$

> ***Note*** If A and B are unitary, then AB is also a unitary.

11. Equivalent matrices

Let A and B are two matrices. If B is obtained from A by elementary transformation, then A and B are called equivalent matrices.

12. Rank of a matrix

Rank of a matrix represents the non-zero rows of an equivalent matrix.

Rule to find out the rank of a matrix

Let A be any type of matrix

Case I: When A is a null matrix, then the rank of a matrix is zero.

Case II: When A is a square matrix, then we shall first find the determinant of A.

(i) If A is non-singular (i.e. $|A| \neq 0$), then rank of the matrix = order of the matrix

(ii) If A is singular (i.e. $|A| = 0$), then we shall find the minor along rows:

 (a) If at-least one minor is zero and rest are non-zero, then rank of the matrix = order of the matrix –1.

 (b) If all minor is non-zero, then rank of the matrix = 0.

Case III: When A is a rectangular matrix of order $m \times n$, then we shall find an equivalent matrix of A.

(i) If any one row is zero, then rank of the matrix = Minimum of $\{m-1, n-1\}$

(ii) If any two row is zero, then rank of the matrix = Minimum of $\{m-2, n-2\}$

(iii) If all rows are non-zero, then rank of the matrix = Minimum of $\{m, n\}$.

Exercises

Level I

(Problems based on Fundamentals)

ORDER OF MATRICES

1. Find the number of all possible matrices of order 2×2 with each entry 0 or 1.
2. Find the number of all possible matrices of order 3×3 with each entry either 1 or 2.

ADDITION OF MATRICES

3. If $A = \begin{pmatrix} 2 & 4 \\ 3 & 5 \end{pmatrix}$, find the additive inverse of A.
4. Find a matrix X, if $X + \begin{pmatrix} 2 & 5 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 2 & 7 \end{pmatrix}$.
5. Find X and Y, if $X + Y = \begin{pmatrix} 2 & 5 \\ 3 & -2 \end{pmatrix}$.

   and $X - Y = \begin{pmatrix} 4 & 2 \\ 8 & -2 \end{pmatrix}$.
6. Find a matrix X such that

   $A + 2B + X = \mathbf{O}$,

   where $A = \begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}$.
7. Find x and y, if

   $$\begin{pmatrix} |x| & 2 \\ 5 & |y-2| \end{pmatrix} = \begin{pmatrix} <3 & 2 \\ 5 & <4 \end{pmatrix}$$
8. Find $\Sigma(x+y)$, if

   $$\begin{pmatrix} x^3 - 3x + 2 & 2 \\ 3 & y^3 + 7y^2 - 35 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix}$$
9. Find x, y, z and t satisfying the equations

   $$2\begin{pmatrix} x & y \\ z & t \end{pmatrix} + 3\begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix} = 4\begin{pmatrix} 3 & 5 \\ 4 & 6 \end{pmatrix}$$
10. Find the matrices X and Y, if

    $$2X + 3Y = \begin{pmatrix} 2 & 3 \\ 4 & 0 \end{pmatrix} \text{ and } 3X + 2Y = \begin{pmatrix} -1 & 2 \\ 1 & -5 \end{pmatrix}$$
11. If $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$,

    and $f(x) = 1 + x + x^2 + \ldots + \text{to } \infty$, find $f(A)$.

MULTIPLICATION OF MATRICES

12. Let $A = [1\ 2]$ and $B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

    Find AB and BA.
13. Let $A = [a\ b\ c]$ and $B = \begin{bmatrix} a^2 \\ b^2 \\ c^2 \end{bmatrix}$.

    Find AB and BA.
14. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 4 \\ 5 & 7 \end{pmatrix}$.

    Find AB and BA.

15. If A be a 2×3 matrix and AB a 2×5 matrix, find the order of the matrix B.
16. If A be a $(A)_{2\times3}$ matrix and B a $(B)_{5\times3}$ matrix and C a $(C)_{3\times7}$ matrix, find $2015ABC$.
17 If $A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$,

prove that

$$A^2 + 2A + I_2 = \mathbf{O}$$

18. If $A = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $B = (2 \quad 3 \quad 4)$, AB and BA.
19. If $A = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$, find A^2, A^3, and A^4.
20 Find a 2×2 matrix X such that

$$\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} . X = \begin{pmatrix} 3 & 5 \\ 0 & 6 \end{pmatrix}$$

21. If $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$,

prove that $A^2 - 4A - 51_3 = \mathbf{O}$.

22. Find x, if $(1 \quad x \quad 1)\begin{pmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ x \end{pmatrix} = 0$.
23. If $A = \begin{pmatrix} a & 0 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$, the value of a for which $A^2 = B$ is
    (a) 1 (b) –1
    (c) 4 (d) No real values
24. If $A = \begin{pmatrix} \alpha & 2 \\ 2 & \alpha \end{pmatrix}$ and $|A^3| = 125$, the value of α is
    (a) ± 1 (b) ± 2 (c) ± 3 (d) ± 5
25. If A and B be two square matrices of order 3×3 which satisfy $AB = A$ and $BA = B$, find $(A + B)^7$.
26. Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are two matrices such that $AB = BA$ and $c \neq 0$, find the value of $\dfrac{a-d}{3b-c}$.
27. If A be a square matrix of order 2 such that $A^2 = \mathbf{O}$, find $A(I + A)^{2009}$.
28. If A be a square matrix and I be an identity matrix of the same order such that $A^2 = I$, $(I - A)(I + A)$.
29. If A be a square matrix such that $A^2 = A$, find $(I + A)^3 - 7A$.
30. If $A = \begin{pmatrix} 0 & 5 \\ 0 & 0 \end{pmatrix}$ and $f(x) = \sum_{n=0}^{16} x^n$, find $f(A)$.
31. If A be a square matrix such that $A^2 = A + I$, where I is the identity matrix of the same order, find A^5.
32. If A be a square matrix such that $A^2 = A - I$, where I is the identity matrix of the same order, find A^n.
33. If $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and $n \in N$, find A^n.
34. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$.

    If U_1 and U_2 are column matrices such that $AU_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $AU_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, find $U_1 + U_2$.

SYMMETRIC AND SKEW-SYMMETERIC MATRICES

35. Express the matrix $A = \begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix}$ as a sum of a symmetric and a skew-symmetric matrices.
36. Express the matrix $B = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 3 & 5 \\ 4 & 3 & 6 \end{pmatrix}$ as a sum of a symmetric and a skew-symmetric matrices.

DETERMINANT

37. If A be a skew-symmetric matrix of odd order, find det (A).
38. If A be a skew-symmetric matrix of even order, find det(A).
39. If $A = (a_{ij})$ be a square matrix of order n such that $a_{ij} = \begin{cases} 0: & i = j \\ 1: & i \neq j \end{cases}$, find det$(A)$.
40. If $A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$ be a square matrix of order 2, find det$(A^{2010} + 2A^{2009})$.
41. If A be a square matrix such that $A^2 = A$, find det(A).
42. If A be a square matrix of order 3 such that det$(A) = 8$, find det$(3A)$.
43. If A be a square matrix of order n such that $|A| = 2$, find $|An|$ for $n \in I^+$.
44. Find the maximum value of the determinant

$$\begin{vmatrix} \cos^2\theta & \sin^2\theta \\ \sin^2\theta & \cos^2\theta \end{vmatrix}$$

45. Let P and Q be two square matrices of order 3 such that $P^3 = Q^3$ and $P^2Q = Q^2P$ and $P \neq Q$, find

$$\det(P^2 + Q^2).$$

46. Expand the determinant
$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$$

47. Evaluate: $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$.

48. Evaluate: $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$.

49. Evaluate: $\begin{vmatrix} \sin\alpha & \cos\beta & \cos(\alpha+\theta) \\ \sin\beta & \cos\beta & \cos(\beta+\theta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\theta) \end{vmatrix}$.

50. Prove that the value of $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(a+c) \\ 1 & ab & c(a+b) \end{vmatrix}$ is independent of a, b, c.

51. Prove that
$$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2(a+b+c)^3.$$

52. Prove that $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix} = 4abc.$

53. Prove that $\begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2.$

54. Prove that $\begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3a \end{vmatrix} = a^3.$

55. Prove that $\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$
$= (1+a^2+b^2)^3.$

56. Prove that $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & cb & c^2+1 \end{vmatrix} = (1+a^2+b^2+c^2).$

57. Prove that $\begin{vmatrix} {}^xC_1 & {}^xC_2 & {}^xC_3 \\ {}^yC_1 & {}^yC_2 & {}^yC_3 \\ {}^zC_1 & {}^zC_2 & {}^zC_3 \end{vmatrix} = \dfrac{xyz}{12}(x-y)(y-z)(z-x).$

58. Prove that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$
$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = abc+bc+ca+ab.$$

59. Prove that
$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc(a+b+c)^3.$$

60. Prove that $\begin{vmatrix} (a+1)(a+2) & (a+2) & 1 \\ (a+2)(a+3) & (a+3) & 1 \\ (a+3)(a+4) & (a+4) & 1 \end{vmatrix} = -2.$

61. Prove that $\begin{vmatrix} b+c & c+a & a+b \\ a+b & b+c & c+a \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}.$

62. Solve the following system of equations:
$2x + 3y = 4$
$3x - 2y = 5.$

63. Solve the following system of equations:
$x + 3y = 4$
$2x + 6y = 10.$

64. Solve the following system of equations:
$2x + 5y = 6$
$6x + 15y = 18.$

65. Find the number of triplets of a, b and c for which the system of equations
$ax - by = 2a - b$
$(c + 1)x + cy = 10 - a + 3b.$
has infinitely many solutions.

66. Solve for x, y, z:
$x + y + z = 1$
$ax + by + cz = d$
$a^2x + b^2y + c^2z = d$

67. Solve for x, y, z:
$$\frac{1}{x}+\frac{1}{y}-\frac{1}{z}=\frac{1}{4}$$
$$\frac{2}{x}-\frac{1}{y}+\frac{3}{z}=\frac{9}{4}$$
$$-\frac{1}{x}-\frac{2}{y}+\frac{4}{z}=1.$$

68. Find the equation of the parabola $y = ax^2 + bx + c$, which passes through the points (2, 4), (–1, 1) and (–2, 5).

69. Find the value of k, for which the system of equations
$2x + ky = 5$
$3x - 4y = 7$
has a unique solution.

70. Find the value of λ, for which the system of equations
$$3x + 4y = 5$$
$$\lambda x + 8y = 10$$
give infinitely many solutions.

71. If the system of equations
$$x + 2y - 3z = 1$$
$$(p+2)z = 3$$
$$(2p+1)y + z = 2$$
is inconsistent, find the value of p.

72. If the system of equations
$$2x - y + 2z = 2$$
$$x - 2y + z = -4$$
$$x + 2y + \lambda z = 4$$
has no solutions, find λ.

HOMOGENEOUS EQUATIONS

73. Solve the system of equations:
$$2x + 3y = 0$$
$$4x + 6y = 0.$$

74. Find the number of values of t for which the system of equations
$$(a-t)x + by + cz = 0$$
$$bx + (c-t)y + az = 0$$
$$cx + \text{ay} + (b-t)z = 0$$
has non-trivial solution.

75. Find the value of λ, if the system of equations
$$6x + 5y + \lambda z = 0$$
$$3x - y + 4z = 0$$
$$x + 2y - 3z = 0$$
has a unique solution.

76. If the system of equations $x + ay = 0$, $y + az = 0$ and $z + ax = 0$ has infinite solutions, find a.

MULTIPLICATION OF DETERMINANTS

77. Prove that
$$\begin{vmatrix} a & 0 & c \\ a & b & 0 \\ 0 & b & c \end{vmatrix}^2 = \begin{vmatrix} c^2+a^2 & a^2 & c^2 \\ a^2 & a^2+b^2 & b^2 \\ c^2 & b^2 & b^2+c^2 \end{vmatrix}.$$

78. Prove that
$$\begin{vmatrix} 2 & \alpha+\beta+\gamma+\delta & \alpha\beta+\gamma\delta \\ \alpha+\beta+\gamma+\delta & 2(\alpha+\beta)(\gamma+\delta) & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) \\ \alpha\beta+\gamma\delta & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) & 2\alpha\beta\gamma\delta \end{vmatrix} = 0$$

79. Prove that
$$\begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & 1 \end{vmatrix} = 0$$

80. If $\alpha, \beta \neq 0$ and $f(n) = \alpha^n + \beta^n$ and
$$\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix} = k(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2,$$
find k.

DIFFERENTIATION OF DETERMINANT

81. If $F(x) = \begin{vmatrix} 1 & a & a^2 \\ x & x^2 & x^3 \\ e^{x-a} & e^{x^2-a^2} & e^{x^3-a^3} \end{vmatrix}$,

find the value of $F'(a)$.

82. Let $f(x) = \begin{vmatrix} 3 & 2 & 1 \\ 6x^2 & 2x^3 & x^4 \\ 1 & b & b^2 \end{vmatrix}$,

find $f''(b)$

INTEGRATION OF DETERMINANTS

83. If $f(x) = \begin{vmatrix} \sin^2 x & \log(\sin x) & \dfrac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}} \\ n & \sum_{k=1}^{n}(k) & \prod_{k=1}^{n}(k) \\ \dfrac{8}{15} & \dfrac{\pi}{2}\log\left(\dfrac{1}{2}\right) & \dfrac{\pi}{4} \end{vmatrix}$,

find the value of $\int_0^{\pi/2} f(x)\,dx$.

84. If $\Delta_r = \begin{vmatrix} r & 2012 & \dfrac{n(n+1)}{2} \\ 2r-1 & 2013 & n^2 \\ 3r-2 & 2014 & \dfrac{n(3n-1)}{2} \end{vmatrix}$,

find $\sum_{r=1}^{n} \Delta_r$

85. If $D_r = \begin{vmatrix} 2^{r-1} & 101 & (2^n-1) \\ 3^{r-1} & 102 & \left(\dfrac{3^n-1}{2}\right) \\ 5^{r-1} & 103 & \left(\dfrac{5^n-1}{4}\right) \end{vmatrix}$,

find the value of $\sum_{r=1}^{n} D_r$

ADJOINT OF MATRICES

86. If A be a square matrix of order n, find adj(A') – adj $A)'$.

87. If A be a non-singular matrix of order 3, find det(adj A^3).

88. If A be a square matrix of order 2 such that $A\cdot(\text{adj } A)=\begin{pmatrix}10 & 0\\ 0 & 10\end{pmatrix}$, find det($A$).
89. If A be a square matrix of order 3 such that det(A) = 4, find det(adj A).
90. If A be a square matrix of order n such that $|\text{adj}(\text{adj}A)| = |A|^{16}$, find n.
91. If the adjoint of a matrix P of order 3 is $\begin{pmatrix}1 & 4 & 4\\ 2 & 1 & 7\\ 1 & 1 & 3\end{pmatrix}$, the possible values of the determinant of P is (are)
    (a) –2 (b) –1 (c) 1 (d) 2
92. Let $P = |a_{ij}|$ be a 3 × 3 matrix and let $Q = |b_{ij}|$, where $b_{ij} = 2^{i+j} a_{ij}$ for $1 \le i, j \le 3$.
    If the determinant of P is 2, the determinant of the matrix Q is
    (a) 2^{10} (b) 2^{11} (c) 2^{12} (d) 2^{13}
93. If $P=\begin{pmatrix}1 & \alpha & 3\\ 1 & 3 & 3\\ 2 & 4 & 4\end{pmatrix}$ is the adjoint of a matrix A of order 3 × 3 and $|A| = 4$, find α.
94. Find the inverse of a skew symmetric matrix of odd order.
95. If B be a non-singular matrix and A is a square matrix of the same order, find $|B^{-1} AB|$.
96. If $P=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ and $Q=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$, find the inverse of a matrix R, such that $R = (P\cos\theta + Q\sin\theta)$.
97. If A and B be two non-singular square matrices such that $B^{-1}AB = A^2$, find $B^{-3}AB^3$.
98. If A be a non-singular matrix satisfying $I + A + A^2 + A^3 + \ldots + A^k = \mathbf{O}$, find A^{-1}.
99. If A be a non-singular matrix satisfying $A^2 - A + I = \mathbf{O}$, find A^{-1}.
100. If A be a non-singular square matrix of order 3 × 3 such that det(A) = 5, find det(adj(A^{-1})).
101. If A be a non-singular square matrix of order 3 × 3, find $|A^{-1}\,\text{adj}(A)|$.
102. If $A=\begin{pmatrix}2 & 3\\ 4 & 5\end{pmatrix}$, find the multiplicative inverse of A.
103. If X be a non-singular square matrix of order 2 such that $\begin{pmatrix}1 & 3\\ 0 & 1\end{pmatrix}.X=\begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}$, find X.
104. Let $A=\begin{pmatrix}3 & 2\\ 2 & 1\end{pmatrix}$ and $B=\begin{pmatrix}3 & 1\\ 7 & 3\end{pmatrix}$, find the value of det $(2A^9B^{-1})$.
105. Let P and Q be two square matrices such that $|P| = 1 = Q$. If A and B be two square matrices of the same order such that (adj$B$) = A, find adj(QBP).
106. If A is an 3 × 3 non-singular matrix such that $AA' = A'A$ and $B = A^{-1}A'$, find BB'. **[JEE Main, 2014]**
107. Find the inverse of the matrix, $A=\begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 2\\ 3 & 2 & 4\end{pmatrix}$.
108. If $A=\begin{pmatrix}0 & -\tan\left(\dfrac{\alpha}{2}\right)\\ \tan\left(\dfrac{\alpha}{2}\right) & 0\end{pmatrix}$,
    prove that $(I + A) = (I - A)\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}$.
109. Solve the following system of equations by matrix method.
    $x + y + z = 6$
    $x - y + z = 2$
    $2x + y - z = 1$
110. Compute A^{-1} for the matrix
    $$A=\begin{pmatrix}-1 & 2 & 5\\ 2 & -3 & 1\\ -1 & 1 & 1\end{pmatrix}.$$
    and hence solve the following system of equations
    $$-x + 2y + 5z = 2$$
    $$2x - 3y + z = 15$$
    $$-x + y + z = -3$$
111. Solve the following system of equations by matrix (inverse) method.
    $$5x + 3y + 7z = 4$$
    $$3x + 26y + 2z = 9$$
    $$7x + 2y + 10z = 5$$
112. The number of 3 × 3 matrices A whose entries are either 0 or 1 and for which the system
    $$A\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}$$
    has exactly two solutions, is
    (a) 0 (b) $2^9 - 1$ (c) 168 (d) 12
113. Let a, b, c be positive real numbers. Prove that the system of equations
    $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,\ \frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$
    and $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$
    in x, y, z have a unique solutions.

114. By using elementary row operation, find the inverse of $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$.

RANK OF A MATRIX

115. Find the rank of $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{pmatrix}$.

116. Find the rank of $A = \begin{pmatrix} 2 & 4 & 3 \\ 1 & 2 & -1 \\ -1 & -2 & 6 \end{pmatrix}$.

117. Find the rank of $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{pmatrix}$.

118. Find the rank of $A = \begin{pmatrix} 1 & -2 & 1 & -1 \\ 1 & 1 & -2 & 3 \\ 4 & 1 & -5 & 6 \end{pmatrix}$.

ADVANCE MATRICES

119. Prove that the matrix $A = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$ is idempotent

120. Prove that the matrix $A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -1 \end{bmatrix}$ is periodic.

121. Show that $\begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$ is nilpotent matrix of order 3.

122. Show that the matrix $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$ is involuntary.

123. Prove that the matrix $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$ is orthogonal matrix.

124. Prove that $A = \begin{bmatrix} 3 & 3-4i & 5+2i \\ 3+4i & 5 & -2+i \\ 5-2i & -2-i & 2 \end{bmatrix}$ is an Hermitian matrix.

125. Prove that $A = \begin{bmatrix} 2i & -2-3i & 2+i \\ 2-3i & -i & 3i \\ 2+i & 3i & 0 \end{bmatrix}$ is an skew-hermitian matrix.

126. Prove that the matrix $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is unitary.

Level II

(Mixed Problems)

1. If a, b and c are non-zero real numbers, then $\Delta = \begin{vmatrix} b^2c^2 & bc & b+c \\ c^2a^2 & ca & c+a \\ a^2b^2 & ab & a+b \end{vmatrix} =$
   (a) abc (b) $a^2b^2c^2$
   (c) $ab + bc + ca$ (d) none of these
2. The value of the determinant $\begin{vmatrix} 1/a & a^2 & bc \\ 1/b & b^2 & ca \\ 1/c & c^2 & ab \end{vmatrix}$ is
   (a) abc (b) $1/abc$
   (c) $ab + bc + ca$ (d) 0
3. The value of the determinant $\begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix} =$
   (a) abc (b) $4abc$
   (c) $4a^2b^2c^2$ (d) $a^2b^2c^2$
4. The value of the determinant $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} =$
   (a) $(a + b + c)^2$
   (b) $(a + b + c)^3$
   (c) $(a + b + c)(ab + bc + ca)$
   (d) None of these
5. The value of the determinant $\begin{vmatrix} a-b & b-c & c-a \\ x-y & y-z & z-x \\ p-q & q-r & r-p \end{vmatrix} =$
   (a) $a(x + y + z) + b(p + q + r) + c$
   (b) 0
   (c) $abc + xyz + qr$
   (d) none of these
6. The value of the determinant $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} =$
   (a) $3abc + a^3 + b^3 + c^3$
   (b) $3abc - a^3 - b^3 - c^3$
   (c) $abc - a^3 + b^3 + c^3$
   (d) $abc + a^3 - b^3 - c^3$

7. For non-zero a, b, c, if

$$\Delta = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = 0$$

the value of $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} =$

(a) abc (b) $\frac{1}{abc}$

(c) $-(a+b+c)$ (d) none of these

8. The value of the determinant

$$\begin{vmatrix} a_1 & la_1 + mb_1 & b_1 \\ a_2 & la_2 + mb_2 & b_2 \\ a_3 & la_3 + mb_3 & b_3 \end{vmatrix} \text{ is}$$

(a) 0 (b) l (c) m (d) lm

9. If T_p, T_q, T_r are respectively, the pth, qth and rth terms of an AP, then $\begin{vmatrix} T_p & T_q & T_r \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$ is equal to

(a) 1 (b) –1 (c) 0 (d) $p+q+r$

10. The value of the determinant

$$\begin{vmatrix} 2ac - b^2 & a^2 & c^2 \\ a^2 & 2ab - c^2 & b^2 \\ c^2 & b^2 & 2bc - a^2 \end{vmatrix} =$$

(a) $4abc$ (b) $-4abc$

(c) 0 (d) $(a^3 + b^3 + c^3 - 3abc)^2$

11. If $a \neq b \neq c$ and $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = 0$,

the value of $(a + b + c)$ is

(a) 1 (b) 0 (c) 2 (d) $-a$

12. Let $\Delta = \begin{vmatrix} 1 & \sin\alpha & 1 \\ -\sin\alpha & 1 & \sin\alpha \\ -1 & -\sin\alpha & 1 \end{vmatrix}$,

the value of Δ lies in the interval

(a) [2, 3] (b) [3, 4]

(c) [1, 4] (d) [2, 4]

13. If A, B, C are the angles of triangle, the value of

$$\Delta = \begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix} \text{ is}$$

(a) $\cos A \cos B \cos C$ (b) $\sin A \sin B \sin C$

(c) 0 (d) none of these

14. If a, b, c are in AP, the value of

$$\begin{vmatrix} x+2 & x+3 & x+a \\ x+4 & x+5 & x+b \\ x+6 & x+7 & x+c \end{vmatrix} \text{ is}$$

(a) $x - (a+b+c)$ (b) $9x^2 + a + b + c$

(c) $a + b + c$ (d) 0

15. The value of the determinant

$$\begin{vmatrix} a & a+b & a+2b \\ a+2b & a & a+b \\ a+b & a+2b & a \end{vmatrix} =$$

(a) $9a^2(a+b)$ (b) $9b^2(a+b)$

(c) $a^2(a+b)$ (d) $b^2(a+b)$

16. The value of the determinant

$$\begin{vmatrix} (a^x + a^{-x})^2 & (a^x - a^{-x})^2 & 1 \\ (b^x + b^{-x})^2 & (b^x - b^{-x})^2 & 1 \\ (c^x + c^{-x})^2 & (c^x - c^{-x})^2 & 1 \end{vmatrix} =$$

(a) 0 (b) $2abc$

(c) $a^2b^2c^2$ (d) none of these

17. If $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = k(a+b+c)(a^2 + b^2 + c^2 - bc - ca - ab)$,

the value of k is

(a) 1 (b) 2 (c) –1 (d) –2

18. If the determinant

$$\begin{vmatrix} a & a+d & a+2d \\ a^2 & (a+d)^2 & (a+2d)^2 \\ 2a+3d & 2(a+d) & 2a+d \end{vmatrix} = 0,$$

(a) $d = 0$ (b) $a + d = 0$

(c) $d = 0$ or $a + d = 0$ (d) none

19. The value of the determinant

$$\begin{vmatrix} a+b & a+2b & a+3b \\ a+2b & a+3b & a+4b \\ a+4b & a+5b & a+6b \end{vmatrix} =$$

(a) $a^2 + b^2 + c^2 - 3abc$ (b) $3ab$

(c) $3a + 5b$ (d) 0

20. If $0 < \theta < \frac{\pi}{2}$ and

$$\begin{vmatrix} 1 + \sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1 + \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \cos^2\theta & 1 + 4\sin 4\theta \end{vmatrix} = 0,$$

θ is equal to

(a) $\frac{\pi}{24}, \frac{5\pi}{24}$ (b) $\frac{5\pi}{24}, \frac{7\pi}{24}$

(c) $\frac{7\pi}{24}, \frac{11\pi}{24}$ (d) none of these

21. The value of the determinant $\begin{vmatrix} 1 & \cos(\alpha-\beta) & \cos(\alpha-\gamma) \\ \cos(\alpha-\beta) & 1 & \cos(\beta-\gamma) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & 1 \end{vmatrix}$ is

(a) $\begin{vmatrix} \cos\alpha & \sin\alpha & 1 \\ \cos\beta & \sin\beta & 1 \\ \cos\gamma & \sin\gamma & 1 \end{vmatrix}^2$

(b) $\begin{vmatrix} \sin\alpha & \cos\alpha & 0 \\ \sin\beta & \cos\beta & 0 \\ \sin\gamma & \cos\gamma & 0 \end{vmatrix}^2$

(c) $\begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \sin\beta & 0 & \cos\beta \\ 0 & \cos\gamma & \sin\gamma \end{vmatrix}^2$

(d) none of these.

22. In a ΔABC, if $\begin{vmatrix} 1 & a & b \\ 1 & c & a \\ 1 & b & c \end{vmatrix} = 0$,

$\sin^2 A + \sin^2 B + \sin^2 C =$

(a) $\frac{9}{4}$ (b) $\frac{4}{9}$ (c) 1 (d) $3\sqrt{3}$

23. $a \neq p$, $b \neq q$, $c \neq r$ and

$$\begin{vmatrix} p & b & c \\ p+a & q+b & 2c \\ a & b & r \end{vmatrix} = 0,$$

then $\frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c} =$

(a) 3 (b) 2 (c) 1 (d) 0

24. The value of the determinant $\begin{vmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$, where a, b, c are respectively, the pth, qth, rth terms of an HP is

(a) $ap + bq + cr$ (b) $(a + b + c)(p + q + r)$
(c) 0 (d) none of these

25. If $\sqrt{-1} = i$ and ω be non-real cube root of unity, the value of $\begin{vmatrix} 1 & \omega^2 & 1+i+\omega^2 \\ -i & -1 & -1-i+\omega \\ 1-i & \omega^2-1 & -1 \end{vmatrix} =$

(a) 1 (b) i (c) ω (d) 0

26. Let $f(n) = \begin{vmatrix} n & n+1 & n+2 \\ {}^nP_n & {}^{n+1}P_{n+1} & {}^{n+2}P_{n+2} \\ {}^nC_n & n+1C_{n+1} & {}^{n+2}C_{n+2} \end{vmatrix}$

where the symbols have their usual meanings. Then $f(n)$ is divisible by

(a) $n^2 + n + 1$ (b) $(n + 1)!$
(c) $n!$ (d) none of these

27. If $a + b + c = 0$, one root of

$$\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0 \text{ is}$$

(a) $x = 1$ (b) $x = 2$
(c) $x = a^2 + b^2 + c^2$ (d) $x = 0$

28. $\begin{vmatrix} x+1 & \omega & \omega^2 \\ x+\omega & \omega^2 & 1 \\ x+\omega^2 & 1 & \omega \end{vmatrix} = 3$

is an equation of x, where ω, ω^2 are the complex cube roots of unity, the value of x is

(a) 0 (b) 1 (c) –1 (d) 2

29. If $ab + bc + ca = 0$ and

$$\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix} = 0,$$

one of the value of x is

(a) $(a^2+b^2+c^2)^{\frac{1}{2}}$ (b) $\left[\frac{3}{2}(a^2+b^2+c^2)\right]^{\frac{1}{2}}$

(c) $\left[\frac{1}{2}(a^2+b^2+c^2)\right]^{\frac{1}{2}}$ (d) none of these

30. Let $\begin{vmatrix} 1+x & x & x^2 \\ x & 1+x & x^2 \\ x^2 & x & 1+x \end{vmatrix}$

$= ax^5 + bx^4 + cx^3 + dx^2 + \lambda x + \mu$

be an identity in x, where a, b, c, d, λ, μ are independent of x. Then the value of x is

(a) 3 (b) 2
(c) 4 (d) none of these

31. If the entries in a 3×3 determinant are either 0 or 1, the greatest value of this determinant is

(a) 1 (b) 2 (c) 3 (d) 9

32. If $\Delta = \begin{vmatrix} a & c & b \\ b & b & a \\ c & a & c \end{vmatrix}$ and $\Delta' = \begin{vmatrix} a^2 & c^2 & b^2 \\ b^2 & b^2 & a^2 \\ c^2 & a^2 & c^2 \end{vmatrix}$,

then

(a) $\Delta = a^2b^2c^2\Delta'$ (b) $\Delta' = a^2b^2c^2\Delta$
(c) $\Delta = abc\Delta b$ (d) None of these

33. If $\Delta = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix}$,

then $\begin{vmatrix} 2x+4p & p+6a & a \\ 2y+4q & q+6b & b \\ 2z+4r & r+6c & c \end{vmatrix} =$

(a) 2Δ (b) 4Δ (c) 6Δ (d) Δ

34. If $\Delta_1 = \begin{vmatrix} 1 & 0 \\ a & b \end{vmatrix}$ and $\Delta_2 = \begin{vmatrix} 1 & 0 \\ c & d \end{vmatrix}$,
then $\Delta_2\Delta_1 =$
(a) ac (b) bd
(c) $(b-a)(d-c)$ (d) none of these

35. If $D_r = \begin{vmatrix} 1 & n & n \\ 2r & n^2+n+1 & n^2+n \\ 2r-1 & n^2 & n^2+n+1 \end{vmatrix}$ and $\sum_{r=1}^{n} D_r = 56$,
the value of n is
(a) 4 (b) 6 (c) 7 (d) 8

36. If $f(x) = \begin{vmatrix} \cos x & 1 & 0 \\ 1 & \cos x & 1 \\ 0 & 1 & \cos x \end{vmatrix}$,
then $f'\left(\frac{\pi}{3}\right)$ equals
(a) $\frac{11\sqrt{3}}{8}$ (b) $\frac{5\sqrt{3}}{8}$
(c) $\frac{-5\sqrt{3}}{8}$ (d) none of these

37. If $f(x) = \begin{vmatrix} \sec x & \cos & \sec^2 x + \cot x \operatorname{cosec}^2 x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \operatorname{cosec}^2 x \end{vmatrix}$,
then $\int_0^{\pi/2} f(x)\,dx =$
(a) $\frac{\pi}{4} + \frac{8}{15}$ (b) $\left(-\frac{\pi}{4} + \frac{8}{15}\right)$
(c) $-\left(\frac{\pi}{4} + \frac{8}{15}\right)$ (d) none of these

38. Equations $x + y = 2$, $2x + 2y = 3$ will have
(a) only one solution
(b) infinitely many solutions
(c) no solution
(d) none of these

39. The system of equations $x + y + z = 2$, $3x - y + 2z = 6$ and $3x + y + z = -18$ has
(a) a unique solution
(b) no solution
(c) an infinite number of solutions
(d) zero solution as the only solution

40. $x + y + z = 6$, $x - y + z = 2$ and $2x + y - z = 1$, then x, y, z are respectively
(a) 3, 2, 1 (b) 1, 2, 3
(c) 2, 1, 3 (d) none of these

41. The value of k for which the set of equations
$3x + ky - 2z = 0$, $x + ky + 3z = 0$ and $2x + 3y - 4z = 0$
has a non-trivial solution is
(a) 15 (b) 16 (c) 31/2 (d) 33/2

42. If the system of equations $3x - y + 4z - 3 = 0$,
$x + 2y - 3z + 2 = 0$, $6x + 5y + \lambda z + 3 = 0$
has infinite number of solutions, then $\lambda =$
(a) 7 (b) –7 (c) 5 (d) –5

43. If the system of following equations
$2x + 3y + 5 = 0$, $x + ky + 5 = 0$, and $kx - 12y - 14 = 0$
be constant, then $k =$
(a) $2, \frac{12}{5}$ (b) $-1, \frac{1}{5}$
(c) $-6, \frac{17}{5}$ (d) $6, -\frac{12}{5}$

44. If $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, n \in N$, then A^{4n} equals
(a) $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ (b) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
(c) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

45. Let $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$,
then A^n is equal to
(a) $\begin{bmatrix} a^n & 0 & 0 \\ 0 & a^n & 0 \\ 0 & 0 & a \end{bmatrix}$ (b) $\begin{bmatrix} a^n & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$
(c) $\begin{bmatrix} a^n & 0 & 0 \\ 0 & a^n & 0 \\ 0 & 0 & a^n \end{bmatrix}$ (d) $\begin{bmatrix} na & 0 & 0 \\ 0 & na & 0 \\ 0 & 0 & na \end{bmatrix}$

46. If $A = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}, B = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix}$ and $(A + B)^2 = A^2 + B^2$,
values of a and b are
(a) $a = 4, b = 1$ (b) $a = 1, b = 4$
(c) $a = 0, b = 4$ (d) $a = 2, b = 4$

47. The matrix $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$ is
(a) orthogonal (b) involutory
(c) idempotent (d) nilpotent

48. If $A = \begin{bmatrix} 2 & 2 \\ -3 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$,
then $(B^{-1}A^{-1})^{-1} =$

(a) $\begin{bmatrix} 2 & -2 \\ 2 & 3 \end{bmatrix}$ (b) $\begin{bmatrix} 3 & -2 \\ 2 & 2 \end{bmatrix}$

(c) $\frac{1}{10}\begin{bmatrix} 2 & 2 \\ -2 & 3 \end{bmatrix}$ (d) $\frac{1}{10}\begin{bmatrix} 3 & 2 \\ -2 & 2 \end{bmatrix}$

49. If A be a non-singular matrix of order 3, then adj (adj A) is equal to
(a) $|A|A$ (b) $|A|^2A$
(c) $|A|^{-1}A$ (d) none of these

50. Let the matrix $A = \begin{bmatrix} i & 1-2i \\ -1-2i & 0 \end{bmatrix}$,
then A matrix is
(a) symmetric (b) skew-symmetric
(c) hermitian (d) skew-hermitian

51. If $E(\theta) = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$ and θ and ϕ differ by an odd multiple of $\frac{\pi}{2}$, then $E(\theta)E(\varphi)$ is a
(a) null matrix (b) unit matrix
(c) diagonal matrix (d) none of these

52. The inverse of a symmetric matrix is
(a) symmetric (b) skew-symmetric
(c) diagonal matrix (d) none of these

53. The inverse of a diagonal matrix is
(a) a symmetric matrix
(b) a skew-symmetric matrix
(c) a diagonal matrix
(d) none of these

54. If A be a symmetric matrix and $n \in N$, then A^n is
(a) symmetric (b) skew-symmetric
(c) a diagonal matrix (d) none of these

55. If A be a skew-symmetric matrix and n a positive integer, then A^n is
(a) a symmetric matrix
(b) a skew-symmetric matrix
(c) a diagonal matrix
(d) none of these

56. if A be a skew-symmetric matrix and n odd positive integer, then A^n is
(a) a symmetric matrix
(b) a skew-symmetric matrix
(c) a diagonal matrix
(d) none of these

57. If A be a square matrix of order $n \times n$ and k a scalar, then adj (kA) is equal to
(a) k adj A (b) k^n adj A
(c) k^{n-1}adj A (d) k^{n+1}adj A

58. If A and B are square matrices of order 3 such that $|A| = -1$, $|B| = 3$, then $|3AB|$ equals
(a) –9 (b) –81 (c) –27 (d) 81

59. The number of all possible matrices of order 3×3 with each entry with 1 or 0 is
(a) 27 (b) 18 (c) 81 (d) 512.

60. If A be a square matrix such that $A^2 = A$, then $(1 + A)^3 - 7A$ is
(a) A (b) $I - A$ (c) I (d) $3A$

61. If A be a square matrix of order 3 such that $|A| = 2$, then $|\text{adj } A^{-1}|$ is
(a) 11 (b) 13 (c) 17 (d) 19.

62. If $A = \begin{pmatrix} 1 & 2 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $AB = I$,
then $x + y$ is
(a) 0 (b) –1 (c) 2 (d) 5

63. If the value of $D = \begin{vmatrix} {}^{10}C_4 & {}^{10}C_5 & {}^{11}C_m \\ {}^{11}C_6 & {}^{11}C_7 & {}^{12}C_{m+2} \\ {}^{12}C_8 & {}^{12}C_9 & {}^{13}C_{m+4} \end{vmatrix}$ is zero, then m is
(a) 6 (b) 4 (c) 5 (d) 10

64. If $\begin{vmatrix} x & 2 & x \\ x^2 & x & 6 \\ x & x & 6 \end{vmatrix} = Ax^4 + Bx^3 + Cx^2 + Dx + E$,
the value of $5A + 4B + 3C + 2D + E$ is
(a) 0 (b) –16 (c) 16 (d) none

65. The number of values of t for which the system of equations
$(a - t)x + by + cz = 0$
$bx + (c - t)y + az = 0$
$cx + ay + (b - t)z = 0$
has non-trivial solution is
(a) 3 (b) 4 (c) 5 (d) 6

Level III

(Problems for JEE-Advanced)

1. Prove that
$$\begin{vmatrix} 1 & bc & bc(b+c) \\ 1 & ca & ca(c+a) \\ 1 & ab & ab(a+b) \end{vmatrix}$$ is independent of a, b, c.

2. Prove that
$$\begin{vmatrix} b+c & a & a \\ c & c+a & a \\ b & a & a+b \end{vmatrix} = 4abc.$$

3. Prove that
$$\begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & a^2+c^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2.$$

4. Prove that
$$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2.$$

5 Prove that
$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2.$$

6 Prove that
$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$
$= (1 + a^2 + b^2)$.

7. Prove that
$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$
$= 2abc\,(a + b + c)^3$.

8. If $2s = a + b + c$, prove that
$$\begin{vmatrix} a^2 & (s-a)^2 & (s-a)^2 \\ (s-b)^2 & b^2 & (s-b)^2 \\ (s-c)^2 & (s-c)^2 & c^2 \end{vmatrix}$$
$= 2s^3(s - a)(s - b)(s - c)$.

9. Let α, β be the roots of $x^2 + 5x + 3 = 0$ and $s_n = \beta^n + \beta^n$. Then find the value of
$$\begin{vmatrix} 3 & 1+s_1 & 1+s_2 \\ 1+s_1 & 1+s_2 & 1+s_3 \\ 1+s_2 & 1+s_3 & 1+s_4 \end{vmatrix}.$$

10. Prove that
$$\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix} = (ab+bc+ca)^3$$

11. If $-1 \le x < 0$, $0 \le y < 1$, $1 \le z < 2$, find the value of the determinant
$$\begin{vmatrix} [x+1] & [y] & [z] \\ [x] & [y+1] & [z] \\ [x] & [y] & [z+1] \end{vmatrix}, \text{ where } [,] = \text{GIF}.$$

12. Prove that
$$\begin{vmatrix} a & b-c & b+c \\ a+c & b & c-a \\ a-b & a+b & c \end{vmatrix} = (a+b+c)(a^2+b^2+c^2)$$

13. Prove that
$$\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix}$$
$= (a - b)(b - c)(c - a)(a + b + c)(a^2 + b^2 + c^2)$

14. Evaluate
$$\begin{vmatrix} 3 & a+b+c & a^3+b^3+c^3 \\ a+b+c & a^2+b^2+c^2 & a^4+b^4+c^4 \\ a^2+b^2+c^2 & a^3+b^3+c^3 & a^5+b^5+c^5 \end{vmatrix}$$

15. If A, B and C be the angles of a triangle, prove that
$$\begin{vmatrix} -1+\cos B & \cos B+\cos C & \cos B \\ \cos A+\cos C & -1+\cos A & \cos A \\ -1+\cos B & -1+\cos A & -1 \end{vmatrix} = 0$$

16. Prove that
$$\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ac-b^2 \end{vmatrix}$$
$$= \begin{vmatrix} a^2 & c^2 & 2ac-b^2 \\ 2ab-c^2 & b^2 & a^2 \\ b^2 & 2bc-a^2 & c^2 \end{vmatrix}.$$

17. Prove that
$$\begin{vmatrix} bc & -ca & ab \\ bc & ca & -ab \\ -bc & ca & ab \end{vmatrix} = \begin{vmatrix} c^2+a^2 & a^2 & c^2 \\ a^2 & a^2+b^2 & b^2 \\ c^2 & b^2 & b^2+c^2 \end{vmatrix}.$$

18. Prove that
$$\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ac-b^2 \end{vmatrix} = \begin{vmatrix} u^2 & v^2 & v^2 \\ v^2 & u^2 & v^2 \\ v^2 & v^2 & u^2 \end{vmatrix},$$
where $u^2 = a^2 + b^2 + c^2$, $v^2 = ab + bc + ca$.

19. Let a, b and c are positive real numbers. Prove that the system of equations $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, $\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ in x, y, z have a unique solutions.

20. Let $A=\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are two matrices such that $AB = BA$ and $c \ne 0$, find the value of $\frac{a-d}{3b-c}$.

21. Let $A=\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$.

If U_1 and U_2 are column matrices such that $AU_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $AU_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, find $U_1 + U_2$.

22. If $A = (a_{ij})$ be a square matrix of order n such that $a_{ij} = \begin{cases} 0: & i = j \\ 1: & i \neq j \end{cases}$, find det(A).

23. Find the number of values of t for which the system of equations
$$(a-t)x + by + cz = 0$$
$$bx + (c-t)y + az = 0$$
$$cx + ay + (b-t)z = 0$$
has non-trivial solution.

24. If $A = \begin{pmatrix} 0 & -\tan\left(\frac{\alpha}{2}\right) \\ \tan\left(\frac{\alpha}{2}\right) & 0 \end{pmatrix}$,
prove that $(I + A) = (I - A)\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$

25. Evaluate: $\begin{vmatrix} {}^xC_1 & {}^xC_2 & {}^xC_3 \\ {}^yC_1 & {}^yC_2 & {}^yC_3 \\ {}^zC_1 & {}^zC_2 & {}^zC_3 \end{vmatrix}$. **[Roorkee, 1990]**

26. If f, g, h are differentiable function of x and
$$\Delta = \begin{vmatrix} f & g & h \\ (xf)' & (xg)' & (xh)' \\ (x^2f)'' & (x^2g)'' & (x^2h)'' \end{vmatrix},$$
prove that $\Delta' = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^2f'')' & (x^2g'')' & (x^2h'')' \end{vmatrix}$
[Roorkee, 1991]

27. Find the value of
$$\begin{vmatrix} \sqrt{13}+\sqrt{3} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{26} & 5 & \sqrt{10} \\ 3+\sqrt{65} & \sqrt{15} & 5 \end{vmatrix}.$$ **[Roorkee, 1992]**

> ***Note*** *No questions asked in 1993.*

28. Using determinants, solve the following equations: $x + 2y + 3z = 6$, $2x + 4y + z = 17$, $3x + 2y + 9z = 2$ **[Roorkee, 1994]**

29. For what values of p and q, the system of equations $2x + py + 6z = 8$, $x + 2y + qz = 5$, $x + y + 3z = 4$ has
 (i) no solution
 (ii) a unique solution
 (iii) infinitely many solutions. **[Roorkee, 1995]**

30. Find those values of c for which the equations
$2x + 3y = 3$
$(c + 2)x + (c + 4)y = c + 6$
$(c + 2)^2x + (c + 4)^2y = (c + 6)^2$
are consistent. Also solve above equations for those values of c. **[Roorkee, 1996]**

31. For what real values of k, the system of equations $x + 2y + z = 1$, $x + 3y + 4z = k$ and $x + 5y + 10z = k^2$ has solution. Find the solution in each case. **[Roorkee, 1997]**

32. Using matrix method, find the values of λ and μ so that the system of equations $2x - 3y + 5z = 12$, $3x + y + \lambda z = \mu$ and $x - 7y + 8z = 17$ has
 (i) a unique solution
 (ii) infinite solution
 (iii) no solution **[Roorkee, 1998]**

33. For all values of λ, find the rank of the matrix
$$\begin{pmatrix} 1 & 4 & 5 \\ \lambda & 8 & 8\lambda - 6 \\ 1+\lambda^2 & 8\lambda+4 & 2\lambda+21 \end{pmatrix}$$
[Roorkee, 1999]

34. Find the real values of r for which the following system of linear equations has a non-trivial solution. Also find the non-trivial solutions of $2rx - 2y + 3z = 0$, $x + ry + 2z = 0$, $2x + rz = 0$, and $x + ry + 2z = 0$, $2x + rz = 0$ **[Roorkee, 2000]**

35. Solve for x:
$$\begin{vmatrix} a^2 & a & 1 \\ \sin(n+1)x & \sin nx & \sin(n-1)x \\ \cos(n+1)x & \cos nx & \cos(n-1)x \end{vmatrix} = 0$$
[Roorkee, 2001]

Level IV

(Tougher Problems for JEE-Advanced)

1. Prove that
$$\begin{vmatrix} 1 & \cos x - \sin x & \cos x + \sin x \\ 1 & \cos y - \sin y & \cos y + \sin y \\ 1 & \cos z - \sin z & \cos z + \sin z \end{vmatrix} = 2\begin{vmatrix} 1 & \cos x & \sin x \\ 1 & \cos y & \sin y \\ 1 & \cos z & \sin z \end{vmatrix}.$$

2. Prove that
$$\begin{vmatrix} 2 & \alpha+\beta+\gamma+\delta & \alpha\beta+\gamma\delta \\ \alpha+\beta+\gamma+\delta & 2(\alpha+\beta)(\gamma+\delta) & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) \\ \alpha\beta+\gamma\delta & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) & 2\alpha\beta\gamma\delta \end{vmatrix} = 0$$

3. Prove that
$$\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-Q) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix} = 0$$

4. If $S_r = \alpha^r + \beta^r + \gamma^r$, prove that

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^3 & \beta^3 & \gamma^3 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix} = \begin{vmatrix} S_0 & S_1 & S_2 \\ S_1 & S_2 & S_3 \\ S_3 & S_4 & S_5 \end{vmatrix}.$$

5. Solve the system of equations:

$$x + 2y + 3z = 1$$
$$2x + 3y + 2z = 2$$
$$3x + 3y + 4z = 1$$

6. Solve the system of equations:

$$2x + 3y - 3z = 0$$
$$3x - 3y + z = 0$$
$$3x - 2y - 3z = 0$$

7. Let $A = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, where $a \neq 0$.

Show that for $n \geq 0$,

$$A^n = \begin{pmatrix} a^n & \dfrac{b(a^n - 1)}{a - 1} \\ 0 & 1 \end{pmatrix}$$

8. If A, B, C be three matrices such that

$$A = [x \quad y \quad z],\ B = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix},\ C = \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

find the value of ABC.

9. Find the value of the determinant

$$\begin{vmatrix} 1 & a & a^2 \\ \cos(p - dx) & \cos px & \cos(p + d)x \\ \sin(p - d)x & \sin px & \sin(p + d)x \end{vmatrix}.$$

10. Without expanding the determinant at any stage, show that

$$\begin{vmatrix} x^2 + x & x + 1 & x - 2 \\ 2x^2 + 3x + 1 & 3x & 3x - 3 \\ x^2 + 2x + 3 & 2x - 1 & 2x - 1 \end{vmatrix} = xA + B,$$

where A and B are determinants of order 3 not involving x.

11. Let the three-digit numbers $A28$, $3B9$ and $62C$ where A, B, and C are integers between 0 and 9 be divisible by a fixed integer k.

Show that the determinant $\begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$ is divisible by k.

12. If $a \neq p$, $b \neq q$, $c \neq r$ and $\begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix}$, then find the value of $\dfrac{p}{p - a} + \dfrac{q}{q - b} + \dfrac{r}{r - c}$.

13. Prove that for all values of θ,

$$\begin{vmatrix} \sin\theta & \cos\theta & \sin 2\theta \\ \sin\left(\theta + \frac{2\pi}{3}\right) & \cos\left(\theta + \frac{2\pi}{3}\right) & \sin\left(2\theta + \frac{4\pi}{3}\right) \\ \sin\left(\theta - \frac{2\pi}{3}\right) & \cos\left(\theta - \frac{2\pi}{3}\right) & \sin\left(2\theta - \frac{4\pi}{3}\right) \end{vmatrix} = 0.$$

14. Let a, b, c be real numbers with $a^2 + b^2 + c^2 = 1$, show that $\begin{vmatrix} ax - by - c & bx + ay & cx + a \\ bx + ay & -ax + by - c & cy + b \\ cx + a & cy + b & -ax - by + c \end{vmatrix}$ represents a straight line.

15. If matrix $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$,

where a, b, c are not real positive numbers, $abc = 1$ and $A^TA = 1$, find the value of $a^3 + b^3 + c^3$.

16. Let $a > 0$, $d > 0$, prove that the value of the determinant

$$\begin{vmatrix} \frac{1}{a} & \frac{1}{a(a+d)} & \frac{1}{(a+d)(a+2d)} \\ \frac{1}{a+d} & \frac{1}{(a+d)(a+2d)} & \frac{1}{(a+2d)(1+3d)} \\ \frac{1}{a+2d} & \frac{1}{(a+2d)(a+3d)} & \frac{1}{(a+3d)(a+4d)} \end{vmatrix} = \frac{4d^4}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}$$

17. For a fixed positive integers n, if

$$D = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+2)! & (n+4)! \end{vmatrix},$$

show that $\left(\dfrac{\Delta}{(n!)^3} - 4\right)$ is divisible by n.

18. Let a, b, c, d be real numbers in GP.
If u, v, w satisfy the system of equations

$$u + 2v + 3w = 6$$
$$4u + 5v + 6w = 12$$
$$6u + 9v = 4,$$

show that the roots of the equation

$$\left(\frac{1}{u} + \frac{1}{v} + \frac{1}{w}\right)x^2 + [(b - c)^2 + (c - a)^2 + (a - b)^2]x + (u + v + w) = 0$$

and $20x^2 + 10(a - d)^2x - 9 = 0$ are reciprocals of each other.

19. If x, y, z be not all zeroes such that

$$ax + y + z = 0$$
$$x + by + z = 0$$
$$x + y + cz = 0,$$

find the value of

$$\left(\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} + 4\right).$$

20. If $a \neq b$ and x, y, and z are not all zero and if

$$ax + by + cz = 0$$
$$bx + cy + az = 0$$
$$cx + ay + bz = 0,$$

prove that $x : y : z = 1 : 1 : 1$

21. If $a = \frac{x}{y-z}, b = \frac{y}{z-x}, c = \frac{z}{x-y}$,
where x, y and z are not all zero, show that $1 + ab + bc + ca = 0$.

22. If $bc + qr + = -1 = ca + rp = ab + pq$,

show that $\begin{vmatrix} ap & a & p \\ bq & b & q \\ cr & c & r \end{vmatrix} = 0$

23. Show that $\begin{vmatrix} \operatorname{cosec}\alpha & 1 & 0 \\ 1 & 2\operatorname{cosec}\alpha & 1 \\ 0 & 1 & 2\operatorname{cosec}\alpha \end{vmatrix}$

$$= \frac{1}{2}\left[\tan^3\left(\frac{\alpha}{2}\right) + \cot^3\left(\frac{\alpha}{2}\right)\right]$$

24. If $p + q + r = 0$, show that

$$\begin{vmatrix} pa & qb & rc \\ qc & ra & pb \\ rb & pc & qa \end{vmatrix} = pqr\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$$

25. Prove that

$$\begin{vmatrix} 2bc - a^2 & c^2 & b^2 \\ c^2 & 2ac - b^2 & b^2 \\ b^2 & a^2 & 2ab - c^2 \end{vmatrix}$$

$$= (a^3 + b^3 + c^3 - 3abc)^2$$

26. If none of a, b and c is zero, show that

$$\begin{vmatrix} -bc & b^2 + bc & c^2 + bc \\ a^2 + ac & -ac & c^2 + ac \\ a^2 + ab & b^2 + ab & -ab \end{vmatrix}$$

$$= (ab + bc + ca)^3$$

27. If a, b, c are distinct real numbers and the system of equations

$$ax + a^2y + (a^2 + 1)z = 0$$
$$bx + b^2y + (b^2 + 1)z = 0$$
$$cx + c^2y + (c^2 + 1)z = 0$$

has a non-trivial solutions, prove that $1 + abc = 0$

28. Prove that

$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$

$$= 2abc(a + b + c)^3$$

29. If a, b and c are all different and

$$\begin{vmatrix} a & a^3 & a^4 - 1 \\ b & b^3 & b^4 - 1 \\ c & c^3 & c^4 - 1 \end{vmatrix} = 0,$$

prove that $abc(ab + bc + ca) = (a + b + c)$

30. If a, b, c be the sides of a triangle ABC, prove that

$$\begin{vmatrix} a^2 & b\sin A & c\sin A \\ b\sin A & 1 & \cos A \\ c\sin A & \cos A & 1 \end{vmatrix} = 0$$

Integer Type Questions

1. If a, b, c and d be the roots of $x^4 + 2x^3 + 3x^2 + 6x + 8 = 0$, find the value of

$$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix}.$$

2. If $\begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix} = kxyz$ such that $k^n = 64$, find the value of n.

3. Evaluate the determinant

$$\begin{vmatrix} \cos^2 x & \cos x \cdot \sin x & -\sin x \\ \cos x \cdot \sin x & \sin^2 x & \cos x \\ \sin x & -\cos x & 0 \end{vmatrix}.$$

4. If $\begin{vmatrix} a & b-y & c-z \\ a-x & b & c-z \\ a-x & b-y & c \end{vmatrix} = 0$,

find the value of $\left(\frac{a}{x} + \frac{b}{y} + \frac{c}{z}\right)$

5. If $A = \begin{pmatrix} 1 & 2 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $AB = I_3$,

find the value of $(x + y + 5)$.

6. If $\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix} = \lambda \begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$,
   find the value of $(\lambda + 2)$.
7. If 3^n is a factor of the determinant
   $$\begin{vmatrix} 1 & 1 & 1 \\ {}^nC_1 & {}^{n+3}C_1 & {}^{n+6}C_1 \\ {}^nC_2 & {}^{n+3}C_2 & {}^{n+6}C_2 \end{vmatrix},$$
   find the maximum value of n.
8. If a, b and c be the roots of $x^3 + 2x^2 + 5 = 0$, find the value of $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$.
9. If $\begin{vmatrix} 1+x & x & x^2 \\ x & 1+x & x^2 \\ x^2 & x & 1+x \end{vmatrix}$
   $= ax^5 + bx^4 + cx^3 + dx^2 + ex + f,$
   find the value of $(e + f + 2)$.
10. If the equations $2x + 3y + 1 = 0$, $3x + y - 2 = 0$ and $ax + 2y - b = 0$ are consistent, find the value of $(a - b)$.
11. Find the number of real roots of
    $$\begin{vmatrix} x-1 & 1 & 1 \\ 1 & x-1 & 1 \\ 1 & 1 & x-1 \end{vmatrix} = 0$$
12. Find the number of values of x for which the matrix
    $$\begin{bmatrix} x+3 & 5 & 2 \\ 1 & 7+x & 6 \\ 2 & 5 & x+3 \end{bmatrix}$$
    has the rank 2.
13. If $\begin{vmatrix} x & x-1 & x \\ -2x & x+1 & 1 \\ x+1 & 1 & x \end{vmatrix} = ax^3 + bx^2 + cx + d$,
    find the value of $2 - (a + b + c + d)$.
14. If $D_1 = \begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix}$
    and $D_2 = \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$,
    find $\dfrac{D_1}{D_2}$.
15. Find the number of positive integral solutions of
    $$\begin{vmatrix} x^3+1 & x^2y & x^2z \\ xy^2 & y^3+1 & y^2z \\ xz^2 & yz^2 & z^3+1 \end{vmatrix} = 11$$

Comprehensive Link Passage

Passage I

Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ and U_1, U_2 and U_3 are columns of a 3×3 matrix U. If columns matrices U_1, U_2 and U_3 satisfying $AU_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $AU_2 = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$, $AU_3 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ evaluated as directed in the following questions.

1. The value of $|U|$ is
   (a) 3 (b) –3 (c) 3/2 (d) 2
2. The sum of the elements of the matrix U^{-1} is
   (a) –1 (b) 0 (c) 1 (d) 3
3. The value of $[3 \quad 2 \quad 0]\,U \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ is
   (a) 5 (b) 5/2 (c) 4 (d) 3/2

Passage II

Let A be the set of all 3×3 symmetric matrices all of whose entries are either 0 or 1. Five of these entries are 1 and four of them are zeroes.

1. The number of matrices in A is
   (a) 12 (b) 6 (c) 9 (d) 3
2. The number of matrices A to A for which the system of linear equations, $A\begin{bmatrix} x \\ y \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$
   A has a unique solution, is
   (a) less than 4
   (b) at least 4 but less than 7
   (c) at least 7 but less than 10
   (d) at least 10
3. The number of matrices A in A for which the system of linear equation
   $$A\begin{bmatrix} x \\ y \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
   is inconsistent is
   (a) 0 (b) more than 2
   (c) 2 (d) 1

Passage III

Let $D = \begin{vmatrix} x^2+x+2 & 2x^2+3x+1 & 3x^2+5x+3 \\ 20x^2+20x+59 & 40x^2+60x+20 & 60x^2+100x+70 \\ 2x^2+2x+6 & 4x^2+6x+2 & 6x^2+10x+7 \end{vmatrix}$

$= ax^2 + bx + c.$

1. The value of a is

(a) $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix}$

(c) $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{vmatrix}$ (d) $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix}$

2. The value of b is

(a) $\begin{vmatrix} 1 & 3 & 5 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{vmatrix}$ (b) $\begin{vmatrix} 1 & 3 & 5 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{vmatrix}$

(c) $\begin{vmatrix} 1 & 3 & 5 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix}$ (d) $\begin{vmatrix} 1 & 3 & 5 \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix}$

3. The value of c is

(a) $\begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{vmatrix}$ (b) $\begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{vmatrix}$

(c) $\begin{vmatrix} 2 & 1 & 1 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{vmatrix}$ (d) $\begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & 1 & 0 \end{vmatrix}$

Passage IV

Let p be an odd prime number and let T_p the following set of 2×2 matrices

$$T_p = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c \in \{0, 1, 2, \ldots, (p-1)\} \right\}$$

1. The number of A in T_p such that A is either symmetric or skew-symmetric or both, and det (A) is divisible by p, is
(a) $(p-1)^2$ (b) $2(p-1)$
(c) $(p-1)^2 + 1$ (d) $2p - 1$
2. The number of A in T_p such that the trace of A is not divisible by p but det (A) is divisible by p, is
(a) $(p-1)(p^2-p+1)$ (b) $p^3-(p-1)^2$
(c) $(p-1)^2$ (d) $(p+1)(p-1)^2$
3. The number of A in T_p such that det (A) is not divisible by p, is
(a) $2p^2$ (b) $p^3 - 5p$
(c) $p^3 - 3p$ (d) $p^3 - p^2$

Matrix Match

1. Match the following columns.

	Column I		Column II
(A)	The value of $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$ is	(P)	1
(B)	The value of $\begin{vmatrix} 1 & ab & bc+ca \\ 1 & bc & ca+ab \\ 1 & ca & ab+bc \end{vmatrix}$ is	(Q)	$(a-b)(b-c)(c-a)$
(C)	The value of $\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$ is	(R)	-1
(D)	The value of $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is	(S)	$-(a^3+b^3+c^3-3abc)$
		(T)	0

2. Match the following columns.

	Column I		Column II
(A)	Rank of the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$	(P)	3
(B)	Rank of the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$	(Q)	2
(C)	Rank of the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	(R)	1
(D)	Rank of the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$	(S)	0

3. Match the following columns:

	Column I		Column II
(A)	The value of $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$ is	(P)	p
(B)	The value of $\begin{vmatrix} ab+ac & ab & ac \\ ab & bc+ab & bc \\ ac & bc & ac+bc \end{vmatrix}$ is	(Q)	$2abc$

(C)	The value of $\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-Q) \\ \cos(C-P) & \cos(C-Q) & \cos(C-Q) \end{vmatrix}$ is	(R)	$4a^2b^2c^2$
(D)	The value of $\begin{vmatrix} 1 & a & a^2 \\ \cos(P-D)x & \cos Px & \cos(P+D)x \\ \sin(P-D)x & \sin Px & \sin(P+D)x \end{vmatrix}$ is	(S)	0

4. Let $\begin{vmatrix} x^2 & 3x & x+4 \\ x+4 & x^2 & 3x \\ 3x & x+4 & x \end{vmatrix}$

$= ax^5 + bx^4 + cx^3 + dx^2 + ex + f.$

Then match the following columns.

Column I		Column II	
(A)	the value of f is	(P)	–6
(B)	the value of e is	(Q)	48
(C)	the value of $b + d$ is	(R)	64
(D)	the value of $a + c$ is	(S)	2

Assertion and Reason

Codes

(A) Both A and R are true and R is the correct explanation of A.
(B) Both A and R are true and R is not the correct explanation of A.
(C) A is true but R is False.
(D) A is false but R is true.

1. *Assertion (A):* If $A = \begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$, then Adj(Adj$A$) = A

Reason (R): $|\text{Adj}(\text{Adj}A)| = |A|^{(n-1)^2}$, where A be n-rowed non singular matrix.

(a) A (b) B (c) C (d) D

2. *Assertion (A):* If $A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$,

then $A^{-1} = \begin{pmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{pmatrix}$

Reason (R): The inverse of a diagonal matrix is also a diagonal matrix

(a) A (b) B (c) C (d) D

3. *Assertion (A):* The inverse of the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{pmatrix}$ does not exist.

Reason (R): The matrix A is singular

(a) A (b) B (c) C (d) D

4. *Assertion (A):* The inverse of the matrix $A = \begin{pmatrix} 1 & 2 & 4 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{pmatrix}$ does not exist

Reason (R): The matrix A is non-singular

(a) A (b) B (c) C (d) D

5. *Assertion (A):* The value of a determinant $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$ is non zero

Reason (R): If any two columns of a determinant are identical, then the value of a determinant is zero.

(a) A (b) B (c) C (d) D

6. *Assertion (A):* The rank of a matrix $A = \begin{pmatrix} 1 & 2 & 4 \\ 6 & 3 & 2 \\ 3 & 4 & 5 \end{pmatrix}$ is 2.

Reason (R): If the determinant of a matrix of order 3 is non zero, then the rank of the matrix is 3

(a) A (b) B (c) C (d) D

7. *Assertion (A):* Let $A = \begin{pmatrix} 0 & 2 & 4 \\ -2 & 0 & 5 \\ -4 & -5 & 0 \end{pmatrix}$. Then A is skew symmetric matrix.

Reason (R): The diagonal of every skew-symmetric matrix is zero.

(a) A (b) B (c) C (d) D

8. *Assertion (A):* Let A = $\begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$.

Then A is idempotent

Reason (R): A matrix A is idempotent if $A^2 = A$.

(a) A (b) B (c) C (d) D

Questions asked in Previous Years' JEE-Advanced Examinations

1. Given $x = cy + bz$, $y = az + cx$ and $z = bx + ay$ where x, y and z are not all zero, prove that $a^2 + b^2 + c^2 + 2abc = 1$. **[IIT-JEE, 1978]**

2. The equations

$$x + 2y + 2z = 1$$
$$2x + 4y + 4z = 9$$

have

(a) only one solution

(b) only two solutions
(c) infinite number of solutions
(d) none of these. **[IIT-JEE, 1979]**

3. For what values of k does the following system of equations posses a non-trivial solution over the set of rationals.
$$x + ky + 2z = 0$$
$$3x + ky - 2z = 0$$
$$2x + 3y - 4z = 0$$
For what value of k, find all the solutions. **[IIT-JEE, 1979]**

4. For what values of m does the system of equations $3x + my = m$ and $2x - 5y = 20$ has a solution satisfying the conditions $x > 0, y > 0$. **[IIT-JEE, 1980]**

5. Find the solution set of the system
$$x + 2y + z = 1$$
$$2x - 3y - w = 2$$
$$x \geq 0, y \geq 0, z \geq 0, w \geq 0.$$
[IIT-JEE, 1980]

6. Let a, b, c be positive and not all equal, show that the value of the determinant $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$ is negative. **[IIT-JEE, 1981]**

7. Let $p\lambda^2 + q\lambda^3 + r\lambda^2 + s\lambda + t$
$$= \begin{vmatrix} \lambda^2 + 3\lambda & \lambda - 1 & \lambda + 3 \\ \lambda + 1 & -2\lambda & \lambda - 4 \\ \lambda - 3 & \lambda + 4 & 3\lambda \end{vmatrix}$$
be an identity in λ, where p, q, r, s, t are constants. Then the value of t is...... **[IIT-JEE, 1981]**

8. The solution set of the equation
$$\Delta = \begin{vmatrix} 1 & 4 & 2 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$$
is............ **[IIT-JEE, 1981]**

9. Without expanding the determinant, at any stage, show that
$$\Delta = \begin{vmatrix} x^2 + x & x + 1 & x - 2 \\ 2x^2 + 3x - 1 & 3x & 3x - 3 \\ x^2 + 2x + 3 & 2x - 1 & 2x - 1 \end{vmatrix} = xA + B,$$
where A and B are determinants of order 3 not involving x. **[IIT-JEE, 1982]**

10. Show that the system of equations
$$3x - y + 4z = 3$$
$$x + 2y - 3z = -2$$
$$6x + 5y + \lambda z = -3$$
has at least one solution for any real number $\lambda \neq -5$. Find the set of solutions if $\lambda = -5$. **[IIT-JEE, 1983]**

11. The determinants
$$\Delta = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} \text{ and } \Delta' = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$
are not identically equal. **[IIT-JEE, 1983]**

12. The system of equations
$$\lambda x + y + z = 0$$
$$-x + \lambda y + z = 0$$
$$-x - y + \lambda z = 0$$
will have a non-zero solution if real value of λ are given by... **[IIT-JEE, 1984]**

13. Let α be a repeated root of a quadratic equation $f(x) = 0$ and $A(x)$, $B(x)$, $C(x)$ be polynomials of degree 3, 4 and 5, respectively. Show that
$$F(x) = \begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$
is divisible by $f(x)$, where prime denotes the derivatives. **[IIT-JEE, 1984]**

14. Show that $\Delta = \begin{vmatrix} {}^xC_r & {}^xC_{r+1} & {}^xC_{r+2} \\ {}^yC_r & {}^yC_{r+1} & {}^yC_{r+2} \\ {}^zC_r & {}^zC_{r+1} & {}^zC_{r+2} \end{vmatrix}$
$$= \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+2}C_{r+2} \end{vmatrix}$$
[IIT-JEE, 1985]

15. If $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$,
the two triangles with vertices
(x_1, y_1), (x_2, y_2), (x_3, y_3) and
(a_1, b_1), (a_2, b_2), (a_3, b_3) must be congruent. **[IIT-JEE, 1985]**

16. Consider the system of equations in x, y, z
$$(\sin 3\theta)x - y + z = 0$$
$$(\cos 2\theta)x + 4y + 3z = 0$$
$$2x + 7y + 7z = 0.$$
Find the values of θ for which the system has a non-trivial solution. **[IIT-JEE, 1986]**

17. The determinant

$$\Delta = \begin{vmatrix} a & b & a\alpha + b \\ b & c & b\alpha + c \\ a\alpha + b & b\alpha + c & 0 \end{vmatrix}$$

is zero, if

(a) a, b, c are in AP
(b) a, b, c are in GP
(c) a, b, c are in HP
(d) α is a root of the equation $ax^2 + bx + c = 0$
(e) $(x - \alpha)$ is a factor of $ax^2 + 2bx + c$

[IIT-JEE, 1986]

18. Let

$$f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}.$$

Then $\int_0^{\pi/2} f(x)\,dx = \ldots$ **[IIT-JEE, 1987]**

19. The value of the determinant

$$\begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix}$$ is...

[IIT-JEE, 1988]

20. The value of $\theta = 0$ and $\theta = \frac{\pi}{2}$ and satisfying the equation $\Delta = \begin{vmatrix} 1 + \sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1 + \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \cos^2\theta & 1 + 4\sin 4\theta \end{vmatrix} = 0$

are

(a) $\frac{7\pi}{24}$ (b) $\frac{5\pi}{24}$ (c) $\frac{11\pi}{24}$ (d) $\frac{\pi}{24}$

[IIT-JEE, 1988]

21. Let $\Delta_a = \begin{vmatrix} a - 1 & n & 6 \\ (a-1)^2 & 2n^2 & 4n - 2 \\ (a-1)^3 & 3n^3 & 3n^2 - 3n \end{vmatrix}$.

Show that $\sum_{a=1}^{n} \Delta_a = \text{Constant}$

[IIT-JEE, 1989]

22. Let the three-digit numbers $A28$, $3B9$ and $62C$ where A, B, C are integers between 0 and 9 be divided a fixed number k. Show that the determinant $\begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$ is divisible by k. **[IIT-JEE, 1990]**

23. If $a \neq p$, $b \neq q$, $c \neq r$ and

$$\Delta = \begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0.$$

Find the value of

$$E = \frac{p}{p-a} + \frac{q}{q-b} + \frac{r}{r-c}$$ **[IIT-JEE, 1991]**

24. For a fixed positive integer n, if

$$D = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}.$$

Show that $\left(\frac{D}{(n!)^3} - 4\right)$ is divisible by n.

[IIT-JEE, 1992]

25. Let λ and α be real. Find the set of all values of λ for which the system of equations

$\lambda x + (\sin\alpha)y + (\cos\alpha)z = 0$
$x + (\cos\alpha)y + (\sin\alpha)z = 0$
$-x + (\sin\alpha)y - (\cos\alpha)z = 0$

has a non-trivial solution. For $\lambda = 1$, find all the values of α. **[IIT-JEE, 1993]**

26. For positive numbers x, y and z the numerical value of the determinant

$$\Delta = \begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$$ is...

[IIT-JEE, 1993]

27. For all A, B, C, P, Q, R, show that

$$\Delta = \begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix} = 0$$

[IIT-JEE, 1994]

28. If $\omega(\neq 1)$ is a cube root of unity, then

$$\Delta = \begin{vmatrix} 1 & 1 + i + \omega^2 & \omega^2 \\ 1 - i & -1 & \omega^2 - 1 \\ -i & -i + \omega - 1 & -1 \end{vmatrix}$$ is

(a) 0 (b) 1 (c) i (d) ω

[IIT-JEE, 1995]

29. Let a, b, c be the real numbers. The following system of equations in x, y and z

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$
$$-\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

has
(a) no solution
(b) unique solution
(c) infinitely many solutions
(d) finitely many solutions

[IIT-JEE, 1995]

30. For $a > 0$, $d > 0$, find the value of the determinant

$$\Delta = \begin{vmatrix} \frac{1}{a} & \frac{1}{a(a+d)} & \frac{1}{(a+d)(a+2d)} \\ \frac{1}{a+d} & \frac{1}{(a+d)(a+2d)} & \frac{1}{(a+2d)(a+3d)} \\ \frac{1}{a+2d} & \frac{1}{(a+2d)(a+3d)} & \frac{1}{(a+3d)(a+4d)} \end{vmatrix}.$$

[IIT-JEE, 1996]

31. Find the value of the determinant $\begin{vmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$ where a, b and c are respectively the pth, qth and rth terms of a harmonic progression.

[IIT-JEE, 1997]

32. The determinant

$$\Delta = \begin{vmatrix} xp+y & x & y \\ yp+z & y & z \\ 0 & xp+y & yp+z \end{vmatrix} = 0,$$

if
(a) x, y, z are in AP (b) x, y, z are in GP
(c) x, y, z are in HP (d) xy, yz, zx are in AP.

[IIT-JEE, 1997]

33. The parameter on which the value of the determinant

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ \cos(p-d)x & \cos(p)x & \cos(p+d)x \\ \cos(p-d)x & \sin(p)x & \sin(p+d)x \end{vmatrix}$$

does not depend upon is
(a) a (b) p (c) d (d) x

[IIT-JEE, 1997]

34. Let $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$,

where p is a constant. Then $f'''(0)$ equals
(a) p (b) $p + p^2$
(c) $p + p^3$ (d) independent of p

[IIT-JEE, 1998]

35. If $\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} = x + iy$,

then
(a) $x = 3, y = 1$ (b) $x = 1, y = 3$
(c) $x = 0, y = 3$ (d) $x = 0, y = 0$

[IIT-JEE, 1998]

36. If $f(x) = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & x(x+1) \\ 3x(x-1) & x(x-1)(x-2) & x(x+1)(x-1) \end{vmatrix}$,

then $f(100)$ equals
(a) 0 (b) 1
(c) 100 (d) –100

[IIT-JEE, 1999]

37. Prove that for all values of θ

$$\begin{vmatrix} \sin\theta & \cos\theta & \sin\theta \\ \sin\left(\theta+\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) & \sin\left(\theta+\frac{4\pi}{3}\right) \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(\theta-\frac{4\pi}{3}\right) \end{vmatrix} = 0$$

[IIT-JEE, 2000]

38. If the system of equations
$x - ky - z = 0$, $kx - y - z = 0$, $x + y - z = 0$
has a non-zero solution, the possible values of k are
(a) –1, 2 (b) 1, 2
(c) 0, 1 (d) –1, 1

[IIT-JEE, 2000]

39. The number of distinct real roots of $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$ in $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$,

is
(a) 0 (b) 2 (c) 1 (d) 3

[IIT-JEE, 2001]

40. Let a, b, c be real number with $a^2 + b^2 + c^2 = 1$, show that

$$\begin{vmatrix} ax-by-c & bx+ay & cx+a \\ bx+ay & -ax+by-c & cy+b \\ cx+a & cy+b & -ax-by+c \end{vmatrix}$$

represents a straight line.

[IIT-JEE, 2001]

41 Let $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. The value of the determinant

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix}$$

is
(a) 3ω (b) $3\omega(\omega - 1)$
(c) $3\omega^2$ (d) $3\omega(1 - \omega)$

[IIT-JEE, 2002]

42. The number of values of k for which the system of equations
$(k + 1)x + 8y = 4k$; $kx + (k + 3)y = 3k - 1$
has infinitely many solutions is
(a) 0 (b) 1 (c) 2 (d) infinite

[IIT-JEE, 2002]

43. If $A=\begin{pmatrix} a & 0 \\ 1 & 1 \end{pmatrix}$ and $B=\begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$, the value of a for which $A^2 = B$ is
(a) 1 (b) –1
(c) 4 (d) no real values
[IIT-JEE, 2003]

44. If the system of equations $x + ay = 0$, $ax + z = 0$, $az + y = 0$ has infinite solutions, the value of a is
(a) –1 (b) 1
(c) 0 (d) no real values
[IIT-JEE, 2003]

45. If $A=\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$, where a, b, c are real positive numbers, $abc = 1$ and $A^TA = I$, find the value of $a^2 + b^2 + c^2$.
[IIT-JEE, 2003]

46. Given $2x - y + 2z = 2$, $x - 2y + z = -4$, $x + y + \lambda z = 2$, the value of λ such that the given system of equation has no solution is
(a) 3 (b) 1 (c) 0 (d) –3
[IIT-JEE, 2004]

47. If $A=\begin{pmatrix} \alpha & 2 \\ 2 & \alpha \end{pmatrix}$ and $|A^3| = 125$, the value of α is
(a) ±1 (b) ±2 (c) ±3 (d) ±5
[IIT-JEE, 2004]

48. If $A=\begin{pmatrix} a & 0 & 1 \\ 1 & c & b \\ 1 & d & b \end{pmatrix}, B=\begin{pmatrix} a & 1 & 1 \\ 0 & d & c \\ 1 & d & b \end{pmatrix}$

and $U=\begin{pmatrix} f \\ g \\ h \end{pmatrix}, V=\begin{pmatrix} a^2 \\ 0 \\ 0 \end{pmatrix}, X=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$.

If $AX = U$ has infinitely many solutions, prove that $BX = V$ cannot have a unique solution.
If $afd \neq 0$, prove that $BX = V$ has no solution.
[IIT-JEE, 2004]

49. If $A=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{pmatrix}$ and $I=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and

$A^{-1}=\left[\frac{1}{6}(A^2+cA+dI)\right]$, the value of c and d are
(a) (–6, –1) (b) (6, 11)
(c) (–6, 11) (d) (6, –11)
[IIT-JEE, 2005]

50. If $P=\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$ and $A=\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

and $Q = PAP^T$ and $x = P^T Q^{2005} P$, then x is equal to

(a) $\begin{pmatrix} 1 & 2005 \\ 0 & 1 \end{pmatrix}$

(b) $\begin{pmatrix} 4+2005\sqrt{3} & 6015 \\ 2005 & 4-2005\sqrt{3} \end{pmatrix}$

(c) $\frac{1}{4}\begin{pmatrix} 2+\sqrt{3} & 1 \\ -1 & 2-\sqrt{3} \end{pmatrix}$

(d) $\frac{1}{4}\begin{pmatrix} 2005 & 2-\sqrt{3} \\ 2+\sqrt{3} & 2005 \end{pmatrix}$
[IIT-JEE, 2005]

51. Let $A=\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$ and U_1, U_2 and U_3 are columns of a 3 × 3 matrix U. If columns matrices U_1, U_2 and U_3 satisfying

$$AU_1=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, AU_2=\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, AU_3=\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$$

evaluated as directed in the following questions.
(i) The value of $|U|$ is
(a) 3 (b) –3 (c) 3/2 (d) 2
(ii) The sum of the elements of the matrix U^{-1} is
(a) –1 (b) 0 (c) 1 (d) 3
(iii) The value of $[3 \quad 2 \quad 0]\,U\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$ is
(a) 5 (b) 5/2 (c) 4 (d) 3/2
[IIT-JEE, 2006]

52. *No questions asked in IIT-JEE, 2007.*

53. Consider the system of equations
$x - 2y + 3z = -1$, $-x + y - 2z = k$, $x - 3y + 4z = 1$
Assertion: The system of equation has no solution for $k \neq 3$
Reason: The determinant

$$\begin{vmatrix} 1 & 3 & -1 \\ -1 & -2 & k \\ 1 & 4 & 1 \end{vmatrix} \neq 0 \text{ for } k \neq 3.$$

[IIT-JEE, 2008]

54. Let A be the set of all 3 × 3 symmetric matrices all of whose entries are either 0 or 1. Five of these entries are 1 and four of them are zeroes.
(i) The number of matrices in A is
(a) 12 (b) 6 (c) 9 (d) 3
(ii) The number of matrices A to A for which the system of linear equations $A\begin{bmatrix} x \\ y \\ x \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$,

A has a unique solution, is

(a) less than 4.
(b) at least 4 but less than 7.
(c) at least 7 but less than 10.
(d) at least 10.

(iii) The number of matrices A in A for which the system of linear equation

$$A\begin{bmatrix} x \\ y \\ x \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

is inconsistent is

(a) 0 (b) more than 2
(c) 2 (d) 1

[IIT-JEE, 2009]

55. Let k be a positive real number and let

$$A = \begin{pmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{pmatrix}$$

and $$B = \begin{pmatrix} 0 & 2k-1 & \sqrt{k} \\ 1-2k & 0 & 2\sqrt{k} \\ -\sqrt{k} & -2\sqrt{k} & 0 \end{pmatrix}.$$

If det(adj A) + det(adj B) = 10^6, then $[k]$ is

[IIT-JEE, 2009]

56. The number of 3×3 matrices A whose entries are either 0 or 1 and for which the system $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has exactly two solutions, is

(a) 0 (b) $2^9 - 1$ (c) 168 (d) 12

[IIT-JEE, 2010]

57. Let p be an odd prime number and let T_p be the following set of 2×2 matrices:

$$T_p = \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c \in \{0, 1, 2, (p-1)\} \right\}$$

(i) The number of A in T_p such that A is either symmetric or skew-symmetric or both, and det (A) is divisible by p, is

(a) $(p-1)^2$ (b) $2(p-1)$
(c) $(p-1)^2 + 1$ (d) $2p - 1$

(ii) The number of A in T_p such that the trace of A is not divisible by p but det (A) is divisible by p, is

(a) $(p-1)(p^2 - p + 1)$ (b) $p^3 - (p-1)^2$
(c) $(p-1)^2$ (d) $(p+1)(p-1)^2$

(iii) The number of A in T_p such that det (A) is not divisible by p, is

(a) $2p^2$ (b) $p^3 - 5p$
(c) $p^3 - 3p$ (d) $p^3 - p^2$

[IIT-JEE, 2010]

58. Let M and N be two 3×3 non-singular skew-symmetric matrices such that $MN = NM$. If P^T denotes the transpose of P, then $M^2N^2(M^TN)^{-1}(MN^{-1})^T$ is

(a) M^2 (b) $-N^2$ (c) $-M^2$ (d) MN

[IIT-JEE, 2011]

59. Let M be a square matrix of order 3 such that

$$M\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}, M\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \text{ and } M\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 12 \end{pmatrix}$$

Then the sum of the diagonal entries of M is

[IIT-JEE, 2011]

60. If P be a matrix of order 3 such that $P^T = 2P + I$, where P^T is the transpose of P and I the identity matrix of order 3, there exists a column matrix X such that

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

(a) $PX = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ (b) $PX = X$
(c) $PX = 2X$ (d) $PX = -X$

[IIT-JEE, 2012]

61. If the adjoint of a matrix P of order 3 is

$$\begin{pmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{pmatrix},$$

the possible values of the determinant of P is(are)

(a) –2 (b) –1 (c) 1 (d) 2

[IIT-JEE, 2012]

62. Let $P = [a_{ij}]$ be a 3×3 matrix and let $Q = [b_{ij}]$, where $b_{ij} = 2^{i+j}a_{ij}$ for $1 \le i, j \le 3$.

If the determinant of P is 2, the determinant of the matrix Q is

(a) 2^{10} (b) 2^{11} (c) 2^{12} (d) 2^{13}

[IIT-JEE, 2012]

63. For 3×3 matrices M and N, which of the following statement(s) is (are) NOT correct?

(a) N^TMN is symmetric or skew-symmetric, according as M is symmetric or skew-symmetric.
(b) $MN - NM$ is skew-symmetric for all symmetric matrices M and N.
(c) MN is symmetric for all symmetric matrices M and N.
(d) (adj M) (adj N) = adj(MN) for all invertible matrices M and N.

[IIT-JEE, 2013]

64. Let ω be a complex cube root of unity with $\omega \neq 1$ and $P = [P_{ij}]$ be a $n \times n$ matrix with $p_{ij} = \omega^{i+j}$. Then $P^2 \neq 0$, when n =

(a) 57 (b) 55 (c) 58 (d) 56

[IIT-JEE, 2013]

65. Let M be a 2×2 symmetric matrix with integer entries. Then M is invertible if
   (a) the first column of M is the transpose of the second row of M
   (b) the second row of M is the transpose of the first column of M
   (c) M is a diagonal matrix with non-zero entries in the main diagonal
   (d) the product of entries in the main diagonal of M is not the square of an integer.
   [IIT-JEE, 2014]
66. Let M and N be two 3×3 matrices such that $M \neq N^2$ and $M^2 = N^4$, then
   (a) determinant of $(M^2 + MN^2)$ is 0
   (b) there is a 3×3 non-zero matrix U such that $(M^2 + MN^2)U$ is the zero matrix
   (c) determinant of $(M^2 + MN^2) \geq 1$
   (d) for a 3×3 matrix U, if $(M^2 + MN^2)U$ equals the zero matrix then U is the zero matrix.
   [IIT-JEE, 2014]
67. Let X and Y be two arbitrary, 3×3, non-zero, skew-symmetric matrices and Z be an arbitrary 3×3, non-zero, symmetric matrix.
   Which of the following matrices is (are) skew-symmetric?
   (a) $Y^3Z^4 - Z^4Y^3$ (b) $X^{44} + Y^{44}$
   (c) $X^4Z^3 - Z^3X^4$ (d) $X^{23} + Y^{23}$
   [IIT-JEE, 2015]
68. Which of the following values of α satisfy the equation
$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix} = -648\alpha?$$
   (a) –4 (b) 9 (c) –9 (d) 4
   [IIT-JEE, 2015]

Answers

Level II

1. (d)	2. (d)	3. (c)	4. (b)	5. (b)
6. (b)	7. (d)	8. (a)	9. (c)	10. (d)
11. (b)	12. (d)	13. (c)	14. (d)	15. (b)
16. (a)	17. (c)	18. (c)	19. (d)	20. (c)
21. (b)	22. (a)	23. (b)	24. (c)	25. (d)
26. (c)	27. (d)	28. (b)	29. (a)	30. (a)
31. (b)	32. (d)	33. (a)	34. (b)	35. (c)
36. (b)	37. (d)	38. (c)	39. (a)	40. (b)
41. (d)	42. (d)	43. (c)	44. (c)	45. (c)
46. (b)	47. (a)	48. (a)	49. (a)	50. (d)
51. (a)	52. (a)	53. (c)	54. (a)	55. (d)
56. (b)	57. (c)	58. (a)	59. (d)	60. (c)
61. (d)	62. (a)	63. (c)	64. (d)	65. (a)

Integer Type Questions

1. 2	2. 3	3. 1	4. 2	5. 5
6. 6	7. 3	8. 8	9. 5	10. 2
11. 2	12. 3	13 7	14. 2	15. 3

Comprehensive Link Passages

Passgage I:	1. (d)	2. (b)	3. (a)
Passgage II:	1. (a)	2. (b)	3. (b)
Passgage III:	1. (c)	2. (a)	3. (b)
Passgage IV:	1. (d)	2. (c)	3. (d)

Match Matrices

1. (A) → (T); (B) → (T); (C) → (Q); (D) → (S)
2. (A) → (Q); (B) → (R); (C) → (P); (D) → (P)
3. (A) → (Q); (B) → (R); (C) → (S); (D) → (P)
4. (A) → (R); (B) → (Q); (C) → (P); (D) → (S)

Assertion and Reason

1. (b)	2. (b)	3. (a)	4. (d)
5. (c)	6. (d)	7. (b)	8. (a)

Hints and Solutions

Level I

1. Each element of the given matrices of order 2×2 can be filled in 2 ways, i.e. either 1 or 0.
   Thus, the number of possible matrices of order 2×2
   $= 2 \times 2 \times 2 \times 2 = 16$
2. Each element of the given matrices of order 3×3 can be filled in 2 ways, i.e. either 1 or 0.
   Thus, the number of possible matrices of order 3×3 is
   $= 2 \times 2 \times 2 \times \ldots$ up to 9 times
   $= 2^9 = 512$
3. As we know that $A + (-A) = \mathbf{O}$, then $-A$ is the additive inverse of A.
   Therefore, the additive inverse of A,
$$-A = \begin{pmatrix} -2 & -4 \\ -3 & -5 \end{pmatrix}$$

4. We have,

$$X + \begin{pmatrix} 2 & 5 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 2 & 7 \end{pmatrix}$$

$$\Rightarrow \quad X = \begin{pmatrix} 3 & 6 \\ 2 & 7 \end{pmatrix} - \begin{pmatrix} 2 & 5 \\ 3 & -2 \end{pmatrix}$$

$$\Rightarrow \quad X = \begin{pmatrix} 1 & 1 \\ -1 & 9 \end{pmatrix}$$

5. Clearly, $X = \frac{1}{2}\left(\begin{pmatrix} 2 & 5 \\ 3 & -2 \end{pmatrix} + \begin{pmatrix} 4 & 2 \\ 8 & -2 \end{pmatrix}\right)$

$$\Rightarrow \quad X = \frac{1}{2}\begin{pmatrix} 6 & 7 \\ 11 & -4 \end{pmatrix} = \begin{pmatrix} 3 & 7/2 \\ 11/2 & -2 \end{pmatrix}$$

and $Y = \frac{1}{2}\left(\begin{pmatrix} 2 & 5 \\ 3 & -2 \end{pmatrix} - \begin{pmatrix} 4 & 2 \\ 8 & -2 \end{pmatrix}\right)$

$$\Rightarrow \quad Y = \frac{1}{2}\begin{pmatrix} -2 & 3 \\ -5 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 3/2 \\ -5/2 & 0 \end{pmatrix}$$

6. Given $A + 2B + X = \mathbf{O}$

$\Rightarrow \quad X = -(A + 2B)$

$$= -\left(\begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix} + 2\begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}\right)$$

$$= -\left(\begin{pmatrix} 2 & -1 \\ 3 & 5 \end{pmatrix} + \begin{pmatrix} -2 & 2 \\ 0 & 4 \end{pmatrix}\right)$$

$$= -\begin{pmatrix} 0 & 1 \\ 3 & 9 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -3 & -9 \end{pmatrix}$$

7. Given $\begin{pmatrix} |x| & 2 \\ 5 & |y-2| \end{pmatrix} = \begin{pmatrix} < 3 & 2 \\ 5 & < 4 \end{pmatrix}$

$\Rightarrow \quad |x| < 2, |y - 2| < 3$

$\Rightarrow \quad -2 < x < 2, -3 < (y - 2) < 3$

$\Rightarrow \quad -2 < x < 2, -1 < y < 5$

8. We have

$$\begin{pmatrix} x^3 - 3x + 2 & 2 \\ 3 & y^3 + 7y^2 - 35 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix}$$

$x^3 - 3x + 2 = 0, \quad y^3 + 7y^2 - 35 = 1$

$x^3 - 3x + 2 = 0, \quad y^3 + 7y^2 - 36 = 1$

Now, $x^3 - 3x + 2 = 0$

$\Rightarrow \quad x^3 - x^2 + x^2 - x - 2x + 2 = 1$

$\Rightarrow \quad x^2(x - 1) + x(x - 1) - 2(x - 1) = 0$

$\Rightarrow \quad (x - 1)(x^2 + x - 2) = 0$

$\Rightarrow \quad (x - 1)(x + 2)(x - 1) = 0$

$\Rightarrow \quad (x - 1)^2(x + 2) = 0$

$\Rightarrow \quad x = 1, -2$

Also, $y^3 + 7y^2 - 36 = 0$

$\Rightarrow \quad y^3 - 2y^2 + 9y^2 - 18y + 18y - 36 = 1$

$\Rightarrow \quad y^2(y - 2) + 9y(y - 2) + 18(y - 2) = 0$

$\Rightarrow \quad (y - 2)(y^2 + 9y + 18) = 0$

$\Rightarrow \quad (y - 2)(y + 3)(y + 6) = 0$

$\Rightarrow \quad y = 2, -3, -6$

Thus,

$$\Sigma(x + y) = (1 - 2 + 2 - 3 - 6) = -8$$

9. We have

$$2\begin{pmatrix} x & y \\ z & t \end{pmatrix} + 3\begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix} = 4\begin{pmatrix} 3 & 5 \\ 4 & 6 \end{pmatrix}$$

$$\Rightarrow \quad 2\begin{pmatrix} x & y \\ z & t \end{pmatrix} = 4\begin{pmatrix} 3 & 5 \\ 4 & 6 \end{pmatrix} - 3\begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix}$$

$$\Rightarrow \quad \begin{pmatrix} 2x & 2y \\ 2z & 2t \end{pmatrix} = \begin{pmatrix} 12 & 20 \\ 16 & 24 \end{pmatrix} - \begin{pmatrix} 3 & -6 \\ 0 & 12 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 26 \\ 16 & 12 \end{pmatrix}$$

$\Rightarrow \quad x = 9/2, y = 13, z = 8, t = 6$

10. Let $A = \begin{pmatrix} 2 & 3 \\ 4 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 2 \\ 1 & -5 \end{pmatrix}$

Solving, we get

$$X = \frac{1}{5}(3B - 2A) \text{ and } Y = \frac{1}{5}(3A - 2B)$$

Thus, $X = \frac{1}{5}\left(\begin{pmatrix} -3 & 6 \\ 3 & -15 \end{pmatrix} - \begin{pmatrix} 4 & 6 \\ 8 & 0 \end{pmatrix}\right)$

$$\Rightarrow \quad X = \frac{1}{5}\begin{pmatrix} -7 & 0 \\ -5 & -15 \end{pmatrix} = \begin{pmatrix} -7/5 & 0 \\ -1 & -3 \end{pmatrix}$$

and $Y = \frac{1}{5}\left(\begin{pmatrix} 6 & 9 \\ 12 & 0 \end{pmatrix} - \begin{pmatrix} -2 & 4 \\ 2 & -10 \end{pmatrix}\right)$

$$\Rightarrow \quad Y = \frac{1}{5}\begin{pmatrix} 8 & 5 \\ 10 & 10 \end{pmatrix} = \begin{pmatrix} 8/5 & 1 \\ 2 & 2 \end{pmatrix}$$

11. Given $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

$$= 2\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 2I$$

We have,

$$f(x) = 1 + x + x^2 + \ldots \text{ to } \infty$$

$$= \frac{1}{1 - x}$$

$$\Rightarrow \quad f(A) = \frac{\mathrm{I}}{I - A}$$

$$= \frac{I}{I - 2I}$$

$$= -\frac{I}{I}$$

$$= -\frac{I^2}{I}$$

$$= -I$$

12. Then $AB = [1 \times 2 + 2 \times 3] = [8]$

13. Then $AB = [a \times a^2 + b \times b^2 + c \times c^2]$
$= [a^3 + b^3 + c^3]$

14. Then $AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \times \begin{pmatrix} 2 & 4 \\ 5 & 7 \end{pmatrix}$

$= \begin{pmatrix} 1.2 + 2.5 & 1.4 + 2.7 \\ 3.2 + 4.5 & 3.4 + 4.7 \end{pmatrix}$

$= \begin{pmatrix} 12 & 18 \\ 26 & 40 \end{pmatrix}$

Also,

$$\begin{pmatrix} 2 & 4 \\ 5 & 7 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2.1 + 4.3 & 2.2 + 4.7 \\ 5.1 + 7.3 & 5.2 + 7.4 \end{pmatrix}$$

$$= \begin{pmatrix} 14 & 32 \\ 26 & 38 \end{pmatrix}$$

Thus, $AB \neq BA$

Clearly, the matrix multiplication is not commutative.

15. Given A is a 2×3 matrix and AB is a 2×5 matrix. Thus, B is a matrix of 3×5.

16. Clearly, the matrix multiplication is not defined.

17. Given $A = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$

Now, $A^2 = A.A$

$= \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$

Also

$$A^2 - 2A + I_2 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} - 2\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} + \begin{pmatrix} -2 & 0 \\ 2 & -2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= 0$$

Hence, the result

18. We have,

$$AB = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} (2 \quad 3 \quad 4)$$

$$= \begin{pmatrix} 2 & 3 & 4 \\ 4 & 6 & 8 \\ 6 & 9 & 12 \end{pmatrix}$$

Also,

$$BA = (2 \quad 3 \quad 4) \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

$$= (2 + 6 + 12) = (20)$$

19. We have,

$$A^2 = A.A = \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 + 12 & 6 + 3 \\ 8 + 4 & 12 + 1 \end{pmatrix} = \begin{pmatrix} 16 & 9 \\ 12 & 13 \end{pmatrix}$$

Now, $A^3 = A^2.A$

$$= \begin{pmatrix} 16 & 9 \\ 12 & 13 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 32 + 36 & 48 + 9 \\ 24 + 52 & 36 + 39 \end{pmatrix} = \begin{pmatrix} 68 & 57 \\ 76 & 69 \end{pmatrix}$$

Also, $A^4 = A^3.A$

$$= \begin{pmatrix} 68 & 57 \\ 76 & 69 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 136 + 228 & 204 + 57 \\ 152 + 276 & 228 + 69 \end{pmatrix} = \begin{pmatrix} 364 & 261 \\ 428 & 297 \end{pmatrix}$$

20. Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

So, $\begin{pmatrix} 2 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 0 & 6 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 2a + c & 2b + d \\ a + 4c & b + 4d \end{pmatrix} = \begin{pmatrix} 3 & 5 \\ 0 & 6 \end{pmatrix}$

$2a + c = 3 \quad 2b + d = 5$

$\Rightarrow a + 4c = 0 \quad b + 4d = 6$

$\Rightarrow -8c + c = 3, 2(6 - 4d) + d = 5$

$\Rightarrow c = -\frac{3}{7}, -7d = 5 - 12 = -7$

$\Rightarrow c = -\frac{3}{7}, d = 1$

Therefore, $a = \frac{12}{7}, b = 2$

Thus, the matrix $X = \begin{pmatrix} 12/7 & 2 \\ -3/7 & 1 \end{pmatrix}$.

21. We have,

$$A^2 = A.A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix}$$

Now, $A^2 - 4A - 5I_3$

$$= \begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix} - 4\begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{pmatrix} - \begin{pmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{pmatrix} - \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

22. We have

$$(1 \quad x \quad 1)\begin{pmatrix} 1 & 3 & 2 \\ 2 & 5 & 1 \\ 15 & 3 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ x \end{pmatrix} = 0$$

$$\Rightarrow \quad (1 \quad x \quad 1)\begin{pmatrix} 1+6+2x \\ 2+10+x \\ 15+6+2x \end{pmatrix} = 0$$

$$\Rightarrow \quad (1 \quad x \quad 1)\begin{pmatrix} 7+2x \\ 12+x \\ 21+2x \end{pmatrix} = 0$$

$\Rightarrow \quad (7 + 2x) + x(12 + x) + (21 + x) = 0$

$\Rightarrow \quad (7 + 2x) + 12x + x^2 + (21 + x) = 0$

$\Rightarrow \quad x^2 + 15x + 28 = 0$

$$\Rightarrow \quad x = \frac{-15 \pm \sqrt{225-112}}{2} = \frac{-15 \pm \sqrt{113}}{2}$$

Hence, the solution set is

$$\left\{\frac{-15+\sqrt{113}}{2}, \frac{-15+\sqrt{113}}{2}\right\}.$$

23. We have

$$A^2 = A \cdot A = \begin{pmatrix} a & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a^2 & 0 \\ a+1 & 1 \end{pmatrix}$$

Given relation is

$A^2 = B$

$$\Rightarrow \quad \begin{pmatrix} a^2 & 0 \\ a+1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$$

$\Rightarrow \quad a^2 = 1, a + 1 = 5$

$\Rightarrow \quad a = \pm 1, a = 4$

24. We have $A^2 = A.A$

$$= \begin{pmatrix} \alpha & 2 \\ 2 & \alpha \end{pmatrix} \cdot \begin{pmatrix} \alpha & 2 \\ 2 & \alpha \end{pmatrix}$$

$$= \begin{pmatrix} \alpha^2+4 & 4\alpha \\ 4\alpha & \alpha^2+4 \end{pmatrix}$$

Now, $A^3 = A^2.A$

$$= \begin{pmatrix} \alpha^2+4 & 4\alpha \\ 4\alpha & \alpha^2+4 \end{pmatrix} \cdot \begin{pmatrix} \alpha & 2 \\ 2 & \alpha \end{pmatrix}$$

$$= \begin{pmatrix} \alpha^3+12\alpha & 6\alpha^2+8 \\ 6\alpha^2+8 & \alpha^3+12\alpha \end{pmatrix}$$

Given $|A^3| = 125$

$$\Rightarrow \quad \begin{vmatrix} \alpha^3+12\alpha & 6\alpha^2+8 \\ 6\alpha^2+8 & \alpha^3+12\alpha \end{vmatrix} = 125$$

$\Rightarrow \quad (\alpha^3 + 12\alpha)^2 - (6\alpha^2 + 8)^2 = 125$

$\Rightarrow \quad (\alpha^3 + 12\alpha + 6\alpha^2 + 8)(\alpha^3 + 12\alpha - 6\alpha^2 - 8) = 125$

$\Rightarrow \quad (\alpha^3 + 6\alpha^2 + 12\alpha + 8)(\alpha^3 + 6\alpha^2 + 12\alpha - 8) = 125$

$\Rightarrow \quad (\alpha + 2)^3(\alpha - 2)^3 = 125$

$\Rightarrow \quad \{(\alpha + 2)(\alpha - 2)\}^3 = (5)^3$

$\Rightarrow \quad (\alpha + 2)(\alpha - 2) = 5$

$\Rightarrow \quad \alpha^2 - 4 = 5$

$\Rightarrow \quad \alpha^2 = 9$

$\Rightarrow \quad \alpha = \pm 3$

25. Given $AB = A$ and $BA = B$

Now $A = AB$

$\Rightarrow \quad A^2 = ABA = AB = A$

Similarly $B^2 = B$

We have,

$(A + B)^2 = (A^2 + AB + BA + B^2)$

$= (A + A + B + B)$

$= 2(A + B)$

$\Rightarrow \quad (A + B)^4 = 4(A + B)^2 = 8(A + B)$

and $(A + B)^3 = (A + B)^2.(A + B)$

$= 2(A + B).(A + B) = 2(A + B)^2$

$= 4(A + B)$

Thus,

$(A + B)^7 = (A + B)^4.(A + B)^3$

$= 32(A + B)^2 = 64(A + B)$

26. Given $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Now, $AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$= \begin{pmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{pmatrix}$$

Also, $BA = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$= \begin{pmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{pmatrix}$

It is given that

$AB = BA$

$\Rightarrow \begin{pmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{pmatrix} = \begin{pmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{pmatrix}$

$a+2c = a+3b \quad \Rightarrow \quad 2C = 3b$

$b+2d = 2a+4b$

$3a+4c = c+3d$

$\Rightarrow \quad 3b+4d = 2c+4d$

$2c = 3b$

$\Rightarrow \quad 2a-2d = -3b$

Now, $\dfrac{a-d}{3b-c} = \dfrac{-\frac{3}{2}b}{2c-c} = -\dfrac{\frac{3}{2}b}{c} = -\dfrac{\frac{3}{2}b}{\frac{3}{2}b} = -1$

27. Given $A^2 = \mathbf{O}$

$A^2 = \mathbf{O} = A^3 = A^4 = \ldots A^{2009}$

Now, $A(I+A)^{2009}$

$= A\left({}^{2009}C_0 \cdot I^n + {}^{2009}C_1 \cdot I^{n-1} \cdot A + {}^{2009}C_2 \cdot I^{n-2} \cdot A^2 + \ldots + {}^{2009}C_{2009} \cdot I^{n-n} A^{2008}\right)$

$= A(I + 2009A + \mathbf{O} + \mathbf{O} + \ldots + \mathbf{O})$

$= A(I + 2009A)$

$= A.I + 2009\,A^2$

$= A + \mathbf{O}$

$= A$

28. Given $A^2 = I$.

Now,

$(I-A)(I+A) = I^2 + IA - AI - A^2$

$= I + A - A - A^2$

$= I + A - A - I$

$= \mathbf{O}$

29. We have,

$(I+A)^3 - 7A = (I^3 + 3I^2A + 3IA^2 + A^3) - 7A$

$= (I + 3A + 3A^2 + A^3) - 7A$

$= (I + 3A + 3A + A) - 7A$

$= I + 7A - 7A$

$= I$

30. Here, $A^2 = A.A$.

$= \begin{pmatrix} 0 & 5 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 5 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathrm{O}$

Thus, $A^2 = \mathbf{O} = A^3 = A^4 = \ldots = A^{16}$.

We have $f(x) = \sum_{n=0}^{16} x^n$

$= 1 + x + x^2 + x^3 + \ldots + x^{16}$

Now, $f(A) = I + A + A^2 + A^3 + \ldots + A^{16}$

$= I + A + \mathbf{O} + \mathbf{O} + \ldots + \mathbf{O}$

$= I + A$

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 5 \\ 0 & 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$

31. Given,

$A^2 = A + I$

$\Rightarrow \quad A^3 = A^2 + A = A + I + A = 2A + I$

$\Rightarrow \quad A^4 = 2A^2 + A = 2(A + I) + A = 3A + 2I$

$\Rightarrow \quad A^5 = 3A^2 + 2A = 3(A + I) + 2A = 5A + 3I$

32. Given,

$A^2 = 2A - I$

$\Rightarrow \quad A^3 = 2A^2 - A = 2(2A - I) - A = 3A - 2I$

$\Rightarrow \quad A^4 = 3A^2 - 2A = 3(2A - I) - 2A = 4A - 3I$

$\Rightarrow \quad A^5 = 4A^2 - 3A = 4(2A - I) - 3A = 5A - 4I$

Thus, by symmetry, we can say that

$A^n = nA - (n-1)I$

33. Given,

$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$\Rightarrow \quad A^2 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = 2\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = 2A$

$\Rightarrow \quad A^3 = 2A^2 = 2(2A) = 4A = 2^2A$

$\Rightarrow \quad A^4 = 2^2A^2 = 2^2(2A) = 2^2A$

Thus, by symmetry, we can say that

$A^n = 2^{n-1}A$

34. Let $U_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $U_2 = \begin{pmatrix} d \\ e \\ f \end{pmatrix}$

Given $AU_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow \quad \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow \quad \begin{pmatrix} a \\ 2a+b \\ 3a+2b+c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$\Rightarrow \quad U_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

Similarly, we can easily find that

$U_2 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$

Thus, $U_1 + U_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

37. Let $A = \begin{pmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{pmatrix}$

Then $|A| = \begin{vmatrix} 0 & 2 & 3 \\ -2 & 0 & 4 \\ -3 & -4 & 0 \end{vmatrix}$

$= \begin{vmatrix} 0 & 4 \\ -4 & 0 \end{vmatrix} - 2\begin{vmatrix} -2 & 4 \\ -3 & 0 \end{vmatrix} + 3\begin{vmatrix} -2 & 0 \\ -3 & -4 \end{vmatrix}$

$= 0 - 2(0 + 12) + 3(8 - 0)$

$= -24 + 24 = 0$

Thus, the determinant of skew-symmetric matrix of odd order is zero.

38. Let $A = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$

Then $|A| = \begin{vmatrix} 0 & b \\ -b & 0 \end{vmatrix} = b^2$

Thus, the determinant of skew-symmetric matrix of even order is a perfect square.

39. Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Then $|A| = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1.$

$= (2 - 1)(-1)^{2-1}$

Again, let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$= \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

Thus, $|A| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$

$= 0 - \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 + 1 = 2$

$= (3-1)(-1)^{3-1}$

Therefore, in general

$|A| = \begin{vmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 0 \end{vmatrix}$

$= (n-1)(-1)^{n-1}$

40. We have,

$(A^{2016} + 2A^{2015}) = A^{2015}(A + 2I_2)$

$= A^{2015}\left[\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right]$

$= A^{2015}\begin{pmatrix} 4 & 5 \\ 1 & 5 \end{pmatrix}$

Thus,

$|A^{2016} + 2A^{2015}| = \left|A^{2015}\begin{pmatrix} 4 & 5 \\ 1 & 5 \end{pmatrix}\right|$

$= |A^{2015}|\begin{vmatrix} 4 & 5 \\ 1 & 5 \end{vmatrix}$

$= |A|^{2015} \times \begin{vmatrix} 4 & 5 \\ 1 & 5 \end{vmatrix}$

$= (1)^{2015} \times (20 - 5)$

$= 15$

41. Given,

$A = A^2$

$\Rightarrow \quad |A| = |A^2|$

$\Rightarrow \quad |A| = |A|^2$

$\Rightarrow \quad |A|(|A| - 1) = 0$

$\Rightarrow \quad |A| = 0$ or $|A| = 1$

42. We have,

$\det(3A) = 3^3 \times \det(3A)$

$= 3^3 \times 8 = 216$

43. We have,

$|A^n| = |A|^n = 2^n$

44. We have,

$\begin{vmatrix} \cos^2\theta & \sin^2\theta \\ \sin^2\theta & \cos^2\theta \end{vmatrix} = \cos^4\theta - \sin^2\theta$

$= (\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta)$

$= (\cos^2\theta - \sin^2\theta)$

$= \cos(2\theta)$

Thus, the maximum value is 1.

45. Given $P^3 = Q^3$ and $P^2Q = Q^2P$

We have,

$P^3 - P^2Q = Q^3 - Q^2P$

$\Rightarrow \quad P^2(P - Q) = Q^2(Q - P)$

$\Rightarrow \quad P^2(P - Q) = -Q^2(P - Q)$

$\Rightarrow \quad (P^2 + Q^2)(P - Q) = \mathbf{O}$

$\Rightarrow \quad (P^2 + Q^2) = \mathbf{O}$, since $(P - Q) \neq \mathbf{O}$

Thus, $\det(P^2 + Q^2) = \det(\mathbf{O}) = 0$

46. We have,

$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = a\begin{vmatrix} c & a \\ a & b \end{vmatrix} - b\begin{vmatrix} b & a \\ c & b \end{vmatrix} + c\begin{vmatrix} b & c \\ c & a \end{vmatrix}$

$= a(bc - a^2) - b(b^2 - ac) + c(ab - c^2)$

$= abc - a^3 - b^3 + abc + abc - c^3$

$= 3abc - a^3 - b^3 - c^3$

$= -(a^3 + b^3 + c^3 - 3abc)$

47 The given determinant $= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$

$$= \begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix} \quad \begin{pmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$$

$$= \begin{vmatrix} b-a & b^2-a^2 \\ c-a & c^2-a^2 \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$$

$= (b-a)(c-a)(c+a-b-a)$
$= (b-a)(c-a)(c-b)$
$= (a-b)(b-c)(c-a)$

48. The given determinant $= \begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

$$= \begin{vmatrix} 1 & a & a+b+c \\ 1 & b & b+c+a \\ 1 & c & c+a+b \end{vmatrix} \quad (C_3 \to C_2 + C_3)$$

$$= (a+b+c)\begin{vmatrix} 1 & a & 1 \\ 1 & b & 1 \\ 1 & c & 1 \end{vmatrix}$$

$= (a+b+c) \times 0$
$= 0$

49. The given determinant is

$$\begin{vmatrix} \sin\alpha & \cos\beta & \cos(\alpha+\theta) \\ \sin\beta & \cos\beta & \cos(\beta+\theta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\theta) \end{vmatrix}$$

$$= \begin{vmatrix} \sin\alpha & \cos\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ \sin\gamma & \cos\gamma & 0 \end{vmatrix}$$

$[C_3 \to C_3 - (C_1 \cos\theta + C_2 \sin\theta)]$

$= 0$

50. The given determinant $= \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(a+c) \\ 1 & ab & c(a+b) \end{vmatrix}$

$$= \begin{vmatrix} 1 & bc & bc+ab+ac \\ 1 & ca & ac+ab+bc \\ 1 & ab & ab+ac+bc \end{vmatrix} \quad (C_3 \to C_2 + C_3)$$

$$= (ab+bc+ca) \times \begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix}$$

$= (ab+bc+ca) \times 0$
$= 0$

51. The given determinant

$$= \begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix}$$

$$= \begin{vmatrix} 2(a+b+c) & a & b \\ 2(a+b+c) & b+c+2a & b \\ 2(a+b+c) & a & c+a+2b \end{vmatrix}$$

$(C_1 \to C_1 + C_2 + C_3)$

$$= 2(a+b+c) \times \begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

$$= 2(a+b+c) \times \begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 0 & 0 & c+a+b \end{vmatrix}$$

$\begin{pmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$

$$= 2(a+b+c)^3 \times \begin{vmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$= 2(a+b+c)^3 \times 1$
$= 2(a+b+c)^3$

52. The given determinant

$$= \begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

$$= \begin{vmatrix} 2(b+c) & 2(c+a) & 2(a+b) \\ b & c+a & b \\ c & c & a+\mathrm{b} \end{vmatrix}$$

$(R_1 \to R_1 + R_2 + R_3)$

$$= 2\begin{vmatrix} (b+c) & (c+a) & (a+b) \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$$

$$= 2\begin{vmatrix} b+c & (c+a) & (a+b) \\ -c & 0 & -a \\ -b & -a & 0 \end{vmatrix}$$

$\begin{pmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$

$$= 2\begin{vmatrix} 0 & \mathrm{c} & \mathrm{b} \\ -c & 0 & -a \\ -b & -a & 0 \end{vmatrix} \quad (R_1 \to R_1 + R_2 + R_3)$$

$= 2[-c(-ab-0) + b(ca-0)]$

$= 2(abc + abc)$
$= 4abc$

53. The given determinant

$$= \begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$$

$$= \begin{vmatrix} 2(b^2+c^2) & 2(c^2+a^2) & 2(a^2+b^2) \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$$

$(R_1 \to R_1 + R_2 + R_3)$

$$= 2\begin{vmatrix} (b^2+c^2) & (c^2+a^2) & (a^2+b^2) \\ b^2 & c^2+a^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$$

$$= 2\begin{vmatrix} b^2+c^2 & (c^2+a^2) & (a^2+b^2) \\ -c^2 & 0 & -a^2 \\ -b^2 & -a^2 & 0 \end{vmatrix}$$

$\begin{pmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$

$$= 2\begin{vmatrix} 0 & c^2 & b^2 \\ -c^2 & 0 & -a^2 \\ -b^2 & -a^2 & 0 \end{vmatrix} \quad (R_1 \to R_1 + R_2 + R_3)$$

$= 2(-c^2(-a^2b^2 - 0) + b^2(c^2a^2 - 0))$
$= 2(a^2b^2c^2 + a^2b^2c^2)$
$= 4a^2b^2c^2$

54. The given determinant

$$= \begin{vmatrix} a & a+b & a+b+c \\ 2a & 3a+2b & 4a+3b+2c \\ 3a & 6a+3b & 10a+6b+3c \end{vmatrix}$$

$$= \begin{vmatrix} a & a+b & a+b+c \\ 0 & a & 2a+b \\ 0 & 3a & 7a+3b \end{vmatrix} \quad \begin{pmatrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{pmatrix}$$

$$= a\begin{vmatrix} a & 2a+b \\ 3a & 7a+3b \end{vmatrix}$$

$= a(7a^2 + 3ab - 6a^2 - 3ab)$
$= a(a^2) = a^3$

55. The given determinant

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1+a^2-b^2+2b^2 & 2ab-2ab & -2b \\ 2ab-2ab & 1-a^2+b^2+2a^2 & 2a \\ 2b-b(1-a^2-b^2) & -2a+a(1-a^2-b^2) & 1-a^2-b^2 \end{vmatrix}$$

$\begin{pmatrix} C_1 \to C_1 - bC_3 \\ C_2 \to C_2 + aC_3 \end{pmatrix}$

$$= \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b(1+a^2+b^2) & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix}$$

$$= (1+a^2+b^2)^2\begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$$

$$= (1+a^2+b^2)^2\begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ 0 & -a & 1-a^2+b^2 \end{vmatrix}$$

$(R_3 \to R_3 - bR_1)$

$$(1+a^2+b^2)^2\begin{vmatrix} 1 & 2a \\ -a & 1-a^2+b^2 \end{vmatrix}$$

$= (1 + a^2 + b^2)^2\,(1 - a^2 + b^2 + 2a^2)$
$= (1 + a^2 + b^2)^3$

56. The given determinant

$$= \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & cb & c^2+1 \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} a(a^2+1) & a^2b & a^2c \\ ab^2 & b(b^2+1) & b^2c \\ ac^2 & c^2b & c(c^2+1) \end{vmatrix}$$

$$= \frac{abc}{abc}\begin{vmatrix} (a^2+1) & a^2 & a^2 \\ b^2 & (b^2+1) & b^2 \\ c^2 & c^2 & (c^2+1) \end{vmatrix}$$

$$= \begin{vmatrix} (1+a^2+b^2+c^2) & (1+a^2+b^2+c^2) & (1+a^2+b^2+c^2) \\ b^2 & (b^2+1) & b^2 \\ c^2 & c^2 & (c^2+1) \end{vmatrix}$$

$(R_1 \to R_1 + R_2 + R_3)$

$$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & 1 & 1 \\ b^2 & (b^2+1) & b^2 \\ c^2 & c^2 & (c^2+1) \end{vmatrix}$$

$$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & 0 & 0 \\ b^2 & 1 & 0 \\ c^2 & 0 & 1 \end{vmatrix} \quad \begin{pmatrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{pmatrix}$$

$$= (1 + a^2 + b^2 + c^2)\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$= (1 + a^2 + b^2 + c^2)$

57. The given determinant

$$\begin{vmatrix} {}^xC_1 & {}^xC_2 & {}^xC_3 \\ {}^yC_1 & {}^yC_2 & {}^yC_3 \\ {}^zC_1 & {}^zC_2 & {}^zC_3 \end{vmatrix}$$

$$= \begin{vmatrix} x & \frac{x(x-1)}{2} & \frac{x(x-1)(x-2)}{6} \\ y & \frac{y(y-1)}{2} & \frac{y(y-1)(y-2)}{6} \\ z & \frac{z(z-1)}{2} & \frac{z(z-1)(z-2)}{6} \end{vmatrix}$$

$$= \frac{xyz}{12}\begin{vmatrix} 1 & (x-1) & (x-1)(x-2) \\ 1 & (y-1) & (y-1)(y-2) \\ 1 & (z-1) & (z-1)(z-2) \end{vmatrix}$$

$$= \frac{xyz}{12}\begin{vmatrix} 1 & (x-1) & (x-1)(x-2) \\ 0 & (y-x) & (y^2-x^2)-3(y-x) \\ 0 & (x-x) & (z^2-x^2)-3(z-x) \end{vmatrix}$$

$$\begin{pmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$$

$$= \frac{xyz}{12}\begin{vmatrix} (y-x) & (y^2-x^2)-3(y-x) \\ (z-x) & (z^2-x^2)-3(z-x) \end{vmatrix}$$

$$= \frac{xyz(y-x)(z-x)}{12}\begin{vmatrix} 1 & y+x-3 \\ 1 & z+x-3 \end{vmatrix}$$

$$= \frac{xyz(y-x)(z-x)(z-y)}{12}$$

$$= \frac{xyz(x-y)(y-z)(z-x)}{12}$$

58. The given determinant

$$= \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$$

$$= abc\begin{vmatrix} 1+\frac{1}{a} & \frac{1}{a} & \frac{1}{a} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$

$$= abc\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 1 & 1 \\ \frac{1}{b} & 1+\frac{1}{b} & \frac{1}{b} \\ \frac{1}{c} & \frac{1}{c} & 1+\frac{1}{c} \end{vmatrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & 0 & 0 \\ \frac{1}{b} & 1 & 0 \\ \frac{1}{c} & 0 & 1 \end{vmatrix}$$

$$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

59. The given determinant is

$$= \begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$

$$= \begin{vmatrix} (b+c)^2-a^2 & 0 & a^2 \\ 0 & (c+a)^2-b^2 & b^2 \\ c^2-(a+b)^2 & c^2-(a+b)^2 & (a+b)^2 \end{vmatrix}$$

$$\begin{pmatrix} C_1 \to C_1 - C_3 \\ C_2 \to C_2 - C_3 \end{pmatrix}$$

$$= (a+b+c)^2\begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix}$$

$$= (a+b+c)^2\begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix}$$

$$R_3 \to R_3 - (R_1 + R_2)$$

$[= 2ab(c + a)\ (b + c - a) + 2a^2b(c + a - b)$

$= 2ab(c + a)\ (b + c - a) + (c + a - b)$

$= 2ab[(b\ (c + a) + (c^2 - a^2) - ac + a^2 - ab$

$= 2ab(bc + c^2 + ac)$

$= 2abc(a + b + c)$

$= 2abc(a + b + c)^3$

60. The given determinant is

$$\begin{vmatrix} (a+1)(a+2) & (a+2) & 1 \\ (a+2)(a+3) & (a+3) & 1 \\ (a+3)(a+4) & (a+4) & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a^2+3a+2 & (a+2) & 1 \\ a^2+5a+6 & (a+3) & 1 \\ a^2+7a+12 & (a+4) & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a^2+3a+2 & (a+2) & 1 \\ 2a+4 & 1 & 0 \\ 4a+10 & 2 & 0 \end{vmatrix}$$

$$\begin{pmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$$

$$= \begin{vmatrix} 2a+4 & 1 \\ 4a+10 & 2 \end{vmatrix}$$

$= (4a + 8) - (4a + 10)$
$= -2$

61. The given determinant is

$$\begin{vmatrix} b+c & c+a & a+b \\ a+b & b+c & c+a \\ c+a & a+b & b+c \end{vmatrix}$$

$$= \begin{vmatrix} 2(a+b+c) & c+a & a+b \\ 2(a+b+c) & b+c & c+a \\ 2(a+b+c) & a+b & b+c \end{vmatrix}$$

$$(C_1 \to C_1 + C_2 + C_3)$$

$$= 2\begin{vmatrix} (a+b+c) & c+a & a+b \\ (a+b+c) & b+c & c+a \\ (a+b+c) & a+b & b+c \end{vmatrix}$$

$$= 2\begin{vmatrix} (a+b+c) & -b & -c \\ (a+b+c) & -a & -b \\ (a+b+c) & -c & -a \end{vmatrix} \quad \begin{pmatrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{pmatrix}$$

$$= 2\begin{vmatrix} a & -b & -c \\ c & -a & -b \\ b & -c & -a \end{vmatrix} \quad (C_1 \to C_1 + C_2 + C_3)$$

$$= 2\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

62 Here, $D = \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} = -4 - 9 = -13$

$$D_1 = \begin{vmatrix} 4 & 3 \\ 5 & -2 \end{vmatrix} = -8 - 15 = -23$$

$$D_2 = \begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} = 10 - 12 = -2$$

Thus, $x = \frac{D_1}{D} = \frac{-23}{-13} = \frac{23}{13}$

and $y = \frac{D_2}{D} = \frac{-2}{-13} = \frac{2}{13}$

$$D = \begin{vmatrix} 1 & 3 \\ 2 & 6 \end{vmatrix} = 6 - 6 = 0$$

63. Here, $D_1 = \begin{vmatrix} 4 & 3 \\ 10 & 6 \end{vmatrix} = 24 - 30 = -6 \neq 0$

We know that, if $D = 0$ and any one of D_1 and D_2 is non-zero, it has no solution.
So the system of equation is inconsistent.

64. Here, $D = \begin{vmatrix} 2 & 5 \\ 6 & 15 \end{vmatrix} = 30 - 30 = 0$

$$D_1 = \begin{vmatrix} 6 & 5 \\ 18 & 15 \end{vmatrix} = 90 - 90 = 0$$

$$D_2 = \begin{vmatrix} 2 & 6 \\ 6 & 18 \end{vmatrix} = 36 - 36 = 0$$

As we know that, if $D = 0 + D_1 = D_2$, the system of equations has infinitely many solutions.
Let $y = k$

Then $x = \frac{6-5k}{2}, k \in R.$

65. Since the system of equations has infinitely many solutions, so $D = 0 + D_1 = D_2$

Thus, $\begin{vmatrix} a & -b \\ (c+1) & c \end{vmatrix} = 0, \begin{vmatrix} 2a-b & -b \\ 10-a+3b & c \end{vmatrix} = 0$

and $\begin{vmatrix} a & 2a-\text{b} \\ c+1 & 10-a+3b \end{vmatrix} = 0$

66. We have,

$$D = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-\text{c})(c-a)$$

$$D_1 = \begin{vmatrix} 1 & 1 & 1 \\ d & b & c \\ d^2 & b^2 & c^2 \end{vmatrix} = (d-b)(b-c)(c-d)$$

$$D_2 = \begin{vmatrix} 1 & 1 & 1 \\ a & d & c \\ a^2 & d^2 & c^2 \end{vmatrix} = (a-d)(d-c)(\text{c}-a)$$

$$D_3 = \begin{vmatrix} 1 & 1 & 1 \\ a & b & d \\ a^2 & b^2 & d^2 \end{vmatrix} = (a-b)(b-d)(d-a)$$

Thus,

$$x = \frac{D_1}{D} = \frac{(d-b)(b-c)(c-d)}{(a-b)(b-c)(c-a)} = \frac{(d-b)(c-d)}{(a-b)(c-a)}$$

$$y = \frac{D_2}{D} = \frac{(a-d)(d-c)(c-a)}{(a-b)(b-c)(c-a)} = \frac{(a-d)(d-c)}{(a-b)(b-c)}$$

$$z = \frac{D_3}{D} = \frac{(a-b)(b-d)(d-a)}{(a-b)(b-c)(c-a)} = \frac{(b-d)(d-a)}{(b-c)(c-a)}$$

67. Let $u = \frac{1}{x}, v = \frac{1}{y}, w = \frac{1}{z}$

The given system of equations reduces to

$$u + v - w = \frac{1}{4},$$

$$2u - v + 3w = \frac{9}{4}$$

and $-u - 2v + 4w = 1$

Here, $D = \begin{vmatrix} 1 & 1 & -1 \\ 2 & -1 & 3 \\ -1 & -2 & 4 \end{vmatrix}$

$= 1(-4 + 6) - 1(8 + 3) - 1(-4 - 1)$
$= 2 - 11 + 5 = -4$

$$D_1 = \begin{vmatrix} 1/4 & 1 & -1 \\ 9/4 & -1 & 3 \\ 1 & -2 & 4 \end{vmatrix}$$

$$= \frac{1}{4} \cdot 2 - (9-3) - \left(-\frac{9}{2} + 1\right)$$

$$= \frac{1}{2} + \frac{7}{2} - 6 = 4 - 6 = -2$$

Also, $D_2 = \begin{vmatrix} 1 & 1/4 & -1 \\ 2 & 9/4 & 3 \\ -1 & 1 & 4 \end{vmatrix}$

$$= (9-3) - \frac{11}{4} - \left(2 + \frac{9}{4}\right)$$

$= 4 - 5 = -1$

Again, $D_3 = \begin{vmatrix} 1 & 1 & 1/4 \\ 2 & -1 & 9/4 \\ -1 & -2 & 1 \end{vmatrix}$

$$= \left(-1 + \frac{9}{2}\right) - 1\left(2 + \frac{9}{4}\right) - \frac{5}{4}$$

$$= \frac{7}{2} - 2 - \frac{14}{4} = -2$$

Now, $u = \frac{D_1}{D} = \frac{-4}{-2} = 2 \Rightarrow x = \frac{1}{2}$

$$v = \frac{D_2}{D} = \frac{-1}{-2} = \frac{1}{2} \Rightarrow y = 2$$

and $w = \frac{D_3}{D} = \frac{-2}{-2} = 1 \Rightarrow z = 1$

68. Given parabola is $y = ax^2 + bx + c$, which is passing through (2, 4), (–1, 1) and (–2, 5), so,

$4a + 2b + c = 4$
$a - b + c = 1$
$4a - 2b + c = 5$

From Cramers rule,

$$a = \frac{D_1}{D} = \frac{15}{12} = \frac{5}{4}$$

$$b = \frac{D_2}{D} = \frac{1}{12} \text{ and}$$

$$c = \frac{D_3}{D} = \frac{2}{12} = \frac{1}{6}$$

where $D = \begin{vmatrix} 4 & 2 & 1 \\ 1 & -1 & 1 \\ 4 & -2 & 1 \end{vmatrix} = 12$

$$D_1 = \begin{vmatrix} 4 & 2 & 1 \\ 1 & -1 & 1 \\ 5 & -2 & 1 \end{vmatrix} = 15$$

$$D_2 = \begin{vmatrix} 4 & 4 & 1 \\ 1 & 1 & 1 \\ 4 & 5 & 1 \end{vmatrix} = 1$$

$$D_3 = \begin{vmatrix} 4 & 2 & 4 \\ 1 & -1 & 1 \\ 4 & -2 & 5 \end{vmatrix} = 2$$

Hence, the required equation of the parabola is

$y = ax^2 + bx + c$

$$\Rightarrow \quad y = \frac{5}{4}x^2 + \frac{x}{12} + \frac{1}{6}$$

69. Since the system of equations has a unique solution, so,

$$\frac{2}{3} \neq \frac{k}{-4}$$

$$\Rightarrow \quad k \neq -\frac{8}{3}$$

Therefore, the value of k is $k \in R - \left\{-\frac{8}{3}\right\}$.

70. Since the system of equations has infinitely many solutions, so

$$\frac{3}{\lambda} = \frac{4}{8} = \frac{5}{10}$$

$$\Rightarrow \quad \frac{3}{\lambda} = \frac{1}{2}$$

$\Rightarrow \quad \lambda = 6$

Hence, the value of λ is 6.

71. We know that, the system of equations has no solution Only when, if $D = 0$, but any one of D_1, D_2, D_3 is non-zero.

Now, $D = 0$ gives

$$\begin{vmatrix} 1 & 2 & -3 \\ 0 & 0 & (p+2) \\ 0 & (2p+1) & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad (p + 2)(2p + 1) = 0$

$\Rightarrow \quad p = -2, -\frac{1}{2}$

Now, $D_1 \neq 0$ gives $p \neq 1, -1$

$D_2 \neq 0$ gives $p \neq -\frac{1}{2}$

and $D_3 \neq 0$ gives $p \neq -\frac{1}{2}$

Hence, the value of p is -2.

72. Since the given system of equations has no solution, so $D = 0$ and at-least any one of D_1, D_2, D_3 is non-zero.

Now $D = 0$, gives

$$D = \begin{vmatrix} 2 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & 2 & \lambda \end{vmatrix} = 0$$

$\Rightarrow \quad 2(-\lambda - 2) + (\lambda - 1) + 2(2 + 2) = 0$

$\Rightarrow \quad -3\lambda - 4 - 1 + 8 = 0$

$\Rightarrow \quad 3\lambda = 3$

$\Rightarrow \quad \lambda = 1$

Also $D_1 \neq 0$ gives

$$\begin{vmatrix} 2 & -1 & 2 \\ -4 & -2 & 1 \\ 4 & 2 & \lambda \end{vmatrix} \neq 0$$

$\Rightarrow \quad 2(-2\lambda - 2) + 2(-4\lambda - 4) \neq 0$

$\Rightarrow \quad -4\lambda - 4 - 8\lambda - 8 \neq 0$

$\Rightarrow \quad \lambda + 1 \neq 0$

$\Rightarrow \quad \lambda \neq -1$

Again, $D_2 \neq 0$ gives $\lambda \neq 1$.

Hence, the value of λ is φ.

73. Here, $D = \begin{vmatrix} 2 & 3 \\ 4 & 6 \end{vmatrix} = 12 - 12 = 0$

So the system of equations has infinitely many solutions.

Let $y = k$

Then $x = -\frac{3k}{2}, k \in R$

74. Since the system of equations has non-trivial solution, so $D = 0$

$$\begin{vmatrix} a-t & b & c \\ b & c-t & a \\ c & a & a-t \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a+b+c-t & b & c \\ a+b+c-t & c-t & a \\ a+b+c-t & a & a-t \end{vmatrix} = 0$$

$$\Rightarrow (a+b+c-t)\begin{vmatrix} 1 & b & c \\ 1 & c-t & a \\ 1 & a & a-t \end{vmatrix} = 0$$

$$\Rightarrow (a+b+c-t)\begin{vmatrix} 1 & b & c \\ 0 & c-b-t & a-c \\ 0 & a-b & a-c-t \end{vmatrix} = 0$$

$$\Rightarrow (a+b+c-t)\begin{vmatrix} c-b-t & a-c \\ a-b & a-c-t \end{vmatrix} = 0$$

$$\Rightarrow (a+b+c-t) = 0, \begin{vmatrix} c-b-t & a-c \\ a-b & a-c-t \end{vmatrix} = 0$$

$\Rightarrow \quad t = (a + b + c), t^2 + (a + b)t + (2ac - c^2) = 0$

$\Rightarrow \quad t = (a + b + c)$

$$t = \frac{-(a+b) \pm \sqrt{(a+b)^2 - 4(2ac - c^2)}}{2}$$

Thus, the number of values of t is 3.

75. Since the system of equations has a unique solution, so $D_1 \neq 0$

$$\Rightarrow \begin{vmatrix} 6 & 5 & \lambda \\ 3 & -1 & 4 \\ 1 & 2 & -3 \end{vmatrix} \neq 0$$

$\Rightarrow \quad 6(3 - 8) - 5(-9 - 4) + \lambda(6 + 1) \neq 0$

$\Rightarrow \quad -30 + 65 + 7\lambda \neq 0$

$\Rightarrow \quad 7\lambda + 35 \neq 0$

$\Rightarrow \quad \lambda + 5 \neq 0$

$\Rightarrow \quad \lambda \neq -5$

Hence, the value of λ is $R - \{-5\}$.

76. Since the system of equations has infinite solutions, so

$$\begin{vmatrix} 1 & a & 0 \\ 0 & 1 & a \\ a & 0 & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad 1 - a(0 - a^2) = 0$

$\Rightarrow \quad 1 - a^3 = 0$

$\Rightarrow \quad (a + 1)(a^2 - a + 1) = 0$

$\Rightarrow \quad (a + 1) = 0, (a^2 - a + 1) = 0$

$$\Rightarrow \quad a = -1, a = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm i\sqrt{3}}{2}$$

Hence, the values of a are $\left\{-1, \frac{1 \pm i\sqrt{3}}{2}\right\}$.

77. We have,

$$\begin{vmatrix} a & 0 & c \\ a & b & 0 \\ 0 & b & c \end{vmatrix}^2 = \begin{vmatrix} a & 0 & c \\ a & b & 0 \\ 0 & b & c \end{vmatrix} \times \begin{vmatrix} a & 0 & c \\ a & b & 0 \\ 0 & b & c \end{vmatrix}$$

$$= \begin{vmatrix} a.a+0.0+c.c & a.a+0.0+0.0 & a.0+0.b+c.c \\ a.a+b.0+0.c & a.a+b.b+0.0 & a.0+b.b+0.c \\ 0.a+b.0+c.c & 0.a+b.b+c.0 & 0.0+b.b+c.c \end{vmatrix}$$

$$= \begin{vmatrix} c^2+a^2 & a^2 & c^2 \\ a^2 & a^2+b^2 & b^2 \\ c^2 & b^2 & b^2+c^2 \end{vmatrix}$$

78. We have,

$$\begin{vmatrix} 2 & \alpha+\beta+\gamma+\delta & \alpha\beta+\gamma\delta \\ \alpha+\beta+\gamma+\delta & 2(\alpha+\beta)(\gamma+\delta) & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) \\ \alpha\beta+gd & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) & 2\alpha\beta\gamma\delta \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 0 \\ \alpha+\beta & \gamma+\delta & 0 \\ \alpha\beta & \gamma\delta & 0 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 0 \\ \gamma+\delta & \alpha+\beta & 0 \\ \gamma\delta & \alpha\beta & 0 \end{vmatrix}$$

$= 0 \times 0$

$= 0$

79. We have

$$\begin{vmatrix} 1 & \cos(\beta-\alpha) & \cos(\gamma-\alpha) \\ \cos(\alpha-\beta) & 1 & \cos(\gamma-\beta) \\ \cos(\alpha-\gamma) & \cos(\beta-\gamma) & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix} \times \begin{vmatrix} \cos\alpha & \sin\alpha & 0 \\ \cos\beta & \sin\beta & 0 \\ \cos\gamma & \sin\gamma & 0 \end{vmatrix}$$

$= 0 \times 0$

$= 0$

80. We have,

$$\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}^2$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 0 & \alpha-1 & \beta-1 \\ 0 & \alpha^2-1 & \beta^2-1 \end{vmatrix}^2$$

$$= \begin{vmatrix} \alpha-1 & \beta-1 \\ \alpha^2-1 & \beta^2-1 \end{vmatrix}^2$$

$$= (\alpha-1)^2(\beta-1)^2 \begin{vmatrix} 1 & 1 \\ \alpha+1 & \beta+1 \end{vmatrix}^2$$

$= (\alpha-1)^2(\beta-1)^2(\beta-\alpha)^2$

$= (1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$

Hence, the value of k is 1.

81. We have,

$$F(x) = \begin{vmatrix} 0 & 0 & 0 \\ x & x^2 & x^3 \\ e^{x-a} & e^{x^2-a^2} & e^{x^3-a^3} \end{vmatrix}$$

$$+ \begin{vmatrix} 1 & a & a^2 \\ 1 & 2x & 3x^2 \\ e^{x-a} & e^{x^2-a^2} & e^{x^3-a^3} \end{vmatrix}$$

$$+ \begin{vmatrix} 1 & a & a^2 \\ x & x^2 & x^3 \\ e^{x-a} & 2x\times e^{x^2-a^2} & 3x^2\times e^{x^3-a^3} \end{vmatrix}$$

$$\Rightarrow \quad F'(a) = \begin{vmatrix} 1 & a & a^2 \\ 1 & 2a & 3a^2 \\ 1 & 1 & 1 \end{vmatrix} + \begin{vmatrix} 1 & a & a^2 \\ a & a^2 & a^3 \\ 1 & 2a & 3a^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & 2a & 3a^2 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & 2a & 3a^2 \\ 1 & 1 & 1 \end{vmatrix}$$

$= 2a - 3a^2 - a + 3a^3 + a^2 - 2a^3$

$= a - 2a^2 + a^{3.}$

82. Given, $f(x) = \begin{vmatrix} 3 & 2 & 1 \\ 6x^2 & 2x^3 & x^4 \\ 1 & b & b^2 \end{vmatrix}$

$$\Rightarrow \quad f'(x) = \begin{vmatrix} 3 & 2 & 1 \\ 12x & 6x^2 & 4x^3 \\ 1 & b & b^2 \end{vmatrix}$$

$$\Rightarrow \quad f''(x) = \begin{vmatrix} 3 & 2 & 1 \\ 12 & 12x & 12x^2 \\ 1 & b & b^2 \end{vmatrix}$$

$$\Rightarrow \quad f''(b) = \begin{vmatrix} 3 & 2 & 1 \\ 12 & 12b & 12b^2 \\ 1 & b & b^2 \end{vmatrix}$$

$$\Rightarrow \quad f''(b) = 12\begin{vmatrix} 3 & 2 & 1 \\ 1 & b & b^2 \\ 1 & b & b^2 \end{vmatrix} = 0$$

83. $\int_0^{\pi/2} f(x)\,dx$

$$= \begin{vmatrix} \int_0^{\pi/2} \sin^2 x\,dx & \int_0^{\pi/2} \log(\sin x)\,dx & \int_0^{\pi/2} \left(\frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\right) dx \\ n & \sum_{k=1}^{n} (k) & \prod_{k=1}^{n} (k) \\ \frac{8}{15} & \frac{\pi}{2}\log\left(\frac{1}{2}\right) & \frac{\pi}{4} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{8}{15} & \frac{\pi}{2}\log\left(\frac{1}{2}\right) & \frac{\pi}{4} \\ n & \sum_{k=1}^{n} (k) & \prod_{k=1}^{n} (k) \\ \frac{8}{15} & \frac{\pi}{2}\log\left(\frac{1}{2}\right) & \frac{\pi}{4} \end{vmatrix}$$

$= 0$

84. We have,

$$\Delta_r = \begin{vmatrix} r & 2012 & \frac{n(n+1)}{2} \\ 2r-1 & 2013 & n^2 \\ 3r-2 & 2014 & \frac{n(3n-1)}{2} \end{vmatrix}$$

$$\Rightarrow \quad \sum_{r=1}^{n} \Delta_r = \begin{vmatrix} \sum_{r=1}^{n} r & 2012 & \frac{n(n+1)}{2} \\ \sum_{r=1}^{n} (2r-1) & 2013 & n^2 \\ \sum_{r=1}^{n} (3r-2) & 2014 & \frac{n(3n-1)}{2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{n(n+1)}{2} & 2012 & \frac{n(n+1)}{2} \\ n^2 & 2013 & n^2 \\ \frac{\mathrm{n}(3n-1)}{2} & 2014 & \frac{n(3n-1)}{2} \end{vmatrix}$$

$= 0$

85. We have $D_r = \begin{vmatrix} 2^{r-1} & 101 & (2^n - 1) \\ 3^{r-1} & 102 & \left(\frac{3^n-1}{2}\right) \\ 5^{r-1} & 103 & \left(\frac{5^n-1}{4}\right) \end{vmatrix}$

Then $\sum_{r=1}^{n} D_r = \begin{vmatrix} \sum_{r=1}^{n} 2^{r-1} & 101 & (2^n - 1) \\ \sum_{r=1}^{n} 3^{r-1} & 102 & \left(\frac{3^n-1}{2}\right) \\ \sum_{r=1}^{n} 5^{r-1} & 103 & \left(\frac{5^n-1}{4}\right) \end{vmatrix}$

$$= \begin{vmatrix} (2^n - 1) & 101 & (2^n - 1) \\ \left(\frac{3^n-1}{2}\right) & 102 & \left(\frac{3^n-1}{2}\right) \\ \left(\frac{5^n-1}{4}\right) & 103 & \left(\frac{5^n-1}{4}\right) \end{vmatrix}$$

$= 0$

Then $\sum_{r=1}^{n} \Delta_r = \begin{vmatrix} \sum_{r=1}^{n} f(r) & \sum_{r=1}^{n} g(r) & \sum_{r=1}^{n} h(r) \\ a & b & c \\ p & q & r \end{vmatrix}$

where a, c, p, q, r are constants.

86. We have,

adj $(A') = (\text{adj } A)'$

$\Rightarrow$ adj $(A') - (\text{adj } A') = \mathbf{O}$

87. We know that,

$|\text{adj } (A)| = |A|^{n-1}$

Replace A by A^3 and n by 3, we get

$$|\text{adj}(A^3)| = |A^3|^{3-1} = |A|^6$$

88. We know that,

given $A \cdot \text{adj}(A) = |A| \cdot I_n$

$$= \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}$$

$$\Rightarrow \quad |A| \cdot I_2 = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}$$

$$= 10\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$= 10 \cdot I_2$

$\Rightarrow \quad |\mathrm{A}| = 10$

89. We know that,

$|\text{adj } (A)| = |A|^{n-1}$

$\Rightarrow$ $|\text{adj } (A)| = |A|^{3-1} = |A|^2 = 16$

90. We know that,

$$|[\text{adj}\{\text{adj}(A)\}]| = |A|^{(n-1)^2}$$

$\Rightarrow$ $|A|^{(n-1)^2} = |A|^{16}$

$\Rightarrow$ $(n-1)^2 = 16$

$\Rightarrow$ $(n-1) = 4$

$\Rightarrow$ $n = 5$

91. Given $\text{adj}(P) = \begin{pmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{pmatrix}$

$\Rightarrow$ $|\text{adj}(P)| = \begin{vmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{vmatrix} = 4$

As we know that,

$$|\text{adj}(P)| = |P|^{3-1} = |P|^2$$

Thus, $|P|^2 = 4$

$\Rightarrow$ $|P| = 2, -2$

92. Let $P = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, where $|P| = 2$

Now, $Q = \begin{pmatrix} 2^2 a_{11} & 2^3 a_{12} & 2^4 a_{13} \\ 2^3 a_{21} & 2^4 a_{22} & 2^5 a_{23} \\ 2^4 a_{31} & 2^5 a_{32} & 2^6 a_{33} \end{pmatrix}$

$\Rightarrow$ $|Q| = \begin{vmatrix} 2^2 a_{11} & 2^3 a_{12} & 2^4 a_{13} \\ 2^3 a_{21} & 2^4 a_{22} & 2^5 a_{23} \\ 2^4 a_{31} & 2^5 a_{32} & 2^6 a_{33} \end{vmatrix}$

$$= 2^2 \cdot 2^3 \cdot 2^4 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 2a_{21} & 2a_{22} & 2a_{23} \\ 2^2 a_{31} & 2^2 a_{32} & 2^2 a_{33} \end{vmatrix}$$

$$= 2^2 \cdot 2^3 \cdot 2^4 \cdot 2 \cdot 2^2 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= 2^{12} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= 2^{12} \cdot 2 = 2^{13}$$

93. We know that,

$|\text{adj}(A)| = |A|^{n-1}$

$\Rightarrow$ $|\text{adj}(A)| = |A|^2$

$= 16$

$\Rightarrow$ $1(12-12) - \alpha(4-6) + 3(4-6) = 16$

$\Rightarrow$ $2\alpha - 6 = 16$

$\Rightarrow$ $2\alpha = 16 + 6 = 22$

$\Rightarrow$ $\alpha = 1$

94. Since the determinant of a skew-symmetric matrix is zero, so the inverse does not exist

95. We know that,

$BB^{-1} = 1$

$\Rightarrow$ $|BB^{-1}| = |1|$

$\Rightarrow$ $|B||B^{-1}| = |I|$

$\Rightarrow$ $|B^{-1}| = \dfrac{1}{|B|}$

Now,

$|B^{-1}AB| = |B^{-1}||A||B|$

$= \dfrac{1}{|B|}|A||B| = |A|$

96. We have,

$R = (P\cos\theta + Q\sin\theta)$

$= \begin{pmatrix} \cos\theta & 0 \\ 0 & \cos\theta \end{pmatrix} + \begin{pmatrix} 0 & \sin\theta \\ -\sin\theta & 0 \end{pmatrix}$

$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

Now, $R^{-1} = \dfrac{\text{adj}(R)}{|R|} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$.

97. Given $B^{-1}AB = A^2$

Now,

$B^{-3}AB^3 = B^{-2}(B^{-1}AB)B^2$

$= B^{-2}(A^2)B^2$

$= B^{-1}(B^{-1}(A^2)B)B$

$= B^{-1}(A^4)B$

$= (A^4)^2 = A^8$

98. Given,

$I + A + A^2 + A^3 + \ldots + A^k = \mathbf{O}$

$\Rightarrow$ $A^{-1}(I + A + A^2 + A^3 + \ldots + A^k) = \mathbf{O}$

$\Rightarrow$ $(A^{-1} + I + A + A^2 + \ldots + A^{k-1}) = \mathbf{O}$

$\Rightarrow$ $A^{-1} + (-A^k) = \mathbf{O}$

$\Rightarrow$ $A^{-1} = A^k$

99. Given,

$A^{-2} - A + I = \mathbf{O}$

$\Rightarrow$ $A^{-1}(A^2 - A + I) = A^{-1}\mathbf{O} = \mathbf{O}$

$\Rightarrow$ $(A^{-1}A^2 - A^{-1}A + A^{-1}) = \mathbf{O}$

$\Rightarrow$ $(A - I + A^{-1}) = \mathbf{O}$

$\Rightarrow$ $A^{-1} = I - A$

100. We know that,

$|\text{adj}(A)| = |A|^{n-1}$

$= |A|^{3-1} = |A|^2$

$\Rightarrow$ $|\text{adj}(A^{-1})| = |A^{-1}|^2$

$\Rightarrow$ $|\text{adj}(A^{-1})| = \left(\dfrac{1}{|A|}\right)^2 = \dfrac{1}{|A|^2}$

$= \dfrac{1}{|A|^2} = \dfrac{1}{25}$

101. We have,

$|A^{-1}\text{adj}(A)| = |A^{-1}||\text{adj}(A)|$

$$= \frac{1}{|A|}|\text{adj}(A)|$$
$$= \frac{1}{|A|} \times |A|^{n-1}$$
$$= \frac{1}{|A|} \times |A|^2$$
$$= |A|.$$

102. Given,

$$A = \begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$$

Now, $|A| = \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = 10 - 12 = -2$

Since $|A| \neq 0$, so its inverse exists.
Thus,

$$A^{-1} = \frac{\text{adj}(A)}{|A|}$$
$$= -\frac{1}{2}\begin{pmatrix} 5 & -3 \\ -4 & 2 \end{pmatrix}$$
$$= \begin{pmatrix} -5/2 & 3/2 \\ 2 & -1 \end{pmatrix}$$

103. Given $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}. X = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

It can be written as $AX = B$, where

$$A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

Now, $|A| = \begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$

So, its inverse exists.
Thus,

$$X = A^{-1}B$$
$$= \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 4 \\ 0 & -1 \end{pmatrix}.$$

104. We have,

$$B^{-1} = \frac{\text{adj}(B)}{|B|} = \frac{1}{2}\begin{pmatrix} 3 & -1 \\ -7 & 3 \end{pmatrix}$$

$$\Rightarrow \quad 2B^{-1} = \begin{pmatrix} 3 & -1 \\ -7 & 3 \end{pmatrix}$$

Now

$$\det(2A^9B^{-1}) = \det(A^9\, 2B^{-1})$$
$$= \det(A^9) \times \det(2B^{-1})$$
$$= |A|^9 \times |2B^{-1}|$$
$$= (-1)^9 \times (9 - 7)$$
$$= -2$$

105. We know that,

$$B^{-1} = \frac{\text{adj}(B)}{|B|} = \frac{A}{|B|}$$

$\Rightarrow \quad \text{adj}(B) = |B|B^{-1}$

Replacing B by QBP, we get

$\text{adj}(QBP) = |QBP|(QBP)^{-1}$

$\Rightarrow \quad \text{adj}(QBP) = |Q||B||P|P^{-1}B^{-1}Q^{-1}$

$\Rightarrow \quad \text{adj}(QBP) = |B|P^{-1}B^{-1}Q^{-1}$

$\Rightarrow \quad \text{adj}(QBP) = P^{-1}|B|B^{-1}Q^{-1}$

$\Rightarrow \quad \text{adj}(QBP) = P^{-1}(A)Q^{-1}$

106. We have,

$B = A^{-1}A'$

$\Rightarrow \quad AB = A'$

Now,

$ABB' = A'B'$

$\Rightarrow \quad ABB' = (BA)' = (A^{-1}AA)'$

$\Rightarrow \quad = (IA')' = (A')' = A$

$\Rightarrow \quad BB' = 1$

107. Now,

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 4 \end{vmatrix}$$
$$= 0 - 2(8 - 6) + 3(4 - 3)$$
$$= -4 - 3$$
$$= -7 \neq 0$$

So, its inverse exists.
We have,

$$\text{adj}(A) = \begin{pmatrix} \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 3 & 4 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 3 \\ 3 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix} \\ \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} \end{pmatrix}^T$$
$$= \begin{pmatrix} 0 & -2 & 1 \\ -2 & -5 & 4 \\ 1 & 4 & -3 \end{pmatrix}^T$$
$$= \begin{pmatrix} 0 & -2 & 1 \\ -2 & -5 & 4 \\ 1 & 4 & -3 \end{pmatrix}$$

Thus, $A^{-1} = \dfrac{\text{adj}(A)}{|A|}$

$$= -\frac{1}{7}\begin{pmatrix} 0 & -2 & 1 \\ -2 & -5 & 4 \\ 1 & 4 & -3 \end{pmatrix}$$

108. We have,

$$(I+A)=\begin{pmatrix}1 & -\tan\left(\frac{\alpha}{2}\right)\\ \tan\left(\frac{\alpha}{2}\right) & 1\end{pmatrix}$$

and

$$(I-A)=\begin{pmatrix}1 & \tan\left(\frac{a}{2}\right)\\ -\tan\left(\frac{\alpha}{2}\right) & 1\end{pmatrix}$$

Let $I-A=B$

Now, $B^{-1}=\frac{\text{adj}(B)}{|B|}$

$$=\frac{1}{\sec^2\left(\frac{\alpha}{2}\right)}\begin{pmatrix}1 & -\tan\left(\frac{\alpha}{2}\right)\\ \tan\left(\frac{\alpha}{2}\right) & 1\end{pmatrix}$$

Thus,

$(I-A)^{-1}(I+A)=B^{-1}(I+A)$

$$=\frac{1}{\sec^2\left(\frac{\alpha}{2}\right)}\begin{pmatrix}1 & -\tan\left(\frac{\alpha}{2}\right)\\ \tan\left(\frac{\alpha}{2}\right) & 1\end{pmatrix}\begin{pmatrix}1 & -\tan\left(\frac{\alpha}{2}\right)\\ \tan\left(\frac{\alpha}{2}\right) & 1\end{pmatrix}$$

$$=\frac{1}{\sec^2\left(\frac{\alpha}{2}\right)}\times\begin{pmatrix}1-\tan^2\left(\frac{\alpha}{2}\right) & -2\tan\left(\frac{\alpha}{2}\right)\\ 2\tan\left(\frac{\alpha}{2}\right) & 1-\tan^2\left(\frac{\alpha}{2}\right)\end{pmatrix}$$

$$=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}$$

109. The given system of equations can be written in the matrix form as

$AX=B$, where

$$A=\begin{pmatrix}1 & 1 & 1\\ 1 & -1 & 1\\ 2 & 1 & -1\end{pmatrix}, X=\begin{pmatrix}x\\ y\\ z\end{pmatrix} \text{ and } B=\begin{pmatrix}6\\ 2\\ 1\end{pmatrix}$$

$$\text{Now, } |A|=\begin{vmatrix}1 & 1 & 1\\ 1 & -1 & 1\\ 2 & 1 & -1\end{vmatrix}=6\neq 0$$

Thus, A^{-1} exists

Now, $A^{-1}=\frac{\text{adj}(A)}{|A|}$

$$=\frac{1}{6}\begin{pmatrix}0 & 2 & 2\\ 3 & -3 & 0\\ 3 & 1 & -2\end{pmatrix}$$

Therefore, $X=A^{-1}B$

$$=\frac{1}{6}\begin{pmatrix}0 & 2 & 2\\ 3 & -3 & 0\\ 3 & 1 & -2\end{pmatrix}\begin{pmatrix}6\\ 2\\ 1\end{pmatrix}$$

$$=\frac{1}{6}\begin{pmatrix}6\\ 12\\ 18\end{pmatrix}=\begin{pmatrix}1\\ 2\\ 3\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}1\\ 2\\ 3\end{pmatrix}$$

$\Rightarrow$ $x=1, y=2$ and $z=3$.

110. Given matrix is

$$A=\begin{pmatrix}-1 & 2 & 5\\ 2 & -3 & 1\\ -1 & 1 & 1\end{pmatrix}$$

$$\Rightarrow |A|=\begin{vmatrix}-1 & 2 & 5\\ 2 & -3 & 1\\ -1 & 1 & 1\end{vmatrix}=11\neq 0$$

$\Rightarrow$ A^{-1} exists.

Thus,

$$A^{-1}=\frac{\text{adj}(A)}{|A|}$$

$$=\frac{1}{11}\begin{pmatrix}7 & 2 & -6\\ -2 & 1 & -3\\ -4 & 2 & 5\end{pmatrix}$$

Also, the given system of equations can be written in matrix form as

$$\begin{pmatrix}-1 & 2 & 5\\ 2 & -3 & 1\\ -1 & 1 & 1\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}2\\ 15\\ 3\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}-1 & 2 & 5\\ 2 & -3 & 1\\ -1 & 1 & 1\end{pmatrix}^{-1}\begin{pmatrix}2\\ 15\\ 3\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}x\\ y\\ z\end{pmatrix}=\frac{1}{11}\begin{pmatrix}7 & 2 & -6\\ -2 & 1 & -3\\ -4 & 2 & 5\end{pmatrix}\begin{pmatrix}2\\ 15\\ 3\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}x\\ y\\ z\end{pmatrix}=\frac{1}{11}\begin{pmatrix}44\\ -33\\ 11\end{pmatrix}=\begin{pmatrix}4\\ -3\\ 1\end{pmatrix}$$

$\Rightarrow$ $x=4, y=-3$ and $z=1$.

111. The given system of equations can be written in matrix form as

$$\begin{pmatrix}5 & 3 & 7\\ 3 & 26 & 2\\ 7 & 2 & 10\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}4\\ 9\\ 5\end{pmatrix}$$

$\Rightarrow \quad AX = B$, where

$$A = \begin{pmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, B = \begin{pmatrix} 4 \\ 9 \\ 5 \end{pmatrix}$$

Now, $|A| = \begin{vmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{vmatrix} = 0$

So, inverse does not exist.

Also,

$$\text{adj}(A) = \begin{pmatrix} 256 & -16 & -176 \\ -16 & 1 & 11 \\ -176 & 11 & 121 \end{pmatrix}$$

Thus,

$$\text{adj}(A)B = \begin{pmatrix} 256 & -16 & -176 \\ -16 & 1 & 11 \\ -176 & 11 & 121 \end{pmatrix}\begin{pmatrix} 4 \\ 9 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{O}$$

Therefore, the given system of equations have infinitely many solutions.

112. The given system of equations can be written in matrix form as

$AX = B$

It has either
 (i) a unique solution, or
 (ii) infinite solutions, or
 (iii) no solution.

Thus, there cannot exist any matrix A such that

$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has two distinct solutions.

113. Let $\frac{x^2}{a^2} = X, \frac{y^2}{b^2} = Y$ and $\frac{z^2}{c^2} = Z$.

The given system of equations reduces to

$$X + Y - Z = 1$$
$$X - Y + Z = 1$$
$$X + Y + Z = 1$$

It can be written in matrix form as

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Let $AX' = B$, where

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, X' = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Now,

$$|A| = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -4 \neq 0$$

Thus, the system of equations have a unique solution.

114. We have,

$A = I.A$

$\Rightarrow \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot A$

$\Rightarrow \begin{pmatrix} 1 & 2 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix} \cdot A$ $\quad (R_2 \to R_2 - 2R_1)$

$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -3 & 1 \end{pmatrix} \cdot A$ $\quad (R_1 \to R_1 + 2R_2)$

$\Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix} \cdot A$ $\quad \left[R_2 \to \left(\frac{-1}{2}\right) \times R_2\right]$

Thus, $A^{-1} = \begin{pmatrix} -2 & 1 \\ 3/2 & -1/2 \end{pmatrix}$.

115. We have,

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{vmatrix}$$

$= 1(40 - 42) - 4(20 - 21) + 3(12 - 12)$
$= -2 + 2 + 0 = 0$

Since the determinant of A is zero, so the rank of the given matrix is 2.

116. We have,

$$|A| = \begin{vmatrix} 2 & 4 & 3 \\ 1 & 2 & -1 \\ -1 & -2 & 6 \end{vmatrix}$$

$= 2(12 - 2) - 4(6 - 1) + 3(-2 + 2)$
$= 20 - 20 + 0$
$= 0$

Since the determinant of A is zero, so the rank of the given matrix is 2.

117. We have,

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 4 & 5 & 6 \end{vmatrix}$$

$= 1(18 - 20) - 2(12 - 12) + 3(10 - 12)$
$= -2 - 6 = -8$

Since the determinant of A is non-zero, so the rank of the given matrix is 3.

118. We have,

$$A = \begin{pmatrix} 1 & -2 & 1 & -1 \\ 1 & 1 & -2 & 3 \\ 4 & 1 & -5 & 6 \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} 1 & -2 & 1 & -1 \\ 0 & 3 & -3 & 4 \\ 0 & 9 & -9 & 12 \end{pmatrix}$$

$$(R_2 \to R_2 - R_1 \cdot R_3 \to R_3 - 4R_1)$$

$$\Leftrightarrow \begin{pmatrix} 1 & -2 & 1 & -1 \\ 0 & 1 & -1 & 4/3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$(R_3 \to R_3 - 9R_2)$$

Since the number of non-zero rows is 2, so the rank of the given matrix is 2.

119. We have,

$$A^2 = A \cdot A = \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix} \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 4+3-5 & -6-12+15 & -10-15+20 \\ -2-4+5 & 3+16-15 & 5+20-20 \\ 2+3-4 & -3-12+12 & -5-15+16 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -3 & -5 \\ -1 & 4 & 5 \\ 1 & -3 & -4 \end{bmatrix} = A$$

Thus, A is idempotent.

120. We have,

$$A^2 = A.A$$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -1 \end{bmatrix} \times \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -1 \end{bmatrix} = A.$$

Hence, A is periodic.

121. Let $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$.

Now, $A^2 = A \cdot A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix}$$

Also, $A^3 = A^2 \cdot A$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 3 & 3 & 9 \\ -1 & -1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= O$$

Hence A is nilpotent of order 3.

122. Let $A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$

Now, $A^2 = A \cdot A$

$$= \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence A is involuntary.

123. Give, $A = \frac{1}{3} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$

For orthogonal matrix $AA^T = A^T A = I$

$$\therefore \quad AA^T = \frac{1}{9} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence A is an orthogonal matrix.

124. We have,

$$A^T = \begin{bmatrix} 3 & 3+4i & 5-2i \\ 3-4i & 5 & -2-i \\ 5+2i & -2+i & 2 \end{bmatrix}$$

$$\Rightarrow \quad (\overline{A^T}) = \begin{bmatrix} 3 & 3-4i & 5+2i \\ 3+4i & 5 & -2+i \\ 5-2i & -2-i & 2 \end{bmatrix}$$

$$= A$$

Thus, A is a hermitian matrix.

125 We have,

$$\Rightarrow \quad A^T = \begin{bmatrix} 2i & -2-3i & 2+i \\ -2-3i & -i & 3i \\ 2+i & 3i & 0 \end{bmatrix}$$

$$(\overline{A^T}) = \begin{bmatrix} -2i & 2+3i & 2-\text{i} \\ -2+3i & -i & -3i \\ -2-i & -3i & 0 \end{bmatrix}$$

$$\Rightarrow \quad (\overline{A^T}) = -\begin{bmatrix} 2i & -2-3i & -2+i \\ 2-3i & -i & 3i \\ 2+i & 3i & 0 \end{bmatrix} = -A$$

Hence A is an skew Hermitian matrix.

126. Let $A = \dfrac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-\text{i} & -1 \end{bmatrix}$

$$\Rightarrow \quad A^{\theta} = \begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$\Rightarrow \quad AA^{\theta} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix} \times \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

Hence A is unitary matrix.

Level III

1. We have,

$$\begin{vmatrix} 1 & bc & bc(b+c) \\ 1 & ca & ca(c+a) \\ 1 & ab & ab(a+b) \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} a & abc & abc(b+c) \\ b & bca & bca(c+a) \\ c & cab & abc(a+b) \end{vmatrix}$$

$$= \frac{(abc)^2}{abc}\begin{vmatrix} a & 1 & (b+c) \\ b & 1 & (c+a) \\ c & 1 & (a+b) \end{vmatrix}$$

$$= (abc)\begin{vmatrix} a & 1 & (b+c) \\ b & 1 & (c+a) \\ c & 1 & (a+b) \end{vmatrix}$$

$$= (abc)\begin{vmatrix} a & 1 & (a+b+c) \\ b & 1 & (b+c+a) \\ c & 1 & (a+b+c) \end{vmatrix} \qquad (C_1 \to C_1 + C_3)$$

$$= (abc)(a+b+c)\begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix}$$

$$= 0$$

which is independent of a, b and c.

2. We have,

$$\begin{vmatrix} b+c & a & a \\ c & c+a & a \\ b & a & a+b \end{vmatrix} = \begin{vmatrix} 0 & -2a & -2a \\ c & c+a & a \\ b & a & a+b \end{vmatrix}$$

$[R_1 \to R_1 - (R_2 + R_3)]$

$$= -2a\begin{vmatrix} 0 & 1 & 1 \\ c & c+a & a \\ b & a & a+b \end{vmatrix}$$

$$= -2a\begin{vmatrix} 0 & 0 & 1 \\ c & c & a \\ b & -b & a+b \end{vmatrix} \qquad (C_2 \to C_2 - C_3)$$

$= -2a\,(-bc - bc)$

$= -2a \times -2bc$

$= 4abc$

3. We have,

$$\begin{vmatrix} b^2+c^2 & a^2 & a^2 \\ b^2 & a^2+c^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$$

$$= \begin{vmatrix} 2(b^2+c^2) & 2(a^2+c^2) & 2(a^2+b^2) \\ b^2 & a^2+c^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$$

$(R_3 \to R_1 + R_2 + R_3)$

$$= 2\begin{vmatrix} (b^2+c^2) & (a^2+c^2) & (a^2+b^2) \\ b^2 & a^2+c^2 & b^2 \\ c^2 & c^2 & a^2+b^2 \end{vmatrix}$$

$$= 2\begin{vmatrix} (b^2+c^2) & (a^2+c^2) & (a^2+b^2) \\ -c^2 & 0 & -a^2 \\ -b^2 & -a^2 & 0 \end{vmatrix}$$

$(R_2 \to R_2 - R_1, R_3 \to R_3 - R_1)$

$$= 2\begin{vmatrix} 0 & c^2 & b^2 \\ -c^2 & 0 & -a^2 \\ -b^2 & -a^2 & 0 \end{vmatrix}$$

$= 2(a^2b^2c^2 + a^2b^2c^2)$

$= 4(a^2b^2c^2)$

4. We have,

$$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$$

$$= abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

$$= abc\begin{vmatrix} 2(a+c) & c & a+c \\ 2(a+b) & b & a \\ 2(b+c) & b+c & c \end{vmatrix}$$

$(C_3 \to C_1 + C_2 + C_3)$

$$= 2abc\begin{vmatrix} (a+c) & c & a+c \\ (a+b) & b & a \\ (b+c) & b+c & c \end{vmatrix}$$

$$= 2abc\begin{vmatrix} (a+c) & -a & 0 \\ (a+b) & -a & -b \\ (b+c) & 0 & -b \end{vmatrix}$$

$(C_2 \to C_2 - C_1, C_3 \to C_3 - C_1)$

$$= 2abc\begin{vmatrix} c & -a & 0 \\ 0 & -a & -b \\ c & 0 & -b \end{vmatrix}$$

$= 2abc(abc + abc)$

$= 4a^2b^2c^2$

5. We have,

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} a(a^2+1) & a^2b & a^2c \\ ab^2 & b(b^2+1) & b^2c \\ ac^2 & bc^2 & c(c^2+1) \end{vmatrix}$$

$$= \frac{abc}{abc}\begin{vmatrix} (a^2+1) & a^2 & a^2 \\ b^2 & (b^2+1) & b^2 \\ c^2 & c^2 & (c^2+1) \end{vmatrix}$$

$$= \begin{vmatrix} (a^2+1) & a^2 & a^2 \\ b^2 & (b^2+1) & b^2 \\ c^2 & c^2 & (c^2+1) \end{vmatrix}$$

$$= \begin{vmatrix} (a^2+b^2+c^2+1) & (a^2+b^2+c^2+1) & (a^2+b^2+c^2+1) \\ b^2 & (b^2+1) & b^2 \\ c^2 & c^2 & (c^2+1) \end{vmatrix}$$

$(R_3 \to R_1 + R_2 + R_3)$

$$= (a^2+b^2+c^2+1)\begin{vmatrix} 1 & 1 & 1 \\ b^2 & (b^2+1) & b^2 \\ c^2 & c^2 & (c^2+1) \end{vmatrix}$$

$$= (a^2+b^2+c^2+1)\begin{vmatrix} 1 & 0 & 0 \\ b^2 & 1 & 0 \\ c^2 & 0 & 1 \end{vmatrix}$$

$(C_2 \to C_2 - C_1, C_3 \to C_3 - C_1)$

$= (a^2 + b^2 + c^2 + 1)$

6. The given determinant is

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1+a^2-b^2+2b^2 & 2ab-2ab & -2b \\ 2ab-2ab & 1-a^2+b^2+2a^2 & 2a \\ 2b-b(1-a^2-b^2) & -2a+a(1-a^2-b^2) & 1-a^2-b^2 \end{vmatrix}$$

$\begin{pmatrix} C_1 \to C_1 - bC_3 \\ C_2 \to C_2 + aC_3 \end{pmatrix}$

$$= \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b(1+a^2+b^2) & -a(1+a^2+b^2) & 1-a^2-b^2 \end{vmatrix}$$

$$= (1+a^2+b^2)^2\begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$$

$$= (1+a^2+b^2)^2\begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ 0 & -a & 1-a^2+b^2 \end{vmatrix}$$

$(R_3 \to R_3 - bR_1)$

$$= (1+a^2+b^2)^2\begin{vmatrix} 1 & 2a \\ -a & 1-a^2+b^2 \end{vmatrix}$$

$= (1 + a^2 + b^2)^2 [(1 - a^2 + b^2 + 2a^2)]$

$= (1 + a^2 + b^2)^3$

7. The given determinant is

$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$

$$= \begin{vmatrix} (b+c)^2-a^2 & 0 & a^2 \\ 0 & (c+a)^2-b^2 & b^2 \\ c^2-(a+b)^2 & c^2-(a+b)^2 & (a+b)^2 \end{vmatrix}$$

$\begin{pmatrix} C_1 \to C_1 - C_3 \\ C_2 \to C_2 - C_3 \end{pmatrix}$

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ c-a-b & c-a-b & (a+b)^2 \end{vmatrix}$$

$$= (a+b+c)^2 \begin{vmatrix} b+c-a & 0 & a^2 \\ 0 & c+a-b & b^2 \\ -2b & -2a & 2ab \end{vmatrix}$$

$(R_3 \to R_3 - (R_1 + R_2))$

$([= 2ab(c+a)(b+c-a) + 2a^2b(c+a-b)$

$= 2ab[(c+a)(b+c+a) + a(c+a-b)]$

$= 2ab[b(c+a)(b+c-a) + a(c+a-b)]$

$= 2ab[bc + c^2 + ac]$

$= 2abc(a+b+c)])$

$= 2abc(a+b+c)^3$

8. Let $s-a=p$, $s-b=q$, $s-c=r$

Then $q+r = 2s-(b+c) = a$

Similarly, $r+p=b$, $p+q=c$

and $p+q+r = 3s-(a+b+c) = 3s-2s = s$

The given determinant reduces to

$$\begin{vmatrix} (q+r)^2 & p^2 & p^2 \\ q^2 & (r+p)^2 & q^2 \\ r^2 & r^2 & (p+q)^2 \end{vmatrix}$$

$= 2pqr(p+q+r)^3$

$= 2(s-a)(s-b)(s-c)s^3$

$= 2s^3(s-a)(s-b)(s-c)$

9. Given $\alpha+\beta = -5$, $\alpha\beta = 3$.

We have,

$$\begin{vmatrix} 3 & 1+s_1 & 1+s_2 \\ 1+s_1 & 1+s_2 & 1+s_3 \\ 1+s_2 & 1+s_3 & 1+s_4 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & (1+\alpha+\beta) & (1+\alpha^2+\beta^2) \\ (1+\alpha+\beta) & (1+\alpha^2+\beta^2) & (1+\alpha^3+\beta^3) \\ (1+\alpha^2+\beta^2) & (1+\alpha^3+\beta^3) & (1+\alpha^4+\beta^4) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix}$$

$= \{(\alpha-\beta)(1-\beta)(\alpha-\beta)\}^2$

$= \{(1+(\alpha+\beta)+\alpha\beta)\sqrt{(\alpha+\beta)^2 - 4\alpha\beta}\}^2$

$= \{(1-5+3)\sqrt{25-4.3}\}^2$

$= 13$

10. We have,

$$\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} -abc & ab^2+abc & ac^2+abc \\ a^2b+abc & -abc & c^2b+abc \\ a^2b+abc & b^2c+abc & -abc \end{vmatrix}$$

$$= \frac{abc}{abc}\begin{vmatrix} -bc & ab+ac & ac+ab \\ ab+bc & -ac & cb+ab \\ ab+bc & bc+ac & -ab \end{vmatrix}$$

$$= \begin{vmatrix} -bc & ab+ac & ac+ab \\ ab+bc & -ac & cb+ab \\ ab+bc & bc+ac & -ab \end{vmatrix}$$

$$= \begin{vmatrix} (ab+bc+ca) & (ab+bc+ca) & (ab+bc+ca) \\ ab+bc & -ac & cb+ab \\ ab+bc & bc+ac & -ab \end{vmatrix}$$

$$= (ab+bc+ca)\begin{vmatrix} 1 & 1 & 1 \\ ab+bc & -ac & cb+ab \\ ab+bc & bc+ac & -ab \end{vmatrix}$$

$$= (ab+bc+ca)\begin{vmatrix} 1 & 0 & 0 \\ ab+bc & -(ab+bc+ca) & 0 \\ ab+bc & 0 & -(ab+bc+ca) \end{vmatrix}$$

$$= (ab+bc+ca)\begin{vmatrix} -(ab+bc+ca) & 0 \\ 0 & -(ab+bc+ca) \end{vmatrix}$$

$= (ab+bc+ca)^3$

11. We have,

$$\begin{vmatrix} [x+1] & [y] & [z] \\ [x] & [y+1] & [z] \\ [x] & [y] & [z+1] \end{vmatrix}$$

$$= \begin{vmatrix} [x]+1 & [y] & [z] \\ [x] & [y]+1 & [z] \\ [x] & [y] & [z]+1 \end{vmatrix}$$

$$= \begin{vmatrix} ([x]+[y]+[z]+1) & [y] & [z] \\ ([x]+[y]+[z]+1) & [y]+1 & [z] \\ ([x]+[y]+[z]+1) & [y] & [z]+1 \end{vmatrix}$$

$(C_1 \to C_1 + C_2 + C_3)$

$$= ([x]+[y]+[z]+1)\begin{vmatrix} 1 & [y] & [z] \\ 1 & [y]+1 & [z] \\ 1 & [y] & [z]+1 \end{vmatrix}$$

$$= ([x]+[y]+[z]+1)\begin{vmatrix} 1 & [y] & [z] \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$(R_2 \to R_2 - R_1, R_3 \to R_3 - R_1)$

$= ([x] + [y] + [z] + 1)$
$= (-1 + 0 + 1 + 1)$
$= 1$
$= [z]$

12. We have,

$$\begin{vmatrix} a & b-c & b+c \\ a+c & b & c-a \\ a-b & a+b & c \end{vmatrix} = \begin{vmatrix} a & b-c & b+c \\ c & c & -(a+b) \\ -b & a+c & -b \end{vmatrix}$$

$(R_2 \to R_2 - R_1, R_3 \to R_3 - R_1)$

$$= \begin{vmatrix} a & b-c-a & b+c-a \\ c & 0 & -(a+b+c) \\ -b & a+c+b & 0 \end{vmatrix}$$

$(C_2 \to C_2 - C_1, C_3 \to C_3 - C_1)$

$= b(a+b+c)(b-c-a)$
$\quad + (a+b+c)\{a(a+b+c) + c(b+c-a)\}$
$= (a+b+c)(b^2 - bc - ab + a^2 + ab + ac + bc + c^2 - ac)$
$= (a+b+c)(a^2 + b^2 + c^2)$

13. We have,

$$\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix}$$

$$= \begin{vmatrix} b^2+c^2 & a^2 & bc \\ c^2+a^2 & b^2 & ca \\ a^2+b^2 & c^2 & ab \end{vmatrix} \quad (C_1 \to C_1 - 2C_3)$$

$$= \begin{vmatrix} a^2+b^2+c^2 & a^2 & bc \\ b^2+c^2+a^2 & b^2 & ca \\ a^2+b^2+c^2 & c^2 & ab \end{vmatrix} \quad (C_1 \to C_1 + C_2)$$

$$= (a^2+b^2+c^2)\begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ca \\ 1 & c^2 & ab \end{vmatrix}$$

$$= (a^2+b^2+c^2)\begin{vmatrix} 1 & a^2 & bc \\ 0 & b^2-a^2 & c(a-b) \\ 0 & c^2-a^2 & b(a-c) \end{vmatrix}$$

$(R_2 \to R_2 - R_1, R_3 \to R_3 - R_1)$

$$= (a^2+b^2+c^2)\begin{vmatrix} b^2-a^2 & c(a-b) \\ c^2-a^2 & b(a-c) \end{vmatrix}$$

$$= (a^2+b^2+c^2)(b-a)(c-a)\begin{vmatrix} (b+a) & -c \\ (c+a) & -b \end{vmatrix}$$

$= (a^2+b^2+c^2)(b-a)(c-a)(c^2 + ac - b^2 - ab)$
$= (a^2+b^2+c^2)(b-a)(c-a)(c-b)(c+a+b)$
$= (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)$

14. We have,

$$\begin{vmatrix} 3 & a+b+c & a^3+b^3+c^3 \\ a+b+c & a^2+b^2+c^2 & a^4+b^4+c^4 \\ a^2+b^2+c^2 & a^3+b^3+c^3 & a^5+b^5+c^5 \end{vmatrix}$$

$$= \begin{vmatrix} 3 & a+b+c & a^3+b^3+c^3 \\ a+b+c & a^2+b^2+c^2 & a^4+b^4+c^4 \\ a^2+b^2+c^2 & a^3+b^3+c^3 & a^5+b^5+c^5 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$$

$= (a-b)(b-c)(c-a)$
$\quad \times (a-b)(b-c)(c-a)(a+b+c)$
$= (a-b)^2(b-c)^2(c-a)^2(a+b+c)$

15. We have,

$$\begin{vmatrix} -1+\cos B & \cos B + \cos C & \cos B \\ \cos A + \cos C & -1+\cos A & \cos A \\ -1+\cos B & -1+\cos A & -1 \end{vmatrix}$$

$$= \begin{vmatrix} -1 & \cos C & \cos B \\ \cos C & -1 & \cos A \\ \cos B & \cos A & -1 \end{vmatrix}$$

$(C_2 \to C_2 - C_1, C_3 \to C_3 - C_1)$

$= -(1-\cos^2 A) - \cos C(-\cos C - \cos A \cos B)$
$\quad + \cos B(\cos A \cos C + \cos B)$
$= -\sin^2 A - \cos C[\cos(A+B) - \cos A \cos B]$
$\quad + \cos B[\cos A \cos C - \cos(A+C)]$
$= -\sin^2 A - \cos C(-\sin A \sin B)$
$\quad + \cos B(\sin A \sin C)$
$= -\sin^2 A + \sin A(\sin B \cos C + \cos B \sin C)$
$= -\sin^2 A + \sin A[\sin(B+C)]$
$= -\sin^2 A + \sin A \sin(\pi - A)$
$= -\sin^2 A + \sin A \cdot \sin A$
$= -\sin^2 A + \sin^2 A$
$= 0$

16. Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

The determinant of the co-factors of Δ is

$$\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ ca-b^2 & ab-c^2 & bc-a^2 \\ ab-c^2 & bc-a^2 & ac-b^2 \end{vmatrix}$$

$= \Delta^{3-1} = \Delta^2$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & -c & b \\ b & -a & c \\ c & -b & a \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & b & c \\ -c & -a & -b \\ b & c & a \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & c^2 & 2ac - b^2 \\ 2ab - c^2 & b^2 & a^2 \\ b^2 & 2bc - a^2 & c^2 \end{vmatrix}$$

Hence, the result.

17. Let $\Delta = \begin{vmatrix} a & 0 & c \\ a & b & 0 \\ 0 & b & c \end{vmatrix}$

The determinant of the co-factors of Δ is

$$\begin{vmatrix} bc & -ca & ab \\ bc & ca & -ab \\ -bc & ca & ab \end{vmatrix}$$

$$= \Delta^{3-1} = \Delta^2$$

$$= \begin{vmatrix} a & 0 & c \\ a & b & 0 \\ 0 & b & c \end{vmatrix}^2$$

$$= \begin{vmatrix} a & 0 & c \\ a & b & 0 \\ 0 & b & c \end{vmatrix} \times \begin{vmatrix} a & 0 & c \\ a & b & 0 \\ 0 & b & c \end{vmatrix}$$

$$= \begin{vmatrix} c^2 + a^2 & a^2 & c^2 \\ a^2 & a^2 + b^2 & b^2 \\ c^2 & b^2 & b^2 + c^2 \end{vmatrix}$$

18. Let $\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

The determinant of the co-factors of Δ is

$$\begin{vmatrix} bc - a^2 & ca - b^2 & ab - c^2 \\ ca - b^2 & ab - c^2 & bc - a^2 \\ ab - c^2 & bc - a^2 & ac - b^2 \end{vmatrix}$$

$$= \Delta^{3-1} = \Delta^2$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= \begin{vmatrix} a^2 + b^2 + c^2 & ab + bc + ca & ab + bc + ca \\ ab + bc + ca & a^2 + b^2 + c^2 & ab + bc + ca \\ ab + bc + ca & ab + bc + ca & a^2 + b^2 + c^2 \end{vmatrix}$$

$$= \begin{vmatrix} u^2 & v^2 & v^2 \\ v^2 & u^2 & v^2 \\ v^2 & v^2 & u^2 \end{vmatrix}$$

19. Let $\frac{x^2}{a^2} = X, \frac{y^2}{b^2} = Y$ and $\frac{z^2}{c^2} = Z$.

The given system of equations reduces to

$$X + Y - Z = 1$$
$$X - Y + Z = 1$$
$$X + Y + Z = 1$$

It can be written in matrix form as

$$\begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$AX' = B$, where

$$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, X' = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, B = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Now, $|A| = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = -4 \neq 0$

Thus, the system of equations have a unique solution.

20. Given $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Now, $AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$= \begin{pmatrix} a + 2c & b + 2d \\ 3a + 4c & 3b + 4d \end{pmatrix}$$

Also, $BA = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$$= \begin{pmatrix} a + 3b & 2a + 4b \\ c + 3d & 2c + 4d \end{pmatrix}$$

It is given that

$$AB = BA$$

$$\Rightarrow \begin{pmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{pmatrix} = \begin{pmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{pmatrix}$$

$$\Rightarrow \begin{aligned} a+2c &= a+3b \\ b+2d &= 2a+4b \\ 3a+4c &= c+3d \\ 3b+4d &= 2c+4d \end{aligned}$$

$$\Rightarrow \begin{aligned} 2c &= 3b \\ 2a-2d &= -3b \end{aligned}$$

Now,

$$\frac{a-d}{3b-c} = \frac{-\frac{3}{2}b}{2c-c} = -\frac{\frac{3}{2}b}{c} = -\frac{\frac{3}{2}b}{\frac{3}{2}b} = -1$$

21. Let $U_1 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $U_2 = \begin{pmatrix} d \\ e \\ f \end{pmatrix}$

Given $AU_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} a \\ 2a+b \\ 3a+2b+c \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow U_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Similarly, we can easily find that,

$$U_2 = \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}$$

Thus, $U_1 + U_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$

22. Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Then $|A| = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1$

$= (2-1)(-1)^{2-1}$

Again, let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$\Rightarrow A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

Thus, $|A| = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$

$$= 0 - \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1 + 1 = 2$$

$$= (3-1)(-1)^{3-1}$$

Therefore, in general

$$|A| = \begin{vmatrix} 0 & 1 & \dots & 1 \\ 1 & 0 & \dots & 1 \\ \vdots & & & \\ 1 & 1 & \dots & 0 \end{vmatrix}$$

$$= (n-1)(-1)^{n-1}$$

23. Since the system of equations has non-trivial solution, so $D = 0$

$$\Rightarrow \begin{vmatrix} a-t & b & c \\ b & c-t & a \\ c & a & a-t \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a+b+c-t & b & c \\ a+b+c-t & c-t & a \\ a+b+c-t & a & a-t \end{vmatrix} = 0$$

$$\Rightarrow (a+b+c-t)\begin{vmatrix} 1 & b & c \\ 1 & c-t & a \\ 1 & a & a-t \end{vmatrix} = 0$$

$$\Rightarrow (a+b+c-t)\begin{vmatrix} 1 & b & c \\ 0 & c-b-t & a-c \\ 0 & a-b & a-c-t \end{vmatrix} = 0$$

$$\Rightarrow (a+b+c-t)\begin{vmatrix} c-b-t & a-c \\ a-b & a-c-t \end{vmatrix} = 0$$

$$\Rightarrow (a+b+c-t) = 0, \begin{vmatrix} c-b-t & a-c \\ a-b & a-c-t \end{vmatrix} = 0$$

$$\Rightarrow t = (a+b+c),\ t^2 + (a+b)t + (2ac - c^2) = 0$$

$$\Rightarrow t = (a+b+c),$$

$$t = \frac{-(a+b) \pm \sqrt{(a+b)^2 - 4(2ac - c^2)}}{2}$$

Thus, the number of values of t is 3.

24. We have,

$$(I + A) = \begin{pmatrix} 1 & -\tan\left(\frac{\alpha}{2}\right) \\ \tan\left(\frac{\alpha}{2}\right) & 1 \end{pmatrix}$$

and

$$(I - A) = \begin{pmatrix} 1 & \tan\left(\frac{\alpha}{2}\right) \\ -\tan\left(\frac{\alpha}{2}\right) & 1 \end{pmatrix}$$

Let $I - A = B$.

Now, $B^{-1} = \dfrac{\text{adj}\,(B)}{|B|}$

$$= \frac{1}{\sec^2\left(\frac{\alpha}{2}\right)}\begin{pmatrix} 1 & -\tan\left(\frac{\alpha}{2}\right) \\ \tan\left(\frac{\alpha}{2}\right) & 1 \end{pmatrix}$$

Thus,
$(I - A)^{-1}\,(I + A) = B^{-1}(I + A)$

$$= \frac{1}{\sec^2\left(\frac{\alpha}{2}\right)}\begin{pmatrix} 1 & -\tan\left(\frac{\alpha}{2}\right) \\ \tan\left(\frac{\alpha}{2}\right) & 1 \end{pmatrix}\begin{pmatrix} 1 & -\tan\left(\frac{\alpha}{2}\right) \\ \tan\left(\frac{\alpha}{2}\right) & 1 \end{pmatrix}$$

$$= \frac{1}{\sec^2\left(\frac{\alpha}{2}\right)} \times \begin{pmatrix} 1-\tan^2\left(\frac{\alpha}{2}\right) & -2\tan\left(\frac{\alpha}{2}\right) \\ 2\tan\left(\frac{\alpha}{2}\right) & 1-\tan^2\left(\frac{\alpha}{2}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

25. The given determinant is

$$\begin{vmatrix} {}^xC_1 & {}^xC_2 & {}^xC_3 \\ {}^yC_1 & {}^yC_2 & {}^yC_3 \\ {}^zC_1 & {}^zC_2 & {}^zC_3 \end{vmatrix}$$

$$= \begin{vmatrix} x & \frac{x(x-1)}{2} & \frac{x(x-1)(x-2)}{6} \\ y & \frac{y(y-1)}{2} & \frac{y(y-1)(y-2)}{6} \\ z & \frac{z(z-1)}{2} & \frac{z(z-1)(z-2)}{6} \end{vmatrix}$$

$$= \frac{xyz}{12}\begin{vmatrix} 1 & (x-1) & (x-1)(x-2) \\ 1 & (y-1) & (y-1)(y-2) \\ 1 & (z-1) & (z-1)(z-2) \end{vmatrix}$$

$$= \frac{xyz}{12}\begin{vmatrix} 1 & (x-1) & (x-1)(x-2) \\ 0 & (y-x) & (y^2-x^2)-3(y-x) \\ 0 & (x-x) & (z^2-x^2)-3(z-x) \end{vmatrix}$$

$$\begin{pmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$$

$$= \frac{xyz}{12}\begin{vmatrix} (y-x) & (y^2-x^2)-3(y-x) \\ (z-x) & (z^2-x^2)-3(z-x) \end{vmatrix}$$

$$= \frac{xyz(y-x)(z-x)}{12}\begin{vmatrix} 1 & y+x-3 \\ 1 & z+x-3 \end{vmatrix}$$

$$= \frac{xyz(y-x)(z-x)(z-y)}{12}$$

$$= \frac{xyz(x-y)(y-z)(z-x)}{12}$$

26. We have,

$$\Delta = \begin{vmatrix} f & g & h \\ (xf)' & (xg)' & (xh)' \\ (x^2f)'' & (x^2g)'' & (x^2h)'' \end{vmatrix}$$

$$= \begin{vmatrix} f & g & h \\ f + xf' & g + xg' & h + xh' \\ 4xf' + x^2f' & 4xg' + x^2g' & 4xh' + x^2h' \end{vmatrix}$$

$$= \begin{vmatrix} f & g & h \\ xf' & xg' & xh' \\ x^2f'' & x^2g'' & x^2h'' \end{vmatrix}$$

$$= x^3 \times \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix}$$

Differentiating w.r.t x, we get

$$\Delta' = 3x^2 \times \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{vmatrix} + x^3 \times \begin{vmatrix} f & g & h \\ f' & g' & h' \\ f''' & g''' & h''' \end{vmatrix}$$

$$= \begin{vmatrix} f & g & h \\ f' & g' & h' \\ 3x^2f'' & 3x^2g'' & 3x^2h'' \end{vmatrix} + \begin{vmatrix} f & g & h \\ f' & g' & h' \\ x^3f'' & 3x^3g'' & 3x^3h'' \end{vmatrix}$$

$$= \begin{vmatrix} f & g & h \\ f' & g' & h' \\ 3x^2f'' + x^3f''' & 3x^2g'' + x^3g''' & 3x^2h'' + x^3h''' \end{vmatrix}$$

$$= \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3f'')' & (x^3g'')' & (x^3f'')' \end{vmatrix}$$

27. The given determinant is

$$\begin{vmatrix} \sqrt{13}+\sqrt{3} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{15}+\sqrt{26} & 5 & \sqrt{10} \\ 3+\sqrt{65} & \sqrt{15} & 5 \end{vmatrix}$$

$$= \begin{vmatrix} \sqrt{13} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{26} & 5 & \sqrt{10} \\ \sqrt{65} & \sqrt{15} & 5 \end{vmatrix} + \begin{vmatrix} \sqrt{3} & 2\sqrt{5} & \sqrt{5} \\ \sqrt{15} & 5 & \sqrt{10} \\ 3 & \sqrt{15} & 5 \end{vmatrix}$$

$$= \sqrt{13}\cdot\sqrt{5}\cdot\sqrt{5}\begin{vmatrix} 1 & 2 & 1 \\ \sqrt{2} & \sqrt{5} & \sqrt{2} \\ \sqrt{5} & \sqrt{3} & \sqrt{5} \end{vmatrix}$$

$$+ \sqrt{3}\cdot\sqrt{5}\cdot\sqrt{5}\begin{vmatrix} 1 & 2 & 1 \\ \sqrt{5} & \sqrt{5} & \sqrt{2} \\ \sqrt{3} & \sqrt{3} & \sqrt{5} \end{vmatrix}$$

$$= 0 + \sqrt{3}\cdot\sqrt{5}\cdot\sqrt{5}\begin{vmatrix} 1 & 2 & 1 \\ \sqrt{5} & \sqrt{5} & \sqrt{2} \\ \sqrt{3} & \sqrt{3} & \sqrt{5} \end{vmatrix}$$

$$= 5\sqrt{3}\begin{vmatrix} 1 & 2 & 1 \\ 0 & -\sqrt{5} & \sqrt{2}-\sqrt{5} \\ 0 & -\sqrt{3} & \sqrt{5}-\sqrt{3} \end{vmatrix}$$

$$= -5\sqrt{3}\begin{vmatrix} 1 & 2 & 1 \\ 0 & \sqrt{5} & \sqrt{2}-\sqrt{5} \\ 0 & \sqrt{3} & \sqrt{5}-\sqrt{3} \end{vmatrix}$$

$$= -5\sqrt{3}[(5-\sqrt{15})-(\sqrt{6}-\sqrt{15})]$$

$$= -5\sqrt{3}(5-\sqrt{6})$$

$$= 5\sqrt{3}(\sqrt{6}-5)$$

28. We have

$$D = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 1 \\ 3 & 2 & 9 \end{vmatrix} = -20$$

$$D_1 = \begin{vmatrix} 6 & 2 & 3 \\ 17 & 4 & 1 \\ 2 & 2 & 9 \end{vmatrix} = -20$$

$$D_2 = \begin{vmatrix} 1 & 6 & 3 \\ 2 & 17 & 1 \\ 3 & 2 & 9 \end{vmatrix} = -80$$

$$D_3 = \begin{vmatrix} 1 & 2 & 6 \\ 2 & 4 & 17 \\ 3 & 2 & 2 \end{vmatrix} = 20$$

Thus,

$$x = \frac{D_1}{D} = \frac{-20}{-20} = 1$$

$$y = \frac{D_2}{D} = \frac{-80}{-20} = 4$$

$$z = \frac{D_3}{D} = \frac{20}{-20} = -1$$

29. We have,

$$D = \begin{vmatrix} 2 & p & 6 \\ 1 & 2 & q \\ 1 & 1 & 3 \end{vmatrix}$$

$$D_1 = \begin{vmatrix} 8 & p & 6 \\ 5 & 2 & q \\ 4 & 1 & 3 \end{vmatrix}$$

$$D_2 = \begin{vmatrix} 2 & 8 & 6 \\ 1 & 5 & q \\ 1 & 4 & 3 \end{vmatrix}$$

$$D_3 = \begin{vmatrix} 2 & p & 8 \\ 1 & 2 & 5 \\ 1 & 1 & 4 \end{vmatrix}$$

(i) The given system of equations has no solution if $D = 0$ and any one of D_1, D_2 and D_3 is non-zero.
Now, $D = 0$ gives

$$\begin{vmatrix} 2 & p & 6 \\ 1 & 2 & q \\ 1 & 1 & 3 \end{vmatrix} = 0$$

$\Rightarrow$ $2(6-q) - p(3-q) + 6(1-2) = 0$

$\Rightarrow$ $12 - 2q - 3p + pq - 6 = 0$

$\Rightarrow$ $6 - 2q - 3p + pq = 0$

$\Rightarrow$ $(3-q)(2-p) = 0$

$\Rightarrow$ $p = 2,\ q = 3$

Also, $D_1 = \begin{vmatrix} 8 & p & 6 \\ 5 & 2 & q \\ 4 & 1 & 3 \end{vmatrix} = (2-p)(15-4q)$

Thus, $D_1 \neq 0$ gives
$(2-p)(15-4q) \neq 0$

$\Rightarrow$ $p \neq 2,\ q \neq \frac{15}{4}$

Hence, the given system of equations has no solution, if $p \neq 2$, $q \neq 3$.

(ii) The given system of equations has a unique solution, if $D \neq 0$.

$\Rightarrow$ $(p-2)(q-3) \neq 0$

$\Rightarrow$ $p \neq 2$ and $q \neq 3$

(iii) The given system of equations has infinitely many solutions, if $D = 0 = D_1 + D_2 = D_3$

Here $D = (p-2)(q-3)$,
$D_1 = (p-2)(4q-15)$
$D_2 = 0, D_3 = (p-2)$

Thus, the system of equations has infinitely many solutions if $p = 2, q = 3, \frac{15}{4}$.

30. Since the given system of equations are consistent, so

$$\begin{vmatrix} 2 & 3 & -3 \\ (c+2) & (c+4) & -(c+6) \\ (c+2)^2 & (c+4)^2 & -(c+6)^2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2 & 0 & -3 \\ (c+2) & -2 & -(c+6) \\ (c+2)^2 & -4(c+5) & -(c+6)^2 \end{vmatrix} = 0$$

$(C_2 \rightarrow C_2 + C_3)$

$$\Rightarrow -2\begin{vmatrix} 2 & 0 & -3 \\ (c+2) & 1 & -(c+6) \\ (c+2)^2 & 2(c+5) & -(c+6)^2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2 & 0 & -1 \\ (c+2) & 1 & -4 \\ (c+2)^2 & 2(c+5) & -8(c+4) \end{vmatrix} = 0$$

$(C_3 \rightarrow C_3 + C_1)$

$$\Rightarrow \begin{vmatrix} 0 & 0 & -1 \\ (c-6) & 1 & -4 \\ (c^2-12c-60) & 2(c+5) & -8(c+4) \end{vmatrix} = 0$$

$(C_1 \rightarrow C_2 + 2C_3)$

$$\Rightarrow \begin{vmatrix} (c-6) & 1 \\ (c^2-12c-60) & 2(c+5) \end{vmatrix} = 0$$

$\Rightarrow 2(c+5)(c-6) - (c^2 - 12c - 60) = 0$
$\Rightarrow 2(c^2 - c - 30) - (c^2 - 12c - 60) = 0$
$\Rightarrow c^2 + 10c = 0$
$\Rightarrow c(c+10) = 0$
$\Rightarrow c = 0, c = -10$

Here, $c = 0$ is not the solution

So $c = -10$

Solving, we get,

$$x = -\frac{1}{2}, y = \frac{1}{3}.$$

31. The system of equations has a solution, if

$D = 0 = D_1 = D_2 = D_3$.

Here, $D = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & 4 \\ 1 & 5 & 10 \end{vmatrix} = 0$

$$D_1 = \begin{vmatrix} 1 & 2 & 1 \\ k & 3 & 4 \\ k^2 & 5 & 10 \end{vmatrix} = (k-1)(k-2)$$

$$D_2 = \begin{vmatrix} 1 & 1 & 1 \\ 1 & k & 4 \\ 1 & k^2 & 10 \end{vmatrix} = (k-1)(k-2)$$

$$D_3 = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 3 & k \\ 1 & 5 & k^2 \end{vmatrix} = (k-1)(k-2)$$

Thus, the system of equations has a solutions, if $k = 1$ and $k = 2$

When $k = 1$, $x = 1 + 5t$, $y = -3t$, $t \in R$

When $k = 2$, $x = -1 + 5t$, $y = 1 - 3t$, $t \in R$

32. (i) The system of equations have a unique solution if

$$\begin{vmatrix} 2 & -3 & 5 \\ 3 & 1 & \lambda \\ 1 & -7 & 8 \end{vmatrix} \neq 0$$

$\Rightarrow 2(8 + 7\lambda) + 3(24 - \lambda) + 5(-21 - 1) = 0$
$\Rightarrow 16 + 14\lambda + 72 - 3\lambda - 110 \neq 0$
$\Rightarrow 11\lambda - 22 \neq 0$
$\Rightarrow \lambda \neq 2$

(ii) The system of equations has infinitely many solutions, if $D = 0 = D_1 = D_2 = D_3$

We have,

$$D = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 1 & \lambda \\ 1 & -7 & 8 \end{vmatrix} = 0 \Rightarrow \lambda = 2$$

$$D_1 = \begin{vmatrix} 12 & -3 & 5 \\ \mu & 1 & \lambda \\ 17 & -7 & 8 \end{vmatrix} = 0 \Rightarrow \mu = 7$$

$$D_2 = \begin{vmatrix} 2 & 12 & 5 \\ 3 & \mu & \lambda \\ 1 & 17 & 8 \end{vmatrix} = 0$$

and

$$D_3 = \begin{vmatrix} 2 & -3 & 12 \\ 3 & 1 & \mu \\ 1 & -7 & 17 \end{vmatrix} = 0 \Rightarrow \mu = 7$$

Thus, the system of equations has infinitely many solutions, if $\lambda = 2$, $\mu = 7$.

(iii) The given system of equations has no solution if $D = 0$ and any one of D_1, D_2, D_3 is non-zero.

Thus, $D = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 1 & \lambda \\ 1 & -7 & 8 \end{vmatrix} = 0 \Rightarrow \lambda = 2$

and

$$D_1 = \begin{vmatrix} 12 & -3 & 5 \\ \mu & 1 & \lambda \\ 17 & -7 & 8 \end{vmatrix} \neq 0 \Rightarrow \mu \neq 7$$

Hence, the system of equations has no solution, if $\lambda = 2, \mu \neq 7$

33. We have,

$$\begin{vmatrix} 1 & 4 & 5 \\ \lambda & 8 & 8\lambda - 6 \\ 1+\lambda^2 & 8\lambda + 4 & 2\lambda + 21 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 4 & 5 \\ \lambda & 8 & 8\lambda - 6 \\ 0 & 0 & 16 - 8\lambda^2 + 8\lambda \end{vmatrix}$$

$$= (16 - 8\lambda^2 + 8\lambda)\begin{vmatrix} 1 & 4 \\ \lambda & 8 \end{vmatrix}$$

$$= -8(\lambda^2 - \lambda - 2)(8 - 4\lambda)$$

$$= 32(\lambda^2 - \lambda - 2)(\lambda - 2)$$

$$= 32(\lambda - 2)(\lambda + 1)(\lambda - 2)$$

$$= 32(\lambda - 2)^2(\lambda + 1)$$

Thus, the rank of the matrix is 3, when λ –1, 2 and the rank of the matrix is 1 when $\lambda = 2$ and also, the rank of the matrix is 2, when $\lambda = -1$.

34. The given system of equations has a non-trivial solution, if

$$\begin{vmatrix} 2r & -2 & 3 \\ 1 & r & 2 \\ 2 & 0 & r \end{vmatrix} = 0$$

$\Rightarrow \quad 2(-4 - 3r) + r(2r^2 + 2) = 0$
$\Rightarrow \quad 2r^3 + 2r - 6r - 8 = 0$
$\Rightarrow \quad 2r^3 - 4r - 8 = 0$
$\Rightarrow \quad r^3 - 2r - 4 = 0$
$\Rightarrow \quad r = 2$

Thus, the system of equations becomes

$4x - 2y + 3z = 0$
$x + 2y + 2z = 0$
$2x + 2z = 0$
$\Rightarrow \quad x = 2y = -z$

Hence, the non-trivial solutions are

$$\left(-k, \frac{k}{2}, k\right), k \in R - \{0\}.$$

35. $$\begin{vmatrix} a^2 & a & 1 \\ \sin(n+1)x & \sin nx & \sin(n-1)x \\ \cos(n+1)x & \cos nx & \cos(n-1)x \end{vmatrix} = 0$$

$\Rightarrow \quad a^2[\sin\{nx - (n-1)x\}] - a[\sin\{(n+1)x - (n-1)x\}] + [\sin\{(n+1)x - nx\}] = 0$

$\Rightarrow \quad a^2 \sin x - a \sin(2x) + \sin x = 0$
$\Rightarrow \quad (a^2 + 1)\sin x - a\sin(2x) = 0$
$\Rightarrow \quad (a^2 + 1)\sin x - 2a\sin x\cos x = 0$
$\Rightarrow \quad \sin x((a^2 + 1) - 2a\cos x) = 0$
$\Rightarrow \quad \sin x = 0, ((a^2 + 1) - 2a\cos x) = 0$

$$\Rightarrow \quad \sin x = 0, \cos x = \left(\frac{a^2+1}{2a}\right)$$

$$\Rightarrow \quad x = n\pi, x = 2m\pi \pm \cos^{-1}\left(\frac{a^2+1}{2a}\right), m, n \in I$$

Level IV

1. The given determinant is

$$\begin{vmatrix} 1 & \cos x - \sin x & \cos x + \sin x \\ 1 & \cos y - \sin y & \cos y + \sin y \\ 1 & \cos z - \sin z & \cos z + \sin z \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 2\cos x & \cos x + \sin x \\ 1 & 2\cos y & \cos y + \sin y \\ 1 & 2\cos z & \cos z + \sin z \end{vmatrix} \qquad (C_2 \to C_2 + C_3)$$

$$= 2\begin{vmatrix} 1 & \cos x & \cos x + \sin x \\ 1 & \cos y & \cos y + \sin y \\ 1 & \cos z & \cos z + \sin z \end{vmatrix}$$

$$= 2\begin{vmatrix} 1 & \cos x & \sin x \\ 1 & \cos y & \sin y \\ 1 & \cos z & \sin z \end{vmatrix} \qquad (C_3 \to C_3 - C_2)$$

2. The given determinant is

$$\begin{vmatrix} 2 & \alpha+\beta+\gamma+\delta & \alpha\beta+\gamma\delta \\ \alpha+\beta+\gamma+\delta & 2(\alpha+\beta)(\gamma+\delta) & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) \\ \alpha\beta+\gamma\delta & \alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta) & 2\alpha\beta\gamma\delta \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 0 \\ \alpha+\beta & \gamma+\delta & 0 \\ \alpha\beta & \gamma\delta & 0 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 0 \\ \gamma+\delta & \alpha+\beta & 0 \\ \gamma\delta & \alpha\beta & 0 \end{vmatrix}$$

$$= 0 \times 0$$

$$= 0$$

3. The given determinant is

$$\begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-Q) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix}$$

$$= \begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix} \times \begin{vmatrix} \cos P & \sin P & 0 \\ \cos Q & \sin Q & 0 \\ \cos R & \sin R & 0 \end{vmatrix}$$

$$= 0 \times 0$$

$$= 0$$

4. The given determinant is

$$\begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^3 & \beta^3 & \gamma^3 \end{vmatrix} \times \begin{vmatrix} 1 & 1 & 1 \\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1+1+1 & \alpha+\beta+\gamma & \alpha^2+\beta^2+\gamma^2 \\ \alpha+\beta+\gamma & \alpha^2+\beta^2+\gamma^2 & \alpha^3+\beta^3+\gamma^3 \\ \alpha^3+\beta^3+\gamma^3 & \alpha^4+\beta^4+\gamma^4 & \alpha^5+\beta^5+\gamma^5 \end{vmatrix}$$

$$= \begin{vmatrix} S_0 & S_1 & S_2 \\ S_1 & S_2 & S_3 \\ S_3 & S_4 & S_5 \end{vmatrix}$$

Hence, the result.

5. Do yourself.
6. Do yourself.
7. Let $A = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, where $a \neq 0$.

Show that for $n \geq 0$,

$$A^n = \begin{pmatrix} a^n & \dfrac{b(a^n-1)}{a-1} \\ 0 & 1 \end{pmatrix}$$

8. We have,
$A(BC)$

$$= [x\ y\ z]\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= [x\ y\ z]\begin{bmatrix} ax+hy+gz \\ hx+by+fz \\ gx+fy+cz \end{bmatrix}$$

$$= [x(ax+hy+gz)+y(hx+by+fz)+z(gx+fy+cz)]$$

$$= (ax^2+by^2+cz^2+2hxy+2fyz+2gzx)$$

9. We have,

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ \cos(p-d)x & \cos(p)x & \cos(p+d)x \\ \cos(p-d)x & \sin(p)x & \sin(p+d)x \end{vmatrix}$$

$= \sin(p+d-p)x - a\sin(p+d-p+d)x + a^2\sin(p-p+d)x$

$= \sin(d)x - a\sin(2d)x + a^2\sin(d)x$

10. We have

$$\Delta = \begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix}$$

$$= \begin{vmatrix} x^2+x & x+1 & x-2 \\ x-1 & x-2 & x+1 \\ x+3 & x-2 & x+1 \end{vmatrix}\begin{pmatrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$$

$$= \begin{vmatrix} x^2+x & x+1 & x-2 \\ x-1 & x-2 & x+1 \\ 4 & 0 & 0 \end{vmatrix} (R_3 \to R_3 - R_2)$$

$$= 4\begin{vmatrix} x+1 & x-2 \\ x-2 & x+1 \end{vmatrix}$$

$$= 4 \times \begin{vmatrix} x+1 & -3 \\ x-2 & 3 \end{vmatrix} (C_2 \to C_2 - C_1)$$

$= 4[3(x+1) + 3(x-2)]$
$= 4(6x-3)$
$= 24x - 12.$
$= Ax + B$, where $A = 24$ and $B = -12$.

11. Given

$A28 = 100A + 20 + 8 = km_1$
$3B9 = 300 + 10B + 9 = km_2$
$62C = 600 + 20 + C = km_3$

We have,

$$\begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$$

$$= \begin{vmatrix} A & 3 & 6 \\ 100A+20+8 & 300+10B+9 & 600+20+C \\ 2 & B & 2 \end{vmatrix}$$

$(R_2 \to R_2 + 100R_1 + 10R_3)$

$$= \begin{vmatrix} A & 3 & 6 \\ km_1 & km_2 & km_3 \\ 2 & B & 2 \end{vmatrix}$$

$$= k \times \begin{vmatrix} A & 3 & 6 \\ m_1 & m_2 & m_3 \\ 2 & B & 2 \end{vmatrix}$$

which is divisible by k.

12. Given $\Delta = \begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} p-a & b-q & 0 \\ 0 & q-b & c-r \\ a & b & r \end{vmatrix} = 0$$

$\Rightarrow (p-a)\{r(q-b) - b(c-r)\} + a(b-q)(c-r) = 0$

$\Rightarrow r(p-a)(q-b) - b(p-a)(c-r) + a(b-q)(c-r) = 0$

$\Rightarrow r(p-a)(q-b) + b(p-a)(r-c) + a(q-b)(r-c) = 0$

Dividing both the sides by $(p-a)(q-b)(r-c)$, we get

$$\frac{r}{r-c} + \frac{b}{q-b} + \frac{a}{p-a} = 0$$

$$\Rightarrow \frac{a}{p-a} + \frac{b}{q-b} + \frac{r}{r-c} = 0$$

$$\Rightarrow \quad \left(\frac{a}{p-a}+1\right)+\left(\frac{b}{q-b}+1\right)+\frac{r}{r-c}=2$$

$$\Rightarrow \quad \frac{p}{p-a}+\frac{q}{q-b}+\frac{r}{r-c}=2$$

$\Rightarrow \quad E = 2$

13. We have,

$$\begin{vmatrix} \sin\theta & \cos\theta & \sin\theta \\ \sin\left(\theta+\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) & \sin\left(\theta+\frac{4\pi}{3}\right) \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(\theta-\frac{4\pi}{3}\right) \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta & \cos\theta & \sin\theta \\ 2\sin\theta\cos\left(\frac{2\pi}{3}\right) & 2\cos\theta\cos\left(\frac{2\pi}{3}\right) & 2\sin 2\theta\cos\left(\frac{4\pi}{3}\right) \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(\theta-\frac{4\pi}{3}\right) \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta & \cos\theta & \sin\theta \\ -\sin\theta & -\cos\theta & -\sin 2\theta \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(\theta-\frac{4\pi}{3}\right) \end{vmatrix}$$

$$= \begin{vmatrix} \sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 0 \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(\theta-\frac{4\pi}{3}\right) \end{vmatrix}$$

$(R_2 \to R_2 + R_1)$

$= 0$

14. The given determinant is

$$\begin{vmatrix} ax-by-c & bx+ay & cx+a \\ bx+ay & -ax+by-c & cy+b \\ cx+a & cy+b & -ax-by+c \end{vmatrix}$$

Applying $C_2 \to aC_1 + bC_2 + cC_3$, we get

$$\frac{1}{a}\begin{vmatrix} (a^2+b^2+c^2)x & bx+ay & cx+a \\ (a^2+b^2+c^2)x & -ax+by-c & cy+b \\ (a^2+b^2+c^2) & cy+b & -ax-by+c \end{vmatrix}$$

$$\frac{1}{a}\begin{vmatrix} x & bx+ay & cx+a \\ y & -ax+by-c & cy+b \\ 1 & cy+b & -ax-by+c \end{vmatrix}$$

Applying $\begin{pmatrix} C_2 \to C_2 - bC_1 \\ C_3 \to C_3 - cC_1 \end{pmatrix}$, we get

$$\frac{1}{a}\begin{vmatrix} x & ay & a \\ y & -ax-c & b \\ 1 & cy & -ax-by \end{vmatrix}$$

Applying $R_1 \to (xR_1 + yR_2 + R_3)$, we get

$$\frac{1}{ax}\begin{vmatrix} x^2+y^2+1 & 0 & 0 \\ y & -ax-c & b \\ 1 & cy & -ax-by \end{vmatrix}$$

$$= \frac{1}{ax}(x^2+y^2+1)((ax)^2+acx+abxy+bcy-bcy)$$

$$= \frac{1}{ax}(x^2+y^2+1)((ax)^2+acx+abxy)$$

$= (x^2 + y^2 + 1)\,(ax + c + by)$

$= (x^2 + y^2 + 1)\,(ax + by + c)$

If the given determinant is zero, then

$(ax + by + c) = 0$, $((x^2 + y^2 + 1) \neq 0)$

Thus, $(ax + by + c) = 0$ represents a straight line.

15. We have,

$$A^TA = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$$

$$= \begin{pmatrix} a^2+b^2+c^2 & ab+bc+ca & ab+bc+ca \\ ab+bc+ca & a^2+b^2+c^2 & ab+bc+ca \\ ab+bc+ac & ab+bc+ca & a^2+b^2+c^2 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha & \beta & \beta \\ \beta & \alpha & \beta \\ \beta & \beta & \alpha \end{pmatrix}$$

where $a^2 + b^2 + c^2 = \alpha$, $ab + bc + ca = \beta$.

Since $AA^T = I$, so, $a^2 + b^2 + c^2 = 1$ and $ab + bc + ca = 0$.

Now,

$a^3 + b^3 + c^3 - 3abc$

$= (a + b + c)\,(a^2 + b^2 + c^2 - ab - bc - ca)$

$= (a + b + c)(1 - 0)$

$= (a + b + c)$

Also, $(a + b + c)^2 = (a^2 + b^2 + c^2 + 2(ab + bc + ca))$

$\Rightarrow \quad (a + b + c)^2 = 1 + 2.0 = 1$

$\Rightarrow \quad (a + b + c) = 1$, since a, b, c are all positive

Thus,

$a^3 + b^3 + c^3 - 3abc = 1$

$\Rightarrow \quad a^3 + b^3 + c^3 - 3abc + 1 = 3 + 1 = 4.$

16. We have,

$$\Delta = \begin{vmatrix} \frac{1}{a} & \frac{1}{a(a+d)} & \frac{1}{(a+d)(a+2d)} \\ \frac{1}{a+d} & \frac{1}{(a+d)(a+2d)} & \frac{1}{(a+2d)(a+3d)} \\ \frac{1}{a+2d} & \frac{1}{(a+2d)(a+3d)} & \frac{1}{(a+3d)(a+4d)} \end{vmatrix}$$

$$= \frac{1}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}$$

$$\times \begin{vmatrix} (a+d)(a+2d) & a+2d & a \\ (a+2d)(a+3d) & (a+3d) & (a+d) \\ (a+3d)(a+4d) & (a+4d) & (a+2d) \end{vmatrix}$$

$$= \frac{1}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)} \times \begin{vmatrix} 0 & 2d & a \\ d(a+3d) & 2d & (a+d) \\ 2d(a+4d) & 2d & (a+2d) \end{vmatrix} \begin{pmatrix} C_1 \to C_1-(a+d)C_2 \\ C_2 \to C_2 - C_3 \end{pmatrix}$$

$$= \frac{2d^2}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)} \times \begin{vmatrix} 0 & 1 & a \\ (a+3d) & 1 & (a+d) \\ 2(a+4d) & 1 & (a+2d) \end{vmatrix}$$

$$= \frac{2d^2}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)} \times \begin{vmatrix} 0 & 1 & a \\ (a+3d) & 0 & d \\ 2(a+4d) & 0 & 2d \end{vmatrix} \begin{pmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$$

$$= \frac{4d^4}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}$$

17. $$D = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+2)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$$

$$= (n)!(n+1)!(n+2)! \begin{vmatrix} 1 & (n+1) & (n+2)(n+1) \\ 1 & (n+2) & (n+3)(n+2) \\ 1 & (n+3) & (n+4)(n+3) \end{vmatrix}$$

$$= (n)!(n+1)!(n+2)! \begin{vmatrix} 1 & (n+1) & (n+2)(n+1) \\ 0 & 1 & 2(n+2) \\ 0 & 1 & 2(n+3) \end{vmatrix}$$

$$= (n)!(n+1)!(n+2)! \begin{vmatrix} 1 & 2(n+2) \\ 1 & 2(n+3) \end{vmatrix}$$

$$= (n)!(n+1)!(n+2)! \times 2 \begin{vmatrix} 1 & (n+2) \\ 1 & (n+3) \end{vmatrix}$$

$$= 2 \times (n)!(n+1)!(n+2)!$$

Now,

$$\left(\frac{D}{(n!)^3} - 4\right) = \left(\frac{2 \times (n)!(n+1)!(n+2)!}{(n!)^3} - 4\right)$$

$$= \left(\frac{2 \times (n!)^3 (n+1)^2 (n+2)}{(n!)^3} - 4\right)$$

$$= (2 \times (n+1)^2(n+2) - 4)$$

$$= (2(n^3 + 4n^2 + 5n + 2) - 4)$$

$$= 2n(n^2 + 4n + 5)$$

which is divisible by n.

18. Solving, we get

$$u = -\frac{1}{3}, v = \frac{2}{3}, w = \frac{5}{3}$$

It is given that a, b, c, d are in GP, so, let

$b = ar, c = ar^2, d = ar^3$

We have

$$[(b-c)^2 + (c-a)^2 + (d-b)^2]$$
$$= (ar - ar^2)^2 + (ar^2 - a)^2 + (ar^3 - ar)^2$$
$$= (a^2r^2 - 2a^2r^3 + a^2r^4) + (a^2r^4 - 2a^2r^2 + a^2) + (a^2r^6 - 2a^2r^4 + a^2r^2)$$
$$= a^2(r^2 - 2r^3 + r^4 + r^4 - r^2 + 1 + r^6 - 2r^4 + r^2)$$
$$= a^2(r^6 - 2r^3 + 1)$$
$$= a^2(r^3 - 1)^2$$
$$= (ar^3 - a)^2$$
$$= (d-a)^2 = (a-d)^2$$

Thus, the given quadratic equation reduces to

$$-\frac{9}{10}x^2 + (a-d)^2 x + 2 = 0$$

$\Rightarrow \quad 9x^2 - 10(a-d)^2 x - 20 = 0$

Replace x by $1/x$, we get

$$9\left(\frac{1}{x}\right)^2 - 10(a-d)^2\left(\frac{1}{x}\right) - 20 = 0$$

$\Rightarrow \quad 9 - 10(a-d)^2 x - 20x^2 = 0$

$\Rightarrow \quad 20x^2 + 10(a-d)^2 x - 9 = 0.$

Hence, the result.

19. The given system of equations will provide us non-trivial solutions if

$\Delta = 0$

$$\Rightarrow \begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a & 1-a & 1-a \\ 1 & b-1 & 0 \\ 1 & 0 & c-1 \end{vmatrix} = 0$$

$$a(b-1)(c-1) + (1-a)(1-c) + (1-a)(1-b) = 0$$
$$a(1-b)(1-c) + (1-a)(1-c) + (1-a)(1-b) = 0$$

Dividing both sides by $(1-a)(1-b)(1-c)$, we get

$$\frac{a}{1-a} + \frac{1}{(1-b)} + \frac{1}{(1-c)} = 0$$

$$\Rightarrow \quad 1 + \frac{a}{1-a} + \frac{1}{(1-b)} + \frac{1}{(1-c)} = 1$$

$$\Rightarrow \quad \frac{1}{1-a} + \frac{1}{(1-b)} + \frac{1}{(1-c)} = 1$$

$$\Rightarrow \quad \frac{1}{1-a} + \frac{1}{(1-b)} + \frac{1}{(1-c)} + 4 = 1 + 4 = 5$$

20. The given system of equations will provide us a non-trivial solutions if

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

$\Rightarrow \quad (a^3 + b^3 + c^3 - 3abc) = 0$

$\Rightarrow \quad (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = 0$

$\Rightarrow \quad (a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega) = 0$

$\Rightarrow \quad (a + b + c) = 0,$

$(a + b\omega + c\omega^2) = 0$

and $(a + b\omega^2 + c\omega) = 0$

When $(a + b + c) = 0$

$\Rightarrow \quad c = -(a + b)$

Now, first two given equations can be written as

$ax + by - (a + b)z = 0$

$bx - (a + b)y + az = 0$

Solving, we get

$$\frac{x}{ab-(a+b)^2} = \frac{y}{-b(a+b)-a^2} = \frac{z}{-a(a+b)-b^2}$$

$$\Rightarrow \frac{x}{-(a^2+ab+b^2)} = \frac{y}{-(a^2+ab+b^2)} = \frac{z}{-(a^2+ab+b^2)}$$

$$\Rightarrow \frac{x}{1} = \frac{y}{1} = \frac{z}{1}$$

$\Rightarrow x : y : z = 1 : 1 : 1$

Hence, the result.

21. The given equations can be written as

$x - ay + az = 0$

$bx + y - bz = 0$

$cx - cy - z = 0$

For a non-trivial solutions,

$\Delta = 0$

$$\begin{vmatrix} 1 & -a & a \\ b & 1 & -b \\ c & -c & -1 \end{vmatrix} = 0$$

$\Rightarrow \quad 1(-1 - bc) + a(-b + bc) + a(-bc - c) = 0$

$\Rightarrow \quad -1 - bc - ab + abc - abc - ac = 0$

$\Rightarrow \quad -1 - bc - ab - ac = 0$

$\Rightarrow \quad 1 + ab + bc + ca = 0$

22. The given system of equations can be written as

$1 + bc + qr = 0$

$1 + ca + rp = 0$

$1 + ab + pq = 0$

Multiply Eq. (i) by ap, Eq. (ii) by bq and Eq. (iii) by cr, we get

$ap + (abc)p + (pqr)a = 0$

$bq + (abc)q + (pqr)b = 0$

$cr + (abc)r + (pqr)c = 0$

Put $abc = x$ and $pqr = y$, we get

$ap + px + ay = 0$

$bq + qx + by = 0$

$cr + rx + cy = 0$

The above system of equations will be consistent,

if $\quad \Delta = 0$

$$\Rightarrow \begin{vmatrix} ap & p & a \\ bq & q & b \\ cr & c & r \end{vmatrix} = 0$$

Hence, the result.

23. The given determinant is

$$\begin{vmatrix} \operatorname{cosec}\alpha & 1 & 0 \\ 1 & 2\operatorname{cosec}\alpha & 1 \\ 0 & 1 & 2\operatorname{cosec}\alpha \end{vmatrix}$$

$$= \operatorname{cosec}\alpha(4\operatorname{cosec}^2\alpha - 1) - 2\operatorname{cosec}\alpha$$

$$= \operatorname{cosec}\alpha(4\operatorname{cosec}^2\alpha - 3)$$

$$= \operatorname{cosec}\alpha\left(4\frac{\cos^2\alpha}{\sin^2\alpha} + 1\right)$$

$$= \left(\frac{4\cos^2\alpha + \sin^2\alpha}{\sin^3\alpha}\right)$$

$$= \left(\frac{4 - 3\sin^2\alpha}{\sin^3\alpha}\right)$$

$$= \frac{4 - 12\sin^2\left(\frac{\alpha}{2}\right)\cos^2\left(\frac{\alpha}{2}\right)}{8\sin^3\left(\frac{\alpha}{2}\right)\cos^3\left(\frac{\alpha}{2}\right)}$$

$$= \frac{1}{2} \times \left(\frac{1 - 3\sin^2\left(\frac{\alpha}{2}\right)\cos^2\left(\frac{\alpha}{2}\right)}{\sin^3\left(\frac{\alpha}{2}\right)\cos^3\left(\frac{\alpha}{2}\right)}\right)$$

$$= \frac{1}{2} \times \left(\frac{\sin^6\left(\frac{\alpha}{2}\right) + \cos^6\left(\frac{\alpha}{2}\right)}{\sin^3\left(\frac{\alpha}{2}\right)\cos^3\left(\frac{\alpha}{2}\right)}\right)$$

$$= \frac{1}{2} \times \left(\frac{\sin^3\left(\frac{\alpha}{2}\right)}{\cos^3\left(\frac{\alpha}{2}\right)} + \frac{\cos^3\left(\frac{\alpha}{2}\right)}{\sin^3\left(\frac{\alpha}{2}\right)}\right)$$

$$= \frac{1}{2} \times \left(\tan^3\left(\frac{\alpha}{2}\right) + \cot^3\left(\frac{\alpha}{2}\right)\right)$$

24. We have,

$$\begin{vmatrix} pa & qb & rc \\ qc & ra & pb \\ rb & pc & qa \end{vmatrix}$$

$= pa(qra^2 - p^2bc) - qb(q^2ac - prb^2) + rc(pqc^2 - r^2ab)$

$= pqr(a^3 + b^3 + c^3) - abc(p^3 - q^3 + r^3)$

$= pqr(a^3 + b^3 + c^3 - 3abc) - abc(p^3 - q^3 + r^3 - 3pqr)$

$= pqr(a^3 + b^3 + c^3 - 3abc) - 0$, since $p + q + r = 0$

$= pqr(a^3 + b^3 + c^3 - 3abc)$

$$= pqr\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

Hence, the result.

25. We have,

$$\begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ac-b^2 & b^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} -a & c & b \\ -b & a & c \\ -c & b & a \end{vmatrix}$$

$$= -\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & c & b \\ b & a & c \\ c & b & a \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \times \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2$$

$$= (a^3 + b^3 + c^3 - 3abc)^2$$

26. We have,

$$\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} -abc & ab^2+abc & ac^2+abc \\ a^2b+abc & -abc & bc^2+abc \\ a^2c+abc & b^2c+abc & -abc \end{vmatrix}$$

$$= \frac{abc}{abc}\begin{vmatrix} -bc & ab+ac & ac+ab \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & -ab \end{vmatrix}$$

$$= \begin{vmatrix} -bc & ab+ac & ac+ab \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & -ab \end{vmatrix}$$

$$= \begin{vmatrix} ab+bc+ca & ab+bc+ca & ab+bc+ca \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & -ab \end{vmatrix}$$

$$(R_1 \to R_1 + R_2 + R_3)$$

$$= (ab+bc+ca)\begin{vmatrix} 1 & 1 & 1 \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ac & -ab \end{vmatrix}$$

$$= (ab + bc + ca)$$

$$\times\begin{vmatrix} 1 & 0 & 0 \\ ab+bc & -(ab+bc+ca) & 0 \\ ac+bc & 0 & -(ab+bc+ca) \end{vmatrix}$$

$$= (ab+bc+ca)\begin{vmatrix} -(ab+bc+ca) & 0 \\ 0 & -(ab+bc+ca) \end{vmatrix}$$

$$= (ab + bc + ca)^3$$

Hence, the result.

27. Since the given system of equations has non-trivial solutions, so

$$\Delta = 0$$

$$\Rightarrow \begin{vmatrix} a & a^2 & (a^3+1) \\ b & b^2 & (b^3+1) \\ c & c^2 & (c^3+1) \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} + \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow abc\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} + \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}(abc+1) = 0$$

$\Rightarrow \quad (a - b)(b - c)(c - a)(abc + 1) = 0$

$\Rightarrow \quad (abc + 1) = 0 \ (\because a \neq b \neq c)$

$\Rightarrow \quad (abc + 1) = 0$

28. We have,

$$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix}$$

$$= \begin{vmatrix} (b+c)^2 & a^2-(b+c)^2 & a^2-(b+c)^2 \\ b^2 & (c+a)^2-b^2 & 0 \\ c^2 & 0 & (a+b)^2-c^2 \end{vmatrix}$$

$$\begin{pmatrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{pmatrix}$$

$$= (a+b+c)^2\begin{vmatrix} (b+c)^2 & (a-b-c) & (a-b-c) \\ b^2 & (c+a-b) & 0 \\ c^2 & 0 & (a+b-c) \end{vmatrix}$$

$$= (a+b+c)^2 \begin{vmatrix} 2bc & -2c & -2b \\ b^2 & (c+a-b) & 0 \\ c^2 & 0 & (a+b-c) \end{vmatrix}$$

$$= 2(a+b+c)^2 \begin{vmatrix} bc & -c & -b \\ b^2 & (c+a-b) & 0 \\ c^2 & 0 & (a+b-c) \end{vmatrix}$$

$$= 2(a+b+c)^2 \begin{vmatrix} bc & 0 & 0 \\ b^2 & (c+a) & \frac{b^2}{c} \\ c^2 & \frac{c^2}{b} & (a+b) \end{vmatrix}$$

$$\begin{pmatrix} C_2 \to C_2 + \frac{1}{b}C_1 \\ C_3 \to C_3 + \frac{1}{c}C_1 \end{pmatrix}$$

$$= 2(a+b+c)^2 bc \begin{vmatrix} (c+a) & \frac{b^2}{c} \\ \frac{c^2}{b} & (a+b) \end{vmatrix}$$

$= 2(a+b+c)^2 bc\ \{(a+b)(a+c) - b\}$

$= 2(a+b+c)^2 bc\ \{a^2 + ab + ac + bc - bc\}$

$= 2(a+b+c)^2 bc\ (a^2 + ab + ac)$

$= 2abc(a+b+c)^3$

29. We have

$$\begin{vmatrix} a & a^3 & a^4-1 \\ b & b^3 & b^4-1 \\ c & c^3 & c^4-1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} a & a^3 & a^4 \\ b & b^3 & b^4 \\ c & c^3 & c^4 \end{vmatrix} - \begin{vmatrix} a & a^3 & 1 \\ b & b^3 & 1 \\ c & c^3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow abc\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} - \begin{vmatrix} a & a^3 & 1 \\ b & b^3 & 1 \\ c & c^3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow abc\begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} - \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = 0$$

$$\Rightarrow abc\begin{vmatrix} 1 & a^2 & a^3 \\ 0 & b^2-a^2 & b^3-a^3 \\ 0 & c^2-a^2 & c^3-a^3 \end{vmatrix} - \begin{vmatrix} 1 & a & a^3 \\ 0 & b-a & b^3-a^3 \\ 0 & c-a & c^3-a^3 \end{vmatrix} = 0$$

$$\Rightarrow abc\begin{vmatrix} b^2-a^2 & b^3-a^3 \\ c^2-a^2 & c^3-a^3 \end{vmatrix} - \begin{vmatrix} b-a & b^3-a^3 \\ c-a & c^3-a^3 \end{vmatrix} = 0$$

$$\Rightarrow abc(b-a)(c-a)\begin{vmatrix} b+a & b^2+ab+a^2 \\ c+a & c^2+ca+a^2 \end{vmatrix}.$$

$$-(b-a)(c-a)\begin{vmatrix} 1 & a^2+ab+b^2 \\ 1 & a^2+ac+c^2 \end{vmatrix} = 0$$

$$\Rightarrow abc(b-a)(c-a)\begin{vmatrix} b+a & b^2+ab+a^2 \\ c-b & c^2-b^2+ca-ab \end{vmatrix}$$

$$-(b-a)(c-a)(ac+c^2-ab-b^2) = 0$$

$$\Rightarrow abc(b-a)(c-a)(c-a)\begin{vmatrix} b+a & b^2+ab+a^2 \\ 1 & c+b+c \end{vmatrix}$$

$$-(b-a)(c-a)(c-b)(a+b+c) = 0$$

$\Rightarrow abc(b-a)(c-a)(c-b)(ab+bc+ca) = 0$

$-(b-a)(c-a)(c-b)(a+b+c) = 0$

$\Rightarrow (a-b)(b-c)(c-a)\{abc(ab+bc+ca)$

$-(a+b+c)\} = 0$

$\Rightarrow abc(ab+bc+ca) - (a+b+c) = 0$

$\Rightarrow abc(ab+bc+ca) = (a+b+c)\}$

30. We have,

$$\begin{vmatrix} a^2 & b\sin A & c\sin A \\ b\sin A & 1 & \cos A \\ c\sin A & \cos A & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a^2 & bak & cak \\ bak & 1 & \cos A \\ cak & \cos A & 1 \end{vmatrix}$$

$$= a^2\begin{vmatrix} 1 & bk & ck \\ bk & 1 & \cos A \\ ck & \cos A & 1 \end{vmatrix}$$

$$= a^2\begin{vmatrix} 1 & \sin B & \sin C \\ \sin B & 1 & \cos A \\ \sin C & \cos A & 1 \end{vmatrix}$$

$$= a^2\begin{vmatrix} 1 & 0 & 0 \\ \sin B & 1-\sin^2 B & \cos A - \sin B\sin C \\ \sin C & \cos A - \sin B\sin C & 1-\sin^2 C \end{vmatrix}$$

$= a^2[\cos^2 B\cos^2 C - (\cos A - \sin B\sin C)^2]$

$= a^2[\cos^2 B\cos^2 C - \{\cos(B+C) + \sin B\sin C\}^2]$

$= a^2[\cos^2 B\cos^2 C - \cos^2 B\cos^2 C]$

$= 0$

Integer Type Questions

1. Since a, b, c and d are the roots of
$x^4 + 2x^3 + 3x^2 + 6x + 8 = 0$
so,
$\Sigma a = -2, \Sigma ab = 3, \Sigma abc = -6, \text{ and } \Sigma abcd = 8$
We have,
$$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix}$$
$$= abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$
$$= (abcd + (bcd + acd + abd + abc))$$
$$= (8 - 6)$$
$$= 2$$

2. We have,
$$\begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix} = 4xyz$$
Now, $4^n = 64 = 4^3$
$\Rightarrow \quad n = 3$
Hence, the value of n is 3.

3. We have,
$$\begin{vmatrix} \cos^2 x & \cos x \cdot \sin x & -\sin x \\ \cos x \cdot \sin x & \sin^2 x & \cos x \\ \sin x & -\cos x & 0 \end{vmatrix}$$
$$= \cos^2 x(\cos^2 x) - \cos x \cdot \sin x\,(-\cos x \cdot \sin x) - \sin x\,(-\cos^2 x \cdot \sin x - \sin^3 x)$$
$$= \cos^4 x + \cos^2 x \cdot \sin^2 x + \sin^2 x \cdot \cos^2 x + \sin^4 x$$
$$= \cos^4 x + 2\cos^2 x \cdot \sin^2 x + \sin^4 x$$
$$= (\cos^2 x + \sin^2 x)^2$$
$$= 1$$

4. We have,
$$\begin{vmatrix} a & b-y & c-z \\ a-x & b & c-z \\ a-x & b-y & c \end{vmatrix} = 0$$
$$\Rightarrow \begin{vmatrix} a & b-y & c-z \\ -x & y & 0 \\ -x & 0 & z \end{vmatrix} = 0 \quad \begin{pmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$$
$$\Rightarrow \quad ayz + (b-y)xz + (c-z)xy = 0$$
$$\Rightarrow \quad \frac{a\,yz}{xyz} + \frac{(b-y)\,xz}{xyz} + \frac{(c-z)\,xy}{xyz} = 0$$
$$\Rightarrow \quad \frac{a}{x} + \frac{(b-y)}{y} + \frac{(c-z)}{z} = 0$$
$$\Rightarrow \quad \frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$$

5. Given,
$$A = \begin{pmatrix} 1 & 2 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } B = \begin{pmatrix} 1 & -2 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
Now,
$$AB = I_3$$
$$\Rightarrow \begin{pmatrix} 1 & 2 & x \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} 1 & 0 & x+y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$\Rightarrow \quad x + y = 0$$
$$\Rightarrow \quad x + y + 5 = 5$$
Hence, the value of $x + y + 5$ is 5.

6. We have,
$$\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix}$$
$$= \begin{vmatrix} a^2 & b^2 & c^2 \\ 4a & 4b & 4c \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix} \quad (R_2 \to R_2 - R_1)$$
$$= 4\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ (a^2+1) & (b^2+1) & (c^2+1) \end{vmatrix} \quad (R_3 \to R_3 + 2R_2)$$
$$= 4\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} \quad (R_3 \to R_3 - R_1)$$
Thus, $\lambda = 4$
Hence, the value of $(\lambda + 2)$ is 6.

7. We have,
$$\begin{vmatrix} 1 & 1 & 1 \\ {}^nC_1 & {}^{n+3}C_1 & {}^{n+6}C_1 \\ {}^nC_2 & {}^{n+3}C_2 & {}^{n+6}C_2 \end{vmatrix}$$
$$= \begin{vmatrix} 1 & 1 & 1 \\ n & (n+3) & (n+6) \\ \frac{n(n-1)}{2} & \frac{(n+3)(n+2)}{2} & \frac{(n+6)(n+5)}{2} \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ n & (n+3) & (n+6) \\ n(n-1) & (n+3)(n+2) & (n+6)(n+5) \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ n & 3 & 6 \\ n(n-1) & 6(n+1) & 6(2n+5) \end{vmatrix}$$

$$= \frac{1}{2}\begin{vmatrix} 3 & 6 \\ 6(2n+1) & 6(2n+5) \end{vmatrix}$$

$$= \frac{1}{2} \times 6 \times 3 \begin{vmatrix} 1 & 2 \\ (n+1) & (2n+5) \end{vmatrix}$$

$= 9(2n + 5 - 2n - 2)$

$= 9 \times 3 = 3^3$

Hence, the maximum value of n is 3.

8. Since a, b and c are the roots of $x^3 + 2x^2 + 5 = 0$, so,

$a + b + c = -2$

$ab + bc + ca = 0$

$abc = -5$

We have

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$= -(a^3 + b^3 + c^3 - 3abc)$

$= -(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$= -(a + b + c)[(a + b + c)^2 - 3(ab + bc + ca)]$

$= -(-2)(4 - 0)$

$= 8$

9. We have,

$$\begin{vmatrix} 1+x & x & x^2 \\ x & 1+x & x^2 \\ x^2 & x & 1+x \end{vmatrix}$$

$= ax^5 + bx^4 + cx^3 + dx^2 + ex + f$

Differentiating both sides w.r.t x, we get

$$\begin{vmatrix} 1 & 1 & 2x \\ x & 1+x & x^2 \\ x^2 & x & 1+x \end{vmatrix} + \begin{vmatrix} 1+x & x & x^2 \\ 1 & 1 & 2x \\ x^2 & x & 1+x \end{vmatrix} + \begin{vmatrix} 1+x & x & x^2 \\ x & 1+x & x^2 \\ 2x & 1 & 1 \end{vmatrix}$$

$= 5ax^4 + 4bx^3 + 3cx^2 + 2dx + e$

Putting $x = 1$, we get

$$\Rightarrow \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{vmatrix}$$

$= 5a + 4b + 3c + 2d + e$

$\Rightarrow \quad 0 = 5a + 4b + 3c + 2d + e$

Thus, $e = 0, f = 0$

Hence, the value of $(e + f + 2)$ is 2.

10. Since the equations are consistent, so it has a solution.

Solving

$2x + 3y + 1 = 0$

and $3x + y - 2 = 0$,

we get,

$x = 1$ and $y = -1$

It will satisfy the third equation, so,

$a - 2 - b = 0$

$\Rightarrow \quad (a - b) = 2$

Hence, the value of $(a - b)$ is 2.

11. We have,

$$\begin{vmatrix} x-1 & 1 & 1 \\ 1 & x-1 & 1 \\ 1 & 1 & x-1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x+1 & 1 & 1 \\ x+1 & x-1 & 1 \\ x+1 & 1 & x-1 \end{vmatrix} = 0$$

$(C_1 \to C_1 + C_2 + C_3)$

$$\Rightarrow (x+1)\begin{vmatrix} 1 & 1 & 1 \\ 1 & x-1 & 1 \\ 1 & 1 & x-1 \end{vmatrix} = 0$$

$$\Rightarrow (x+1)\begin{vmatrix} 1 & 0 & 0 \\ 1 & x-2 & 0 \\ 1 & 0 & x-2 \end{vmatrix} = 0$$

$$\begin{pmatrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{pmatrix}$$

$\Rightarrow \quad (x + 1)(x - 2)^2 = 0$

$\Rightarrow \quad x = -1, 2$

Hence, the number of real roots is 2.

12. Let $A = \begin{bmatrix} x+3 & 5 & 2 \\ 1 & 7+x & 6 \\ 2 & 5 & x+3 \end{bmatrix}$

Since the given matrix has rank 2, so, det(A) is zero.

$$\text{Thus, } \begin{vmatrix} x+3 & 5 & 2 \\ 1 & x+7 & 6 \\ 2 & 5 & x+3 \end{vmatrix} = 0$$

$\Rightarrow \quad (x + 3)(x^2 + 10x + 21 - 30) - 5(x + 3 - 12) + 2(5 - 2x - 14) = 0$

$\Rightarrow \quad (x + 3)(x^2 + 10x - 9) - 5(x - 9) - 2(2x + 9) = 0$

$\Rightarrow \quad x^3 + 13x^2 + 12x = 0$

$\Rightarrow \quad x(x^2 + 13x + 12) = 0$

$\Rightarrow \quad x(x + 1)(x + 12) = 0$

$\Rightarrow \quad x = 0, -1, -12$

Hence, the values of x are 0, –1, –12.

13. We have,

$$\begin{vmatrix} x & x-1 & x \\ -2x & x+1 & 1 \\ x+1 & 1 & x \end{vmatrix} = ax^3 + bx^2 + cx + d$$

putting $x = 1$, we get

$$(a+b+c+d) = \begin{vmatrix} 1 & 0 & 1 \\ -2 & 2 & 1 \\ 2 & 1 & 1 \end{vmatrix}$$

$$= 1(2-1) + 1(-2-4) = 1 - 6 = -5$$

Hence, the value of

$$2 - (a+b+c+d) = 2 + 5 = 7$$

14. We have,

$$D_1 = \begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix}$$

$$= \begin{vmatrix} 2(a+b+c) & c+a & a+\mathrm{b} \\ 2(p+q+r) & r+p & p+q \\ 2(x+y+z) & z+x & x+y \end{vmatrix} \quad (C_1 \to C_1 + C_2 + C_3)$$

$$= 2\begin{vmatrix} (a+b+c) & c+a & a+b \\ (p+q+r) & r+p & p+q \\ (x+y+z) & z+x & x+y \end{vmatrix}$$

$$= 2\begin{vmatrix} (a+b+c) & -b & -c \\ (p+q+\mathrm{r}) & -q & -r \\ (x+y+z) & -y & -z \end{vmatrix} \quad \begin{pmatrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{pmatrix}$$

$$= 2\begin{vmatrix} a & -b & -c \\ p & -q & -r \\ x & -y & -z \end{vmatrix}$$

$$= 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} \quad (C_1 \to C_1 + C_2 + C_3)$$

$$= 2D_2$$

$$\Rightarrow \quad \frac{D_1}{D_2} = 2$$

15. We have,

$$\begin{vmatrix} x^3+1 & x^2y & x^2z \\ xy^2 & y^3+1 & y^2z \\ xz^2 & yz^2 & z^3+1 \end{vmatrix} = 11$$

$$\Rightarrow \quad \frac{1}{xyz}\begin{vmatrix} (x^3+1)x & x^3y & x^3z \\ xy^3 & (y^3+1)y & y^3z \\ xz^3 & yz^3 & (z^3+1)z \end{vmatrix} = 11$$

$$\Rightarrow \quad \frac{xyz}{xyz}\begin{vmatrix} (x^3+1) & x^3 & x^3 \\ y^3 & (y^3+1) & y^3 \\ z^3 & z^3 & (z^3+1) \end{vmatrix} = 11$$

$$\Rightarrow \quad \begin{vmatrix} (x^3+y^3+z^3+1) & (x^3+y^3+z^3+1) & (x^3+y^3+z^3+1) \\ y^3 & (y^3+1) & y^3 \\ z^3 & z^3 & (z^3+1) \end{vmatrix} = 11$$

$$\Rightarrow \quad (x^3+y^3+z^3+1) \times \begin{vmatrix} 1 & 1 & 1 \\ y^3 & (y^3+1) & y^3 \\ z^3 & z^3 & (z^3+1) \end{vmatrix} = 11$$

$$\Rightarrow \quad (x^3+y^3+z^3+1) \times \begin{vmatrix} 1 & 0 & 0 \\ y^3 & 1 & 0 \\ z^3 & 0 & 1 \end{vmatrix} = 11$$

$\Rightarrow \quad x^3 + y^3 + z^3 + 1 = 11$

$\Rightarrow \quad x^3 + y^3 + z^3 = 10$

Hence, the positive integral solutions are

(1, 1, 2), (1, 2, 1), and (2, 1, 1).

Thus, the number of solutions is 3.

Previous Years' JEE-Advanced Examinations

1. The given system of equations can be written as

$$x - cy - bz = 0$$
$$cx - y + az = 0$$
$$bx + ay - z = 0$$

Since the given system of equations has a non-zero solution, so

$$\begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0$$

$\Rightarrow \quad 1(1 - a^2) + c(-c - ab) - b(ac + b) = 0$

$\Rightarrow \quad 1 - a^2 - c^2 - abc - b^2 - abc = 0$

$\Rightarrow \quad a^2 + b^2 + c^2 + 2abc = 1$

2. Here, $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2} \neq \dfrac{d_1}{d_2}$

So, the system of equations has no solution.

3. Since the system of equations has non-trivial solution, so

$$\begin{vmatrix} 1 & k & 3 \\ 2 & k & -2 \\ 2 & 3 & -4 \end{vmatrix} = 0$$

$\Rightarrow \quad (6 - 4k) - k(-8 + 4) + 3(6 - 2k) = 0$

$\Rightarrow \quad 6 - 4k + 4k + 18 - 6k = 0$

$\Rightarrow \quad 6k = 24$

$\Rightarrow \quad k = 4$

4. $D = \begin{vmatrix} 3 & m \\ 2 & -5 \end{vmatrix}$, $D_1 = \begin{vmatrix} m & m \\ 20 & -5 \end{vmatrix}$, $D_2 = \begin{vmatrix} 3 & m \\ 2 & 20 \end{vmatrix}$

Now, $x = \frac{D_1}{D} = \frac{-(25m)}{-(2m+15)} = \frac{(25m)}{(2m+15)}$

and $y = \frac{D_2}{D} = \frac{-(25m)}{(60-2m)} = \frac{(25m)}{(2m-60)}$

since $x > 0, y > 0$, so we have

$\Rightarrow \quad m > 0, 2m - 60 > 0, 2m + 15 < 0$

$\Rightarrow \quad m > 30$ and $m < -\frac{15}{2}$

$\Rightarrow \quad m \in \left(-\infty, -\frac{15}{2}\right) \cup (30, \infty)$

5. The given system of equations can be written as

$1 - (x + y) = z \geq 0$

$2x - 3y - 2 = w \geq 0$

$(x + y) \leq 1$, $2x - 3y \geq 2$ and $x \geq 0$, $y \geq 0$

Thus $x = 1$ and $y = 0$.

For these values, $z = 0$ and $w = 0$.

6. We have,

$$\Delta = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & b & c \\ a+b+c & c & a \\ a+b+c & a & b \end{vmatrix} \quad (C_1 \to C_1 + C_2 + C_3)$$

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & c & a \\ 1 & a & b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & c-b & a-c \\ 0 & a-b & b-c \end{vmatrix} \begin{pmatrix} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{pmatrix}$$

$$= (a+b+c)\begin{vmatrix} c-b & a-c \\ a-b & b-c \end{vmatrix}$$

$= (a + b + c)\{-(b - c)^2 - (a - b)(a - c)$

$= -(a + b + c)\{(b - c)^2 + (a - b)(a - c)\}$

$= -(a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$= -(a^2 + b^3 + c^3 - 3abc)$

< 0, since $a, b, c \in R^+$.

7. Put $\lambda = 0$, we get

$$t = \begin{vmatrix} 0 & -1 & 3 \\ 1 & 0 & -4 \\ -3 & 4 & 0 \end{vmatrix}$$

$\Rightarrow \quad t = 0$.

8. The given equation is

$$\Delta = \begin{vmatrix} 1 & 4 & 2 \\ 1 & -2 & 5 \\ 1 & 2x & 5x^2 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 4 & 2 \\ 0 & -6 & 3 \\ 0 & 2x+2 & 5x^2-5 \end{vmatrix} = 0$$

$$\begin{pmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_2 \end{pmatrix}$$

$$\Rightarrow \begin{vmatrix} -6 & 3 \\ 2(x+1) & 5(x^2-1) \end{vmatrix} = 0$$

$$\Rightarrow (x+1)\begin{vmatrix} -6 & 3 \\ 2 & 5(x-1) \end{vmatrix} = 0$$

$\Rightarrow \quad (x + 1)–30(x –1) – 6) = 0$

$\Rightarrow \quad -6(x + 1)(5(x - 1) + 1) = 0$

$\Rightarrow \quad (x + 1)(5x - 4) = 0$

$\Rightarrow \quad x = -1, \frac{4}{5}$

9. We have,

$$\Delta = \begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix}$$

$$= \begin{vmatrix} x^2+x & x+1 & x-2 \\ x-1 & x-2 & x+1 \\ x+3 & x-2 & x+1 \end{vmatrix} \begin{pmatrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$$

$$= \begin{vmatrix} x^2+x & x+1 & x-2 \\ x-1 & x-2 & x+1 \\ 4 & 0 & 0 \end{vmatrix} \quad (R_3 \to R_3 - R_2)$$

$$= 4\begin{vmatrix} x+1 & x-2 \\ x-2 & x+1 \end{vmatrix}$$

$$= 4 \times \begin{vmatrix} x+1 & -3 \\ x-2 & 3 \end{vmatrix} \quad (C_2 \to C_2 - C_1)$$

$= 4[3(x + 1) + 3(x - 2)]$

$= 4(6x - 3)$

$= 24x - 12$

$= Ax + B$, where $A = 24$ and $B = -12$.

10. From first two equations, we get

$$x = \frac{1}{7}(4 - 5z);\ y = \frac{1}{7}(13z - 9)$$

Putting the values of x and y in the third equations, we get

$-3 + (\lambda + 5)z = -3$

$\Rightarrow \quad (\lambda + 5)z = 0$

$\Rightarrow \quad z = 0$

Thus, for all real $\lambda \neq -5$, we have

$$x = \frac{4}{7}, y = -\frac{9}{7}, z = 0$$

Now, for $\lambda = -5$, then

$$z = k, x = \frac{1}{7}(4 - 5k), y = \frac{1}{7}(13k - 9)$$

11. We have

$$\Delta = \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & \text{abc} \\ c & c^2 & abc \end{vmatrix}$$

$$= \frac{abc}{abc} \times \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= \Delta'$$

12. Since the system of equations has a non-zero solution,

$$\begin{vmatrix} \lambda & 1 & 1 \\ -1 & \lambda & 1 \\ -1 & -1 & \lambda \end{vmatrix} = 0$$

$\Rightarrow \quad \lambda(\lambda^2 + 1) - (1 - \lambda) + (1 + \lambda) = 0$

$\Rightarrow \quad \lambda^3 + \lambda - 1 + \lambda + 1 + \lambda = 0$

$\Rightarrow \quad \lambda^3 + 3\lambda = 0$

$\Rightarrow \quad (\lambda^3 + 3)\lambda = 0$

$\Rightarrow \quad \lambda = 0$

13. Since α is a repeated root of the quadratic equation $f(x) = 0$, we can write

$$f(x) = a(x - \lambda)^2, a \in R$$

Also, it is given that, $F(x)$ is a polynomial of degree atmost 5 and $F(\alpha) = 0$.

Thus,

$$F'(x) = \begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$

$\Rightarrow \quad F'(\alpha) = 0$

Thus, λ is a repeated root of the polynomial $F(x)$.
Therefore, $F(x)$ is divisible by $f(x)$.

14. We have,

$$\Delta = \begin{vmatrix} {}^xC_r & {}^xC_{r+1} & {}^xC_{r+2} \\ {}^yC_r & {}^yC_{r+1} & {}^yC_{r+2} \\ {}^zC_r & {}^zC_{r+1} & {}^zC_{r+2} \end{vmatrix}$$

$$= \begin{vmatrix} {}^xC_r & {}^xC_r + {}^xC_{r+1} & {}^xC_{r+1} + {}^xC_{r+2} \\ {}^yC_r & {}^yC_r + {}^yC_{r+1} & {}^yC_{r+1} + {}^yC_{r+2} \\ {}^zC_r & {}^zC_r + {}^zC_{r+1} & {}^zC_{r+1} + {}^zC_{r+2} \end{vmatrix}$$

$$\begin{pmatrix} C_2 \to C_2 + C_1 \\ C_3 \to C_3 + C_2 \end{pmatrix}$$

$$= \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+1}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+1}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+1}C_{r+2} \end{vmatrix}$$

$$= \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+1}C_{r+1} + {}^{x+1}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+1}C_{r+1} + {}^{y+1}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+1}C_{r+1} + {}^{z+1}C_{r+2} \end{vmatrix}$$

$$(C_3 \to C_3 + C_2)$$

$$= \begin{vmatrix} {}^xC_r & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+2} \\ {}^yC_r & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+2} \\ {}^zC_r & {}^{z+1}C_{r+1} & {}^{z+2}C_{r+2} \end{vmatrix}$$

Hence, the result.

15. We have,

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$$

$$\Rightarrow \quad \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2}\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$$

$\Rightarrow \quad \Delta_1 = \Delta_2$

Areas of two triangles are the same but they may not be congruent.

16. Since the system of equations has a non-trivial solution, so

$$\begin{vmatrix} \sin(3\theta) & -1 & 1 \\ \cos(2\theta) & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} \sin(3\theta) & 0 & 1 \\ \cos(2\theta) & 7 & 3 \\ 2 & 14 & 7 \end{vmatrix} = 0$$

$$(C_2 \to C_2 + C_3)$$

$$\Rightarrow \quad (49 - 42)\sin(3\theta) + 14(\cos(2\theta) - 1) = 0$$

$$\Rightarrow \quad 7\sin(3\theta) + 14(\cos(2\theta) - 1) = 0$$

$$\Rightarrow \quad \sin(3\theta) + 2(\cos(2\theta) - 1) = 0$$

$$\Rightarrow \quad 3\sin\theta - 4\sin^3\theta - 4\sin^2\theta = 0$$

$$\Rightarrow \quad \sin\theta(3 - 4\sin^2\theta - 4\sin\theta) = 0$$

$$\Rightarrow \quad \sin\theta(4\sin^2\theta + 4\sin\theta - 3) = 0$$

$$\Rightarrow \quad \sin\theta(4\sin^2\theta + 6\sin\theta - 2\sin\theta - 3) = 0$$

$$\Rightarrow \quad \sin\theta\{2\sin\theta(\sin\theta + 3) - 1(2\sin\theta + 3)\} = 0$$

$$\Rightarrow \quad \sin\theta(2\sin\theta - 1)(\sin\theta + 3) = 0$$

$$\Rightarrow \quad \sin\theta(2\sin\theta - 1) = 0$$

$$\Rightarrow \quad \sin\theta = 0,\ \sin\theta = \frac{1}{2}$$

$$\Rightarrow \quad \theta = n\pi, \theta = m\pi + (-1)^n\left(\frac{\pi}{6}\right), m, n \in I$$

17. The given determinant is

$$\Delta = \begin{vmatrix} a & b & a\alpha + b \\ b & c & b\alpha + c \\ a\alpha + b & b\alpha + c & 0 \end{vmatrix}$$

$$= \begin{vmatrix} a & b & 0 \\ b & c & 0 \\ a\alpha + b & b\alpha + c & -((a\alpha + b)\alpha + (b\alpha + c)) \end{vmatrix}$$

$$= \begin{vmatrix} a & b & 0 \\ b & c & 0 \\ a\alpha + b & b\alpha + c & -(a\alpha^2 + 2b\alpha + c) \end{vmatrix}$$

$$= (a\alpha^2 + 2b\alpha + c)(b^2 - ac)$$

$$= 0\begin{cases} \text{if } a, b, c \text{ are in GP} \\ \text{or } (x - \alpha) \text{ is a factor of } a\alpha^2 + 2b\alpha + c \end{cases}$$

18. We have,

$$f(x) = \begin{vmatrix} \sec x & \cos x & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$$

$$= \cos^2 x\begin{vmatrix} \sec x & \sec x & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & 1 & \operatorname{cosec}^2 x \\ 1 & 1 & \cos^2 x \end{vmatrix}$$

$$= \cos^2 x\begin{vmatrix} \sec x & 0 & \sec^2 x + \cot x \operatorname{cosec} x \\ \cos^2 x & \sin^2 x & \operatorname{cosec}^2 x \\ 1 & 0 & \cos^2 x \end{vmatrix}$$

$(C_2 \to C_2 - C_1)$

$$= \cos^2 x \cdot \sin^2 x\begin{vmatrix} \sec x & \sec^2 x + \operatorname{cosec} x \cot x \\ 1 & \cos^2 x \end{vmatrix}$$

$$= \cos^2 x \cdot \sin^2 x\left(\cos x - \sec^2 x - \frac{\cos x}{\sin^2 x}\right)$$

$$= \cos^3 x \sin^2 x - \sin^2 x - \cos^3 x$$

$$= \cos^3 x(\sin^2 x - 1) - \sin^2 x$$

$$= -\cos^5 x - \sin^2 x$$

Thus,

$$\int_0^{\pi/2} f(x)\,dx = \int_0^{\pi/2} (-\cos^5 x - \sin^2 x)\,dx$$

$$= -\frac{4}{5}\cdot\frac{2}{3} - \frac{1}{2}\cdot\frac{\pi}{2}$$

$$= -\left(\frac{\pi + 8}{15}\right)$$

19. The given determinant is

$$\begin{vmatrix} 1 & a & a^2 - bc \\ 1 & b & b^2 - ca \\ 1 & c & c^2 - ab \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \frac{1}{abc}\begin{vmatrix} a & a^2 & abc \\ b & b^2 & abc \\ c & c^2 & abc \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \frac{abc}{abc}\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= 0$$

20. The given equation is

$$\Delta = \begin{vmatrix} 1 + \sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1 + \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \cos^2\theta & 1 + 4\sin 4\theta \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} 2 + 4\sin 4\theta & \cos^2\theta & 4\sin 4\theta \\ 2 + 4\sin 4\theta & 1 + \cos^2\theta & 4\sin 4\theta \\ 2 + 4\sin 4\theta & \cos^2\theta & 1 + 4\sin 4\theta \end{vmatrix} = 0$$

$$\Rightarrow \quad (2+4\sin 4\theta)\begin{vmatrix} 1 & \cos^2\theta & 4\sin 4\theta \\ 1 & 1+\cos^2\theta & 4\sin 4\theta \\ 1 & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0$$

$$\Rightarrow \quad (2+4\sin 4\theta)\begin{vmatrix} 1 & \cos^2\theta & 4\sin 4\theta \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \quad (2+4\sin 4\theta) = 0$$

$$\Rightarrow \quad \sin 4\theta = -\frac{1}{2}$$

$$\Rightarrow \quad \sin 4\theta = \begin{cases} \sin\left(\pi + \dfrac{\pi}{6}\right) \\ \sin\left(2\pi - \dfrac{\pi}{6}\right) \end{cases}$$

$$\Rightarrow \quad 4\theta = \begin{cases} \dfrac{7\pi}{6} \\ \dfrac{11\pi}{6} \end{cases}$$

$$\Rightarrow \quad \theta = \begin{cases} \dfrac{7\pi}{24} \\ \dfrac{11\pi}{24} \end{cases}$$

21. Given $\Delta_a = \begin{vmatrix} a-1 & n & 6 \\ (a-1)^2 & 2n^2 & 4n-2 \\ (a-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$

We have,

$$\sum_{a=1}^{n}\Delta_a = \begin{vmatrix} \sum_{a=1}^{n}(a-1) & n & 6 \\ \sum_{a=1}^{n}(a-1)^2 & 2n^2 & 4n-2 \\ \sum_{a=1}^{n}(a-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$$

$$= \begin{vmatrix} \dfrac{n(n-1)}{2} & n & 6 \\ \dfrac{n(n-1)(2n-1)}{6} & 2n^2 & 4n-2 \\ \dfrac{n^2(n-1)^2}{4} & 3n^3 & 3n^2-3n \end{vmatrix}$$

$$= \frac{n(n-1)}{12}\begin{vmatrix} 6 & n & 6 \\ 2(2n-1) & 2n^2 & 4n-2 \\ 3n(n-1) & 3n^3 & 3n^2-3n \end{vmatrix}$$

$$= \frac{n(n-1)}{12}\begin{vmatrix} 6 & n & 6 \\ 4n-2 & 2n^2 & 4n-2 \\ 3n^2-3n & 3n^3 & 3n^2-3n \end{vmatrix}$$

$$= \frac{n(n-1)}{12} \times 0$$

$$= 0$$

22. Given

$A28 = 100A + 20 + 8 = \text{km}_1$

$3B9 = 300 + 10B + 9 = \text{km}_2$

$62C = 600 + 20 + C = \text{km}_3$

We have,

$$\begin{vmatrix} A & 3 & 6 \\ 8 & 9 & C \\ 2 & B & 2 \end{vmatrix}$$

$$= \begin{vmatrix} A & 3 & 6 \\ 100A+20+8 & 300+10B+9 & 600+20+C \\ 2 & B & 2 \end{vmatrix}$$

$(R_2 \to R_2 + 100R_1 + 10R_3)$

$$= \begin{vmatrix} A & 3 & 6 \\ \text{km}_1 & \text{km}_2 & \text{km}_3 \\ 2 & B & 2 \end{vmatrix}$$

$$= k \times \begin{vmatrix} A & 3 & 6 \\ \text{m}_1 & \text{m}_2 & \text{m}_3 \\ 2 & B & 2 \end{vmatrix}$$

which is divisible by k.

23. Given $\Delta = \begin{vmatrix} p & b & c \\ a & q & c \\ a & b & r \end{vmatrix} = 0$

$$\Rightarrow \quad \begin{vmatrix} p-a & b-q & 0 \\ 0 & q-b & c-r \\ a & b & r \end{vmatrix} = 0$$

$\Rightarrow (p-a)\{r(q-b) - b(c-r)\} + a(b-q)(c-r) = 0$

$\Rightarrow r(p-a)(q-b) - b(p-a)(c-r) + a(b-q)(c-r) = 0$

$\Rightarrow r(p-a)(q-b) + b(p-a)(r-c) + a(q-b)(r-c) = 0$

Dividing both the sides by $(p-a)(q-b)(r-c)$, we get,

$$\frac{r}{r-c} + \frac{b}{q-b} + \frac{a}{p-a} = 0$$

$$\Rightarrow \quad \frac{a}{p-a} + \frac{b}{q-b} + \frac{r}{r-c} = 0$$

$$\Rightarrow \quad \left(\frac{a}{p-a} + 1\right) + \left(\frac{b}{q-b} + 1\right) + \frac{r}{r-c} = 2$$

$$\Rightarrow \quad \frac{p}{p-a}+\frac{q}{q-b}+\frac{r}{r-c}=2$$

$\Rightarrow \quad E = 2$

24. We have,

$$\Delta = \begin{vmatrix} n! & (n+1)! & (n+2)! \\ (n+1)! & (n+3)! & (n+3)! \\ (n+2)! & (n+3)! & (n+4)! \end{vmatrix}$$

$$= (n)!(n+1)!(n+2)!\begin{vmatrix} 1 & (n+1) & (n+2)(n+1) \\ 1 & (n+2) & (n+3)(n+2) \\ 1 & (n+3) & (n+4)(n+3) \end{vmatrix}$$

$$= (n)!(n+2)!(n+2)!\begin{vmatrix} 1 & (n+1) & (n+2)(n+1) \\ 0 & 1 & 2(n+2) \\ 0 & 1 & 2(n+3) \end{vmatrix}$$

$$= (n)!(n+1)!(n+2)!\begin{vmatrix} 1 & 2(n+2) \\ 1 & 2(n+3) \end{vmatrix}$$

$$= (n)!(n+1)!(n+2)! \times 2\begin{vmatrix} 1 & (n+2) \\ 1 & (n+3) \end{vmatrix}$$

$$= 2 \times (n)!(n+1)!(n+2)!$$

Now,

$$\left(\frac{D}{(n!)^3}-4\right)=\left(\frac{2\times(n)!(n+1)!(n+2)!}{(n!)^3}-4\right)$$

$$=\left(\frac{2\times(n!)^3(n+1)^2(n+2)}{(n!)^3}-4\right)$$

$$= (2 \times (n+1)^2(n+2) - 4$$

$$= (2(n^3 + 4n^2 + 5n + 2) - 4)$$

$$= 2n(n^2 + 4n + 5)$$

which is divisible by n.

25. Since the given system of equations has a non-trivial solutions, so

$$\begin{vmatrix} \lambda & \sin\alpha & \cos\alpha \\ 1 & \cos\alpha & \sin\alpha \\ -1 & \sin\alpha & -\cos\alpha \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda(-\cos^2\alpha - \sin^2\alpha) - \sin\alpha(\sin\alpha - \cos\alpha)$$
$$+\cos\alpha(\sin\alpha+\cos\alpha)=0+\cos\alpha(\sin\alpha+\cos\alpha)=0$$

$$\Rightarrow \quad -\lambda - \sin^2\alpha + 2\sin\alpha\cos\alpha + \cos^2\alpha = 0$$

$$\Rightarrow \quad \lambda = \cos 2\alpha + \sin 2\alpha$$

Thus, the system of equations has a non-trivial solution if $-\sqrt{2} \le \lambda \le \sqrt{2}$.

Also, when $\lambda = 1$

$$\Rightarrow \quad \cos 2\alpha + \sin 2\alpha = 1$$

$$\Rightarrow \quad \frac{1}{\sqrt{2}}\cos 2\alpha + \frac{1}{\sqrt{2}}\sin 2\alpha = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \quad \cos\left(2\alpha+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}=\cos\left(\frac{\pi}{4}\right)$$

$$\Rightarrow \quad \left(2\alpha+\frac{\pi}{4}\right)=2n\pi\pm\frac{\pi}{4}, n\in I$$

$$\Rightarrow \quad 2\alpha = 2n\pi \pm \frac{\pi}{4}-\frac{\pi}{4}, n \in I$$

$$\Rightarrow \quad 2\alpha = 2n\pi, 2n\pi - \frac{\pi}{2}, n\in I$$

$$\Rightarrow \quad \alpha = n\pi, n\pi - \frac{\pi}{4}, n \in I$$

26. We have

$$\Delta = \begin{vmatrix} 1 & \log_x y & \log_x z \\ \log_y x & 1 & \log_y z \\ \log_z x & \log_z y & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \log_x x & \log_x y & \log_x z \\ \log_y x & \log_y y & \log_y z \\ \log_z x & \log_z y & \log_z z \end{vmatrix}$$

$$= \frac{1}{\log x \times \log y \times \log z}\begin{vmatrix} \log x & \log y & \log z \\ \log x & \log y & \log z \\ \log x & \log y & \log z \end{vmatrix}$$

$$= 0$$

27. We have,

$$\Delta = \begin{vmatrix} \cos(A-P) & \cos(A-Q) & \cos(A-R) \\ \cos(B-P) & \cos(B-Q) & \cos(B-R) \\ \cos(C-P) & \cos(C-Q) & \cos(C-R) \end{vmatrix}$$

$$= \begin{vmatrix} \cos A & \sin A & 0 \\ \cos B & \sin B & 0 \\ \cos C & \sin C & 0 \end{vmatrix} \times \begin{vmatrix} \cos P & \sin P & 0 \\ \cos Q & \sin Q & 0 \\ \cos R & \sin R & 0 \end{vmatrix}$$

$$= 0$$

Hence, the result.

28. The given determinant is

$$\Delta = \begin{vmatrix} 1 & 1+i+\omega^2 & \omega^2 \\ 1-i & -1 & \omega^2-1 \\ -i & -i+\omega-1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 1-i & \omega^2+\omega & \omega^2-1 \\ 1-i & -1 & \omega^2-1 \\ -i & -i+\omega-1 & -1 \end{vmatrix}$$

$(R_1 \to R_1 + R_1)$

$$= \begin{vmatrix} 0 & \omega^2+\omega+1 & 0 \\ 1-i & -1 & \omega^2-1 \\ -i & -i+\omega-1 & -1 \end{vmatrix}$$

$(R_1 \to R_1 - R_2)$

$$= \begin{vmatrix} 0 & 0 & 0 \\ 1-\text{i} & -1 & \omega^2 - 1 \\ -i & -i+\omega-1 & -1 \end{vmatrix}$$

$= 0$

29. Adding the first two equations, we get

$$\frac{2x^2}{a^2} = 2$$

$\Rightarrow \quad x = \pm a$

Similarly, we can easily find that

$y = \pm b, z = \pm c$

30. We have,

$$\Delta = \begin{vmatrix} \frac{1}{a} & \frac{1}{a(a+d)} & \frac{1}{(a+d)(a+2d)} \\ \frac{1}{a+d} & \frac{1}{(a+d)(a+2d)} & \frac{1}{(a+2d)(a+3d)} \\ \frac{1}{a+2d} & \frac{1}{(a+2d)(a+3d)} & \frac{1}{(a+3d)(a+4d)} \end{vmatrix}$$

$$= \frac{1}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)} \times \begin{vmatrix} (a+d)(a+2d) & a+2d & a \\ (a+2d)(a+3d) & (a+3d) & (a+d) \\ (a+3d)(a+4d) & (a+4d) & (a+2d) \end{vmatrix}$$

$$= \frac{1}{a(a+\text{d})^2(a+2d)^3(a+3d)^2(a+4d)} \times \begin{vmatrix} 0 & 2d & a \\ d(a+3d) & 2d & (a+d) \\ 2d(a+4d) & 2d & (a+2d) \end{vmatrix}$$

$$\begin{pmatrix} C_1 \to C_1 - (a+d)C_2 \\ C_2 \to C_2 - C_3 \end{pmatrix}$$

$$= \frac{2d^2}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)} \times \begin{vmatrix} 0 & 1 & a \\ (a+3d) & 1 & (a+d) \\ 2(a+4d) & 1 & (a+2d) \end{vmatrix}$$

$$= \frac{2d^2}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)} \times \begin{vmatrix} 0 & 1 & a \\ (a+3d) & 0 & d \\ 2(a+4d) & 0 & 2d \end{vmatrix} \begin{pmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$$

$$= \frac{4d^4}{a(a+d)^2(a+2d)^3(a+3d)^2(a+4d)}$$

31. Here $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are the pth, qth and rth terms of an A.P

Let first term be A and the common difference be D.

Then $\frac{1}{a} = A + (p-1)D$

$\frac{1}{b} = A + (q-1)D$

and $\frac{1}{c} = A + (r-1)D$

Then given determinant is

$$= \begin{vmatrix} bc & ca & ab \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

$$= abc \times \begin{vmatrix} \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

$$= abc \times \begin{vmatrix} A+(p-1)D & A+(q-1)D & A+(r-1)D \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

Applying $R_1 \to R_1 - (A-D)R_3 - DR_2$, we get

$$= abc \times \begin{vmatrix} 0 & 0 & 0 \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix}$$

$= 0$

32. We have

$$\Delta = \begin{vmatrix} xp+y & x & y \\ yp+z & y & z \\ 0 & xp+y & yp+z \end{vmatrix}$$

$$= \begin{vmatrix} 0 & x & y \\ 0 & y & z \\ -(p(xp+y)+yp+z) & xp+y & yp+z \end{vmatrix}$$

$$= -(p(xp+y)+yp+z)(xz-y^2)$$

Now, $\Delta = 0$ gives

$(xz - y^2) = 0$

$\Rightarrow \quad xz = y^2$

$\Rightarrow \quad x, y$ and z are in GP

33. We have

$$\Delta = \begin{vmatrix} 1 & a & a^2 \\ \cos(p-d)x & \cos(p)x & \cos(p+d)x \\ \cos(p-d)x & \sin(p)x & \sin(p+d)x \end{vmatrix}$$

$$= \sin(p+d-p)x - a\sin(p+d-p+d)x + a^2\sin(p-p+d)x$$

$$= \sin(d)x - a\sin(2d)x + a^2\sin(d)x$$

which is independent of p.

34. We have

$$f(x)=\begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

$$\Rightarrow \quad \frac{d}{dx}[f(x)]=\begin{vmatrix} 3x^2 & \cos x & -\sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

$$=\begin{vmatrix} 6x & -\sin x & -\cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

$$\Rightarrow \quad \frac{d^3}{dx^3}[f(x)]=\begin{vmatrix} 6 & -\cos x & -\sin x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

Now,

$$\left(\frac{d^3}{dx^3}[f(x)]\right)_{x=0}=\begin{vmatrix} 6 & -1 & 0 \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$$

$$=0$$

which is independent of p.

35. We have,

$$\begin{vmatrix} 6i & -3i & 1 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix}=\begin{vmatrix} 26i & 0 & 0 \\ 4 & 3i & -1 \\ 20 & 3 & i \end{vmatrix} \qquad (R_1 \to R_1 + iR_3)$$

$$= 26i(-3 + 3)$$
$$= 0$$

Thus, $x + iy = 0$

$\Rightarrow \quad x = 0, y = 0$

36. We have,

$$f(x)=\begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & x(x+1) \\ 3x(x-1) & x(x-1)(x-2) & x(x+1)(x-1) \end{vmatrix}$$

$$=x^2(x-1)\begin{vmatrix} 1 & x & x+1 \\ 2 & (x-1) & (x+1) \\ 3 & (x-2) & (x+1) \end{vmatrix}$$

$$=x^2(x-1)\begin{vmatrix} 1 & x & x+1 \\ 1 & -1 & 0 \\ 2 & -2 & 0 \end{vmatrix}\begin{pmatrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{pmatrix}$$

$$=x^2(x^2-1)\begin{vmatrix} 1 & -1 \\ 2 & -2 \end{vmatrix}$$

$$=0$$

37. We have,

$$\begin{vmatrix} \sin\theta & \cos\theta & \sin\theta \\ \sin\left(\theta+\frac{2\pi}{3}\right) & \cos\left(\theta+\frac{2\pi}{3}\right) & \sin\left(\theta+\frac{4\pi}{3}\right) \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(\theta-\frac{4\pi}{3}\right) \end{vmatrix}$$

$$=\begin{vmatrix} \sin\theta & \cos\theta & \sin\theta \\ 2\sin\theta\cos\left(\frac{2\pi}{3}\right) & 2\cos\theta\cos\left(\frac{2\pi}{3}\right) & 2\sin 2\theta\cos\left(\frac{4\pi}{3}\right) \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(\theta-\frac{4\pi}{3}\right) \end{vmatrix}$$

$$=\begin{vmatrix} \sin\theta & \cos\theta & \sin\theta \\ -\sin\theta & -\cos\theta & -\sin 2\theta \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(\theta-\frac{4\pi}{3}\right) \end{vmatrix}$$

$$=\begin{vmatrix} \sin\theta & \cos\theta & \sin\theta \\ 0 & 0 & 0 \\ \sin\left(\theta-\frac{2\pi}{3}\right) & \cos\left(\theta-\frac{2\pi}{3}\right) & \sin\left(\theta-\frac{4\pi}{3}\right) \end{vmatrix}$$

$(R_2 \to R_2 + R_1)$

$$= 0$$

38. Since the given system of equation has a non zero solution, then

$$\begin{vmatrix} 1 & -k & -1 \\ k & -1 & -1 \\ 1 & 1 & -1 \end{vmatrix}=0$$

$\Rightarrow \quad 1(2) + k(-k + 1) - (k + 1) = 0$

$\Rightarrow \quad 2 - k^2 + k - k - 1 = 0$

$\Rightarrow \quad 2 - k^2 - 1 = 0$

$\Rightarrow \quad k^2 = 1$

$\Rightarrow \quad k = \pm 1$

Hence, the values of k are 1 and –1.

39. The given equation is

$$\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix}=0$$

$$\Rightarrow \quad \begin{vmatrix} \sin x+2\cos x & \cos x & \cos x \\ \sin x+2\cos x & \sin x & \cos x \\ \sin x+2\cos x & \cos x & \sin x \end{vmatrix}=0$$

$$\Rightarrow \quad (\sin x+2\cos x)\begin{vmatrix} 1 & \cos x & \cos x \\ 1 & \sin x & \cos x \\ 1 & \cos x & \sin x \end{vmatrix}=0$$

$$\Rightarrow \quad (\sin x+2\cos x)\begin{vmatrix} 1 & \cos x & \cos x \\ 0 & \sin x-\cos x & 0 \\ 0 & 0 & \sin x-\cos x \end{vmatrix}=0$$

$\Rightarrow$ $(\sin x + 2\cos x)(\sin x - \cos x)^2 = 0$
$\Rightarrow$ $(\sin x + 2\cos x) = 0, (\sin x - \cos x) = 0$
$\Rightarrow$ $\tan x = -2, \tan x = 1$

since $-\frac{\pi}{4} \le x \le \frac{\pi}{4}$

$\Rightarrow$ $-1 \le \tan x \le 1$

Hence, the number of solutions is 1 at $x = \frac{\pi}{4}$.

40. The given determinant is

$$\begin{vmatrix} ax-by-c & bx+ay & cx+a \\ bx+ay & -ax+by-c & cy+b \\ cx+a & cy+b & -ax-by+c \end{vmatrix}$$

Applying $C_1 \to aC_1 + bC_2 + cC_3$, we get

$$\frac{1}{a}\begin{vmatrix} (a^2+b^2+c^2)x & bx+ay & cx+a \\ (a^2+b^2+c^2)y & -ax+by-c & cy+b \\ (a^2+b^2+c^2) & cy+b & -ax-by+c \end{vmatrix}$$

$$= \frac{1}{a}\begin{vmatrix} x & bx+ay & cx+a \\ y & -ax+by-c & cy+b \\ 1 & cy+b & -ax-by+c \end{vmatrix}$$

Applying $\begin{pmatrix} C_2 \to C_2 - bC_1 \\ C_3 \to C_3 - cC_1 \end{pmatrix}$, we get

$$\frac{1}{a}\begin{vmatrix} x & ay & a \\ y & -ax-c & b \\ 1 & cy & -ax-by \end{vmatrix}$$

Applying $R_1 \to (xR_1 + yR_2 + R_3)$, we get

$$\frac{1}{ax}\begin{vmatrix} x^2+y^2+1 & 0 & 0 \\ y & -ax-c & b \\ 1 & cy & -ax-by \end{vmatrix}$$

$$= \frac{1}{ax}(x^2+y^2+1)[(ax)^2 + acx + abxy + bcy - bcy]$$

$$= \frac{1}{ax}(x^2+y^2+1)[(ax)^2 + acx + abxy]$$

$$= (x^2+y^2+1)(ax+by+c)$$

If the given determinant is zero, then

$(ax + by + c) = 0, [(x^2 + y^2 + 1) \neq 0]$

Thus, $(ax + by + c) = 0$ represents a straight line.

41. The given determinant is

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1-\omega^2 & \omega^2 \\ 1 & \omega^2 & \omega^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix}$$

$$= \begin{vmatrix} 3 & 1 & 1 \\ 0 & \omega & \omega^2-1 \\ 0 & \omega^2 & \omega \end{vmatrix} \quad (C_1 \to C_1 + C_2 + C_3)$$

$= 3(\omega^2 - \omega^4)$
$= 3(\omega^2 - \omega)$
$= 3\omega(\omega - 1)$

42. Given system of equations are

$(k+1)x + 8y = 4k;\ kx + (k+3)y = 3k - 1$

has infinitely many solutions, if

$$\frac{(k+1)}{k} = \frac{8}{(k+3)} = \frac{4k}{(3k-1)}$$

$\Rightarrow$ $\frac{(k+1)}{k} = \frac{8}{(k+3)}$
$\Rightarrow$ $(k+1)(k+3) = 8k$
$\Rightarrow$ $k^2 + 4k + 3 - 8k = 0$
$\Rightarrow$ $k^2 - 4k + 3 = 0$
$\Rightarrow$ $(k-1)(k-3) = 0$
$\Rightarrow$ $k = 1, 3$

43. We have

$$A^2 = A \cdot A = \begin{pmatrix} a & 0 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} a^2 & 0 \\ a+1 & 1 \end{pmatrix}$$

Given relation is

$A^2 = B$

$\Rightarrow$ $\begin{pmatrix} a^2 & 0 \\ a+1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}$
$\Rightarrow$ $a^2 = 1, a + 1 = 5$
$\Rightarrow$ $a = \pm 1, a = 4$

44. The given system of equations has infinitely many solutions, if

$$\begin{vmatrix} 1 & a & 0 \\ a & 0 & 1 \\ 0 & 1 & a \end{vmatrix} = 0$$

$\Rightarrow$ $1(0 - 1) - a(a^2 - 0) = 0$
$\Rightarrow$ $-1 - a^3 = 0$
$\Rightarrow$ $a^3 = -1$
$\Rightarrow$ $a = -1$

45. We have,

$$A^T A = \begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}\begin{pmatrix} a & b & c \\ b & c & a \\ c & a & b \end{pmatrix}$$

$$= \begin{pmatrix} a^2+b^2+c^2 & ab+bc+ca & ab+bc+ca \\ ab+bc+ca & a^2+b^2+c^2 & ab+bc+ca \\ ab+bc+ac & ab+bc+ca & a^2+b^2+c^2 \end{pmatrix}$$

$$= \begin{pmatrix} \alpha & \beta & \beta \\ \beta & \alpha & \beta \\ \beta & \beta & \alpha \end{pmatrix}$$

where $a^2 + b^2 + c^2 = \alpha$, $ab + bc + ca = \beta$

Since $AA^T = I$, so,

$a^2 + b^2 + c^2 = 1$

and $ab + bc + ca = 0$

Now, $a^3 + b^3 + c^3 - 3abc$

$= (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$

$= (a + b + c)(1 - 0)$

$= (a + b + c)$

Also, $(a + b + c)^2 = (a^2 + b^2 + c^2 + 2(ab + bc + ca))$

$\Rightarrow \quad (a + b + c)^2 = 1 + 2.0 = 1$

$\Rightarrow \quad (a + b + c) = 1$, since a, b, c are all positive

Thus, $a^3 + b^3 + c^3 - 3abc = 1$

$\Rightarrow \quad a^3 + b^3 + c^3 - 3abc + 1 - 3 + 1 - 4$

46. The given system of equations has no solution, if

$$\begin{vmatrix} 2 & -1 & 2 \\ 1 & -2 & 1 \\ 1 & 1 & \lambda \end{vmatrix} = 0$$

$\Rightarrow \quad 2(-2\lambda - 1) + (\lambda - 1) + 2(1 + 2) = 0$

$\Rightarrow \quad -4\lambda - 2 + \lambda - 1 + 6 = 0$

$\Rightarrow \quad -3\lambda + 3 = 0$

$\Rightarrow \quad \lambda = 1$

47. We have

$A^2 = A \cdot A$

$$= \begin{pmatrix} \alpha & 2 \\ 2 & \alpha \end{pmatrix} \cdot \begin{pmatrix} \alpha & 2 \\ 2 & \alpha \end{pmatrix}$$

$$= \begin{pmatrix} \alpha^2 + 4 & 4\alpha \\ 4\alpha & \alpha^2 + 4 \end{pmatrix}$$

Now,

$A^3 = A^2 A$

$$= \begin{pmatrix} \alpha^2 + 4 & 4\alpha \\ 4a & \alpha^2 + 4 \end{pmatrix} \cdot \begin{pmatrix} \alpha & 2 \\ 2 & \alpha \end{pmatrix}$$

$$= \begin{pmatrix} \alpha^3 + 12\alpha & 6\alpha^2 + 8 \\ 6\alpha^2 + 8 & \alpha^3 + 12\alpha \end{pmatrix}$$

Given,

$|A^3| = 125$

$$\Rightarrow \quad \begin{vmatrix} \alpha^3 + 12\alpha & 6\alpha^2 + 8 \\ 6\alpha^2 + 8 & \alpha^3 + 12\alpha \end{vmatrix} = 125$$

$\Rightarrow \quad (\alpha^3 + 12\alpha)^2 - (6\alpha^2 + 8)^2 = 125$

$\Rightarrow \quad (\alpha^3 + 12\alpha + 6\alpha^2 + 8)(\alpha^3 + 12\alpha - 6\alpha^2 - 8) = 125$

$\Rightarrow \quad (\alpha^3 + 6\alpha^2 + 12\alpha + 8)(\alpha^3 - 6\alpha^2 + 12\alpha - 8) = 125$

$\Rightarrow \quad (\alpha + 2)^3 (\alpha - 2)^3 = 125$

$\Rightarrow \quad \{(\alpha + 2)(\alpha - 2)\}^3 = (5)^3$

$\Rightarrow \quad (\alpha + 2)(\alpha - 2) = 5$

$\Rightarrow \quad \alpha^2 - 4 = 5$

$\Rightarrow \quad \alpha^2 = 9$

$\Rightarrow \quad \alpha = \pm 3$

49. We have $6A^{-1} = A^2 + cA + dI$

$\Rightarrow \quad 6I = A^3 + cA^2 + dA$

Now,

$A^2 = A.A$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 5 \\ 0 & -10 & 14 \end{pmatrix}$$

and

$A^3 = A^2 A$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 5 \\ 0 & -10 & 14 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -11 & 19 \\ 0 & -38 & 46 \end{pmatrix}$$

Now,

$6I = A^3 + cA^2 + dA$ gives

$6 = 1 + c + d, \qquad 0 = 19 + 5c + d$

$6 = -11 - c + 5d, \qquad 0 = -38 - 10c - 2d$

$6 = 46 + 14c + 4d$

Thus, $c = -6$ and $d = 11$.

50. Given

$$P = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\pi}{6}\right) & \sin\left(\frac{\pi}{6}\right) \\ -\sin\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right) \end{pmatrix}$$

$$\Rightarrow \quad P^T = \begin{pmatrix} \cos\left(\frac{\pi}{6}\right) & -\sin\left(\frac{\pi}{6}\right) \\ \sin\left(\frac{\pi}{6}\right) & \cos\left(\frac{\pi}{6}\right) \end{pmatrix}$$

Since $PP^T = I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow \quad P^T = P^{-1}$

We have $Q = PAP^T = PAP^{-1}$

$\Rightarrow \quad Q^{2005} = (PAP^{-1})^{2005} = PA^{2005}P^T$

$\Rightarrow \quad P^T Q^{2005} P = P^{-1}(A^{2005}P^T)P$

$\Rightarrow \quad P^T Q^{2005} P = (P^{-1}P)(A^{2005})(P^{-1}P)$

$\Rightarrow \quad P^T Q^{2005} P = (A^{2005})$

Now, $A = I + B$,

where $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Since $B^2 = O$, we get

$$B^r = O \ \forall\ r \geq 2$$

Thus, $A^{2005} = I + 2005B = \begin{pmatrix} 1 & 2005 \\ 0 & 1 \end{pmatrix}$

51. Given,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

$\Rightarrow \quad |A| = 1 \neq 0$

$\Rightarrow \quad A^{-1}$ exists

Thus,

$$A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}$$

Now, $U_1 = A^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$$U_2 = A^{-1}\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -4 \end{pmatrix}$$

$$U_3 = A^{-1}\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ -3 \end{pmatrix}$$

(i) Thus, $U = \begin{pmatrix} 1 & 2 & 2 \\ -2 & -1 & -1 \\ 1 & -4 & -3 \end{pmatrix}$

$$\Rightarrow \quad |U| = \begin{vmatrix} 1 & 2 & 2 \\ -2 & -1 & -1 \\ 1 & -4 & -3 \end{vmatrix} = 3$$

(ii) Given, $U = \begin{pmatrix} 1 & 2 & 2 \\ -2 & -1 & -1 \\ 1 & -4 & -3 \end{pmatrix}$

$$\Rightarrow \quad U^{-1} = \frac{1}{3}\begin{pmatrix} -1 & -7 & 9 \\ -2 & -5 & 6 \\ 0 & -3 & 3 \end{pmatrix}$$

Thus, the sum of the elements of U^{-1}

$$= \frac{1}{3}(-18 + 18) = 0$$

(iii) We have,

$$[3 \ \ 2 \ \ 0]\,U\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

$$= (3 \ \ 2 \ \ 0)\begin{pmatrix} 1 & 2 & 2 \\ -2 & -1 & -1 \\ 1 & -4 & -3 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}$$

$$= (3 \ \ 2 \ \ 0)\begin{pmatrix} 7 \\ -8 \\ -5 \end{pmatrix}$$

$$= (21 - 16) = (5)$$

53. We have,

$$D = \begin{vmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = 0$$

and the determinant

$$D_1 = \begin{vmatrix} 1 & 3 & -1 \\ -1 & -2 & k \\ 1 & 4 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & k-1 \\ -1 & -2 & k \\ 0 & 3 & k+1 \end{vmatrix} = 3 - k$$

$$\begin{pmatrix} R_1 \to R_1 + R_2 \\ R_3 \to R_2 + R_3 \end{pmatrix}$$

If $k \neq 3$, $D = 0$, $D \neq 0$, therefore, the system of equations has no solutions.

54. (i) Total number of matrices = 9 + 3 = 12

We have

$$A_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$A_4 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, A_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, A_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$A_7 = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, A_8 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, A_9 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$A_{10} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, A_{11} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, A_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Alternate method: If two zeros are the entries in the diagonal, then

$$^3C_2 \times {}^3C_1$$

If all the entries in the principle diagonal is 1, then $^3C_1$.

Thus, the total number of matrices = 12.

(ii) Here, $|A_2| \neq 0$, $|A_3| \neq 0$, $|A_4| \neq 0$

and $|A_5| \neq 0$, $|A_7| \neq 0$, $|A_9| \neq 0$,

Thus, there are six matrices A such that

$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ has a unique solution.

(iii) Let $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Thus, $A_1X = B$ has infinite number of solutions
$A_6X = B$ has no solution
$A_8X = B$ has no solution
$A_{10}X = B$ has no solution
$A_{12}X = B$ has infinite number of solutions

55.
We know that,
$$\det(\text{Adj}A) = (\det A)^{n-1}$$
and B is a skew-symmetric matrix, so
$$\det(B) = 0$$
$$\det(\text{adj } B) = \det B)^{3-1} = (\det B)^2 = 0$$
now,
$$|A| = \begin{vmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{vmatrix}$$
$$= \begin{vmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 0 & 1+2k & -(1+2k) \\ -2\sqrt{k} & 2k & -1 \end{vmatrix} (R_2 \to R_2 + R_3)$$
$$= \begin{vmatrix} 2k-1 & 2\sqrt{k} & 4\sqrt{k} \\ 0 & 1+2k & 0 \\ -2\sqrt{k} & 2k & 2k-1 \end{vmatrix} \quad (C_3 \to C_3 + C_2)$$
$$= (2k-1)(4k^2-1) + 8k(1+2k)$$
$$= (2k+1)[(2k-1)^2 + 8k]$$
$$= (2k+1)[4k^2 + 4k + 1]$$
$$= (2k+1)(2k+1)^2$$
$$= (2k+1)^3$$
It is given that,
$$\det(\text{adj } A) + \det(\text{adj } B) = 10^6$$
$\Rightarrow$ $(2k+1)^6 + 0 = 10^6$
$\Rightarrow$ $(2k+1)^6 = 10^6$
$\Rightarrow$ $(2k+1) = 10$
$\Rightarrow$ $k = 4.5$
Thus, the value of $[k] = [4.5] = 4$

56. The given system of equations can be written in matrix form as
$$AX = B$$
It has either
(i) a unique solution
or (ii) infinite solutions
or (iii) no solution.
Thus, there can not exist any matrix A such that
$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has two distinct solutions.

58. Given
$$MN = NM$$
Now, $M^2N^2(M^TN)^{-1}(MN^{-1})^T$
$= M^2N^2N^{-1}(M^TN)^{-1}(MN^{-1})^T$
$= M^2N \cdot (M^T)^{-1}(N^{-1})^TM^T$
$= -M^2N \cdot (M)^{-1}(N^T)^{-1}M^T$
$= +M^2N \cdot (M)^{-1}N^{-1}M^T$
$= -MNMM^{-1}N^{-1}M$
$= -MNN^{-1}M$
$= -M^2$

59. Let $M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

Now, $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$

$\Rightarrow$ $b = -1, e = 2, h = 3$

Also, $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$\Rightarrow$ $a = 0, d = 3, g = 2$

and, $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 12 \end{pmatrix}$

$\Rightarrow$ $g + h + i = 12$
$\Rightarrow$ $i = 7$
Thus, sum of the main diagonal
$= a + e + i = 0 + 2 + 7 = 9$

60. Given
$$P^T = 2P + I$$
$\Rightarrow$ $(P^T)^T = (2P + I)^T$
$\Rightarrow$ $P = (2P^T + I)$
$\Rightarrow$ $P = 2(2P + I) + I$
$\Rightarrow$ $3(P + I) = O$
$\Rightarrow$ $(P + I) = O$
$\Rightarrow$ $PX + X = O$
$\Rightarrow$ $PX = -X$

61. Given $\text{adj}(P) = \begin{pmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{pmatrix}$

$\Rightarrow$ $|\text{adj}(P)| = \begin{vmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{vmatrix} = 4$

As we know that,
$$|\text{adj}(P)| = |P|^{3-1} = |P|^2$$
Thus,
$$|P|^2 = 4$$
$\Rightarrow$ $|P| = 2, -2$

62. Let $P = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, where $|P| = 2$

Now, $Q = \begin{pmatrix} 2^2 a_{11} & 2^3 a_{12} & 2^4 a_{13} \\ 2^3 a_{21} & 2^4 a_{22} & 2^5 a_{23} \\ 2^4 a_{31} & 2^5 a_{32} & 2^6 a_{33} \end{pmatrix}$

$$\Rightarrow \quad |Q| = \begin{vmatrix} 2^2 a_{11} & 2^3 a_{12} & 2^4 a_{13} \\ 2^3 a_{21} & 2^4 a_{22} & 2^5 a_{23} \\ 2^4 a_{31} & 2^5 a_{32} & 2^6 a_{33} \end{vmatrix}$$

$$= 2^2 \cdot 2^3 \cdot 2^4 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 2a_{21} & 2a_{22} & 2a_{23} \\ 2^2 a_{31} & 2^2 a_{32} & 2^2 a_{33} \end{vmatrix}$$

$$= 2^2 \cdot 2^3 \cdot 2^4 \cdot 2 \cdot 2^2 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= 2^{12} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$= 2^{12} \cdot 2 = 2^{13}$

63. (a) $(N^T MN)^T = N^T M^T N$

$$= \begin{cases} N^T MN & : \text{ if } M \text{ is symmetric} \\ -N^T MN & : \text{ if } M \text{ is skew-symmetric} \end{cases}$$

(b) $(MN - NM)^T = N^T M^T - M^T N^T$

$= NM - MN = -(MN - NM)$

Thus, $(MN - NM)$ is skew-symmetric.

(c) $(MN)^T = N^T M^T = NM \neq MN$

if M and N are symmetric.

So, MN is not symmetric.

(d) (adj M)(adj N) = adj(NM) $\neq$ adj(MN)

64. We have

$$P = \begin{pmatrix} \omega^2 & \omega^3 & \omega^4 & \cdots & \omega^{n+2} \\ \omega^3 & \omega^4 & \omega^5 & \cdots & \omega^{n+3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \omega^{n+2} & \omega^{n+3} & \omega^{n+4} & \cdots & \omega^{2n+4} \end{pmatrix}$$

Now

$P^2 = P.P$

$$= \begin{pmatrix} \omega^2 & \omega^3 & \omega^4 & \ldots \omega^{n+2} \\ \omega^3 & \omega^4 & \omega^5 & \ldots \omega^{n+3} \\ \vdots & \vdots & \vdots & \vdots \quad \vdots \\ \omega^{n+2} & \omega^{n+3} & \omega^{n+4} & \ldots \omega^{2n+4} \end{pmatrix} \begin{pmatrix} \omega^2 & \omega^3 & \omega^4 & \ldots \omega^{n+2} \\ \omega^3 & \omega^4 & \omega^5 & \ldots \omega^{n+3} \\ \vdots & \vdots & \vdots & \vdots \quad \vdots \\ \omega^{n+2} & \omega^{n+3} & \omega^{n+4} & \ldots \omega^{2n+4} \end{pmatrix}$$

$$= \begin{pmatrix} \omega^4 + \omega^6 + \cdots. & \omega^5 + \omega^7 + \omega^9 + \cdots \cdots \cdots & \cdots \\ \omega^5 + \omega^7 + \omega^9 + \cdots & \cdots & \cdots\cdots & \cdots \\ \vdots & \vdots & \vdots \ \vdots & \vdots \\ \omega^{n+4} + \omega^{n+6} + \cdots & \cdots & \cdots\cdots \ \omega^{2n+4} + \omega^{2n+6} \end{pmatrix}$$

Thus, P^2 is a null matrix if n is a multiple of 3.

65. Let $M = \begin{pmatrix} a & c \\ c & b \end{pmatrix}$ where $a, b, c, d \in I$.

Then $|M| = \begin{vmatrix} a & c \\ c & b \end{vmatrix} = ab - c^2$.

If $a = b = c$, then $|M| = 0$

If $c = 0$, $a, b \neq 0$, then $|M| \neq 0$

If $ab \neq$ square of integer, then $|M| \neq 0$

66. Given

$M^2 = N^4$

$\Rightarrow \quad M^2 - N^4 = O$

$\Rightarrow \quad (M - N^2)(M + N^2) = O$

As M, N commute.

Also, $M \neq N^2$

$\Rightarrow \quad \det((M - N^2)(M + N^2)) = 0$

As $(M - N^2)$ is not null.

$\Rightarrow \quad \text{Det}(M^2 + N^2) = 0$

Also, $\text{Det}(M + N^2)$

$= (\text{Det}M)(\text{Det}(M + N^2)) = 0$

There exists a non-null U such that

$(M^2 + MN^2)U = O$

67. (a) $(Y^3Z^4 - Z^4Y^3)$

$= (-Y^3Z^4 - Z^4Y^3)$

$\Rightarrow \quad (Y^3Z^4 - Z^4Y^3)$ is skew-symmetric.

(c) $(X^4Z^3 - Z^3X^4)^T = ((X^4Z^3)^T - (Z^3X^4)^T)$

$= (Z^T)^3(X^T)^4 - (X^T)^4(Z^T)^3$

$= (Z^3X^4 - X^4Z^3)$

$= -(X^4Z^3 - Z^3X^4)$

(d) $(X^{23} + Y^{23})^T$

$= -X^{23} - Y^{23}$

$\Rightarrow (X^{23} + Y^{23})$ is skew-symmetric

68. We have,

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix} = -648\alpha?$$

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ 3+2\alpha & 3+4\alpha & 3+6\alpha \\ 5+2\alpha & 5+4\alpha & 5+6\alpha \end{vmatrix} = -648\alpha$$

$(R_3 \to R_3 - R_2, R_2 \to R_2 - R_1)$

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ 3+2\alpha & 3+4\alpha & 3+6\alpha \\ 2 & 2 & 2 \end{vmatrix} = -648\alpha$$

$(R_3 \to R_3 - R_2)$

$$\begin{vmatrix} (1+\alpha)^2 & \alpha(2+3\alpha) & \alpha(2+5\alpha) \\ 3+2\alpha & 2\alpha & 2\alpha \\ 2 & 0 & 0 \end{vmatrix} = -648\alpha$$

$$\Rightarrow \quad \alpha^2 \begin{vmatrix} (1+\alpha)^2 & (2+3\alpha) & (2+5\alpha) \\ 3+2\alpha & 2 & 2 \\ 2 & 0 & 0 \end{vmatrix} = -648\alpha$$

$\Rightarrow \quad 2\alpha^2(4 + 6\alpha - 4 - 10\alpha) = -64\alpha$

$\Rightarrow \quad \alpha^2 \times -4\alpha = -324\alpha$

$\Rightarrow \quad \alpha^3 = 81\alpha$

$\Rightarrow \quad \alpha(\alpha^2 - 81) = 0$

$\Rightarrow \quad \alpha = 0, (\alpha^2 - 81) = 0$

$\Rightarrow \quad \alpha = 0, \pm 9$

CHAPTER

8

Probability

CONCEPT BOOSTER

1. Introduction

The probability theory is originated from the game of chance, related to gambling, namely, tossing a coin, throwing a die, drawing a card from a pack of well-shuffled 52 cards, etc., in which the outcome of a trial is uncertain. Actual appearance of one of these results is not predicted.

The word 'probability' or chance is very commonly used in day-to-day conversation and also, generally, people have a rough idea about its meaning.

For example, the statements, 'Probably it will rain today', 'It is likely that Principal Sir may not come for taking his class today', ' The chance of passing of Nidhi in the examination is good', 'It is possible that the captain of Indian cricket team will be replaced next year', etc.

All these terms—probably, likely, chance, probable, etc. —convey in the same sense of uncertainty. So, the objective of the probability theory is to make such statements precise by giving them numerical measures.

2. Sample Space

The set of all possible outcomes of a random experiment is called a sample space or a probability space. It is denoted by S.

Each element of a sample space is called a sample point. It is denoted by the symbol ω (Omega).

For example,

(i) On tossing a coin, the sample space is

$S = \{H, T\}$,

where H and T are sample points.

(ii) On tossing a coin two times, the sample space is $S = \{\text{HH, HT, TH, TT}\}$.

(iii) On tossing a coin three times, the sample space is

$S = \{\text{HHH, HHT, HTH, THH, TTT, TTH, THT, HTT}\}$

(iv) On throwing a die, the sample space is

$S = \{1, 2, 3, 4, 5, 6\}$,

where 1, 2, 3, 4, 5 and 6 are sample points.

(v) On throwing a die two times, the sample space is

	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Note: If the number of sample points is finite, the sample space is called a *discrete sample space*, otherwise it is called a *continuous sample space*.

Probability theory is used to estimate whether a missile hit its target or not, to determine premium of insurance policies and to make important business decisions such as where to locate a super market and how many clerks to employ so that customers will not be kept waiting in line too long. Various sampling techniques, which are used in opinion polls and in the quality control of mass-produced items, are based on the the theory of probability. In fact, it has become an indispensable tool for all types of formal studies that involve uncertainty. It should be noted that the concept of probability is employed not only for various types of investigations but also for many problems in everyday life.

Now we shall define some important terms which are essential for the study of the concept of the theory of the probability.

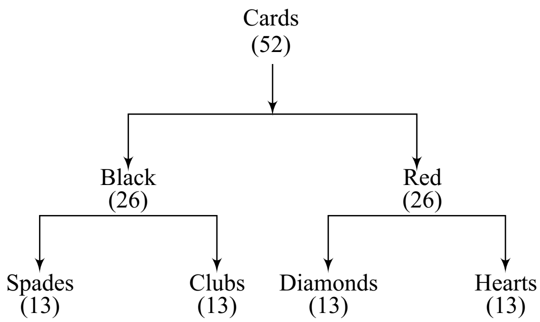
3. Random Experiment

In an experiment, if all the possible outcomes are known in advance and none of them can be predicted with certainty, the experiment is known as random experiment.

For example, throwing a die, tossing of a fair coin, drawing a card from a pack of 52 cards, etc.

In the study of probability, we will use the words like coin, die, cards, etc. Let us try to give some brief descriptions about these words.

1. **Coin:** It is a piece of stamped metal, used as money and has two faces known as H (head) or T (tail). On tossing a coin, we can get either H or T.
2. **Die:** It is a solid cube, used for playing ludo and has six marked faces (1, 2, 3, 4, 5, 6). On throwing a die, we can get any of the numbers from 1 to 6. The plural of die is dice.
3. **Cards:** A pack of playing cards has 52 cards in two colours and divided equally in four suits. They are Spades, Clubs (black cards) and Hearts and Diamonds (red cards). Each suit consists of 13 cards, namely, an ace, a king, a queen, a jack and 9 number cards (from 2 to 10).

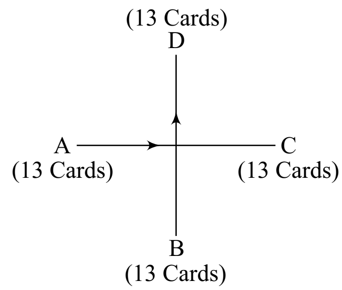


(i) **Face Cards**
The kings, the queens and the jacks are called the face cards. Obviously they are 12 in numbers (3 in each suit).

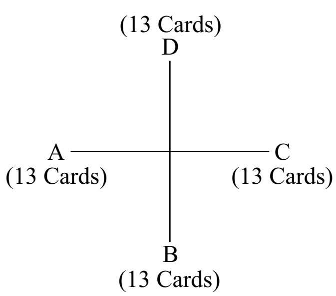
(ii) **Court Cards**
The aces, the kings, the queens and the jacks are called the court cards. Obviously they are 16 in numbers (4 in each suit).

(iii) **Honours Cards**
The aces, the kings and the queens are called honours cards. Obviously they are 12 in numbers (3 in each suit).

(iv) **Game of Bridge**
It is played by 4 players. They will play individually and each player will get 13 cards.



(v) **Game of Whist**
It is also played by 4 players. But they will play it into two groups. Group members will sit opposite to each other and also each player will get 13 cards.



4. Event Space

Any subset of a sample space is called an event space or simply an event. It is generally denoted by capital letters of English alphabet, say A, B, C, etc.

An event having only one element is called a *simple event*.

An event which can never occur when a certain random experiment is performed is called an *impossible event*. It is denoted by ϕ or $\{\}$.

For example, on throwing of a die, the occurrence of 7 is an impossible event.

An event which may or may not occur while performing a certain random experiment is called a *random event*.

For example, on throwing of a die, the occurrence of 1 or 2 or 3 or 4 or 5 or 6 is a random event.

Equally Likely Events

Two or more events are said to be equally likely if the chance of their happening is equal on tossing of a fair coin.

For example, head and tail are equally likely events. Similarly, 1, 2, 3, 4, 5 and 6 are equally likely events on throwing of an unbiased die.

Exhaustive Events

The total number of possible outcomes of a random experiment is known as exhaustive event.

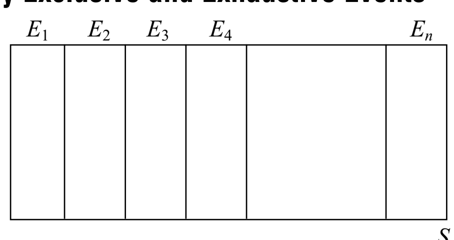
For example, on tossing of a fair coin, exhaustive event is 2, on throwing of a die, exhaustive event is 6. On drawing a card from a pack of 52 cards, exhaustive event is 52.

Mutually Exclusive Events

Two or more events are said to be mutually exclusive if they cannot occur simultaneously in a single trial. All simple events are mutually exclusive.

If a sample space consists of $E_1, E_2, E_3, \ldots, E_n$ events, then $E_1 \cap E_2 \cap E_3 \cap \ldots E_{n-1} \cap E_n = \varphi$

Mutually Exclusive and Exhaustive Events



A set of events $E_1, E_2, \ldots, E_n$ of a sample space S is said to be mutually exclusive and exhaustive event if

(i) $E_1 \cap E_2 \cap E_3 \cap \ldots E_{n-1} \cap E_n = \varphi$

(ii) $E_1 \cap E_2 \cap E_3 \cap \ldots E_{n-1} \cap E_n = S$.

To explain further, we use some examples.

Example 1: If we throw an unbiased die, then $S = \{1, 2, \ldots, 6\}$ in which $E_1 = \{2, 4, 6\}$ and $E_2 = \{1, 3, 5\}$.

Clearly $E_1 \cup E_2 = S$ and $E_1 \cup E_2 = \varphi$.

Thus E_1 and E_2 are mutually exclusive and exhaustive events.

Example 2: Let a die is thrown. The sample is $S = \{1, 2, 3, 4, 5, 6\}$.

Let $A = \{2, 4, 6\}$ and $B = \{1, 2, 3, 5\}$

Then A and B are not mutually exclusive, i.e. $A \cap B \neq \varphi$

But $A \cup B = \{1, 2, 3, 4, 5, 6\} = S$.

Thus A and B are exhaustive events.

From the above examples, we can conclude that, mutually exclusive events can be a exhaustive events and its converse is also true.

Independent Events

Two or more events are said to be independent when the occurrence of one does not affect the other.

For example, if we toss a coin twice, the occurrence of the second toss will, in no way, be affected by the outcome of the first toss.

But if we throw a die, the sample space is

$S = \{1, 2, \ldots, 6\}$

Let E be the event of getting an even number and F be the event of getting an odd number, then

$E = \{2, 4, 6\}$ and $F = \{1, 3, 5\}$ clearly $E_1 \cap E_2 = \varphi$.

Thus E and F are mutually exclusive events. But these events are not independent.

The event of getting a tail on the first coin and the event of getting a tail on the second coin in a simultaneous throw of two coins are independent.

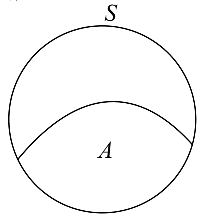
Probability of Occurrence of an Event

Let S be a sample space and A be any event. Then

$$P(A) = \frac{n(A)}{n(S)}$$

$$= \frac{\text{Number of cases favourable to event } A}{\text{total number of cases}}$$

S



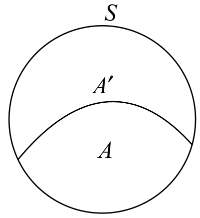
If A be any event and A' be the complement event of A on a sample S, then

$$P(A) + P(A') = 1$$

Proof

Here A and A' are mutually exclusive and exhaustive event of a sample space S.

S



Thus $A \cap A' = \varphi$ and $A \cup A' = S$

Now, $P(A \cup A') = P(S) = 1$

$\Rightarrow \quad P(A) + P(A') = 1.$

$\Rightarrow \quad P(A) = 1 - P(A').$

Odds in Favour and Against of an Event

Let S be a sample space and A be an event. Let A' be the complement of an event A, then

(i) Odds in favour of an event A

$$= \frac{\text{number of cases favourable to the event } A}{\text{number of cases against the event } A}$$

$$= \frac{n(A)}{n(A')} = \frac{n(A)/n(S)}{n(A')/n(S)} = \frac{P(A)}{P(A')}$$

(ii) Odds against an event A

$$= \frac{\text{number of cases against the event } A}{\text{number of cases favourable to the event } A}$$

$$= \frac{n(A')}{n(A)} = \frac{P(A')}{P(A)}$$

5. Axioms of Probability

Let A be any event of a sample space S. Then

Axioms 1: $0 \leq P(A) \leq 1$

Axioms 2: $P(S) = 1$.

Axioms 3: $P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$

6. Addition Theorem on Probability

Theorem: If A and B be any two events of a sample space S, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Notes

(i) If A and B be mutually exclusive events, then
$P(A \cup B) = P(A) + P(B)$.

(ii) If A, B and C be any three events, then
$P(A \cup B \cup C)$
$= P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$

(iii) If A, B and C be three mutually exclusive events, then $P(A \cup B \cup C) = P(A) + P(B) + P(C)$

(iv) The probability of an event at least one of the events A and B is $P(A \cup B)$.

(v) $P(A - B) = P(A) - P(A \cap B)$

(vi) $P(B - A) = P(B) - P(A \cap B)$

7. Inequalities in Probability

Let S be a sample space and A and B be two events associated with the same sample space.

(i) If A be a proper subset of B, i.e. $A \subset B$, then

$$P(A) \leq P(B)$$

(ii) $P(AB) \le P(A) \le P(A \cup B) \le P(A) + P(B)$

(iii) $\max\{P(A), P(B)\} \le P(A \cup B) \le \min\{1, P(A) + P(B)\}$

(iv) **Bon-Ferroni's Inequality**

(a) $P(A) + P(B) \le P(AB) \le \min\{P(A), P(B)\}$

(b) $\max\left\{0, (1 - P(\overline{A}) - P(\overline{B}))\right\} \le P(AB)$

(v) **Boole's Inequality**

For a finite set of events $A_1, A_2, \ldots, A_n$,

$$P(A_1 \cup A_2 \cup \ldots A_n) \le P(A_1) + P(A_2) + \ldots + (A_n)$$

8. Conditional Probability

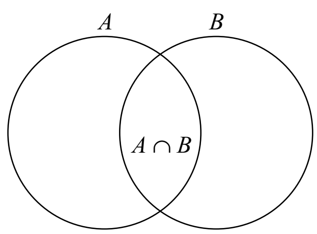
Let A and B be two events associated with a same sample space S. The conditional probability of an event A under B, where B is already occurred, is denoted as

$$P\left(\frac{A}{B}\right)$$

and is defined as

$$P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}, P(B) \ne 0$$

that means we shall find out the probability of $(A \cap B)$ in B.



Properties of Conditional Probability

Property I

If A be any non-empty event of a sample space S, then

$$P(S/A) = 1$$

Property II

If A and B be any two events of a sample space S such that $P(A) \ne 0$, then

$$P\left(\frac{(A \cup B)}{A}\right) = P\left(\frac{A}{A}\right) + P\left(\frac{B}{A}\right) - P\left(\frac{(A \cap B)}{A}\right)$$

Property III

If A and B be any two events of a sample space S such that $P(A) \ne 0$, then

$$P\left(\frac{B'}{A}\right) = 1 - P\left(\frac{B}{A}\right)$$

9. Independent Events

Two events are said to be independent if the occurrence (or non-occurrence) of one event does not affect the other.

A coin is tossed two times. The outcome of first head does not affect the outcome of second head. So the outcome of the first head and the second head are independent events.

Theorem: If A and B be two independent events, then

$$P(A \cap B) = P(A) \cdot P(B).$$

Deduction 1 If A and B be independent events, then A and B' are independent events i.e.

$$P(A \cap B') = P(A) \cdot P(B')$$

Deduction 2 If A and B be two independent events, then A' and B are independent events, i.e.

$$P(A' \cap B) = P(A')\, P(B)$$

Deduction 3 If A and B be two independent events, then A' and B' are independent events, i.e.

$$P(A' \cap B') = P(A') \cdot P(B')$$

Deduction 4 If A and B be independent events such that $P(B) \ne \varphi$, then

$$P\left(\frac{A}{B}\right) + P\left(\frac{A'}{B}\right) = 1.$$

10. Total Probability

Theorem: Let $E_1, E_2, E_3, \ldots, E_n$ be n mutually exclusive and exhaustive events associated with the same sample space S. If A be any event, which occurs either of E_1 or E_2 or E_3 or … or E_n, then

$$P(A) = \sum_{i=1}^{n} P(E_i)\, P\left(\frac{A}{E_i}\right)$$

11. Baye's Theorem

Statement: Let $E_1, E_2, E_3, \ldots, E_n$ be n mutually exclusive and exhaustive events associated with the same sample space S.

If A be any event, which occurs either of E_1, or E_2, or E_3, or …, or E_n and the probabilities $P(A/E_i)$, $i = 1, 2, \ldots, n$, are known, then

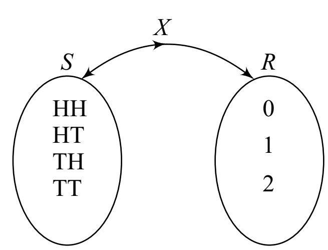
$$P(E_i/A) = \frac{P(E_i)P(A/E_i)}{\sum_{i=1}^{n} P(E_i)P(A/E_i)}.$$

12. Probability Distribution

Before discussing probability distribution, we must define some important terms.

Random Variable

A random variable is a function from a sample space to a real number.



Let X be a random variable.

Then $X : S \to R$ is a function from a sample space S to a real number R. There are two types of random variable—Discrete random variable and continuous random variable.

(i) Consider, tossing a coin two times, we shall count only head.
Thus the range of X is {0, 1, 2}
If the range of X is finite, then it is discrete random variable.

(ii) Consider a coin is tossed till a head occurs.
Thus, the range of X is {1, 2, 3, 4, …}
If the range of X is infinite, then it is also discrete random variable.

Thus, a random variable which can take only finite and countable infinite number of values is called a **discrete random variable**.

A random variable which can take any value between two given limits is called a **continuous random variable**.

Now we shall discuss about the probability distribution.

Consider a coin is tossed two times, we shall count only tail.

Then, S = {HH, TH, HT, TT} and R_X = {0, 1, 2}

$$P(X = 0) = \frac{1}{4}, P(X = 1) = \frac{2}{4}, P(X = 2) = \frac{1}{4}$$

Thus,

x	:	0	1	2
$P(x)$	:	1/4	1/2	1/4

A tabular representation of a random variable with their corresponding probabilities is called a probability distribution.

Mean and Variance of a Probability Distribution

Mean of a probability distribution (P.D.) = $\sum_{i=1}^{n} p_i x_i$

Variance of a probability distribution (P.D.)

$$= \sum_{i=1}^{n} p_i x_i^2 - (\text{Mean})^2$$

13. Binomial Distribution

Let the number of trials of a random experiment is n, where p is the probability of the occurrence of an event and q is the probability of the non-occurrence of an event such that $p + q = 1$.

Let X be the number of successes.

The probability of occurrence of the event exactly r times in n trials is denoted by $P(X = r)$ and is defined as $P(X = r) = {}^nC_r.p^r.q^{n-r}$.

Mean and Variance of a Binomial Distribution

(i) Mean = $\sum_{r=0}^{n} rP(r) = np$

(ii) Variance = npq

14. Geometrical Probability or Probability in Continuum

If we are interested in finding the probability that a point is selected at random in a given region will lie in a specified part of it. The classical definition of probability is modified and extended to what is called geometrical probability or *probability in continuum*. In this case the general expression for probability P

= Measure of specified part of the region/measure of the whole region.

where measures refers to the length, area and the volume of the region if we are dealing with one, two and three-dimensional space, respectively.

Exercises

Level I

(Questions based on Fundamentals)

ABC OF PROBABILITY

1. A coin is tossed twice. Find the probability of getting
   (i) two heads
   (ii) exactly one head
   (iii) exactly two tails.
2. A coin is tossed three times. Find the probability of getting
   (i) exactly 2 heads
   (ii) at least 2 heads
   (iii) exactly 1 head.
3. A die is thrown twice. Find the probability of getting
   (i) a sum of 10.
   (ii) a sum of at least 9.
   (iii) a sum of even numbers
   (iv) a sum of odd numbers
   (v) a sum of perfect numbers
   (vi) both the dice show prime numbers.
4. A die is thrown 3 times. Find the probability of getting
   (i) a sum of 3
   (ii) a sum of 4
   (iii) a sum of 5
   (iv) a sum of 6.
   (v) a sum of 7
   (vi) a sum of at least 15
   (vii) a sum of at most 16.
5. A card is drawn from a pack of 52 cards. Find the probability of getting
   (i) a red card

(ii) a black card
(iv) a king of a red colour
(v) a queen of spade
(vi) a face card.

6. Two cards are drawn at random from a well-shuffled pack of 52 cards. Find the probability of getting
   (i) both are red cards
   (ii) both are kings
   (iii) one is face card and another one is a king of a red colour.
   (iv) one is court card and another one is an ace card
   (v) both are of the same suit
   (vi) both are of the different suit.
7. Four cards are drawn at random from a pack of 52 cards. Find the probability of getting
   (i) 2 red cards and 2 kings of black colour
   (ii) all are of different suit
   (iii) all are of same suit
   (iv) all are honours cards
   (v) all are face cards
   (vi) 3 are court cards
   (vii) 2 are aces
   (viii) 2 slave cards
   (ix) all are number cards
   (x) all are same number cards.
8. A natural number x is chosen at random from first 100 natural numbers. Find the probability of getting
   (i) an even number
   (ii) an odd number
   (iii) a prime number
   (iv) a perfect number
   (v) a number which have only three factors
   (vi) a number which have only four factors
   (vii) a perfect square
   (viii) a perfect cube
   (ix) a perfect fourth power
   (x) a perfect sixth power.
9. Two natural numbers are drawn at random from 100 natural numbers. Find the probability that
   (i) both are divisible by 2 and 3
   (ii) both are divisible by 3 and 4
   (iii) both are divisible by 3 and 7
   (iv) both are divisible by 3 and 5
   (v) one is divisible by 2 and another one is divisible by 5
   (vi) their sum is 10
   (vii) their product is 10
   (viii) both are perfect numbers
   (ix) both the numbers are less than 11 and having a difference of 2
   (x) sum of the squares of both the numbers is a square of another number.
10. Two balls are drawn at random from a bag containing 4 red and 6 black balls. Find the probability that
   (i) both are red
   (ii) both are black
   (iii) one is red and another one is black
   (iv) either both are red or both are black.
11. A natural number x is selected at random from first 100 natural numbers. Find the probability that x satisfies the equation(s)
   (i) $x^2 - 3x \le 30$
   (ii) $\dfrac{(x-40)(x-60)}{(x-20)} < 0$
   (iii) $x^2 - 16 \ge 0$ and $x^2 - 81 \le 0$.
12. In a hand of bridge, what is the chance that the 4 kings are held by a specified player?
13. If 12 persons be seated at a round table. What is the chance that 2 particular persons sit together?
14. What is the chance of getting a total of less than 12 on throwing a die two times?
15. 6 boys and 6 girls sit in a row randomly. Find the probability that all 6 girls sit together.
16. There are 5 addressed envelopes corresponding to 5 letters. If the letters are placed in the envelope at random, what is the probability that all the letters are not placed in the right envelopes?
17 The letters of the word IIT are placed at random. Find the probability that 2 vowels come together.
18. In a random arrangement of the letters of the word AIEEE, what is the chance that 3 Es come together.
19. There are 6 persons are seated at a row. Find the probability that 2 persons always sit together.
20. In a science conference hall, there are 5 Indian Scientists, 10 American Scientists and 8 Latin American Scientists seated in a row. What is the chance that all persons of the same nationality will sit together?
21. In a class, there are 6 boys and 4 girls seated in a row. Find the probability that no boy is in between two girls.
22. In a different arrangement of the letters of the word BANANA, what is the chance that two Ns do not appear adjacently?
23. In a random arrangement of the letters of the word SUCCESS, find the probability that all Ss do not come together.
24. In a class, there are 4 girls and 6 boys. Find the probability that they will be seated in a row so that all the 4 girls are not together.
25. In a class, there 10 boys and 9 girls. Find the probability that they will be seated in a row so that they are alternate.
26. In a class, 5 boys and 5 girls. Find the probability that they will be seated in a row so that boys and girls are alternate.
27. A bag contains 50 tickets numbered {1, 2, 3, …, 50} of which 5 are drawn at random and arranged in ascending order of magnitude ($x_1 < x_2 < x_3 < x_4 < x_5$). Find the probability that $x_3 = 30$.
28. Find the probability that in a random arrangement of the letters of the word UNIVERSITY, so that two Is come together.

29. Find the probability that the birthdays of six different persons will fall in exactly two calendar months.
30. What is the probability that in a group of n persons, at least two of them will have the same birthday?
31. The odds in favour of an event are 3 : 5. Find the probability of occurrence of this event.
32. A card is drawn from an ordinary pack of 52 cards and a gambler bets that, it is a spade or an ace. What are the odds against his winning this bet?
33. Find the probability that a leap year selected randomly have 53 sundays?
34. Find the probability that a non-leap year selected randomly will have 53 sundays?
35. Find the probability that in a year of 22nd Century chosen at random there will be 53 sundays?
36. A natural number x is chosen at random from first 100 natural numbers such that

$$\frac{(x-10)(x-20)}{(x-30)} < 0$$

Find the probability of x.
37. A natural number x is chosen at random from $\{1, 2, \ldots, 10\}$ such that $x^3 - 6x^2 + 11x - 6 = 0$. Find the probability of x.
38. Two distinct natural numbers are chosen at random from the first 100 natural numbers. Find the probability that
    (i) the first one is divisible by 2 and the second one is divisible by 3
    (ii) both are divisible by 2 or 3
    (iii) both are divisible by 2 and 3
    (iv) both are neither divisible by 2 nor 3
    (v) the two numbers x and y satisfy the equation $3x - 4y = 0$.
39. If two of the 64 squares are chosen at random on a chess board. Find the probability that they have one side in common.
40. In a single throw of three dice, find the probability of getting a total of at least 5.
41. In a single throw of three dice, find the probability of getting a total of at least 15.
42 A determinant is chosen at random from the set of all determinants of order 2 with elements 0 or 1 only. Find the probability that the determinant chosen has the value non-negative.
43. Three dice are thrown. Find the probability of getting a sum, which is a perfect square.
44. Three distinct numbers are selected at random from the set $A = \{1, 2, \ldots, 10\}$. Find the probability that the product of two of the numbers is equal to the third.
45. The interior angles of a regular polygon is 150° each. Find the probability of getting of the number of diagonals of the given regular polygon.
46. A function is selected at random from all the functions defined over a given set A consisting of 3 distinct elements. Find the probability that the mapping selected is one-one.
47. In a convex hexagon, two diagonals are drawn at random. Find the probability that the diagonals intersect at an interior point of the hexagon.
48. Three of the six vertices of a regular hexagon are chosen at random. Find the probability that the triangle formed by these vertices is an equilateral.
49. A mapping is selected at random from the set of all mappings of the set $A = \{1, 2, \ldots, n\}$. Find the probability that the mapping selected is bijective.
50. A mapping is selected at random from
$f: \{x_1, x_2, x_3, x_4, x_5\} \to \{y_1, y_2, y_3, y_4, y_5\}$
such that $f(x_i) \neq y_i$, $i = 1, 2, 3, 4, 5$. Find the probability that the selected mapping is one-one.
51. Let a set A has 7 elements and the set B has 5 elements. If one function is selected from all possible defined functions from A to B, find the probability that it is onto.
52. Of all the mappings that can be defined from the set $A = \{1, 2, 3, 4\}$ to the set $B = \{5, 6, 7, 8, 9\}$, a mapping is a selected at random. Find the probability that the selected mapping is strictly monotonic
53. A die is rolled three times. Find the probability of getting a larger number than the previous number.
54. Four small squares on a chessboard are selected at random. Find the probability that they form a square of the size 2×2.
55. Three letters are to be sent to different persons and addresses on the three envelopes are also be written without looking at the address in the letters. Find the probability that all the letters are not placed into the right envelop.
56. If three distinct numbers are chosen randomly from the first 100 natural numbers. Find the probability that all the three of them are divisible by both 2 and 3.

ADDITION THEOREM ON PROBABILITY

57. A natural number is chosen at random from first 100 natural numbers. Find the probability that it is divisible by 2 or 3.
58. A natural number is chosen at random from first 200 natural numbers. Find the probability that it is divisible by 3 or 4 or 5.
59. A card is drawn at random from a pack of 52 cards. Find the probability that it is a court card or a face card.
60. A card is drawn from a pack of 52 cards. Find the probability that the card drawn is a king or a heart card or a red card.
61. A die is thrown twice. Find the probability that the sum of the numbers of the two faces is divisible by 4 or 5.
62. A die is thrown two times. Find the probability that at least one of them shows the number 3.
63. A die is thrown two times. Find the probability of getting an odd number on the first die or a total of 7.

64. If two dice are thrown simultaneously, find the probability of the sum of the numbers coming up is greater than 9.
65 The probability of a student selected in IIT is 1/3 and in AIEEE is 1/2. Find the probability that he/she will get selected either of them.
66 Four cards are drawn from a pack of 52 cards. Find the probability that the drawn cards are of the same suit.
67. The probability of a student passing in mathematics is 1/3 and in physics is 1/4 and the probability of passing in both physics and mathematics is 1/5. Find the probability that the student will pass at least one of the subjects and none of the subject.
68. Two cards are drawn from a pack of 52 cards. What is the probability that either both are red or both are kings?
69. The probability that a person will get an electric contract is 2/5 and the probability that he will not get plumbing contract is 4/7. If the probability of getting at least one contract is 2/3, what is the probability that he will get both the contracts?
70 The probability that Hameed passes in mathematics is 2/3 and the probability that he passes in English is 4/9. If the probability of passing in both the courses is 1/4, what is the probability that Hameed will pass in at least one of these subjects?
71. The probability that a person will travel by plane is 3/5 and that he will travel by train is 1/4. What is the probability that he (she) will travel by plane or train?
72. A card is drawn from a deck of 52 cards. Find the probability of getting a king or a heart or a red card.
73. Two cards are drawn at random from a deck of 52 cards. Find the probability of getting both the cards spade, heart or club.
74. An integer is chosen at random from the numbers ranging from 1 to 50. What is the probability that the integer chosen is a multiple of 2 or 3 or 10?
75 What is the chance of getting 4 or 7 or 12 in a throw of two dice?
76. Let A, B, C be three events. If the probability of occurring exactly one event out of A and B is $1 - x$, out of B and C is $1 - 2x$, out of C and A is $1 - x$, and that of occurring three events simultaneously is x^2, prove the probability that at least one out of A, B, C will occur is greater than 1/2.

INEQUALITY

77. If $P(A) = \frac{3}{5}$ and $P(B) = \frac{2}{3}$,

    prove that

    (i) $P(A \cup B) \geq \frac{2}{3}$

    (ii) $\frac{4}{15} \leq P(A \cap B) \leq \frac{3}{5}$.

78. If $P(A) = \frac{3}{4}$ and $P(B) = \frac{3}{8}$,

    prove that

    (i) $P(A \cup B) \geq \frac{1}{8}$

    (ii) $\frac{1}{8} \leq P(A \cap B) \leq \frac{3}{4}$.

CONDITIONAL PROBABILITY

79. A die is rolled. If the outcome is an odd number, find the probability that it is a prime.
80. A pair of dice is thrown. Find the probability of getting 7 as the sum, if it is known that the second die always exhibits a prime number.
81. Two dice are thrown. Find the probability that sum of the numbers coming up on them is 9, if it is known that the number 5 always occurs on the first die.
82. Two coins are tossed. What is the chance of coming up of two heads, if it is known that at least one head comes up?
83. A pair of dice is thrown. Find the probability of getting the sum 8 or more, if 4 appears on the first die.
84. A pair of dice is thrown. Find the probability of getting 7 as the sum, if it is known that the second die always exhibits a prime number.
85. A pair of dice is thrown. Find the probability of getting 7 as the sum, if it is known that the second die always exhibits an odd number.
86. Two dice are thrown. Find the probability that the numbers appeared has the sum 8, if it is known that the second die always exhibits 4.
87. A dice is thrown twice and the sum of the numbers appearing is observed to be 6. What is the conditional probability that the number 4 has appeared at least once?
88 To test the quality of electric bulbs produced in a factory, two bulbs are randomly selected from a large sample without replacement. If either bulb is defective, the entire lot is rejected. Suppose a sample of 200 bulbs contains 5 defective bulbs. Find the probability that the sample will be rejected.
89. A bag contains 10 white and 15 black balls. Two balls are drawn in succession without replacement. What is the probability that the first ball is white and the second is black?
90. Find the probability of drawing a diamond card in each of the two consecutive draws from a well-shuffled pack of cards, if the card drawn is not replaced after the first draw.
91. A bag contains 5 white, 7 red and 8 black balls. If four balls are drawn one by one without replacement, find the probability of getting all white balls.
92. Two integers are selected at random from integers 1 through 11. If the sum is even, find the probability that both the numbers are odd.

93. A couple has 2 children. Find the probability that both are boys, if it is known that
    (i) one of the children is a boy
    (ii) the older child is a boy.
94. Two coins are tossed. What is the probability of coming up two heads if it is known that at least one head comes up?
95. A card is drawn at random from a pack of 52 cards. What is the chance that it is a king, if it is known that the red card is drawn.

INDEPENDENT EVENTS

96. A bag contains 3 white and 2 black balls and another bag contains 3 white and 5 black balls. If one ball is drawn from each bag, find the probability that
    (i) both are white
    (ii) both are black
    (iii) one is white and one is black.
97. A problem in mathematics is given to 3 students whose chances of solving it are 1/2, 1/3, 1/4 respectively. What is the probability that the problem is solved?
98. *A* speak truth in 75% cases and *B* in 80% of the cases. In what percentage of the cases are they likely to contradict each other in stating the same fact?
99. Three students appear at an examination of JEE-Advanced. The probabilities of their success are 1/3, 1/4, 1/5 respectively. Find the probability of success of at least two.
100. A person is known to hit a target 3 out of 4 times and another person hits the target 2 out of 3 times. Find the probability that the target will be hit when they both try.
101. *A* speaks truth in 60% of the cases and *B* in 90% of the cases. In what percentage of cases are they likely to contradict each other in stating the same fact?
102. *A* can solve 90% of the problems given in a book and *B* can solve 70%. What is the probability that at least one of them will solve the problem selected at random from the book?
103. The probabilities of solving a specific problem independently by *A* and *B* are 1/2 and 1/3 respectively. If both try to solve the problem independently, find the probability that
    (i) the problem is solved
    (ii) exactly one of them solves the problem.
104. Given the probability that *A* can solve a problem is 2/3 and the probability that *B* can solve the same problem is 3/5. Find the probability that none of the two will be able to solve the problem.
105. Three cards are drawn with replacement from a well-shuffled pack of cards. Find the probability that the cards drawn are king, queen and jack.
106. The probability that *A* hits a target is 1/3 and the probability that *B* hits it, is 2/5. What is the probability that the target will be hit, if each one of *A* and *B* shoots at the target?
107. A bag contains 5 white, 7 red and 4 black balls. If four balls are drawn one by one with replacement, what is the probability that none is white?
108. A bag contains 5 white, 7 red and 8 black balls. Four balls are drawn one by one with replacement, what is the probability that at least one is white?
109. *A* can hit a target 4 times in 5 shots, *B* can hit 3 times in 4 shots, and *C* can hit 2 times in 3 shots.
    Calculate the probability that
    (i) *A*, *B*, *C* all may hit
    (ii) *B*, *C* may hit and *A* may lose.
110. A box contains 3 red and 5 blue balls. Two balls are down one by one at a time at random without replacement. Find the probability of getting 1 red and 1 blue ball.
111. Two cards are drawn from a well-shuffled pack of 52 cards without replacement, What is the probability that one is a red queen and the other is a black king?
112. Two cards are drawn without replacement from a well-shuffled pack of 52 cards. Find the probability that one is a spade and other is a queen of red colour.
113. Cards are numbered 1 to 25. Two cards are drawn one after the other. Find the probability that the number on one card is a multiple of 7, on the other it is a multiple of 11.
114. The probability of *A*, *B*, *C* solving a problem are 1/3, 2/7 and 3/8, respectively. If all the three try to solve the problem simultaneously, find the probability that exactly one of them can solve it.
115. The probability of a student *A* passing an examination is 3/7 and of the student *B* passing is 5/7. Assuming the two events '*A* passes', '*B* passes', as independent, find the probability
    (i) only *A* passing the examination
    (ii) only one of them passing the examination.
116. Two persons *A* and *B* throw a die alternately till one of them gets a 'three' and wins the game. Find their respective probabilities of winning, if *A* begins.
117. *A* and *B* throw alternately with a pair of dice. *A* wins if he throws 6 before *B* throws 7 and *B* wins if he throws 7 before *A* throws 6. Find the chance of winning if *A* makes the first throw.
118. Three persons *A*, *B*, *C* throw a die in succession till one gets a 'six' and wins the game. Find their respective probabilities of winning, if *A* begins.

TOTAL PROBABILITY

119. A bag contains 4 red and 3 black balls. A second bag contains 2 red and 4 black balls. One bag is selected at random. From the selected bag, one ball is drawn. Find the probability that the ball drawn is red.
120. One bag contains 4 white and 5 black balls. Another bag contains 6 white and 7 black balls. A ball is trans-

ferred from the first bag to the second bag and then a ball is drawn from the second bag. Find the probability that the ball drawn is white.

121. A lot contains 20 articles. The probability that the lot contains exactly two defective articles is 0.4 and the probability that the lot contains exactly 3 defective articles is 0.6. Articles are drawn from the lot at random one by one, without replacement and are tested till all the defective articles are found. What is the probability that the testing procedure ends at the 12th testing?
122. In a bolt factory, machines A, B and C manufacture respectively 25%, 35% and 40% of the total bolts. Of their output 5, 4 and 2 per cent are, respectively, defective bolts. A bolt is drawn at random from the product, what is the probability that the bolt drawn is defective?
123. Two-thirds of the students in a class are boys and the rest are girls. It is known that the probability of a girl getting a first class is 0.25 and that of a boy getting a first class is 0.28. Find the probability that a student chosen at random will get first class.
124. An urn contains 2 white and 2 black balls. A ball is drawn at random. If it is white, it is not replaced into the urn. Otherwise it is replaced along with another ball of the same colour. The process is repeated. Find the probability that the third ball drawn is black.
125. An unbiased coin is tossed. If the result is head, a pair of unbiased dice is rolled and the number obtained by adding the number on the two faces is noted. What is the probability that the noted number is either 7 or 8?
126. An urn contains m white and n black balls. A ball is drawn at random and is put into the urn along with k additional balls of the same colour as that of the ball drawn. A ball is again drawn at random. What is the probability that the ball drawn now is white?

BAYE'S THEOREM

127. A bag contains 3 red and 4 black balls and 2nd bag contains 5 red and 6 black balls. One ball is drawn at random from one of the bags and is found to be red. Find the probability that it was drawn from second bag.
128. An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accident involving a scooter driver, car driver and truck driver is 0.01, 0.03 and 0.15, respectively. One of the insured person meets an accident. What is the probability that he is a scooter driver?
129. A man is known to speak truth 3 times out of 5 times. He throws a die and reports that it is a six. Find the probability that it is actually six.
130. A speaks the truth 3 out of 4 times, and B speaks truths 5 out of 6 times. What is the probability that they will contradict each other in stating the same fact?
131. A speaks the truth 2 out of 3 times and B speaks truths 4 times out of 5. They agree in the assertion that from a bag containing 6 balls of different colors a red ball has been drawn. Find the probability that the statement is true.
132. A company has two plants to manufacture bicycles. The first plant manufactures 60% of the bicycles and the second plant 40%. Out of that 80% of the bicycles are rated of standard quality at the first plant and 90% of standard quality at the second plant. A bicycle is picked up at random and found to be of standard quality. Find the probability that it comes from the second plant.
133. In a test, an examinee either guess or copies or knows the answer to a multiple-choice questions with four choices. The probability that he makes a guess is 1/3 and the probability that he copies the answer is 1/6. The probability that his answer is correct, given that he copied it is 1/8. Find the probability that he knew the answer to the question, given that he correctly answered it.
134. A pack of playing cards was found to contain only 51 cards. If the first 13 cards which are examined are all red. What is the probability that the missing card is a black card.
135. An employer sends a letter to his employee but he does not receive the reply. (It is certain that the employee would had replied it if he did receive the letter.) It is known that one out of n letters does not reach the destination. Find the probability that the employee does not receive the letter.
136. In a combat between A, B and C, A tries to hit B and C and B and C try to hit A. The probabilities of A, B and C hitting the targets are 2/3, 1/2 and 1/3 respectively. If A is hit, find the probability that B hits and A and C does not.
137. A box contains N coins, m of which are fair and the rest are biased. The probability of getting a head when a fair coin is tossed is 1/2, while it is 2/3 when a biased coin is tossed. A coin is drawn from the box at random and is tossed twice. The first time it shows a head and the second time it shows a tail. What is the probability that the coin drawn is fair?
138. A bag contains 6 white and 12 red balls. Six balls are drawn at random without replacement. If at least four of these are white, find the probability that in the next two draws exactly one white ball is drawn.
139. A person goes to the office either by a car or by a scooter or by bus or by train. The probabilities of his using car, scooter, bus and train are respectively, 1/2, 2/7, 3/7 and 1/7. The probabilities of his reaching late in the office by using these modes of transport are 2/9, 4/9, 1/9 and 1/9, respectively. If the person reaches the office in time, find the probability that he used the car to reach the office.
140. A card from a pack of 52 cards is lost. From the remaining cards of the pack two cards are drawn at random and are found to be spade. Find the probability that the missing card to be a spade.

141. A speaks truth 3 times out of 5 while B speaks truths 7 times out of 10. A ball is drawn at random from a bag containing one black ball and 5 other balls of different colors. Both A and B report that a black ball has been drawn from the bag. Find the probability of their assertion being true.
142. A letter is know to have come from CALCUTTA or TATANAGAR. On the envelope, just two consecutive letters TA are visible. Find the probability that the letter came from CALCUTTA.
143. A letter is known to have come from either MAHARASTRA or MADRAS. On the post mark only consecutive letters RA can be read clearly. What is the chance that the letter came from MAHARASTRA?
144. A can hit a target 4 times in 5 shots, B can hit 3 times in 4 shots and C twice in 3 shots. They fire once each. If two of them hit. What is the chance that C has missed it?
145. A and B are two independent witnesses (i.e. there is no collusion between them). The probability that A and B will speak truths are x and y, respectively. A and B are agree in a certain statement. Prove that the probability that the statement is true is $\dfrac{xy}{1 - x - y + xy}$.
146. There are two bags, one of which contains 3 black and 4 white balls and the other contains 4 black and 3 white balls. A ball is taken out from the first bag, and a ball is taken from the second bag. Find the probability of getting a black ball and that was drawn from the second bag.

PROBABILITY DISTRIBUTION

147. Find the probability distribution of X, the number of heads in two tosses of a coin (or a simultaneous toss of two coins).
148. An urn contains 4 white and 6 red balls. 4 balls are drawn at random from the urn. Find the probability distribution of the number of white balls.
149. An unbiased die is thrown twice. Find the probability distribution of the number of sixes.
150. Two cards are drawn successively with replacement from a well-shuffled pack of 52 cards. Find the probability distribution of the number of kings.
151. Three cards are drawn from a pack of 52 playing cards. Find the probability distribution of the number of aces.
152. From a lot of 30 bulbs which includes 6 defectives, a sample of 4 bulbs is drawn at random with replacement. Find the probability distribution of the number of defective bulbs.
153. A random variable X takes the values –3, –2, 0, 1, 2 and 3 such that

$$P(X = 0) = P(X > 0) = P(X < 0)$$
$$P(X = -3) = P(X - 2) = P(X = -1)$$
$$P(X = 1) = P(X = 2) = P(X = 3)$$

Obtain the probability distribution of X.

154. Let X be a random variable which assumes values x_1, x_2, x_3, x_4 such that
$2P(X = x_1) = 3P(X = x_2) = P(X = x_3) = 5P(X = x_4)$.
Find the probability distribution of X.

BINOMIAL DISTRIBUTION

155. A die is thrown 6 times. If getting a number 6 is a success, find the probability that
    (i) at least 5 successes
    (ii) atmost 5 successes
    (iii) exactly 5 successes
    (iv) no success.
156. A die is thrown 6 times. If 'getting an odd number' is a 'success', what is the probability of
    (i) 5 successes
    (ii) at least 5 successes
    (iii) atmost 5 successes
    (iv) at least one success
    (v) no success.
157. A pair of dice is thrown 7 times. If getting a total of 7 is a success, what is the probability of
    (i) no success
    (ii) 6 successes
    (iii) at least 6 successes
    (iv) atmost 6 successes
158. Two fair dice are thrown simultaneously. If the operation is repeated 5 times, find the probability that the sum will be 7 exactly 4 times.
159. The probability that Pakistan wins a cricket match against India is 1/5. If India and Pakistan play 5 test matches, what is the probability that India will win 4 test matches?
160. A pair of fair dice is thrown 4 times. Find the probability of getting a doublet at least 2 times.
161. If a coin is tossed n times. Find the probability that the head will appear an odd number of times.
162. The probability that a student is not a cricketer is 1/10. Find the probability that out of 10 students, 7 are cricketer.
163. The probability that a student can get a chance in IIT is 3/4. He tries 5 times to get a chance. Find the probability that he tries to get a chance in IIT at least 3 times.
164. The probability of guessing correctly straight objective-type multiple choice questions in IIT is 2/7. Find the probability that a student can give right answer at least 8 questions out of 10 questions.
165. A family has 5 children. Assuming that the probability of a male child and a female child is equal. Find the probability that the family has at least 2 male children.
166. If n chocolates are distributed among m students in MI-IT-JEE Institute. Find the probability that a particular student will get r ($< n$) chocolates.
167. A man takes a step forward with probability 0.4 and backward with probability 0.6. Find the probability that at the end of eleven steps he is just one step away from starting point.

168. The probability of a man hitting a target is 1/4. How many times must he fires so that the probability of his hitting the target at least once is greater than 2/3?
169. How many dice must be thrown so that there is a better chance of obtaining a six?
170. If m things are distributed among a men and b women, show that the probability that the number of things received by men is odd is $\frac{1}{2} \times \left\{ \frac{(b+a)^m - (b-a)^m}{(b+a)^m} \right\}$.
171. Suppose the probability for A to win a game against B is 0.4. If A has an option of playing a 'best of 3 games' or a 'best of 5 games' match against B, which option should A choose so that the probability of his winning the match is higher? (No game ends in a draw)
172. Find the maximum number of tosses of a pair of dice so that the probability of getting the sum of the digits on the dice equal to 7 on at least one toss is greater than 0.95.
173. A die is thrown $(2n + 1)$ times. Prove that the probability of getting 1, 3 or 5 atmost n times is 1/2.
174. In a hurdle race, a player has to cross 10 hurdles. The probability that he will clear each hurdle is 5/6. Find the probability that he will knock down fewer hurdles.
175. Find the mean, variance and the standard deviation of the binomial distribution with parameters 4 and 1/3.
176. The mean and the variance of a binomial distribution are 4 and 4/3 respectively, find $P(X \geq 1)$.
177. If the sum of the mean and the variance of a binomial distribution for 5 trails is 1.8, find the distribution.

GEOMETRICAL PROBABILITY

178. A telephone subscriber has ordered a call. He may get it at any time in the next hour. What is the probability of the call taking place in the last 20 minutes of the hour?
179. Two points are selected in a line AB of length a so as to lie in opposite sides of its middle point C. Find the probability that the distance between them is less than $a/3$.
180. Two points are chosen independently at random from the interval [0, 1]. What is the chance that the two numbers differ by more than 1/2?
181. Two numbers are selected independently at random from the interval [0, 1]. If the smaller one is less than 1/3, find the probability that the larger one is greater than 3/4 is 3/10.
182. A line is divided at random into three parts. What is the chance that they form the sides of a possible triangle?
183. Let the sum of two positive numbers be 24. Find the probability that the product of two numbers is not less than 3/4 times of their greatest product.
184. A circular target is divided into three zones bounded by concentric circles of radii 1/3, 1/2 and 1 respectively. If three shots are fixed at random at the target, what is the probability that exactly one shot lands in each zone?
185. If p is chosen at random in [0, 5], what is the probability of the equation $x^2 + px + \frac{p+2}{4} = 0$ to have real roots?
186. Two real numbers x and y are selected at random, given that $0 \leq x \leq 1$, $0 \leq y \leq 1$.
Find the probability $y^2 \leq x$.
187. Two persons A and B agree to meet at a place between 11 to 12 noon. The first one to arrived, wait for 20 minutes and then leave. If the time of their arrival be independent and at random, what is the probability that A and B meet?
188. Two real numbers x and y are chosen at random and are such that $|x| \leq 3$, $|y| \leq 5$. What is the probability that the fraction x/y being positive?
189. A real number x is selected at random from the solution set $y^2 - y - 6 \leq 0$. What is the probability that $\frac{(x+1)(x-2)}{(x-4)} \geq 0$?
190. Three points P, Q, R are selected at random from the circumference of a circle. Find the probability that the points lie on a semicircle.
191. Two real numbers x and y are randomly chosen on the real number line, such that $0 \leq x, y \leq 0$. Find the probability that the difference between the chosen numbers is not greater than 10.
192. On a straight line of length a, two points are taken at random. Find the probability that the distance between them is greater than b.
193. Two positive numbers x and y are chosen at random for which, each of the two does not exceed 2. Find the probability that $x\,y \leq 1, \frac{y}{x} \leq 2$.
194. Two friends decide to meet at a spot between 2 p.m. and 3 p.m. Whosoever arrives first agrees to wait for 15 minutes for the other. What is the probability that they meet?

LEVEL II

(Mixed Problems)

1. A person draws a card from a pack of playing cards, replaces it and shuffles the pack. He continues doing this until he shows a spade. The chance that he will fail the first two times is
(a) $\frac{9}{64}$ (b) $\frac{1}{64}$ (c) $\frac{1}{16}$ (d) $\frac{9}{16}$
2. Three letters are written to different persons, and their addresses on three envelopes are also written. Without looking at the addresses, one letter is put in each envelope. The probability that the letters go into right envelopes is
(a) $\frac{1}{27}$ (b) $\frac{1}{6}$ (c) $\frac{1}{9}$ (d) $\frac{1}{15}$

3. Three identical dice are rolled. The probability that the same number will appear on each of them is

   (a) $\frac{1}{6}$ (b) $\frac{1}{36}$ (c) $\frac{1}{18}$ (d) $\frac{1}{28}$

4. The probability that a leap year have 52 Tuesdays is

   (a) $\frac{1}{7}$ (b) $\frac{3}{7}$ (c) $\frac{2}{7}$ (d) $\frac{5}{7}$

5. The probability that at least one of the events A and B occurs is 0.6. If A and B occur simultaneously with probability 0.2, then $P(\overline{A})+P(\overline{B})$ is

   (a) 0.4 (b) 0.8 (c) 1.2 (d) 1.4

6. A and B are two independent events. The probability that both A and B occur is 1/6 and the probability that neither of them is 1/3. The probability of the two events are, respectively

   (a) $\frac{1}{2}$ and $\frac{1}{3}$ (b) $\frac{1}{5}$ and $\frac{1}{6}$

   (c) $\frac{1}{2}$ and $\frac{1}{6}$ (d) $\frac{2}{3}$ and $\frac{1}{4}$

7. The probability that a person will hit a target is 0.3. If he shoots 10 times, the probability that the target is hit is

   (a) 1 (b) $1-(0.7)^{10}$

   (c) $(0.7)^{10}$ (d) $(0.3)^{10}$

8. The probability that a man lives after 10 years is 1/4 and the probability of his wife is alive after 10 years is 1/3. The probability that neither of them alive after 10 years is

   (a) 1/2 (b) 1/12 (c) 7/12 (d) 3/4

9. A coin is tossed twice. If the second throw results in a tail, a die is thrown. The number of points in the sample space is

   (a) 5 (b) 9 (c) 10 (d) 14.

10. The probability that three persons have same birthday is

    (a) $\frac{1}{(365)^2}$ (b) $\frac{1}{(365)^3}$

    (c) $\frac{364}{(365)^2}$ (d) $1-\frac{364\times 363}{(365)^2}$

11. A sample space has only 3 points ω_1, ω_2, ω_3. If ω_1 is twice as likely to occur as ω_2 and ω_2 is twice as likely as ω_3, the probability of ω_3 is

    (a) 1/7 (b) 2/7 (c) 3/7 (d) 4/7

12. Three coins are tossed simultaneously. The head may appear in one, two, all the three or in none. The probabilities assigned to these events are respectively

    (a) $\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}$ (b) $\frac{3}{8},\frac{2}{8},\frac{3}{8},0$

    (c) $\frac{1}{8},\frac{2}{8},\frac{3}{8},\frac{2}{8}$ (d) $\frac{3}{8},\frac{3}{8},\frac{1}{8},\frac{1}{8}$

13. P and Q stand in a line with 10 other people. The probability that there will be three persons between P and Q is

    (a) 4/33 (b) 1/66 (c) 2/99 (d) 5/132

14. If p, q, r be the probability of three independent events of the given alternatives, which one is true?

    (a) $p+q+r=1$ (b) $p+q+r\le 3$

    (c) $p+q+r<3$ (d) $p+q+r<4$

15. There are three children in a family, the probability that there is one girl child in the family is

    (a) 2/3 (b) 1/3 (c) 3/28 (d) 3/8.

16. A fair die is thrown and a number greater than 4 appears. The probability that an even number of tosses is needed is

    (a) 1/2 (b) 2/5 (c) 1/5 (d) 2/3

17. A coin is tossed 3 times. Let E be the event that the second toss results is a tail. Then $P(E)$ is

    (a) 1/2 (b) 3/8 (c) 5/8 (d) 1/4

18. Let $P(E)=x$, $P(F)=y$ and $P(\cap F)=z$, then $P(E\cap\overline{F})\cup(\overline{E}\cap F)$ is

    (a) $x+y+z$ (b) $x+y-z$

    (c) $x+y-2z$ (d) $2x+y-2z$

19. A coin is tossed and a die is rolled. The probability that the coin shows the head and die shows 6 is

    (a) 1/2 (b) 1/6 (c) 1/12 (d) 1/24

20. Two natural numbers are randomly chosen from first 40 natural numbers. The probability that the sum of two numbers is odd is

    (a) 14/29 (b) 20/39 (c) 1/2 (d) 1/3

21. Given that a throw of three fair dice shows different faces, the probability that at least one face shows 6 is

    (a) 5/6 (b) 5/18 (c) 1/2 (d) 13/18

22. A single letter is selected at random from the word PROBABILITY. The probability that it is a vowel is

    (a) 3/11 (b) 4/11 (c) 2/11 (d) 5/11

23. From each of three married couples, one of the partners is selected at random. The probability of their being of same sex is

    (a) 1/8 (b) 1/9 (c) 1/4 (d) 1/2

24. Three natural numbers are chosen at random from first 10 natural numbers. The probability that the minimum of the chosen number is 3 and the maximum is 7, is

    (a) 29/40 (b) 11/40 (c) 1/4 (d) 31/40

25. A die is thrown n times. For the probability of a six appearing at least once to be more than 1/2 is

    (a) $n=6$ (b) $n>3$

    (c) $n<4$ (d) $n=2$

26. Four-digit numbers are formed using the digits 1, 2, 3, …, 8 only once. One number is chosen at random. The probability that the selected number contains unity is

    (a) 1/8 (b) 1/4 (c) 1/2 (d) 1/3

27. Six fair dice are thrown. The probability that different numbers will turn up is equal to

    (a) 5/36 (b) 5/324 (c) 1/108 (d) 1/324

28. A fair coin is tossed repeatedly. If tail appears on first four tosses, the probability of head appearing on fifth toss is
(a) 1/2 (b) 1/32 (c) 31/32 (d) 1/5
29. If from each of the three boxes containing 3 white and 1 black, 2 white and 2 black, 1 white and 3 black balls, one ball is drawn at random, the probability that 2 white and 1 black balls will be drawn is
(a) 13/32 (b) 1/4 (c) 1/32 (d) 3/16.
30. Three dice are thrown together. The probability of getting the same digit on each of them will be
(a) 1/6 (b) 1/36 (c) 1/18 (d) 3/28
31. Three dice are rolled. The probability of getting different faces is
(a) 1/3 (b) 1/4 (c) 5/9 (d) 4/9
32. Three distinct natural numbers are chosen at random from first 100 natural numbers. The probability that all the three numbers are divisible by 2 and 3 is
(a) 4/25 (b) 4/35 (c) 4/55 (d) 4/1155
33. A six-faced fair dice is thrown until 1 comes, the probability that 1 comes in even number of trials is
(a) 5/11 (b) 5/6 (c) 6/11 (d) 1/6
34. If x is a positive integer, the probability that 3^x has 3 at unit place is
(a) 1/4 (b) 1/5 (c) 3/10 (d) 1/8
35. A natural number x is chosen at random from first 30 natural numbers. The probability of $x^2 - 10x + 12 < 0$ is
(a) 1/6 (b) 1/10 (c) 1/12 (d) 1/15
36. A matrix of order 2 is formed from the set of {–1, 1}. The probability that the matrix is singular is
(a) 1/2 (b) 1/3 (c) 1/4 (d) 1/5
37. A determinant of order 2 is formed from the set of {0, 1}. The probability that the value of the determinant is non-negative is
(a) 13/16 (b) 3/8 (c) 1/16 (d) 5/16
38. In a class, there are 5 boys and 4 girls seated in a row. The probability that they seat in alternate is
(a) $\frac{(5)!\times(4)!}{(9)!}$ (b) $\frac{(6)!\times(4)!}{(9)!}$
(c) $\frac{(3)!\times(4)!}{(9)!}$ (d) $\frac{2\times(5)!\times(4)!}{(9)!}$
39. In a class, there are 6 boys and 6 girls seated in a row. The probability that they seat in alternate is
(a) $\frac{2\times(6)!\times(5)!}{(12)!}$ (b) $\frac{2\times(6)!\times(6)!}{(12)!}$
(c) $\frac{(6)!\times(6)!}{(12)!}$ (d) $\frac{(5)!\times(6)!}{(12)!}$
40. Two events A and B have probabilities 0.25 and 0.50 respectively. The probability that both A and B occur simultaneously is 0.14, the probability that neither A nor B occurs is
(a) 0.39 (b) 0.25 (c) 0.11 (d) None
41. The probability that an event A happens in one trial of an experiment is 0.4. Three independent trials of these experiments are performed. The probability that the event A happens at least once is
(a) 0.936 (b) 0.784 (c) 0.904 (d) None
42. If A and B be two independent events such that $P(A) > 0$ and $P(B) \neq 1$, then $P\left(\frac{\overline{A}}{\overline{B}}\right)$ is equal to
(a) $1 - P(A/B)$ (b) $1 - P(P/\overline{B})$
(c) $\frac{1 - P(A \cup B)}{P(B)}$ (d) $\frac{P(\overline{A})}{P(\overline{B})}$
43. The probability that at least one of the events A and B occurs is 0.6. If A and B occur simultaneously with probability 0.2, then $P(\overline{A}) + P(\overline{B})$ is
(a) 0.4 (b) 0.8 (c) 1.2 (d) 1.4
44. India plays two matches each with West Indies and Australia. In any match, the probabilities of India getting points 0, 1 and 2 are 0.45, 0.05 and 0.50, respectively. Assuming that the outcomes are independent, the probability of india getting at least 7 points is
(a) 0.8750 (b) 0.0875 (c) 0.0625 (d) 0.0250
45. An unbiased die with faces marked 1, 2, 3, 4, 5 and 6 is rolled four times. Out of four face values obtained, the probability that the minimum face value is not less than 2 and the maximum face value is not greater than 5 is
(a) 16/81 (b) 1/81 (c) 80/81 (d) 65/81
46. The probability of India winning a test match against West Indies is ½. Assuming independence from match to match, the probability that in a 5 match series India's second win occurs at third test is
(a) 1/8 (b) 1/4 (c) 1/2 (d) 2/3
47. Let $0 < P(A) < 1$, $0 < P(B) < 1$ and
$P(A \cup B) = P(A) + P(B) - P(A)P(B)$, then
(a) $P(B/A) = P(B) - P(A)$
(b) $P(A' - B') = P(A') - P(B')$
(c) $P(A \cup B)' = P(A')P(B')$
(d) $P(A/B) = P(A)$
48. Three of the six vertices of a regular hexagon are chosen at random. The probability that the triangle with three vertices is equilateral, equals
(a) 1/2 (b) 1/5 (c) 1/10 (d) 1/20
49. For the three events A, B, C,
P (Exactly one of A or B occurs) = P (Exactly one of B or C occurs) = P (Exactly one of C or A occurs) = p and P (all the three events occurs simultaneously) = p^2, where $0 < p < 1/2$,
the probability that at least one of A, B, C occurring is
(a) $\frac{3p + 2p^2}{2}$ (b) $\frac{p + 3p^2}{4}$
(c) $\frac{p + 3p^2}{2}$ (d) $\frac{3p + 3p^2}{4}$
50. Three numbers are chosen at random without replacement from {1, 2, …, 10}. The probability that the minimum of the chosen numbers is 3, or their maximum is 7, is …
(a) 13/60. (b) 17/60 (c) 19/60 (d) 23/60

51. Seven white balls and three black balls are randomly placed in a row. The probability that no two black balls are placed adjacently equals,
(a) 1/2 (b) 7/15 (c) 2/15 (d) 1/3
52. If from each of the three boxes containing 3 white and 1 black, 2 white and 2 black, 1 white and 3 black balls, one ball is drawn at random, the probability that 2 white and 1 black balls will be drawn is
(a) 13/32 (b) 1/4 (c) 1/32 (d) 3/16
53. If the integers m and n are chosen at random between 1 and 100, the probability that a number of the form $7^m + 7^n$ is divisible by 5 equals
(a) 1/4 (b) 1/7 (c) 1/8 (d) 1/49
54. Two numbers are selected randomly from the set $S = \{1, 2, 3, 4, 5, 6\}$ without replacement one by one. The probability that minimum of the two numbers is less than 4 is
(a) 1/15 (b) 14/15 (c) 1/5 (d) 4/5.
55. If $P(B) = \frac{3}{4}$, $P(A \cap B \cap \overline{C}) = \frac{1}{3}$ and $P(\overline{A} \cap B \cap \overline{C}) = \frac{1}{3}$, then $P(B \cap C)$ is
(a) 1/12 (b) 1/8 (c) 1/15 (d) 1/9
56. A fair die is rolled. The probability that the first time 1 occurs at the even number of trials is
(a) 6/11 (b) 1/6 (c) 5/36 (d) 5/11
57. Let ω be a complex cube root of unity with ω' 1. A fair die is thrown three times. If r_1, r_2 and r_3 are the numbers obtained on the die, the probability that $\omega^{r_1} + \omega^{r_2} + \omega^{r_3} = 0$ is
(a) 1/18 (b) 1/9 (c) 2/9 (d) 1/36
58. A signal which can be green or red with probability 4/5 and 1/5 respectively, is received by station A and then transmitted to station B. The probability of each station receiving the signal correctly is 3/4. If the signal received at station B is green, the probability that the original signal was green is
(a) 3/5 (b) 6/7 (c) 20/21 (d) 9/20.

Level III

(Problems for JEE-Advanced)

1. Three of 6 faced dice are tossed together. Find out the probability of obtaining a sum of 15 of all the three numbers on the dice. **[Roorkee, 1983]**
2. Two cards are drawn one after another from a pack of 52 ordinary cards. Find the probability that the first card drawn is an ace and the second card is an honours card. The first card is not replaced while drawing the second card. **[Roorkee, 1983]**

Note *No question asked in 1984.*

3. Three groups of children contain respectively 3 girls and 1 boy, 2 girls and 2 boys, 1 girls and 3 boys. One child is selected at random from each group. What is the probability that three children selected are 1 girl and two boys. **[Roorkee, 1985]**

Note *No question asked in 1986.*

4. In the manufacturer of light bulbs, filaments, glass casings and bases are manufactured separately and then assembled into final product. From past records we know that 2% of all the filaments are defective, 3 % of all the glass casings are defective, 1% of all the bases are defective. What is the probability that a bulb randomly selected is defective? **[Roorkee, 1987]**
5. Out of 21 tickets marked with numbers from 1 to 21, three tickets are drawn at random. Find the probability that the three numbers on them are in AP. **[Roorkee, 1988]**
6. A five-digit number is formed by the digits 1, 2, 3, 4, 5 without repetition. Find the probability that the number formed is divisible by 4. **[Roorkee, 1989]**
7. Five persons entered the lift cabin on the ground floor of an 8-floor house. Suppose that each of them, independently and with equal probability, can leave the cabin at any floor beginning with the first. Find out the probability of all five persons leaving at different floor. **[Roorkee, 1990]**
8. The probability that a person will hit a target in a shooting practice is 0.3. If he shoots 10 times, find the probability. **[Roorkee, 1991]**
9. A ten-digit number is formed using the digits from 0 to 9, every digit being used exactly once. Find the probability that the number is divisible by four. **[Roorkee, 1991]**
10. The probability that a man lives after 10 years is 1/4. The probability that his wife is alive after 10 years is 1/ 3. Find the probability that neither of them is alive after 10 years. **[Roorkee, 1992]**
11. Three persons A, B, C in order cut a pack of cards, replacing them after each cut, on the condition that the first who cuts a card of spade shall win a prize. Find their respective probabilities. **[Roorkee, 1992]**
12. Find the probability that three persons have same date and month of the birthday. **[Roorkee, 1993]**
13. Find the minimum number of tosses of a pair of dice so that the probability of getting the sum of the dice equal to 7 on at-least one toss is greater than 0.95 [Given, $\log_{10} 2 = 0.3010$ and $\log_{10} 3 = 0.4771$] **[Roorkee, 1993]**
14. Two numbers are selected at random from 1, 2, 3, …, 100 and are multiplied. Find the probability correct to two places of decimals that the product, thus obtained, is divisible by 3. **[Roorkee, 1993]**
15. Three groups A, B and C are competing for positions on the Board of Directors of a company. The probabilities of their winning are 0.5, 0.3, 0.2, respectively. If the group A wins the probability of introducing a new

product is 0.7 and the corresponding probabilities for group B and C are 0.6 and 0.5, respectively. Find the probability that the new product will be introduced. **[Roorkee, 1994]**

16. A factory A produces 10% defective valves and another factory B produces 20% defective. A bag contains 4 valves of factory A and 5 valves of factory B. If two valves are drawn at random from the bag. Find the probability that at least one valve is defective. Give your answer up to two places of decimals. **[Roorkee, 1995]**
17. The probabilities of three events A, B and C are $P(A) = 0.6$, $P(B) = 0.4$ and $P(C) = 0.5$.
If $P(A \cap B) = 0.8$, $P(A \cap B) = 0.3$, $P(A \cap B \cap C) = 0.2$, $P(A \cup B \cup C) \geq 0.85$,
find $P(B \cap C)$. **[Roorkee, 1996]**
18. From each of three married couples, one of the partners is selected at random. Find the probability of their being of same sex. **[Roorkee, 1997]**
19. A die is thrown n times. Find the probability of a six appearing at least once to be more than 1/2. **[Roorkee, 1997]**
20. There is 30% chance that it rains on any particular day. What is the probability that there is at least one rainy day within a period of 7 days? Given that there is at least one rainy day. What is the probability that there are at least two rainy days? **[Roorkee, 1997]**
21. Each co-efficient in the equation $ax^2 + bx + c = 0$ is determined by throwing an ordinary die. Find the probability that the equation will have equal roots. **[Roorkee, 1998]**
22. Four cards are drawn from a pack of 52 playing cards. Find the probability (correct up to two places of decimals) of drawing exactly one pair. **[Roorkee, 1999]**
23. A and B are two independent events. The probability that both occur simultaneously is 1/6, and the probability that neither occurs is 1/3. Find the probabilities of occurrence of the events A and B separately. **[Roorkee, 2000]**
24. Two cards are drawn at random from a pack of 52 playing cards. Find the probability that one card is a heart and the other is an ace. **[Roorkee, 2001]**

Level IV

(Tougher Problems for JEE-Advanced)

1. In a hand of bridge, what is the chance that the four kings are held by a specified player.
2. If a coin is tossed n times, find the probability that heads will appear an odd number of times.
3. Find the minimum number of times, a fair coin must be tossed so that the probability of getting at least one head is at least 0.9.
4. A pair of fair dice is rolled together. Find the probability that a sum of 5 comes before a sum of 7.
5. A die is thrown $(2n + 1)$ times. Find the probability of getting 1, 3 or 5 atmost n times.
6. A second-order determinant is written down at random using the numbers 1 and –1. Find the probability that the value of the determinant is non-zero.
7. A and B throw a dice. Find the probability that A's throw is not greater than B's throw.
8. Three numbers are chosen at random without replacement $\{1, 2, \ldots, 10\}$. Find the probability that the minimum of the chosen number is 4 and their maximum is 8.
9. Let $f(x) = x^3 + ax^2 + bx + c$, where a, b and c are chosen randomly of throwing a die three times. Find the probability that $f(x)$ is strictly increasing function.
10. Find the probability of getting a sum of 12 of throwing a die 4 times.
11. Three six-faced dice are thrown together. Find the probability that the sum of the numbers appearing on the dice is k $(3 \leq k \leq 8)$.
12. A pair of fair dice is tossed. Find the probability that the maximum of the two numbers is greater than 4.
13. A room has three electric lamps. From a collection of 10 electric bulbs of which 6 are good, 3 are selected at random and put in the lamps. Find the probability that the room is lighted.
14. A bomber wants to destroy a bridge. Two bombs are sufficient to destroy it. If four bombs are dropped, what is the probability that it is destroyed, if the chance of a bomb hitting the target is 0.4.
15. Mr Anand plays with Mr Kasparov 3 games of chess. The probability that he wins a game is 0.5, looses with probability 0.3 and ties with probability 0.2. If he plays 3 games, find the probability that he wins at least two games.
16. Three numbers are randomly chosen from first 10 natural numbers. Find the probability that the numbers are in AP.
17. A pack of 52 cards is equally distributed among 4 players. Find the probability that all the 13 spades are received by one player.
18. If four squares are chosen at random in a chessboard, Find the probability that they should lie in a diagonal line.
19. If ten apples are distributed among ten persons, find the probability that at least one of them will not get any apple.
20. Two cards are drawn at random from a pack of 52 cards. Find the probability of getting at least a spade and an ace.
21. If the integers m and n are chosen at random between 1 and 100, find the probability that a number of the form $7^m + 7^n$ is divisible by 5.
22. If n positive integers are taken at random and multiplied together, find the probability that the last digit of the product is 2, 4, 6 or 8.

23. Find the probability that three persons have same date and month for their birthday.
24. A natural number x is chosen at random from $\{0, 1, 2, \ldots, 9\}$. Find the probability of x for which the equation $3^x = 2x^2 + 1$ has a solution.
25. Three integers are chosen at random from the first 20 integers. Find the probability that their product is even.

Integer Type Questions

1. A and B toss a coin each simultaneously 50 times. If the probability that both of them will not get tail at the same toss is $\left(\frac{p}{q}\right)^{50}$, where p and q are relatively prime, find $(p + q + 2)$.
2. A natural number x is chosen at random from first 10 natural numbers. If the probability of x satisfies the in-equation $x^2 - 13x \le 30$ is $\left(\frac{a}{b}\right)$ where a and b are relatively prime, find the value of $(b - a + 2)$.
3. A man is known to speak the truth 3 out of 4 times. He throws a die and reports that it is three. If the probability that it is not a three is $\left(\frac{m}{n}\right)$, where m and n are natural numbers, find the value of $\left(\frac{n}{m+1}\right)$.
4. 3 mangoes and 3 apples are in a box. If fruits are chosen at random such that the probability of one mango and one apple is $\left(\frac{a}{b}\right)$, where a and b are prime numbers, find the value of $(a + b)$.
5. A man alternately tosses a coin and throws a die. If the probability of getting a head on the coin before he gets six on the die is p, find $(7p + 1)$
6. The probability of getting a sum of 12 in four throws of an ordinary dice is $\left(\frac{m}{n}\right)^3 \times \frac{1}{6}$, where m and n are relatively prime numbers, find the value of $(n - m + 2)$.
7. Two numbers a and b are chosen at random from the set of integers $\{1, 2, 3, \ldots, 15\}$. If the probability that the equation $2a - 3b = 0$ is satisfied is p, find the value of $(84p + 2)$.
8. Two numbers x and y are chosen at random from $\{1, 2, 3, \ldots, 9\}$. If the probability of $(x^3 + y^3)$ is divisible by 3 is $\left(\frac{p}{q}\right)$, where p and q are positive integers, find $(p + q + pq)$.
9. A box contains 24 identical balls out of which 8 are black and 16 are white. The balls are drawn at random from the box one at a time with replacement. If the probability that a white ball is drawn for the 4th time on the 7th draw is $\left(\frac{a}{b}\right)^3 \times \frac{40}{81}$, where a and b are relatively prime numbers, find the value of $(a + b + 1)$.
10. Let X be a universal set and A and B be two subsets of it. If the probability of selecting 2 subsets A and B such that $B = \overline{A}$ is 1/127, find the number of elements in the set X.
11. A box contains 3 white, 2 black and 4 red balls. Four balls are drawn at random with replacement. If the probability that the sample contains only one white ball is $2 \times \left(\frac{a}{b}\right)^4$, where a and b are relatively prime numbers, find $(a + b + 1)$.
12. If the probability that a randomly chosen year of the 22nd Century will have 53 Sundays is $\left(\frac{a}{b}\right) \times \frac{1}{7}$, where a and b are relatively prime, find the value of $(a + b - 3)$.
13. Two dice are thrown simultaneously. If the probability that the sum of two numbers will be 5 before 7 is $\left(\frac{m}{n}\right)$, where m and n are prime numbers, find the value of $(m + n)$.

Comprehensive Link Passage

Passage 1

There are four boxes A_1, A_2, A_3 and A_4. Box A_1 has i cards and on each card a number is printed, the numbers are from 0 to i. A box is selected randomly, the probability of selection of box A_i is 1/10 and then a card is drawn. Let E_i denotes the event that a card with number i is drawn.

(i) $P(E_i)$ is equal to
  (a) 1/5 (b) 1/10 (c) 2/5 (d) 1/4

(ii) $P(A_3/E_2)$ is equal to
  (a) 1/4 (b) 1/3 (c) 1/2 (d) 2/3

(iii) Expectation of the number on the card is
  (a) 2 (b) 2.5 (c) 3 (d) 3.5

Passage 2

A biased coin shows head with a probability 3/4 and tails with a probability 1/4. Let P_n denotes the probability that no three or more heads appear consecutively in n throws of the coins.

On the basis of the above information, answer the following questions.

(i) If $P_n = \alpha P_{n-3} + \beta P_{n-2} + \gamma P_{n-3}$, the value of $64(\alpha + \beta + \gamma)$ is
  (a) 37 (b) 23 (c) 121 (d) 119.

(ii) The value of P_4 is
  (a) 121/256 (b) 23/64 (c) 37/64 (d) 119/256

(iii) If the coin is tossed 4 times and let A be the event that three or more head occurs in four tosses and B is the

event that three heads do not occur in first three tosses, $P(A/B)$ is
(a) 27/148 (b) 4/37 (c) 81/148 (d) 5/49

Passage 3

Let U_1 and U_2 be two urns and U_1 contains 3 white and 2 red balls and U_2 contains only 1 white ball. A fair coin is tossed. If head appears then 1 ball is drawn at random from U_1 and put into U_2. However, if tail appears then 2 balls are drawn at random from U_1 and put into U_2. Now 1 ball is drawn at random from U_2.

(i) The probability of the drawn ball from U_2 being white is
(a) 13/30 (b) 23/30 (c) 19/30 (d) 11/30

(ii) Given that the drawn ball from U_2 is white, the probability that head appeared on the coin is
(a) 17/23 (b) 11/23 (c) 15/23 (d) 12/23

Passage 4

A fair die is tossed repeatedly until a six is obtained. Let X denotes the number of tosses required.

(i) The probability that $X = 3$ equals
(a) 25/216 (b) 25/36 (c) 5/36 (d) 125/216

(ii) The probability that $X \geq 3$ equals
(a) 125/216 (b) 25/36 (c) 5/36 (d) 25/216

(iii) The conditional probability that $X \geq 3$ is given $X > 3$ equals
(a) 125/216 (b) 25/36 (c) 5/36 (d) 25/216

Passage 5

There are n urns each containing $n + 1$ balls such that ith urn contains i white balls and $(n + 1 - i)$ red balls. Let be the event of selecting ith urn, $i = 1, 2, 3, \ldots, n$ and w denote the event of getting a white ball.

(i) If $P(u_i) = i$, where $i = 1, 2, 3, \ldots, n$, then $\lim_{n\to\infty} P(w)$ is equal to
(a) 1 (b) 2/3 (c) 3/4 (d) 1/4

(ii) If $P(u_i) = c$, where c is a constant, $P\left(\frac{u_n}{w}\right)$ is equal to
(a) $\frac{2}{(n+1)}$ (b) $\frac{1}{(n+1)}$
(c) $\frac{n}{(n+1)}$ (d) $\frac{1}{2}$

(iii) If n is even and E denotes the event of choosing even-numbered urn $P\left((u_i) = \frac{1}{n}\right)$, the value of $P(w/E)$ is
(a) $\frac{n+2}{2n+1}$ (b) $\frac{n+2}{2(n+1)}$
(c) $\frac{n}{n+1}$ (d) $\frac{1}{(n+1)}$

Matrix Match

1. Match the following columns.

A natural number x is chosen at random from first 100 natural numbers.

Column I		Column II	
(A)	If x satisfies the equation $1^x + 6^x + 8^x = 9^x$, the probability of x is	(P)	1/50
(B)	If x satisfies the equation $3^{\frac{x}{2}} + 2^x = 5^{\frac{x}{2}}$, the probability of x is	(Q)	3/50
(C)	If x satisfies the equation $2^x = x^2 + 1$, the probability of x is	(R)	1/100
(D)	If x satisfies the equation $\sin(\pi x) = \log_e \lvert x \rvert$, the probability of x is	(S)	1/25
		(T)	1/5

2. Match the following columns.

A natural number x is chosen at random from first 200 natural numbers.

Column I		Column II	
(A)	If x satisfies the equation $3^{\lvert x\rvert} \lvert 2 - \lvert x \rvert \rvert = 1$, the probability of x is	(P)	3/100
(B)	If x satisfies the equation $\left[\frac{x}{2}\right] + \left[\frac{x}{3}\right] + \left[\frac{x}{5}\right] = \frac{31}{30}x$, where [,] = G.I.F, the probability of x is	(Q)	1/50
(C)	If x satisfies the equation $\left(\frac{3}{5}\right)^x + \frac{x}{5} = 2^x$, the probability of x is	(R)	0
(D)	If x satisfies the equation $5^x + 5^{-x} = \log_{10} 25$, the probability of x is	(S)	1/200

3. Match the following columns.

Three distinct natural numbers are chosen at random from first 100 natural numbers. The probability of all three of them

Column I		Column II	
(A)	are divisible by 2 and 3 is	(P)	${}^{8}C_3/{}^{100}C_3$
(B)	are divisible by 3 and 4 is	(Q)	$1/{}^{100}C_3$
(C)	are divisible by 5 and 6 is	(R)	$10/{}^{100}C_3$
(D)	are divisible by 3 and 7 is	(S)	${}^{16}C_3/{}^{100}C_3$
		(T)	$4/{}^{100}C_3$

4. **Match the following columns.**

Two distinct natural numbers are chosen at random from first 10 natural numbers. The probability that

Column I		Column II	
(A)	minimum of two numbers is less than 4, is	(P)	1/3
(B)	minimum of two numbers is more than 4, is	(Q)	1/15
(C)	maximum of two numbers is less than 8, is	(R)	2/15
(D)	maximum of two numbers is more than 6, is	(S)	7/15
		(T)	4/15

5. Match the following columns.
A natural number is chosen at random from first 100 natural numbers. The probability that

Column I		Column II	
(A)	the number is a perfect number, is	(P)	1/25
(B)	the number is a perfect cube, is	(Q)	3/100
(C)	the number has only three factors, is	(R)	1/50
(D)	the number is a perfect 4th power, is	(S)	1/20
		(T)	1/10

6. Match the following columns.
A fair die is rolled. The probability that

Column I		Column II	
(A)	the first time 6 occurs at the odd throw, is	(P)	5/11
(B)	the first time 3 occurs at the even throw, is	(Q)	3/11
(C)	the first time 1 occurs at the odd throw, is	(R)	1/11
(D)	the first time 5 occurs at the even throw, is	(S)	5/11

7. Match the following columns:
Five identical pieces of paper have the digits 1 to 5 written on them. The experiment consists in choosing at random three pieces and arranging them in a row from left to right in the order of arrival.

Column I		Column II	
(A)	If A = {the number 123 will appear}, $P(A)$ =	(P)	1/20
(B)	If B = {a number will appear that contains 3}, $P(B)$ =	(Q)	9/10
(C)	If C = {a number will appear that consists of consecutive digits}, $P(C)$ =	(R)	1/60
(D)	If D ={an even number will appear}, $P(D)$ =	(S)	1/5
		(T)	2/5

8. Match the following columns:
An elevator starts with 7 passengers and stops at 10 floors.

Column I		Column II	
(A)	If A = {no two passengers leave at the same floor}, $P(A)$ =	(P)	$(10)^{-7}$
(B)	If B = {all the passengers are discharged on the 5th floor}, $P(B)$ =	(Q)	$(10)^{-6}$
(C)	If C = {all of the passengers are discharged on the same unspecified floor}, $P(C)$ =	(R)	$\frac{(10)\,!}{6\times(10)^7}$
(D)	If D = {exactly three of them are discharged on the 2nd floor}, $P(D)$ =	(S)	$\frac{{}^7C_3(7)^3}{(10)^7}$
		(T)	$\frac{{}^7C_3(7)^4}{(10)^7}$

9. Match the following columns:
Over some time interval an amoeba may perish with probability 1/4, survive with probability 1/4 or divide into two with probability 1/2. Over the next time interval of the same duration, whatever its 'origin' may be, every amoeba has the same things happen to it. At the end of the second interval match the number of amoeba with the probabilities.

Column I		Column II	
(A)	There may exist 1 amoeba with the probability	(P)	1/8
(B)	There may exist 2 amoeba with the probability	(Q)	5/32
(C)	There may exist 3 amoeba with the probability	(R)	7/32
(D)	There may exist 4 amoeba with the probability	(S)	9/32
		(T)	11/32

10. Match the following columns.
If a fair coin is tossed 10 times, the probability of

Column I		Column II	
(A)	exactly six heads is	(P)	53/64
(B)	at least six heads is	(Q)	105/512
(C)	atmost six heads is	(R)	193/512
(D)	no head is	(S)	121/512
		(T)	1/1024

Questions asked in Previous Years' JEE-Advanced Examinations

1. Six boys and six girls are seated in a row at random. Find the probability that
   (i) the six girls sit together
   (ii) the boys and girls sit alternately.
   [IIT-JEE, 1978]
2. A box contains 2 black, 4 white and 3 red balls. One ball is drawn at random from the box and kept aside. From the remaining balls in the box, another ball is drawn at random and kept beside the first. This process is repeated till all the balls drawn are in the sequence of 2 black, 4 white and 3 red balls. **[IIT-JEE, 1979]**
3. Two events A and B have probabilities 0.25 and 0.50, respectively, the probability that both A and B occur simultaneously is 0.14, the probability that neither A nor B occurs is
   (a) 0.39 (b) 0.25 (c) 0.11 (d) none
   [IIT-JEE, 1980]
4. The probability that an event A happens in one trial or an experiment is 0.4. Three independent trials of these experiments are performed. The probability that the event A happens at least once is
   (a) 0.936 (b) 0.784 (c) 0.904 (d) none
   [IIT-JEE, 1980]
5. If A and B are two independent events such that $P(A) > 0$ and $P(B) \neq 1$, then $P\left(\frac{\bar{A}}{\bar{B}}\right)$ is equal to
   (a) $1 - P(A/B)$ (b) $1 - P(A/\bar{B})$
   (c) $\frac{1 - P(A \cup B)}{P(B)}$ (d) $\frac{P(\bar{A})}{P(\bar{B})}$
   [IIT-JEE, 1980]
6. An anticraft gun can take a maximum of four shots at an enemy plane moving away from it. The probabilities of hitting the plane at the 1st, 2nd, 3rd and 4th shots are 0.4, 0.3, 0.2 and 0.1, respectively. What is the probability that the gun hits the plane? **[IIT-JEE, 1981]**
7. For a biased die the probabilities for different faces to turn up are given below:

Face :	1	2	3	4	5	6
Probability :	0.1	0.32	0.21	0.15	0.05	0.17

   This die is tossed and you are told that either face 1 or face 2 has turned up. Then the probability that it is face 1 is **[IIT-JEE, 1981]**
8. A and B are two candidates seeking admission in IIT. The probability that A is selected is 0.5 and the probability that both A and B are selected is atmost 0.3. Is it possible that the probability of B getting selected is 0.9? **[IIT-JEE, 1982]**
9. Fifteen coupons are numbered 1, 2, 3, …, 15, respectively. Seven coupons are selected at random one at a time with replacement. The probability that the largest number appearing on a selected coupon is 9 is
   (a) $\left(\frac{9}{15}\right)^7$ (b) $\left(\frac{8}{15}\right)^7$
   (c) $\left(\frac{3}{5}\right)^7$ (d) none
   [IIT-JEE, 1983]
10. Cards are drawn one by one at random from a well-shuffled full pack of 52 playing cards until 2 aces are obtained for the first time. If N is the number of cards required to be drawn, show that $P(N = n)\frac{(n-1)(n-52)(n-51)}{50 \times 49 \times 17 \times 13}$.
    [IIT-JEE, 1983]
11. If A, B, C are events such that $P(A) = 0.3$, $P(B) = 0.4$, $P(C) = 0.8$, $P(AB) = 0.08$, $P(AC) = 0.28$, $P(ABC) = 0.09$ and $P(A \cup B \cup C) \geq 0.75$, show that $P(BC)$ lies in the interval $0.23 \leq x \leq 0.48$. **[IIT-JEE, 1983]**
12. If the letters of word ASSASSIN are written down at random in a row. The probability that no twos occur together is 1/35. **[IIT-JEE, 1983]**
13. Three identical dice are rolled. The probability that the same number will appear on each of them is
    (a) $\frac{1}{6}$ (b) $\frac{1}{36}$ (c) $\frac{1}{18}$ (d) $\frac{3}{28}$
    [IIT-JEE, 1984]
14. A box contains 24 identical balls out of which 12 are white and 12 are black. The balls are drawn at random from the box one at a time with replacement. The probability that a white ball is drawn for the 4th time on the 7th draw is
    (a) 5/64 (b) 27/32 (c) 5/32 (d) ½
    [IIT-JEE, 1984]
15. If M and N are two events, the probability that exactly one of them occurs is
    (a) $P(N) + P(N) - 2P(M \cap N)$
    (b) $P(M) + P(N) - P(\overline{M \cup N})$
    (c) $P(\bar{M}) + P(\bar{N}) - 2P(\bar{M} \cup \bar{N})$
    (d) $P(M \cap \bar{N}) - P(\bar{M} \cap N)$
    [IIT-JEE, 1984]
16. A and B are two independent events. The probability that both A and B occur is 1/6 and the probability that neither of them occur is 1/3. Find the probability of the occurrence of A. **[IIT-JEE, 1984]**
17. In a certain city only two newspapers A and B are published. It is known that 25% of the city population reads A and 20% reads B. While 8% reads both A and B. It is also known that 30% of those who read A but not B look into advertisement and 40% of those who read B but not A look into the advertisement while 50% of those who read A and B look into advertisement. What is the percentage of the population reads an advertisement. **[IIT-JEE, 1984]**
18. In a multiple-choice question, there are four alternative answers of which one or more are correct. A candidate will get marks in the question only if he ticks the cor-

rect answers. The candidate decide to tick the answers at random. If he is allowed up to three chances to answer the questions, find the probability that he will get marks in the question. **[IIT-JEE, 1985]**

19. If $P(A \cup B) = P(A \cap B)$, if and only if the relation between $P(A)$ and $P(B)$ is…… **[IIT-JEE, 1985]**
20. A box contains 100 tickets numbered 1, 2, …, 100. Two tickets are chosen at random. It is given that the maximum number on the two chosen tickets is not more than 10. The minimum is 5 with probability **[IIT-JEE, 1985]**
21. A lot contains 20 articles. The probability that the lot contains exactly 2 defective articles is 0.4 and the probability that the lot contains exactly 3 defective articles is 0.6. Articles are drawn from the lot at random one by one without replacement and are tested till all defective articles are found. What is the probability that the testing procedure ends at the 12^{th} testing. **[IIT-JEE, 1986]**
22. A student appears for tests I, II and III. The student is successful if he passes either in tests I and III or tests I and II. The probabilities of the student passing in tests I, II and III are p, q and ½, respectively. If the probability that the student is successful is ½, then
    (a) $p = q = 1$ (b) $p = q = ½$
    (c) $p = 1, q = 0$ (d) none
    [IIT-JEE, 1986]
23. If $\frac{1+3p}{3}, \frac{1-p}{4}$ and $\frac{1-2p}{2}$ are the probabilities of three mutually exclusive events, the set of all values of p is…… **[IIT-JEE, 1986]**
24. The probability that at least one of the events A and B occurs is 0.6. If A and B occur simultaneously with probability 0.2, then $P(\bar{A}) + P(\bar{B})$ is
    (a) 0.4 (b) 0.8 (c) 1.2 (d) 1.4
    [IIT-JEE, 1987]
25. A man takes a step forward with probability 0.4 and backwards with probability 0.6. Find the probability that at the end of eleven steps he is one step away from the starting point. **[IIT-JEE, 1987]**
26. An urn contains 2 white and 2 black balls. A ball is drawn at random. If it is white, it is not replaced into the urn. Otherwise, it is replaced along with another ball of the same colour. The process is repeated. Find the probability that the third ball drawn is black. **[IIT-JEE, 1987]**
27. One hundred identical coins, each with probability p of showing up heads are tossed once. If $0 < p < 1$ and the probability of heads showing on 50 coins is equal to that of heads showing on 51 coins, the value of p is
    (a) 1/2 (b) 49/101 (c) 50/101 (d) 51/101
    [IIT-JEE, 1988]
28. For two events A and B, $P(A \cap B)$ is
    (a) not less than $P(A) + P(B) - 1$
    (b) not greater than $P(A) + P(B)$
    (c) equal to $P(A) + P(B) - P(A \cup B)$
    (d) equal to $P(A) + P(B) + P(A \cup B)$
    [IIT-JEE, 1988]
29. A box contains 2 fifty paise coins, 5 twenty-five paise coins and a certain fixed number N (≥ 2) of ten- and five-paise coins. Five coins are taken out of the box at random. Find the probability that the total value of these 5 coins is less than one rupee and fifty paise. **[IIT-JEE, 1988]**
30. Urn A contains 6 red, 4 black balls and urn B contains 4 red and 6 balls. One ball is drawn at random from urn A and placed in urn B. Then one ball is drawn at random from urn B and placed in urn A. If one ball is now drawn at random from urn A, the probability that it is found to be red is … **[IIT-JEE, 1988]**
31. If E and F are independent events such that $0 < P(E)$, $P(F) < 1$, then
    (a) E and F are mutually exclusive
    (b) E and F^c are independent
    (c) E^c and F^c are independent
    (d) $P(E/F) + P(E/F^c) = 1$
    [IIT-JEE, 1989]
32. Suppose the probability for A to win a game against B is 0.4. If A has an option of playing either test of 3 games or a best of 5 games match against B, which option should be chosen so that the probability of his winning the match is higher? **[IIT-JEE, 1989]**
33. If the probability for A to fail in an examination is 0.2 and that for B is 0.3, the probability that either A or B fails is 0.5. Is it true or false? **[IIT-JEE, 1989]**
34. A pair of four dice is rolled together till a sum of either 5 or 7 is obtained. The probability that 5 comes before 7 is **[IIT-JEE, 1989]**
35. A is a set containing n elements. A subset P of A is chosen at random. The set A constructed by replacing the elements of P. A subset Q of A is again chosen at random. Find the probability that P and Q have no common elements. **[IIT-JEE, 1990]**
36. Let A and B between such that $P(A) = 0.3$ and $P(A \cup B) = 0.8$. If A and B are independent events, then $P(B)$ is **[IIT-JEE, 1990]**
37. For any two events A and B in a sample space
    (a) $P(A/B) \geq \frac{P(A)+P(B)-1}{P(B)}$, $P(B) \neq 0$ is always true.
    (b) $P(A \cap \bar{B}) = P(A) - P(A \cap B)$ does not hold.
    (c) $P(A \cup B) = 1 - P(\bar{A})P(\bar{B})$, if A and B are independent.
    (d) $P(A \cup B) = 1 - P(\bar{A})P(\bar{B})$, if A and B are disjoint.
    [IIT-JEE, 1991]
38. In a test, an examination either guesses or copies or knows the answer to a multiple-choice question with four choices. The probability that he make a guess is 1/3 and the probability that he copies the answer is 1/6.

The probability that his answer is correct given that he copied it is 1/8. Find the probability that he knew the answer to the question given that he correctly answered it. **[IIT-JEE, 1991]**

39. If the mean and the variance of a binomial variate X are 2 and 1 respectively, the probability that X takes a value greater than one is equal to ….. **[IIT-JEE, 1991]**

40. India plays two matches each with West Indies and Australia. In any match, the probabilities of India getting points 0, 1 and 2 are 0.45, 0.05 and 0.50, respectively. Assuming that the outcomes are independent, the probability of India getting at least 7 points is
(a) 0.8750 (b) 0.0875 (c) 0.0625 (d) 0.0250
[IIT-JEE, 1992]

41. A lot contains 50 defective and 50 non-defective bulbs. Two bulbs are drawn at random, one at a time, with replacement. The events A, B, C are defined as
A = {The first bulb is defective}
B = {the second bulb is non-defective}
C = {the two bulbs are both defective or both non-defective}.
Determine whether
(i) A, B, C are pair-wise independent
(ii) A, B, C are independent.
[IIT-JEE, 1992]

42. Three faces of a fair die are yellow, two faces red and one blue. The die is tossed three times. The probability that the colours, yellow, red, and blue appear in the first, second and the third tosses, respectively, is …..
[IIT-JEE, 1992]

43. An unbiased die with faces marked 1, 2, 3, 4, 5 and 6 is rolled four times. Out of four face values obtained, the probability that the minimum face value is not less than 2 and the maximum face value is not greater than 5 is then
(a) 16/81 (c) 1/81 (c) 80/81 (d) 65/81
[IIT-JEE, 1993]

44. E and F are two independent events. The probability that both E and F happens is 1/12 and the probability that neither E nor F happens is 1/2, then
(a) $P(E) = 1/3, P(F) = 1/4$
(b) $P(E) = 1/2, P(F) = 1/6$
(c) $P(E) = 1/6, P(F) = 1/2$
(d) $P(E) = 1/4, P(F) = 1/3$
[IIT-JEE, 1993]

45. Numbers are selected at random, one at a time, from the two-digit numbers 00, 01, 02, 03, …, 99 with replacement. An event E occurs if and only if the product of the two digits of a selected number is 18. If four numbers are selected, find the probability that the event E occurs at least 3 times. **[IIT-JEE, 1993]**

46. Let A, B, C be three mutually independent events. Consider the two statements S_1 and S_2:
S_1: A and $B \cup C$ are independent.
S_2 : A and $B \cap C$ are independent.
Then
(a) Both S_1 and S_2 are true
(b) only S_1 is true
(c) only S_2 is true
(d) neither S_1 nor S_2 is true.
[IIT-JEE, 1994]

47. An unbiased coin is tossed. If the result is a head, a pair of unbiased dice is rolled and the number obtained by adding the numbers on the two faces is noted. If the result is a tail, a card from a well-shuffled pack of eleven cards numbered 2, 3, 4, …., 12 is picked and the number is noted. What is the probability that the noted number is 7 or 8. **[IIT-JEE, 1994]**

48. If two events A and B are such that $P(A^c) = 0.3$, $P(B) = 0.4$ and $P(A \cap B^c) = 0.5$, then $P(B\ (A \cup B^c)) =$
[IIT-JEE, 1994]

49. The probability of India winning a test match against West Indies is ½. Assuming independence from match to match, the probability that in a 5-match series India's second win occurs at third test is
(a) 1/8 (b) 1/4 (c) 1/2 (d) 2/3
[IIT-JEE, 1995]

50. Let $0 < P(A) < 1$, $0 < P(B) < 1$ and $P(A \cup B) = P(A) + P(B) - P(A)P(B)$, then
(a) $P(B/A) = P(B) - P(A)$
(b) $p(A' - B') = P(A') - P(B')$
(c) $P(A \cup B)' = P(A')\ P(B')$
(d) $P(A/B) = P(A)$
[IIT-JEE, 1995]

51. Three of the six vertices of a regular hexagon are chosen at random. The probability that the triangle with three vertices is equilateral, equals
(a) 1/2 (b) 1/5 (c) 1/10 (d) 1/20
[IIT-JEE, 1995]

52. For the three events A, B, C,
P (Exactly one of A or B occurs) = P (Exactly one of B or C occurs) = P (Exactly one of C or A occurs) = p and P (all the three events occurs simultaneously) = p^2,
where $0 < p < \frac{1}{2}$,
the probability that at least one of A, B, C occurring is
(a) $\frac{3p + 2p^2}{2}$ (b) $\frac{p + 3p^2}{4}$
(c) $\frac{p + 3p^2}{2}$ (d) $\frac{3p + 2p^2}{2}$
[IIT-JEE, 1996]

53. If p and q are chosen randomly from the set {1, 2, 3, 4, 5, 6, 7, 8, 9, 10} with replacement, determine the probability that the roots of the equation $x^2 + px + q = 0$ are real. **[IIT-JEE, 1997]**

54. Sixteen players $S_1, S_2, ..., S_{16}$ play in a tournament. They are divided into eight pairs at random. From each pair a winner is decided on the basis of a game played between the two players are of equal strength.

(a) Find the probability that the player S_1 is among the eight winners.
(b) Find the probability that exactly one of the two players S_1 and S_2 is among the eight winners.
[IIT-JEE, 1997]

55. Three numbers are chosen at random without replacement from $\{1, 2, \ldots, 10\}$. The probability that the minimum of the chosen numbers is 3, or their maximum is 7, is … **[IIT-JEE, 1997]**

56. Seven white balls and three black balls are randomly placed in a row. The probability that no two black balls are placed adjacently equals
(a) 1/2 (b) 7/15 (c) 2/15 (d) 1/3
[IIT-JEE, 1998]

57. If from each of the three boxes containing 3 white and 1 black, 2 white and 2 black, 1 white and 3 black balls, one ball is drawn at random, then the probability that 2 white and 1 black balls will be drawn is
(a) 13/32 (b) 1/4 (c) 1/32 (d) 3/16
[IIT-JEE, 1998]

58. If $\bar{E}$ and $\bar{F}$ are the complementary events of events E and F, respectively and $0 < P(E), P(F) < 1$, then
(a) $P(E/F) + P(\bar{E}/F) = 1$
(b) $P(E/F) + P(E/\bar{F}) = 1$
(c) $P(\bar{E}/F) + P(E/\bar{F}) = 1$
(d) $P(E/\bar{F}) + P(\bar{E}/\bar{F}) = 1$
[IIT-JEE, 1998]

59. If E and F are events with $P(E) \le P(F)$ and $P(E \cap F) > 0$, then
(a) occurrence of $E \Rightarrow$ Occurrence of F
(b) occurrence of $F \Rightarrow$ Occurrence of E
(c) non-occurrence of $E \Rightarrow$ Non- occurrence of F
(d) none **[IIT-JEE, 1998]**

60. A fair coin is tossed repeatedly. If the tail appears on first four tosses, the probability of the head appearing on the fifth toss equals
(a) 1/2 (b) 1/32 (c) 31/32 (d) 1/5
[IIT-JEE, 1998]

61. There are four machines and it is known that exactly two of them are faulty. They are tested, one by one, in a random order till both the faulty machines are identified. The probability that only two tests are needed, is
(a) 1/3 (b) 1/6 (c) 1/2 (d) 1/4
[IIT-JEE, 1998]

62. Three players A, B and C toss a coin cyclically in that order (i.e. $A, B, C, A, B, C, A, B, \ldots$) till a head shows. Let p be the probability that the coin shows a head. Let α, β and γ be, respectively A, B and C gets the first head. Prove that $\beta = (1 - p)\alpha$.
Determine α, β and γ in terms of p.
[IIT-JEE, 1998]

63. If the integers m and n are chosen at random between 1 and 100, the probability that a number of the form $7^m + 7^n$ is divisible by 5 equals
(a) 1/4 (b) 1/7 (c) 1/8 (d) 1/49
[IIT-JEE, 1999]

64. The probabilities that a student passes in mathematics, physics and chemistry are m, p and c, respectively of these subjects, the student has a 75% change of passing in at least one, a 50% change of passing in at least two, and a 40% change of passing in exactly two. Which of the following relations are true,
(a) $p + m + c = 19/20$ (b) $p + m + c = 27/20$
(c) $pmc = 1/10$ (d) $pmc = 1/4$
[IIT-JEE, 1999]

65. Eight players $P_1, P_2, \ldots, P_8$ play a knock-out tournament. It is known that whenever the players P_i and P_j play, the player P_i will win if $i < j$. Assuming that the player are paired at random in each round, what is the probability that the player P_4 reaches the final?
[IIT-JEE, 1999]

66. A coin has probability of showing head when tossed. It is tossed n times, let p_n denotes the probability that no two (or more) consecutive heads occur. Prove that
$$p_1 = 1, p_2 = 1 - p^2$$
and
$$P_n = (1 - p)P_{n-1} + p(1 - p)P_{n-2},\ n \ge 3.$$
[IIT-JEE, 2000]

67. An urn contains m white and n black balls. A ball is drawn at random and is put back into the urn along with k additional balls of the same colour as that of the ball drawn. A ball is again drawn at random. What is the probability that the ball drawn now is white?
[IIT-JEE, 2001]

68. An unbiased die, with faces numbered 1, 2, 3, 4, 5, 6 is thrown n times and the list of numbers showing up is noted. What is the probability that among the numbers 1, 2, 3, 4, 5, 6 only three numbers appear in this list?
[IIT-JEE, 2001]

69. A box contains N coins, m of which are fair and the rest are biased. The probability of getting a head when a fair coin is tossed is 1/ 2, while it is 2/3 when a biased coin is tossed. A coin is drawn from the box at random and is tossed twice. The first time it shows head and the second time it shows tail. What is the probability that the coin drawn is fair? **[IIT-JEE, 2002]**

70. Two numbers are selected randomly from the set $S = \{1, 2, 3, 4, 5, 6\}$ without replacement one by one. The probability that the minimum of the two numbers is less than 4 is
(a) 1/15 (b) 14/15 (c) 1/5 (d) 4/5
[IIT-JEE, 2003]

71. If $P(B) = \frac{3}{4}$, $P(A \cap B \cap \bar{C}) = \frac{1}{3}$ and
$P(\bar{A} \cap B \cap \bar{C}) = \frac{1}{3}$,
then $P(B \cap C)$ is
(a) 1/12 (b) 1/8 (c) 1/15 (d) 1/9
[IIT-JEE, 2003]

72. A is targeting to B, B and C are targeting to A. Probability of hitting the target by A, B and C are 2/3, 1/2 and 1/3, respectively. If A is hit, find the probability that hits the target and C does not. **[IIT-JEE, 2003]**

73. For a student to qualify, he must pass at least two out of three examinations. The probability that he will pass the 1st exam is p. If he fails in one of examinations, the probability of his passing in the next examinations is $p/2$, otherwise it remains the same. Find the probability that he will qualify? **[IIT-JEE, 2003]**

74. If three distinct numbers are chosen randomly from the first 100 natural numbers, the probability that all the three of them are divisible by both 2 and 3 is
(a) 4/25 (b) 4/35 (c) 4/33 (d) 4/1155.
[IIT-JEE, 2004]

75. A box contains 12 red and 6 white balls. Balls are drawn from the bag, one at a time, without replacement. If in 6 draws, there are at least 4 white balls, find the probability that exactly one white ball is drawn in the next two draws (binomial co-efficients can be left as such). **[IIT-JEE, 2004]**

76. If A and B are two independent events, prove that $P(A \cup B) \cdot P(A' \cup B') \le P(C)$,
where C is an event defined that exactly one of A and B occurs. **[IIT-JEE, 2004]**

77. A fair die is rolled. The probability that the first time 1 occurs at the even number of trials is
(a) 6/11 (b) 1/6 (c) 5/36 (d) 5/11
[IIT-JEE, 2005]

78. Rohan goes to office either by a car, a scooter, a bus or a train, probability of which being 1/7, 3/ 7, 2/7 and 1/7 respectively and probability that he is reaching office late if he takes a car, a scooter, a bus or a train is 2/9, 1/9, 1/7 and 1/9, respectively. Find the probability that he has travelled by car, if he reaches office in time. **[IIT-JEE, 2005]**

Comprehensive Link Passage

Paragraph (81 to 83): There are n urns each containing n + 1 balls such that ith urn contains i white balls and $(n + 1 - i)$ red balls. Let be the event of selecting ith urn, i = 1, 2, 3, …, n and w denotes the event of getting a white ball.
[IIT-JEE, 2006]

79. If $P(u_i) = i$, where i = 1, 2, 3, …, n, then $\lim_{n\to\infty} P(w)$ is equal to
(a) 1 (b) 2/3 (c) 3/4 (d) 1/4

80. If $P(u_i) = c$, where c is a constant, then $P(u_i/w)$ is equal to
(a) $\frac{2}{n+1}$ (b) $\frac{1}{n+1}$ (c) $\frac{n}{n+1}$ (d) $\frac{1}{2}$

81. If n is even and E denotes the event of choosing even-numbered urn $\left(P(u_i) = \frac{1}{n}\right)$, the value of $P(w/E)$ is
(a) $\frac{n+2}{2n+1}$ (b) $\frac{n+2}{2(n+1)}$
(c) $\frac{n}{n+1}$ (d) $\frac{1}{n+1}$

82. One Indian and four American men and their wives are to be seated randomly around a circular table. The conditional probability that the Indian man is seated adjacent to his wife given that each American man is seated adjacent to his wife is
(a) 1/2 (b) 1/3 (c) 2/5 (d) 1/5
[IIT-JEE, 2007]

83. Let E^c denotes the complement of an event E. Let E, F, G be pair-wise independent events with $P(G) > 0$ and
$P(E \cap F \cap G) = 0$.
Then $P(E^c \cap F^c/F)$ equals
(a) $P(E^c) + P(F^c)$ (b) $P(E^c) - P(F^c)$
(c) $P(E^c) - P(F)$ (d) $P(E) - P(F^c)$.
[IIT-JEE, 2007]

84. An experiment has 10 equally-likely outcomes. Let A and B be two non-empty events of the same experiment. If A consists of 4 outcomes, the number of outcomes that B must have so that A and B are independent, is
(a) 2, 4 or 8 (b) 3, 6 or 9
(c) 4 or 8 (d) 5 or 10
[IIT-JEE, 2008]

85. Consider the system of equations
$ax + by = 0$, $cx + dy = 0$,
where $0 \le a, b, c, d \le 1$.
Statement I: The probability that the system of equations has a unique solution.
Statement II: The probability that the system of equations has a solution is 1.
[IIT-JEE, 2008]

86. **Comprehensive Link Passage**
A fair die is tossed repeated until a six is obtained. Let X denotes the number of tosses required.
(i) The probability that $X = 3$ equals
(a) 25/216 (b) 25/36
(c) 5/36 (d) 125/216
(ii) The probability that $X \ge 3$ equals
(a) 125/216 (b) 25/36
(c) 5/36 (d) 25/216.
(iii) The conditional probability that $X \ge 6$ is given $X > 3$ equals
(a) 125/216 (b) 25/36
(c) 5/36 (d) 25/216.
[IIT-JEE, 2009]

87. Let ω be a complex cube root of unity with $\omega \ne 1$. A fair die is thrown three times.
If r_1, r_2 and r_3, are the numbers obtained on the die, the probability that $\omega^{r_1} + \omega^{r_2} + \omega^{r_3} = 0$ is
(a) 1/18 (b) 1/9 (c) 2/9 (d) 1/36
[IIT-JEE, 2010]

88. A signal which can be green or red with probability 4/5 and 1/5, respectively, is received by station *A* and then transmitted to station *B*. The probability of each station receiving the signal correctly is 3/4. If the signal received at station *B* is green, the probability that the original signal was green is
(a) 3/5 (b) 6/7 (c) 20/23 (d) 9/20
[IIT-JEE, 2010]

89. Let *E* and *F* be two independent events. The probability that exactly one of them occurs is 11/ 25 and the probability that none of them occurs is 2/25. If $P(T)$ denotes the probability of occurrence of the event *T*, then
(a) $P(E) = 4/5, P(F) = 3/5$
(b) $P(E) = 2/5, P(F) = 2/5$
(c) $P(E) = 2/5, P(F) = 1/5$
(d) $P(E) = 3/5, P(F) = 4/5$
[IIT-JEE, 2011]

90. **Comprehensive Link passage**
Let U_1 and U_2 be two urns and U_1 contains 3 white and 2 red balls and U_2 contains only 1 white ball. A fair coin is tossed. If head appears then 1 ball is drawn at random from U_1 and put into U_2. However, if tail appears then 2 balls are drawn at random from U_1 and put into U_2. Now 1 ball is drawn at random from U_2.
(i) The probability of the drawn ball from U_2 being white is
(a) 13/30 (b) 23/30 (c) 19/30 (d) 11/30
(ii) Given that the drawn ball from U_2 is white, the probability that head appeared on the coin is
(a) 17/23 (b) 11/23 (c) 15/23 (d) 12/23
[IIT-JEE, 2011]

91. A ship is fitted with three engines E_1, E_2 and E_3. The engines function independently of each other with respective probabilities 1/2, 1/4 and 1/4. For the ship to be operational at least two of its engines must function. Let *X* denotes the event that the ship is operational and X_1, X_2 and X_3 denote, respectively, the events that the engines E_1, E_2 and E_3 are functioning. Which of the following is(are) true?
(a) $P[X_1^c/X] = 3/16$
(b) $P[X/X_2] = 5/16$
(c) $P[X/X_1] = 7/16$
(d) *P* [Exactly two engines of the ship are functioning $|X] = 7/8$
[IIT-JEE, 2012]

92. Four fair dice D_1, D_2, D_3 and D_4, each having six-face numbered 1, 2, 3, 4, 5, and 6, are rolled simultaneously. The probability that D_4 shows a number appearing on one of D_1, D_2 and D_3 is
(a) 91/216 (b) 108/216
(c) 125/216 (d) 127/216
[IIT-JEE, 2012]

93 Let *X* and *Y* be two events such that
$P(X/Y) = 1/2$, $P(Y/X) = 1/3$, $P(X \cap Y) = 16.$
Which of the following is (are) correct?
(a) $P(X \cup Y) = 2/3$
(b) *X* and *Y* are independent
(c) *X* and *Y* are not independent
(d) $P(X^C \cap Y) = 2/3$
[IIT-JEE, 2012]

94 Four persons independently solve a certain problem correctly with probabilities $\frac{1}{2}, \frac{3}{4}, \frac{1}{4}, \frac{1}{8}$ respectively.
Then the probability that the problem is solved correctly by at least one of them is
(a) $\frac{235}{256}$ (b) $\frac{21}{256}$ (c) $\frac{3}{256}$ (d) $\frac{253}{256}$
[IIT-JEE, 2013]

95 Of the three independent events E_1, E_2 and E_3, the probability that only E_1 occurs is α, only E_2 occurs is β, only E_3 occurs is γ.
Let the probability *p* that none of events E_1, E_2, or E_3 satisfies the equations $(\beta - 2\beta)p = \alpha\beta$ and $(\beta - 3\gamma)p = 2\beta\gamma$.
All the given probabilities lie in the interval (0, 1). Then
$$\frac{\text{Probability of occurence of } E_1}{\text{Probability of occurence of } E_2} = \ldots$$ **[IIT-JEE, 2013]**

96 A pack contains *n* card numbered from 1 to *n*. Two consecutive numbered card are removed from the pack and the sum of the numbers on the remaining cards is 1224. If the smaller of the numbers on the removed cards is *k*, then $k - 20 =$.... **[IIT-JEE, 2013]**

97. A box B_1 contains 1 white balls, 3 red balls and 2 black balls. Another box B_2 contains 2 white balls, 3 red balls and 4 black balls. A third box B_3 contains 3 white balls, 4 red balls and 5 black balls.
(i) If 1 ball is drawn from each of the boxes B_1, B_2, B_3, the probability that all 3 drawn balls are of the same colours is
(a) $\frac{82}{648}$ (b) $\frac{90}{648}$ (c) $\frac{558}{648}$ (d) $\frac{566}{648}$
(ii) 2 balls are drawn (without replacement) from a randomly selected box and one of the balls is white and the other ball is red, the probability that these 2 balls are drawn from box B_2 is
(a) $\frac{116}{181}$ (b) $\frac{126}{181}$ (c) $\frac{65}{181}$ (d) $\frac{55}{181}$
[IIT-JEE, 2013]

98. Three boys and two girls stand in a queue. The probability that the number of boys ahead of every girl is at least one more than the number of girls ahead of her, is
(a) 1/2 (b) 1/3 (c) 2/3 (d) 3/4
[IIT-JEE, 2014]

99. Box 1 contains three cards bearing numbers 1, 2, 3; box 2 contains five cards bearing numbers 1, 2, 3, 4, 5; and box 3 contains seven cards bearing numbers 1, 2, 3, 4,

5, 6, 7. A card is drawn from each of the boxes. Let x_i be the number on the card drawn from the ith box, $i = 1, 2, 3$.

(i) The probability that $x_1 + x_2 + x_3$ is odd, is

(a) $\frac{29}{105}$ (b) $\frac{53}{105}$ (c) $\frac{57}{105}$ (d) $\frac{1}{2}$

(ii) The probability that $x_1.x_2.x_3$ are in an arithmetic progression, is

(a) $\frac{9}{105}$ (b) $\frac{10}{105}$ (c) $\frac{11}{105}$ (d) $\frac{7}{105}$

[IIT-JEE, 2014]

Answers

Level I

1. (i) 1/4 (ii) 1/2 (iii) 1/4
2. (i) 3/8 (ii) 1/2 (iii) 3/8
3. (i) 1/12 (ii) 5/18 (iii) 1/2
   (iv) 1/2 (v) 5/36 (vi) 1/6
4. (i) 1/216 (ii) 1/72 (iii) 1/36
   (iv) 1/18 (v) 5/72 (vi) 5/54
   (vii) 5/108
5. (i) 1/2 (ii) 1/2 (iii) 1/13
   (iv) 1/26 (v) 1/52 (vi) 3/13
6. (i) 25/102 (ii) 1/221
   (iii) 4/221 (iv) 32/663
   (v) 4/13 (vi) 13/102
7. (i) $\frac{^{26}C_2 \times {^2C_2}}{^{52}C_4}$ (ii) $\frac{(13)^4}{^{52}C_4}$
   (ii) $\frac{4 \times {^{13}C_4}}{^{52}C_4}$ (iv) $\frac{^{12}C_4}{^{52}C_4}$
   (v) $\frac{^{12}C_4}{^{52}C_4}$ (vi) $\frac{^{16}C_3 \times {^{36}C_1}}{^{52}C_4}$
   (vii) $\frac{^4C_2 \times {^{48}C_2}}{^{52}C_4}$ (viii) $\frac{^{12}C_2 \times {^{40}C_2}}{^{52}C_4}$
   (ix) $\frac{^{36}C_4}{^{52}C_4}$ (x) $\frac{^4C_4 \times 9}{^{52}C_4}$
8. (i) 1/2 (ii) 1/2 (iii) 6/25
   (iv) 1/50 (v) 1/25 (vi) 3/10
   (vii) 1/10 (viii) 3/100
   (ix) 1/50 (x) 1/50
9. (i) 1/165 (ii) 7/2475
   (iii) 1/825 (iv) 1/330
   (v) 10/99 (vi) 2/275
   (vii) 2/2375 (viii) 1/4950
   (ix) 1/2475
10. (i) 2/15 (ii) 1/3 (iii) 8/15
    (iv) 7/15
11. (i) 1/20 (ii) 37/100 (iii) 3/50
12. $\frac{^{48}C_9 \times {^4C_4}}{^{52}C_4}$
13. 35/36
14. 1/6
15. $\frac{7! \times 6!}{(12)!}$
16. 119/120
17. 2/3
18. 3/10
19. 1/3
20. $\frac{5! \times 10! \times 8! \times 3!}{(23)!}$
21. $\frac{7! \times 4!}{(10)!}$
22. 2/3
23. 6/7
24. $\frac{6! \times {^7P_4}}{(10)!}$
25. $\frac{10! \times 9!}{(19)!}$
26. $\frac{2 \times 5! \times 5!}{(10)!}$
27. 551/15134
28. 1/5
29. $\frac{341}{(12)^5}$
30. $\left(1 - \frac{^{365}C_n}{(365)^n}\right)$
31. 3/8
32. 9/4
33. 2/7
34. 1/7
35. 5/28
36. 9/50
37. 3/10
38. (i) 1/25 (ii) 67/100
    (iii) 1/25 (iv) 21/25
    (v) 1/2
39. 1/18
40. 53/54

41. 5/54
42. 13/16
43. 17/108
44. 1/4028
45. 9/11
46. 2/9
47. 5/12
48. 1/10
49. $\left(\dfrac{(n)!}{(n)^n}\right)$
50. 11/30
51. $\left(\dfrac{7!\times 2}{3\times 5^6}\right)$
52. 2/125
53. 5/54
54. 7/90768
55. 5/6
56. 4/1155
57. 67/100
58. 3/5
59. 4/13
60. 7/13
61. 4/9
62. 11/36
63. 7/12
64. 1/6
65. 2/3
66. $\dfrac{4\times {}^{13}C_4}{{}^{52}C_4}$
67. 23/60
68. 17/108
69. 1/4028
70. 31/36
71. 17/20
72. 7/13
73. 3/17
74. 33/50
75. 5/18
79. 2/3
80. 3/5
81. 1/6
82. 1/3
83. 4/1155
84. 1/6
85. 1/6
86. 1/6
87. 2/5
88. 197/3830
89. 1/4
90. 1/17
91. 1/969
92. 3/5
93. (i) 1/3 (ii) 1/2
94. 1/3
95. 1/13
96. (i) 1/4 (ii) 1/4 (iii) 13/24
97. 3/4
98. 7/20
99 1/6
100 11/12
101 21/50
102. 97/100
103. (i) 2/3 (ii) 1/2
104. 2/15
105. 6/2197
106. 3/5
107. $\left(\dfrac{11}{16}\right)^4$
108. $\left(1-\left(\dfrac{3}{4}\right)^4\right)$
109. (i) 2/5 (ii) 1/10
110. 15/28
111. 8/663
112. 1/51
113. 1/50
114. 25/26
115. (i) 6/49 (ii) 26/49
116. 6/11, 5/11
117 30/61, 31/61
118 36/91, 30/91, 25/91
119. 19/42
120. 26/63
121. 99/1900
122. 0.0345
123. 0.27
124. 23/30
125. 2/11
126. $\left(\dfrac{m}{m+n}\right)$
127. 35/68
128. 1/52
129. 3/13
130. 1/3
131. 40/41
132. 3/7
133. 24/29
134. 2/3
135. $\left(\dfrac{n-1}{2n-1}\right)$
136. 1/2

137. $\left(\dfrac{9m}{m+8N}\right)$

138. $\dfrac{20\times {}^6C_4\times {}^{12}C_4+11\times {}^6C_5\times {}^{12}C_1}{66\left[{}^6C_4\times {}^{12}C_2+{}^6C_5\times {}^{12}C_1+1\right]}$

139. 1/7
140. 11/50
141. 7/9
142. 4/11
143. 10/19
144. 6/13
145.
146. 11/21

147. X: 0 1 2
$P(X)$: $\frac{1}{4}$ $\frac{2}{4}$ $\frac{1}{4}$

148. X: 0 1 2 3 4
$P(X)$: $\frac{{}^6C_4}{{}^{10}C_4}$ $\frac{{}^4C_1\times {}^6C_3}{{}^{10}C_4}$ $\frac{{}^4C_2\times {}^6C_2}{{}^{10}C_4}$ $\frac{{}^4C_3\times {}^6C_1}{{}^{10}C_4}$ $\frac{{}^4C_4}{{}^{10}C_4}$

149. X: 0 1 2
$P(X)$: $\frac{25}{36}$ $\frac{10}{36}$ $\frac{1}{36}$

150. X: 0 1 2
$P(X)$: $\frac{144}{169}$ $\frac{24}{169}$ $\frac{1}{169}$

151. X: 0 1 2
$P(X)$: $\frac{144}{169}$ $\frac{24}{169}$ $\frac{1}{169}$

152. X: 0 1 2 3
$P(X)$: $\frac{{}^{48}C_3}{{}^{52}C_3}$ $\frac{{}^4C_1\times {}^{48}C_3}{{}^{52}C_3}$ $\frac{{}^4C_2\times {}^{48}C_2}{{}^{52}C_3}$ $\frac{{}^4C_3}{{}^{52}C_3}$

153. X: 0 1 2 3 4
$P(X)$: $\left(\frac{4}{5}\right)^4$ $\frac{4^3}{5^4}$ $\frac{4^2}{5^4}$ $\frac{4}{5^4}$ $\frac{1}{5^4}$

154. X: 1 2 3 4
$P(X)$: $\frac{15}{61}$ $\frac{10}{61}$ $\frac{30}{61}$ $\frac{6}{61}$

155. (i) $\left(\frac{11}{6^6}\right)$ (ii) $1-\left(\frac{1}{6}\right)^6$
(iii) $\left(\frac{5}{6^6}\right)$ (iv) $6\times\left(\frac{5}{6}\right)^6$

156. (i) $6\times\left(\frac{1}{2}\right)^6$ (ii) $7\times\left(\frac{1}{2}\right)^6$
(iii) $1-\left(\frac{1}{2}\right)^6$ (iv) $1-\left(\frac{1}{2}\right)^6$
(v) $\left(\frac{1}{2}\right)^6$

157. (i) $\left(\frac{5}{6}\right)^7$ (ii) $35\times\left(\frac{1}{6}\right)^7$
(iii) $\left(\frac{1}{6}\right)^5$ (iv) $1-\left(\frac{1}{6}\right)^7$

158. $35\times\left(\frac{1}{6}\right)^4\times\left(\frac{5}{6}\right)^3$

159. 4/625
160. 19/144
161. 1/2

162. ${}^{10}C_7\times\left(\frac{9}{10}\right)^7\times\left(\frac{1}{7}\right)^7$

163. 189/1024

164. $\left(\frac{2}{7}\right)^9\times\left(\frac{1129}{49}\right)$

165. 13/16

166. $\left({}^nC_r\times\frac{(m-1)^{n-r}}{m^n}\right)$

167. ${}^{11}C_5\times\left(\frac{6}{25}\right)^5$

168. $n=4$
169. 1/7
171. 0.352
172. 1/7
173. 3/10
174. 11/16
178. 1/3
179. 2/9
180. 1/4
181. 3/10
182. 1/4
183. 1/2
184. 5/72
185. 3/5
186. 2/3
187. 5/9
188. 1/2

189. $\left(\frac{2}{9}ln2+\frac{1}{12}\right)$

190. 3/4

192. $\left(1-\frac{b}{a}\right)^2$

193. $\left(\frac{3ln2+1}{8}\right)$

194. 7/16

Level II

(Objective Questions)

1 (a)	2 (b)	3. (b)	4. (d)	5. (c)
6. (a)	7. (b)	8. (a)	9. (d)	10. (d)
11. (a)	12. (d)	13. (a)	14. (b)	15. (d)
16. (b)	17. (a)	18. (c)	19. (c)	20. (b)
21. (d)	22. (b)	23. (c)	24. (b)	25. (b)
26. (c)	27. (b)	28. (a)	29. (a)	30. (b)
31. (c)	32. (d)	33. (a)	34. (a)	35. (b)
36. (a)	37. (a)	38. (a)	39. (b)	40. (a)
41. (b)	42. (b)	43. (c)	44. (b)	45. (a)
46. (b)	47. (c)	48. (c)	49. (a)	50. (a)
51. (b)	52. (a)	53. (a)	54. (d)	55. (a)
56. (b)				

Level III

1. 5/108
2. 4/221
3. 13/32
4 3/5
5 10/133
6 1/20
7 $^7P_5/7^5$
8 $(0.3)^D$
9. 0.26
10. 1/ 2
11. $P(A)$= 16/37; $P(B)$ = 12/ 37; $P(C)$ = 9/37
12. $1-\frac{364\times 363}{(365)^2}$
13. $n = 17$
14. .33/50
15. 0.63
16. 0.98
17. $P(B \cup C) \leq 0.35$
18. 1/4
19. $n > 3$
20. $1-{}^7C_0\left(\frac{7}{10}\right)^7$, 0.73
21. 5/216
22. 0.3204
23. $P(A)$ = ½, $P(B)$ = 1/3
24. 191/2652

Level IV

1. $\frac{11}{4165}$
2. $\frac{1}{2}$
3. 4
4. $\frac{2}{5}$
5. $\frac{1}{2}$
6. $\frac{1}{2}$
7. $\frac{7}{12}$
8. $\frac{11}{40}$
9. $\frac{7}{18}$
10. $\frac{125}{1296}$
11. $1-{}^7C_0\left(\frac{7}{10}\right)^7$
12. $\frac{5}{9}$
13. $\frac{117}{125}$
14. $\frac{328}{625}$
15. $\frac{1}{2}$
16. $\frac{1}{6}$
17. $\frac{4}{^{52}C_{13}}$
18. $\frac{91}{158844}$
19. $\left(1-\frac{10!}{(10)^{10}}\right)$
20. $\frac{29}{442}$
21. 1/4
22. $\left(\frac{4}{5}\right)^n$
23. $\frac{1}{(365)^2}$

24. $\frac{3}{10}$
25. $\frac{17}{19}$

INTEGER TYPE QUESTIONS

1. 9 2. 3 3. 8 4. 8 5. 7
6. 3
7. 6
8. 7
9. 6
10. 7
11. 6
12. 6
13. 7

COMPREHENSIVE LINK PASSAGES

Passage I: 1. (c) 2. (b) 3. (a)
Passage II: 1. (a) 2. (a) 3. (c)
Passage III: 1. (b) 2. (d)
Passage IV: 1. (a) 2. (b) 3. (b)
Passage V: 1. (b) 2. (a) 3. (b)

MATRIX MATCH

1. (A) → (R); (*B*) → (R); (*C*) → (P); (D) → (Q)
2. (A) → (Q); (B) → (R); (C) → (S); (D) → (R)
3. (A) → (S); (B) → (P); (C) → (Q); (D) → (T).
4. (A) → (Q); (B) → (P); (C) → (S); (D) → (R)
5. (A) → (R); (B) → (P); (C) → (P); (D) → (Q)
6. (A) → (S); (B) → (P); (C) → (S); (D) → (P)
7. (A) → (R); (B) → (T); (C) → (P); (D) → (T)
8. (A) → (R); (B) → (P); (C) → (Q); (D) → (T)
9. (A) → (P); (B) → (S); (C) → (P); (D) → (P)
10. (A) → (Q); (B) → (R); (C) → (P); (D) → (T)

(QUESTIONS ASKED IN IIT-JEE EXAMINATIONS)

1. (i) 1/32 (ii) 1/462
2. 1/1260
3. (a)
4. (d)
5. (b)
6. 0.6976
7. 5/21
8. No.
9. (d)
12. False
13. (b)
14. (c)
15. (a), (c)
16. 1/2 or 1/3
17. 13.9 %
18. 1/5
19. $P(A) = P(B)$
20. 1/9
21. 11/190
22. (d)
23. $\frac{1}{3} \le p \le \frac{1}{2}$
24. (d)
25. $462 \times (0.24)^5$
26. 23/30
27. (d)
28. (a), (b), (c)
29. $\frac{1-10(N+2)}{N+{}^7C_5}$
30. 32/55
31. (b, c, d)
32. Best of three games
33. False
34. 2/5
35. $\frac{1}{2^r}$
36. 5/7
37. (a, c)
38. 24/29
39. 11/16
40. (b)
41. A, B, C are pair-wise independent but dependent.
42. (a)
43. 2.4832×10^{-4}
44. 2.4832×10^{-4}
45. 1/36
46. 9
47. (a)
48. 193/ 792
49. 1/4.
50. (b)
51. (c)
52. (c)
53. (a)
54. 1/91
55. 0.62
56. (a) 1/2 (b) 8/15
57. 13/60
58. (b)
59. (a)
60. (a)
61. (d)
62. (a)
63. (c)
64. (a)
65. (a)
66. (b, c)

67. 4/35
68. (a)
69. $\dfrac{m}{m+n}$
70. $\dfrac{{}^6C_3 \times [3^n - 3.2^n + 3]}{6^n}$
71. $\dfrac{9m}{m+8N}$
72. (d)
73. (a)
74. 1/2
75. $2p^2 - p^3$
76. (d)
77. $\dfrac{{}^{10}C_1 \times {}^2C_1}{{}^{12}C_2} \cdot \dfrac{{}^{12}C_2 \times {}^6C_4}{{}^{18}C_6} + \dfrac{{}^{11}C_1 \times {}^1C_1}{{}^{12}C_2} \cdot \dfrac{{}^{12}C_1 \times {}^6C_5}{{}^{18}C_6}$
79. (b)
80. 1/7
81 (b)
82. (a)
83. (b)
84. (c)
85. (c)
86. (d)
88. (i) a (ii) b (iii) b
89. (b)
91. (a, d)
92. (i) b (ii) d
94. (a)
95. (a, b)

Hints and Solutions

Level I

1. Here, S = {HH, HT, TH, TT}.
   Then,
   (i) the probability of 2 heads = $\dfrac{1}{4}$
   (ii) the probability of exactly one head, i.e.
   {HT, TH} is = $\dfrac{2}{4} = \dfrac{1}{2}$.
   (iii) the probability of exactly two tails, i.e.
   {TT} = $\dfrac{1}{4}$.
2. Here,
   S = {HHH, HHT, HTH, THH,
   TTT, TTH, THT, HTT}
   Then
   (i) the probability of exactly two heads, i.e.
   {HHT, HTH, THH} = $\dfrac{3}{8}$
   (ii) the probability of at least two heads, i.e.
   {HHT, HTH, THH, HHH} = $\dfrac{4}{8} = \dfrac{1}{2}$
   (iii) the probability of exactly one head, i.e.
   {TTH, THT, HTT} = $\dfrac{3}{8}$.
3. Here, S = {(1, 1), (1, 2), (1, 3), ..., (6, 6)}
   (i) Let A be the event, which shows the outcome, a sum of 10 = {(4, 6), (6, 4), (5, 5)}
   So, the probability of A,
   $$P(A) = \frac{3}{36} = \frac{1}{12}$$
   (ii) Let B be the event, which shows the outcome, a sum of at least 9
   ={a sum of 9, 10, 11 and 12}
   = {(3, 6), (6, 3), (4,5), (5, 4), (4, 6), (6, 4), (5, 5), (5, 6), (6, 5), (6, 6)}
   So, the probability of B,
   $$P(B) = \frac{10}{36} = \frac{5}{18}$$
   (iii) Let C be the event which shows the outcome, a sum of even numbers
   = {a sum of 2, 4, 6, 8, 10, 12}
   = {(1, 1), (1, 3), (3, 1), (2, 2), (1, 5), (5, 1), (2, 4), (4, 2), (3, 3), (2, 6), (6, 2), (3, 5), (5, 3), (4, 4)}
   So, the probability of C,
   $$P(C) = \frac{18}{36} = \frac{1}{2}$$
   (iv) Let D be the event shows the outcome, a sum of odd numbers
   = {a sum of 3, 5, 7, 9, 11}
   = {(1, 2), (2, 1), (1, 4), (4, 1), (2, 3), (3, 2), (1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3), (3, 6), (6, 3), (4, 5), (5, 4), (5, 6), (6, 5)}
   So, the probability of D,
   $$P(D) = \frac{18}{36} = \frac{1}{2}$$
   (v) Let E be the event. A sum of perfect numbers = {a sum of 6}(since only two perfect number exist from 1 to 100 natural numbers, say, 6 and 28}
   = {(1, 5), (5, 1), (2, 4), (4, 2), (3, 3)}

So, the probability of E,

$$P(E)=\frac{5}{36}$$

(vi) Let F be the event, both the dice show prime numbers = {2, 3, 5}
= {(2, 3), (2, 5), (5, 2), (3, 2), (3, 5), (5, 3)}
So, the probability of F,

$$P(F)=\frac{6}{36}=\frac{1}{6}$$

4. Here,
S = {(1, 1, 1), (1, 1, 2), (1, 1, 3), ..., (6, 6, 6)}
(i) Let A be the event of getting a sum of 3, i.e.
{(1, 1, 1)}
So, the probability of A,

$$P(A)=\frac{1}{216}$$

(ii) Let B be the event of getting a sum of 4 $\rightarrow$ (1, 1, 2)
These three numbers can be arranged themselves in $\frac{(3)!}{(2)!}$ ways, i.e. in 3 ways
Hence, the probability of B,

$$P(B)=\frac{3}{216}=\frac{1}{72}$$

(iii) Let C be the event of getting
a sum of 5 $\rightarrow$ (1, 1, 3), (1, 2, 2)
Hence, the probability of C,

$$P(C)=\frac{3+3}{216}=\frac{6}{216}=\frac{1}{36}$$

(iv) Let D be the event of getting
a sum of 6 $\rightarrow$ (1, 2, 3), (1, 1, 4), (2, 2, 2)
Hence, the probability of D,

$$P(D)=\frac{3+6+3}{216}=\frac{12}{216}=\frac{1}{18}$$

(v) Let E be the event of getting
A sum of 7 $\rightarrow$ (1, 1, 5), (1, 2, 4), (1, 3, 3), (2, 2, 3)
Hence, the probability of E,

$$P(E)=\frac{3+6+3+3}{216}=\frac{15}{216}=\frac{5}{72}$$

(vi) Let F be the event of getting
A sum of at least 15 = a sum of {15, 16, 17, 18}
15 $\rightarrow$ (3, 6, 6), (4, 5, 6), (5, 5, 5),
16 $\rightarrow$ (4, 6, 6), (5, 5, 6),
17 $\rightarrow$ (5, 6, 6)
and 18 $\rightarrow$ (6, 6, 6)
Hence, the required probability,

$$P(F)=\frac{10+6+3+1}{216}=\frac{20}{216}=\frac{5}{54}$$

(vii) Let G be the event of getting
A sum of at least
16 = {16, 17, 18}
16 $\rightarrow$ (4, 6, 6), (5, 5, 6),
17 $\rightarrow$ (5, 6, 6)
and 18 $\rightarrow$ (6, 6, 6)
Hence, the required probability,

$$P(G)=\frac{6+3+1}{216}=\frac{10}{216}=\frac{5}{108}$$

5. (i) Required probability of a red card $=\frac{26}{52}=\frac{1}{2}$

(ii) Required probability of a black card $=\frac{26}{52}=\frac{1}{2}$

(iii) Required probability of a king card $=\frac{4}{52}=\frac{1}{13}$

(iv) Required probability of a king of a red colour $=\frac{2}{52}=\frac{1}{26}$

(v) Required probability of a queen of spade $=\frac{1}{52}$

(vi) Required probability of a face card $=\frac{12}{52}=\frac{3}{13}$

6. (i) Required probability of getting both are red cards

$$=\frac{^{26}C_2}{^{52}C_2}=\frac{26\times 25}{52\times 51}=\frac{25}{102}$$

(ii) Required probability of getting both are king cards

$$=\frac{^{4}C_2}{^{52}C_2}=\frac{4\times 3}{52\times 51}=\frac{1}{13\times 17}=\frac{1}{221}$$

(iii) Required probability of getting one is a face card and another one is a king of a red colour.

$$=\frac{^{12}C_1\times {^{2}C_1}}{^{52}C_2}=\frac{12\times 2\times 2}{52\times 51}=\frac{12}{13\times 51}=\frac{12}{663}$$

(iv) Required probability of getting one is a court card and another one is an ace card

$$=\frac{^{16}C_1\times {^{4}C_1}}{^{52}C_2}=\frac{16\times 4\times 2}{52\times 51}=\frac{32}{13\times 51}=\frac{32}{663}$$

(v) Required probability of getting both are of the same suit.

$$=\frac{^{13}C_2+{^{13}C_2}+{^{13}C_2}+{^{13}C_2}}{^{52}C_2}$$

$$=\frac{4\times {^{13}C_2}}{^{52}C_2}=\frac{4\times 13\times 12}{52\times 51}=\frac{4}{13}$$

(vi) Required probability of getting both are of the different suit

$$=\frac{4(^{13}C_1\times {^{13}C_1}+{^{13}C_1}\times {^{13}C_1})}{^{52}C_2}$$

$$=\frac{169}{^{52}C_2}=\frac{169\times 2}{52\times 51}=\frac{13}{102}$$

7. (i) Required probability of getting 2 red cards and 2 kings of black colour.

$$= \frac{^{26}C_2 \times {}^{2}C_2}{^{52}C_4}$$

(ii) Required probability of getting all have come from the different suit

$$= \frac{^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1 \times {}^{13}C_1}{^{52}C_4}$$

(iii) Required probability of getting the cards, all coming from the same suit

$$= \frac{^{13}C_4 + {}^{13}C_4 + {}^{13}C_4 + {}^{13}C_4}{^{52}C_4} = \frac{4 \times {}^{13}C_4}{^{52}C_4}$$

(iv) Required probability of getting all the cards are honours cards

$$= \frac{^{12}C_4}{^{52}C_4}$$

(v) Required probability of getting all are face cards

$$= \frac{^{12}C_4}{^{52}C_4}$$

(vi) Required probability of getting 3 are court cards

$$= \frac{^{16}C_3 \times {}^{36}C_1}{^{52}C_4}$$

(vii) Required probability of getting 2 are aces

$$= \frac{^{4}C_2 \times {}^{48}C_2}{^{52}C_4}$$

(viii) Required probability of getting 2 slave cards

$$= \frac{^{12}C_2 \times {}^{40}C_2}{^{52}C_4}$$

(ix) Required probability of getting all are number cards $= \frac{^{36}C_4}{^{52}C_4}$

(x) Required probability of getting all are same number cards $= \frac{^{4}C_4 \times 9}{^{52}C_4}$.

8. (i) Required probability of getting an even number $= \frac{50}{100} = \frac{1}{2}$

(ii) Required probability of getting an odd number $= \frac{50}{100} = \frac{1}{2}$

(iii) Required probability of getting a prime number $= \frac{24}{100} = \frac{6}{25}$

(iv) Required probability of getting a perfect number $= \frac{2}{100} = \frac{1}{50}$

(v) Required probability of getting a number which have three factors only $= \frac{4}{100} = \frac{1}{25}$

(vi) Required probability of getting a number which have only 4-factors $= \frac{30}{100} = \frac{3}{10}$

(those numbers are 6, 8, 10, 14, 15, 21, 22, 26, 27, 33, 34, 35, 38, 46, 51, 55, 57, 58, 65, 69, 74, 78, 82, 85, 86, 87, 89, 91, 94, 95}

(vii) Required probability of getting a perfect square $= \frac{10}{100} = \frac{1}{10}$

(viii) Required probability of getting a perfect cube $= \frac{3}{100}$

(ix) Required probability of getting a perfect fourth power $= \frac{2}{100} = \frac{1}{50}$

(x) Required probability of getting a perfect sixth power $= \frac{2}{100} = \frac{1}{50}$

9. Here, $S = \{1, 2, 3, 4, \ldots, 100\}$

(i) Let A be the event which is divisible by both 2 and 3

$= \{6, 12, 18, ..., 96\}$

Then the probability of A,

$$P(A) = \frac{n(A)}{n(\text{S})} = \frac{^{16}C_2}{^{100}C_2}$$

$$= \frac{16 \times 15}{100 \times 99} = \frac{1}{5 \times 33} = \frac{1}{165}$$

(ii) Let B be the event which is divisible by both 3 and 4.

$= \{12, 24, ..., 96\}$

Then the probability of B,

$$P(B) = \frac{n(A)}{n(S)} = \frac{^{8}C_2}{^{100}C_2}$$

$$= \frac{8 \times 7}{100 \times 99} = \frac{7}{2475}$$

(iii) Let C be the event which is divisible by 3 and 7

$= \{12, 24, 63, 84\}$

Then the probability of C,

$$P(C) = \frac{n(A)}{n(S)} = \frac{^{4}C_2}{^{100}C_2}$$

$$= \frac{4 \times 3}{100 \times 99} = \frac{1}{25 \times 33} = \frac{1}{825}$$

(iv) Let D be the event which is divisible by both 3 and 5.

= {15, 30, ..., 90}

Then the probability of D,

$$P(D) = \frac{n(A)}{n(S)} = \frac{^{6}C_2}{^{100}C_2}$$

$$= \frac{6 \times 5}{100 \times 99} = \frac{1}{330}$$

(v) Let E be the event, whose first outcome is divisible by 2 and second outcome is divisible by 5.

Then the probability of E,

$$P(E) = \frac{^{50}C_1 \times {}^{10}C_1}{^{100}C_2} = \frac{50 \times 10 \times 2}{100 \times 99} = \frac{10}{99}$$

(vi) Let F be the event, whose sum is 10

= {(1, 9), (2, 8), (3, 7), (4, 6), (5, 5), (6, 4), (7, 3), (8, 2), (9,1)}

Then the probability of F,

$$P(F) = \frac{^{9}C_2}{^{100}C_2} = \frac{9 \times 8}{100 \times 99} = \frac{2}{275}$$

(vii) Let G be the event, whose product of the integer is 10

= {(1, 10), (2, 5), (5, 2), (10, 1)}.

Then the probability of G,

$$P(G) = \frac{^{4}C_2}{^{100}C_2} = \frac{4 \times 3}{100 \times 99} = \frac{1}{825}$$

(viii) Let H be the event, where both integers are perfect numbers

= {6, 28}

Then the probability of H,

$$P(H) = \frac{^{1}C_1}{^{100}C_2} = \frac{1 \times 2}{100 \times 99} = \frac{1}{4950}$$

(ix) Let I be the event, where both integers are less than 11 and their difference is 2

= {(1, 3), (3, 1), (2, 4), (4, 2), (3, 5), (5, 3), (4, 6), (6, 4), (5, 7), (7, 5), (7, 9), (9, 7), (8, 10), (10,8)}

Then the probability of I,

$$P(I) = \frac{^{12}C_2}{^{100}C_2} = \frac{12 \times 11}{100 \times 99} = \frac{3}{25 \times 9} = \frac{1}{75}$$

(x) Let J be the event, where the sum of the squares of both the numbers is a square of another number

= {(3, 4, 5), (6, 8, 10)}

Then the probability of J,

$$P(J) = \frac{2}{^{100}C_2}$$

$$= \frac{2 \times 2}{100 \times 99} = \frac{1}{25 \times 99} = \frac{1}{2475}$$

10. (i) Required probability $= \frac{^{4}C_2}{^{10}C_2} = \frac{4 \times 3}{10 \times 9} = \frac{2}{15}$

(ii) Required probability $= \frac{^{6}C_2}{^{10}C_2} = \frac{6 \times 5}{10 \times 9} = \frac{1}{3}$

(iii) Required probability $= \frac{^{4}C_1 \times {}^{6}C_1}{^{10}C_2}$

$$= \frac{4 \times 6 \times 2}{10 \times 9} = \frac{2 \times 2 \times 2}{5 \times 3} = \frac{8}{15}$$

(iv) Required probability $= \frac{^{4}C_2 + {}^{6}C_2}{^{10}C_2}$

$$= \frac{4 \times 3 + 6 \times 5}{10 \times 9}$$

$$= \frac{12 + 30}{90} = \frac{42}{90} = \frac{7}{15}$$

11. (i) Given in-equation is

$x^2 - 3x \le 10$

$\Rightarrow \quad x^2 - 3x - 10 \le 0$

$\Rightarrow \quad (x - 5)(x + 2) \le 0$

$\Rightarrow \quad -2 \le x \le 5$

$\Rightarrow \quad x = 1, 2, 3, 4, 5$

Hence, the required probability

$$= \frac{5}{100} = \frac{1}{20}$$

(ii) Given in-equation is $\frac{(x-40)(x-60)}{(x-20)} < 0$

$$\Rightarrow \quad \frac{(x-40)(x-60)}{(x-20)} < 0$$

$\Rightarrow \quad x \in (1, 20) \cup (40, 60)$

Hence, the required probability $= \frac{37}{100}$

(iii) Given inequations are

$x^2 - 16 \ge 0$ and $x^2 - 81 \le 0$.

$\Rightarrow \quad (x + 4)(x - 4) \ge 0$ and $(x + 9)(x - 9) \le 0$

$\Rightarrow \quad x \in [1, 4] \cup [1, 9]$

Hence, the required probability $= \frac{13}{100}$.

12. Hence the required probability $= \frac{^{4}C_4 \times {}^{48}C_9}{^{52}C_{13}}$

$$= \frac{^{48}C_9}{^{52}C_{13}}$$

13. Hence the required probability $= \frac{(11)! \times (2)!}{(12)!} = \frac{1}{6}$

14 Hence, the required probability $= 1 - \frac{1}{36} = \frac{35}{36}$

15. Hence, the required probability $= \frac{(7)! \times (6)!}{(12)!}$

16. Hence the required probability $= 1 - \frac{1}{(5)!} = \frac{119}{120}$

17. Hence, the required probability $= \frac{(2)! \times \frac{(2)!}{(2)!}}{\frac{(3)!}{(2)!}} = \frac{4}{6} = \frac{2}{3}$

18. Hence, the required probability $= \frac{(3)! \times \frac{(3)!}{(3)!}}{\frac{(5)!}{(3)!}} = \frac{36}{120} = \frac{3}{10}$

19. Hence, the required probability is $= \frac{(5)! \times (2)!}{(6)!} = \frac{2}{6} = \frac{1}{3}$

20. Hence, the required probability $= \frac{(5)! \times (10)! \times (8)! \times (3)!}{(23)!}$

21. Hence the required probability $= \frac{(7)! \times (4)!}{(10)!}$

22. Hence, the required probability

$$= \frac{\frac{(6)!}{(2)! \times (3)!} - \frac{(5)!}{(3)!}}{\frac{(6)!}{(2)! \times (3)!}} = \frac{60 - 20}{50} = \frac{40}{60} = \frac{2}{3}$$

23. Hence, the required probability

$$= \frac{\frac{(7)!}{(2)! \times (3)!} - \frac{(5)!}{(2)!} \times \frac{(3)!}{(3)!}}{\frac{(7)!}{(2)! \times (3)!}}$$

$$= \frac{7 \times 60 - 60}{7 \times 60} = \frac{6 \times 60}{7 \times 60} = \frac{6}{7}$$

24. Hence, the required probability $= \frac{(6)! \times {}^7P_4}{(10)!}$

25. Hence, the required probability $= \frac{(10)! \times (9)!}{(19)!}$

26. Hence, the required probability $= \frac{(5)! \times (5)! \times 2}{(10)!}$

27. Hence, the required probability $= \frac{{}^{29}C_2 \times {}^1C_1 \times {}^{20}C_2}{{}^{50}C_5}$

28. Hence, the required probability $= \frac{9! \times \frac{2!}{2!}}{\frac{10!}{2!}}$

$$= \frac{9! \times 2!}{10!} = \frac{2}{10} = \frac{1}{5}$$

29. Six persons born in 12 months = 12^6 ways
Now, any two months can be chosen in ${}^{12}C_2$ ways
The six birthdays can fall in these two months in 2^6 ways. Out of these 2^6 ways, there are 2 ways when all the six birthdays fall in one month.
So, the number of possible cases = ${}^{12}C_2 \times (2^6 - 2)$

Hence, the required probability $= \frac{{}^{12}C_2 \times (2^6 - 2)}{12^6}$

$$= \frac{12 \times 11 \times (2^5 - 1)}{12^6}$$

$$= \frac{11 \times 31}{12^5} = \frac{341}{12^5}$$

30. There are 365 days in a year $n(S)$ = the number of ways in which n persons can have birthdays
= $365 \times 365 \times 365 \times \ldots$ n times
= $(365)^n$ ($\because$ each person can have birthday in 365 ways
$n(E)$ = the number of ways in which n persons can have different birthdays
= $(365)^n - {}^{365}C_n$

Hence, the required probability $= \left(\frac{(365)^n - {}^{365}C_n}{(365)^n}\right)$

$$= \left(1 - \frac{{}^{365}C_n}{(365)^n}\right)$$

31. Hence, the required probability $= \frac{3}{3+5} = \frac{3}{8}$

32. We know that, odds against of an event $E = \frac{n(E')}{n(E)}$

$$= \frac{P(E')}{P(E)} = \frac{9/13}{4/13} = \frac{9}{4},$$

where $P(E) = \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$

and $P(E') = 1 - \frac{4}{13} = \frac{9}{13}$

33. Let S = {(Mon, Tue), (Tue, Wed), (Wed, Thu), (Thu, Fri), (Fri, Sat), (Sat, Sun), (Sun, Mon)
and A = {(Sat, Sun), (Sun, Mon)}

Thus, $P(A) = \frac{n(A)}{n(S)} = \frac{2}{7}$

34. Let S = {Mon, Tue, Wed, Thu, Fri, Sat, Sun}
and A = {Sun}

Hence, the required probability = $P(A) = \frac{n(A)}{n(S)} = \frac{1}{7}$

35. Here, $P(\text{leap year}) = \frac{1}{4}$ and

$P(\text{non-leap year}) = \frac{3}{4}$

Let A and B be the events that a non-leap year and a leap year, respectively, have 53 Sundays.

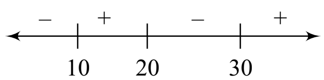
Thus, $P(A) = \frac{1}{4} \times \frac{2}{7}$ and $P(B) = \frac{3}{4} \times \frac{1}{7}$

Hence, the required probability $= P(A \text{ or } B)$

$$= P(A \cup B)$$
$$= P(A) + P(B)$$
$$= \frac{1}{4} \times \frac{2}{7} + \frac{3}{4} \times \frac{1}{7}$$
$$= \frac{5}{28}$$

36. We have,

$$\frac{(x-10)(x-20)}{(x-30)} < 0$$



Thus, $x \in (0, 10) \cup (20, 30)$

Hence, the required probability $= \frac{18}{100} = \frac{9}{50}$

37. We have,

$$x^3 - 6x^2 + 11x - 6 = 0$$
$$\Rightarrow (x-1)(x-2)(x-3) = 0$$
$$\Rightarrow x = 1, 2, 3$$

Hence, the required probability $= \frac{3}{10}$

38. (i) Hence, the required probability

$$= \frac{{}^{50}C_1 \times {}^{33}C_1}{{}^{100}C_2}$$
$$= \frac{50 \times 33 \times 2}{100 \times 99}$$
$$= \frac{1}{3}$$

(ii) Hence, the required probability

$$= \frac{{}^{50}C_2 + {}^{33}C_2 - {}^{16}C_2}{{}^{100}C_2}$$

(iii) Hence, the required probability $= \frac{{}^{16}C_2}{{}^{100}C_2}$

$$= \frac{4}{165}$$

(iv) Hence, the required probability $= 1 - \frac{{}^{16}C_2}{{}^{100}C_2}$

$$= 1 - \frac{4}{165}$$
$$= \frac{161}{165}$$

(v) Hence, the required probability $= \frac{{}^{25}C_1 \times {}^{33}C_1}{{}^{100}C_2}$

$$= \frac{25 \times 33 \times 2}{100 \times 99}$$
$$= \frac{1}{6}$$

39. Hence, the required probability $= \frac{7 \times 8 + 7 \times 8}{{}^{64}C_2}$

$$= \frac{2 \times 112}{64 \times 63}$$
$$= \frac{2 \times 7}{4 \times 63} = \frac{7}{126} = \frac{1}{18}$$

40. Hence, the required probability $= P(S \geq 5)$

$$= 1 - P(S < 5)$$
$$= 1 - \{(P(S = 3) + P(S = 4)\}$$
$$= 1 - \left\{\frac{1}{216} + \frac{3}{216}\right\}$$
$$= \left(1 - \frac{4}{216}\right) = \left(1 - \frac{1}{54}\right) = \frac{53}{54}$$

41. Hence, the required probability $= P(S \geq 15)$

$$= P(S = 15) + P(S = 16) + P(S = 17) + P(S = 18)$$
$$= \frac{10 + 6 + 3 + 1}{216} = \frac{20}{216} = \frac{5}{54}$$

42. Let S be the sample space.

Then $n(S) = 2 \times 2 \times 2 \times 2 = 16$

Let E be the event that the determinant is non-negative

Then E' be the event that the determinant is negative

$$= \left\{\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}\right\}$$

Now, $P(E') = \frac{n(E')}{n(S)} = \frac{3}{16}$

Hence, the required probability, $P(E)$

$$= 1 - P(E') = 1 - \frac{3}{16} = \frac{13}{16}$$

43. When $4 \rightarrow (1, 1, 2)$

$$P(S = 4) = \frac{3!}{2!} = 3$$

$$16 \rightarrow (4, 6, 6), (5, 5, 6)$$

$$P(S = 16) = \frac{3!}{2!} + \frac{3!}{2!} = 6$$

$$9 \rightarrow (1, 2, 6), (1, 3, 5), (2, 3, 4), (1, 4, 4)$$
$$(2, 2, 5), (3, 3, 3)$$

$$P(S = 9) = 3 \times 3! + 2 \times \frac{3!}{2!} + 1 = 25$$

Hence, the required probability

$$= P(S = 4) + P(S = 9) + P(S = 16)$$
$$= \frac{3 + 6 + 25}{216} = \frac{34}{216} = \frac{17}{108}$$

44. Let E be the event, where the product of two numbers is equal to the third number.

$$= \{(2, 5, 10), (2, 4, 8), (2, 3, 6)\}$$

Hence, the required probability $= \frac{3}{^{10}C_3}$

$= \frac{3 \times 6}{10 \times 9 \times 8} = \frac{1}{40}$

45. Let n be the sides of a polygon.

$\therefore$ Each interior angle = 150°

$\Rightarrow \frac{(2n-4)\times 90}{n} = 150$

$\Rightarrow 180n - 360 = 150n$

$\Rightarrow 30n = 360$

$\Rightarrow n = 12$

Hence, the required probability

$$= \frac{\frac{12(12-3)}{2}}{^{12}C_2} = \frac{12 \times 9}{12 \times 11} = \frac{9}{11}$$

46. Hence, the required probability $= \frac{^3P_3}{3^3}$

$= \frac{3!}{3^3} = \frac{6}{27} = \frac{2}{9}$

47. Let E be the event, where the number of selections of two diagonals which intersect at an interior point.

Thus, $n(E)$ = the total number of selection of four vertices = $^6C_1$

Hence, the required probability $= \frac{^6C_4}{^9C_2}$

$= \frac{^6C_2}{^9C_2} = \frac{6 \times 5}{9 \times 8} = \frac{5}{12}$

48. Hence, the required probability $= \frac{2}{^6C_3} = \frac{2}{5 \times 4} = \frac{1}{10}$

49. Hence, the required probability $= \frac{^nP_n}{(n^n)} = \left(\frac{n!}{n^n}\right)$

50. Hence, the required probability

$$= \frac{5! \times \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right)}{^5P_5}$$

$$= \frac{5! \times \left(\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!}\right)}{^5P_5}$$

$$= \frac{120 \times \left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right)}{120}$$

$$= \left(\frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120}\right)$$

$$= \frac{60 - 20 + 5 - 1}{120} = \frac{44}{120} = \frac{11}{30}$$

51. We have,

$7 \rightarrow (1, 1, 1, 1, 3), (1, 1, 1, 2, 2)$

Thus, the number of onto functions

$$= \frac{7!}{3!} \times \frac{5!}{4!} + \frac{7!}{2! \times 2!} \times \frac{5!}{3! \times 2!}$$

$$= \frac{7! \times 5!}{36}$$

Hence, the required probability

$$= \frac{\left(\frac{7! \times 5!}{36}\right)}{5^7} = \frac{7! \times 5!}{5^7 \times 36} = \frac{7! \times 5 \times 24}{5^7 \times 36} = \frac{7! \times 2}{5^6 \times 3}$$

52. Of all the mapping that can be defined from the set $A = \{1, 2, 3, 4\}$ to the set $B = \{5, 6, 7, 8, 9\}$, a mapping is a selected at random. Find the probability that the selected mapping is strictly monotonic.

53. Hence, the required probability

$$= \frac{\frac{6!}{3! \times 3!}}{6 \times 6 \times 6} = \frac{20}{216} = \frac{5}{54}$$

54. Hence, the required probability

$$= \frac{7 \times 7}{^{64}C_4} = \frac{49 \times 24}{64 \times 63 \times 62 \times 61}$$

$$= \frac{7}{24 \times 62 \times 61} = \frac{7}{90768}$$

55. Hence, the required probability

$$\left(1 - \frac{1}{3!}\right) = 1 - \frac{1}{6} = \frac{5}{6}$$

56. Hence, the required probability $= \frac{^{16}C_3}{^{100}C_3}$

$$= \frac{16 \times 15 \times 14}{100 \times 99 \times 98}$$

$$= \frac{4}{1155}$$

57. Here, $S = \{1, 2, 3, \ldots, 100\}$.

Let A be the event, where the natural numbers is divisible by 2 = $\{2, 4, 6, \ldots, 100\}$

and B be the event, where the natural numbers is divisible by both 3 = $\{3, 6, 9, \ldots, 99\}$

and $A \cap B$ is the event, which is divisible by both 2 and 3 = $\{6, 12, 18, \ldots, 96\}$

Hence, the required probability

$= P(A \text{ or } B)$

$= P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$

$$= \frac{50}{100} + \frac{33}{100} - \frac{16}{100} = \frac{67}{100}$$

58. Here, $S = \{1, 2, 3, \ldots, 200\}$.
Let A be the event, where the natural numbers is divisible by $3 = \{3, 6, \ldots, 198\}$.
B be the event, where the natural numbers is divisible by $4 = \{4, 8, 12, \ldots, 200\}$
and C be the event, where the natural numbers is divisible by $5 = \{5, 10, 15, \ldots, 200\}$
Hence, the required probability
$= P(A \text{ or } B \text{ or } C)$
$= P(A \cup B \cup C)$
$= P(A) + P(B) + P(C) - P(A \cap B)$
$- P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$
$$= \frac{66}{200} + \frac{50}{200} + \frac{40}{200} - \frac{16}{200} - \frac{13}{200} - \frac{10}{200} + \frac{3}{200}$$
$$= \frac{120}{200} = \frac{3}{5}$$

59. Hence, the required probability
$= P(C \text{ or } F) = P(C \cup F)$
$= P(C) + P(F) - P(C \cap F)$
$$= \frac{{}^{16}C_1}{{}^{52}C_1} + \frac{{}^{12}C_1}{{}^{52}C_1} - \frac{{}^{12}C_1}{{}^{52}C_1} = \frac{16}{52} = \frac{4}{13}$$

60. Required Probability
$= P(K \cup H \cup R)$
$= P(K) + P(H) + P(R) - P(K \cap H)$
$- P(K \cap R) + P(H \cap R) + P(K \cap H \cap R)$
$$= \frac{4}{52} + \frac{13}{52} + \frac{26}{52} - \frac{1}{52} - \frac{2}{52} - \frac{13}{52} + \frac{1}{52}$$
$$= \frac{28}{52} = \frac{7}{13}$$

61. Here, $S = \{(1, 1), (1, 2), (1, 3), ..., (6, 6)\}$
Let A be the event, where the sum of the numbers of two faces is divisible by 4
$= \{(1, 3), (3, 1), (2, 2), (2, 6), (6, 2)$
$(3, 5), (5, 3), (4, 4), (6, 6)\}$
Let B be the event, where the sum of the numbers of two faces is divisible by 5.
$= \{(1, 4), (4, 1), (2, 3), (3, 2), (4, 6), (6, 4), (5, 5)\}$
Clearly A and B are mutually exclusive events.
Thus, the required probability
$= P(A \cup B) + P(A) + P(B)$
$$= \frac{9}{36} + \frac{7}{36} = \frac{16}{36} = \frac{4}{9}$$

62. Here, $S = \{(1, 1), (1, 2), (1, 3), ..., (6, 6)$
Let A be the event, where the first die always shows 3
$= \{(3, 1), (3, 2), ..., (3, 6)\}$
B be the event where the second die always shows 3
$= \{(1, 3), (2, 3), ..., (6, 3)\}$
and $A \cap B$ be the event, where both the die always shows $3 = \{(3, 3)\}$
Hence, the required probability $= P(A \cup B)$
$= P(A) + P(B) - P(A \cap B)$
$$= \frac{6}{36} + \frac{6}{36} - \frac{1}{36} = \frac{11}{36}$$

63. Here, $S = \{(1, 1), (1, 2), (1, 3), ..., (6, 6)\}$.
Let A be the event, where the first die always shows odd numbers $= \{(1, 1), (1, 2), ..., (1, 6), (3, 1), ...,$
$(3, 6), (5, 1), ..., (5, 6)\}$
Let B be the event, where the sum of two outcome is 7.
$= \{(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)\}$
and $A \cap B$ be the event, where first die is an odd number and sum of the outcome is a 7
$= \{(1, 6), (3, 4), (5, 2)\}$
Hence, the required probability $= P(A \cup B)$
$= P(A) + P(B) - P(A \cap B)$
$$\frac{18}{36} + \frac{6}{36} - \frac{3}{36} = \frac{21}{36} = \frac{7}{12}$$

64. Here, $S = \{(1, 1), (1, 2), (1, 3), \ldots, (6, 6)\}$.
Let A be the event, where the sum of the numbers is coming up more that 9
$= \{(4, 6), (6, 4), (5, 5), (5, 6), (6, 5), (6, 6)\}$
Hence, the required probability $= \frac{6}{36} = \frac{1}{6}$

65. Hence the required probability $= P(I \cup A)$
$= P(I) + P(A) - P(I \cap A)$
$$= \frac{1}{3} + \frac{1}{2} - \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$$

66. Required Probability $= P(S \cup C \cup H \cup D)$
$= P(S) + P(C) + P(H) + P(D)$
$$= \frac{{}^{13}C_4}{{}^{52}C_4} + \frac{{}^{13}C_4}{{}^{52}C_4} + \frac{{}^{13}C_4}{{}^{52}C_4} + \frac{{}^{13}C_4}{{}^{52}C_4}$$
$$= \frac{4 \times {}^{13}C_4}{{}^{52}C_4}$$

67. Hence, the required probability $= P(M \cup P)$
$= P(M) + P(P) - P(P \cap M)$
$$= \frac{1}{3} + \frac{1}{4} - \frac{1}{5} = \frac{20 + 15 - 12}{60} = \frac{23}{60}$$

68. Hence, the required probability
$$= \frac{{}^{26}C_2 + {}^{26}C_2}{{}^{52}C_2}$$
$$= \frac{2 \times {}^{26}C_2}{{}^{52}C_2}$$
$$= \frac{2 \times 26 \times 25}{52 \times 51}$$
$$= \frac{25}{51}$$

69. $P(E \cup P) = P(E) + P(P) - P(E \cap P)$
$$\Rightarrow \quad \frac{2}{3} = \frac{2}{5} + \frac{4}{7} - P(E \cap P)$$
$$\Rightarrow \quad P(E \cap P) = \frac{2}{5} + \frac{4}{7} - \frac{2}{3}$$
$$\Rightarrow \quad P(E \cap P) = \frac{32}{105}$$

70. Hence, the required probability $= P(M \cup E)$
$= P(M) + P(E) - P(M \cap E)$

$= \frac{2}{3} + \frac{4}{9} - \frac{1}{4}$

$= \frac{31}{36}$

71. Hence, the required probability = $P(P \cup T)$

$= P(P) + P(T)$

$= \frac{3}{5} + \frac{1}{4}$

$= \frac{17}{20}$

72. Hence, the required probability

$= P(k \text{ or } H \text{ or } R)$

$= P(k \cup H \cup R)$

$= P(k) + P(H) + P(R) - P(K \cap H)$
$-P(K \cap R) - P(R \cap H) - P(K \cap H \cap R)$

$= \frac{4}{52} + \frac{13}{52} + \frac{26}{52} - \frac{1}{52} - \frac{2}{52} - \frac{13}{52} + \frac{1}{52}$

$= \frac{4}{52} + \frac{26}{52} - \frac{2}{52}$

$= \frac{28}{52} = \frac{7}{13}$

73. Hence, the required probability

$= \frac{{}^{13}C_2 + {}^{13}C_2 + {}^{13}C_2}{{}^{52}C_2}$

$= \frac{3 \times {}^{13}C_2}{{}^{52}C_2}$

$= \frac{3 \times 13 \times 12}{52 \times 51} = \frac{3}{17}$

74. Let A, B and C be the events represents integers which are multiple of 2, 3 and 10.

Hence, the required probability

$= P(A \cup B \cup C)$

$= P(A) + P(B) + PC - P(A \cap B) - P(A \cap C)$
$- P(B \cap C) + P(A \cap B \cap C)$

$= \frac{25}{50} + \frac{16}{50} + \frac{5}{50} - \frac{8}{50} - \frac{5}{50} - \frac{1}{50} + \frac{1}{50}$

$= \frac{25}{50} + \frac{16}{50} - \frac{8}{50}$

$= \frac{34}{50}$

75. $4 \rightarrow (1, 3), (3, 1), (2, 2)$

$7 \rightarrow (1, 6), (6, 1), (2, 5), (5, 2), (4, 3), (3, 4)$

$12 \rightarrow (6, 6)$

Hence, the required probability

$= \frac{10}{36} = \frac{5}{18}$

76. $P((A \cap \overline{B}) \cup (\overline{A} \cap B)) = 1 - x$ ...(i)

$P((B \cap \overline{C}) \cup (\overline{B} \cap C)) = 1 - 2x$ ...(ii)

$P((C \cap \overline{A}) \cup (\overline{C} \cap A)) = 1 - x$ ...(iii)

Adding (i), (ii) and (iii), we get,

$2(P(A) + P(B) + P(C) - P(A \cap B) - P(C \cap B) - P(A \cap C))$
$= 3 - 4x$

Now, $P(A \cup B \cup C)$

$= P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C)$
$- P(C \cap A) + P(A \cap B \cap C)$

$= \frac{3 - 4x}{2} + x^2$

$= \frac{2x^2 - 4x + 3}{2}$

$= \frac{2(x^2 - 2x + 1) + 1}{2}$

$= (x - 1)^2 + \frac{1}{2} \geq \frac{1}{2}$

Hence, the result.

77. (i) From the definition of in-equalities of probability, we can write

$P(A \cup B) \leq P(B) \Rightarrow P(A \cup B) \geq \frac{2}{3}$

(ii) From the definition of in-equalities of probability, we can write

$P(A \cap B) \leq P(A)$

$\Rightarrow P(A \cap B) \leq \frac{3}{5}$

Also,

$P(A \cap B) = P(A) + P(B) - P(A \cap B)$
$\geq P(A) + P(B) - 1$

$\Rightarrow P(A \cap B) \geq \frac{3}{5} + \frac{2}{3} - 1 = \frac{9 + 10 - 15}{15} = \frac{4}{15}$

Hence from the above two relation, we get and $\frac{4}{15} \leq P(A \cap B) \leq \frac{3}{5}$.

78.

78. (i) From the definition of in-equalities of probability, we can write

$P(A \cup B) \geq P(A) + P(B) - 1$

$= \frac{3}{4} + \frac{3}{8} - 1 = \frac{6 + 3 - 8}{8} = \frac{1}{8}$

(ii) From the definition of in-equalities of probability, we can write

$P(A) + P(B) - 1 \leq P(A \cap B) \leq P(A)$

$\Rightarrow \frac{1}{8} \leq P(A \cap B) \leq \frac{3}{4}$

79. Let $S = \{1, 2, 3, 4, 5, 6\}$ and $B = \{1, 3, 5\}$ and $A = \{2, 3, 5\}$.

Required Probability $= P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{2}{3}$

80. Let $S = \{(1, 1), (1, 2), \ldots, (6, 6)\}$

$B = \{(1, 2), \ldots (6, 2), (1, 3), \ldots, (6, 3), (1, 5), \ldots, (6, 6)\}$

and $A = \{(2, 6), (6, 2), (3, 5), (5, 3), (4, 4)\}$.

Required Probability $= \frac{P(A \cap B)}{P(B)} = \frac{3}{5}$

81. Here, $S = \{(1, 1), (1, 2), \ldots, (6, 6)\}$.
Let A be the event, the sum of the numbers coming up is $9 = \{(3, 6), (6, 3), (5, 4), (4, 5)\}$ and
B be the event, where first will shows 5
$=\{(5, 1), (5, 3), (5, 3), \ldots (5, 6)\}$
and $A \cap B$ be the event, where sum of two outcomes is 9 and first die shows $5 = \{(5, 4)\}$
Hence, the required probability
$$= P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{1}{6}$$

82. Here, $S = \{(1, 1), (1, 2), \ldots, (6, 6)\}$.
Let A be the event of coming up two heads = {(H, H)} and B be the event of coming up at least one head
= {(H, H), (H, T), (T, H)}
Hence, the required probability $= P\left(\frac{A}{B}\right)$
$$= \frac{P(A \cap B)}{\mathrm{P}(B)} = \frac{1}{3}$$

83. Here, $S = \{(1, 1), (1, 2), \ldots, (6, 6)\}$.
Let A be the event, the sum of the numbers coming up is 8
$= \{(3, 5), (5, 3), (4, 4), (2, 6), (6, 2)\}$
B be the event, where first will shows 4
$= \{(4, 1), (4, 2), (4, 3), \ldots (4, 6)\}$
and $A \cap B$ be the event, where the sum of two outcomes is 8 and first die shows $4 = \{(4, 4)\}$
Hence, the required probability $= P\left(\frac{A}{B}\right)$
$$= \frac{P(A \cap B)}{P(B)} = \frac{1}{6}$$

84. Here, $S = \{(1, 1), (1, 2), \ldots, (6, 6)\}$
Let A be the event, the sum of the numbers coming up is $7 = \{(1, 6), (6, 1), 1(2, 5), (5, 2), (3, 4), (4, 3)\}$
B be the event, where second die will show a prime number.
$= \{(1, 2), \ldots, (6, 2), (1, 3), \ldots, (6, 3), (1, 5), \ldots, (6, 5)\}$
and $A \cap B$ be the event, where sum of two outcomes is 7 and second die always exhibits a prime number
$= \{(2, 2), (2, 5), (4, 3)\}$
Hence, the required probability
$$= P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{3}{18} = \frac{1}{6}$$

85. Here, $S = \{(1, 1), (1, 2), \ldots, (6, 6)\}$.
Let A be the event, the sum of the numbers coming up is $7 = \{(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)\}$
B be the event, where second die always exhibits an odd number $= \{(1, 1), \ldots, (6, 1), (1, 3), \ldots, (6, 3), (1, 5), \ldots, (6, 5)\}$
and $A \cap B$ be the event, where sum of two outcomes is 7 and second die always exhibits an odd number $= \{(6, 1), (2, 5), (4, 3)\}$
Hence, the required probability
$$= P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{3}{18} = \frac{1}{6}$$

86. Here, $S = \{(1, 1), (1, 2), \ldots, (6, 6)\}$
Let A be the event, the sum of the numbers coming up is $8 = \{(3, 5), (5, 3), (4, 4), (2, 6), (6, 2)\}$
B be the event, where second die will show 4
$= \{(1, 4), (2, 4), \ldots, (6, 4)\}$
and $A \cap B$ be the event, where sum of two outcomes is 8 and second die shows $4 = \{(4, 4)\}$
Hence, the required probability
$$= P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{1}{6}$$

87. Here, $S = \{(1, 1), (1, 2), \ldots, (6, 6)\}$.
Let A be the event, the sum of the numbers coming up is $6 = \{(1, 5), (5, 1), (2, 4), (4, 2), (3, 3)\}$
B be the event, where 4 has appeared at least once.
$= \{(1, 4), (2, 4), (4, 4), \ldots, (6, 4), (4, 1), \ldots, (4, 6)\}$
and $A \cap B$ be the event, where the sum of two outcomes is 6 and 4 has appeared at least once $= \{(2, 4), (4, 2)\}$
Hence, the required probability
$$= P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{2}{11}$$

88. To test the quality of electric bulbs produced in a factory, two bulbs are randomly selected from a large sample without replacement. If either bulb is defective, the entire lot is rejected. Suppose a sample of 200 bulbs contains 5 defective bulbs. Find the probability that the sample will be rejected.

89. Hence, the required probability $= \frac{10}{25} \times \frac{15}{24}$
$$= \frac{2}{5} \times \frac{15}{24}$$
$$= \frac{3}{12} = \frac{1}{4}$$

90. Hence, the required probability $= \frac{13}{52} \times \frac{12}{51}$
$$= \frac{1}{4} \times \frac{12}{51}$$
$$= \frac{3}{51} = \frac{1}{17}$$

91. Hence, the required probability $= \frac{{}^{5}C_4}{{}^{20}C_4}$
$$= \frac{5.4.3.2}{20.19.18.17}$$
$$= \frac{1}{19.3.17}$$

$$= \frac{1}{19 \times 51}$$
$$= \frac{1}{969}$$

92. Let $S = \{1, 2, 3, \ldots, 11\}$.

Hence, the required probability $= \dfrac{^6C_2}{^6C_2 \times {^5C_2}}$

$$= \frac{6.5}{6.5 + 5.4}$$
$$= \frac{30}{30 + 20}$$
$$= \frac{3}{5}$$

93. Let $S = \{B_1B_2, B_1G_2, G_1B_2, G_1G_2\}$.
   (i) Let A be the event, where one children is a boy, i.e. $A = \{B_1B_2, B_1G_2, G_1B_2\}$

   Hence, the required probability $= \dfrac{1}{3}$

   (ii) Let B be the event, where older child is a boy, i.e. $B = \{B_1G_2, B_1B_2\}$
   Hence, the required probability $= \dfrac{1}{2}$

94. Let $S = \{\text{HH, HT, TH, TT}\}$ and $A = \{\text{HH, GT, TH}\}$, $B = \{\text{HH}\}$

Hence, the required probability $= \dfrac{1}{3}$

95. Hence, the required probability $= \dfrac{2}{26} = \dfrac{1}{13}$

95. Let A be the event of drawing a king card and B be the event of drawing a red card.
Hence, the required probability

$$= P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)} = \frac{^2C_1}{^{26}C_1} = \frac{2}{26} = \frac{1}{13}$$

96. We have, Bag 1 $\rightarrow$ (4 W, $2B_1$)
and Bag 2 $\rightarrow$ (3 W, $5B_2$)
Here, W_1 and W_2, and B_1 and B_2 are independents.

   (i) $P(W_1 \cap W_2) = P(W_1).\, P(W_2) = \dfrac{4}{6} \times \dfrac{3}{8} = \dfrac{1}{4}$

   (ii) $P(B_1 \cap B_2) = P(B_1).\, P(B_2) = \dfrac{1}{3} \times \dfrac{5}{8} = \dfrac{5}{24}$

   (iii) $P\{(W_1 \cap B_2) \cup (W_2 \cap B_1)\}$
   $= P(W_1 \cap B_2) + P(W_2 \cap B_1)\}$
   $= P(W_1) \cdot P(B_2) + P(W_2) \cdot P(B_1)$

   $$= \frac{2}{3} \times \frac{5}{8} + \frac{1}{3} \times \frac{3}{8} = \frac{5}{12} + \frac{1}{8} = \frac{13}{24}$$

97. Let three students are A, B, C.
Consider the problem is not solved by any one of them.
Thus,

$$P(\overline{A} \cap \overline{B} \cap \overline{C}) = P(\overline{A}) \cdot P(\overline{B}) \cdot P(\overline{C})$$
$$= \left(1 - \frac{1}{2}\right) \cdot \left(1 - \frac{1}{3}\right) \cdot \left(1 - \frac{1}{4}\right)$$
$$= \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{4}$$

Hence, the required probability is $= 1 - \dfrac{1}{4} = \dfrac{3}{4}$

98. Hence, the required probability

$$= P(A \cap \overline{B} \quad \text{or} \quad \overline{A} \cap B)$$
$$= P(A \cap \overline{B}) + P(\overline{A} \cap B)$$
$$= P(A) \cdot P(\overline{B}) + P(\overline{A}) \cdot P(B)$$
$$= \frac{75}{100} \times \frac{20}{100} + \frac{25}{100} \times \frac{80}{100}$$
$$= \frac{15}{100} + \frac{20}{100} = \frac{7}{20}$$
$$= \frac{35}{100}$$
$$= 35\%$$

99. Let three students are A, B, C respectively.
Hence, the required probability

$$= P(A \cap B \cap \overline{C}) + P(A \cap \overline{B} \cap C) + P(\overline{A} \cap B \cap C) + P(A \cap B \cap C)$$
$$= P(A) \cdot P(B) \cdot P(\overline{C}) + P(A) \cdot P(\overline{B}) \cdot P(C) + P(\overline{A}) \cdot P(B) \cdot P(C) + P(A) \cdot P(B) \cdot P(C)$$
$$= \frac{1}{3} \cdot \frac{1}{4} \cdot \left(1 - \frac{1}{5}\right) + \frac{1}{3} \cdot \left(1 - \frac{1}{4}\right) \cdot \frac{1}{5} + \left(1 - \frac{1}{3}\right) \cdot \frac{1}{4} \cdot \frac{1}{5} + \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{5}$$
$$= \frac{4}{60} + \frac{2}{60} + \frac{3}{60} + \frac{1}{60} = \frac{10}{60} = \frac{1}{6}$$

100. Let two persons be A and B, respectively. Consider none of them hit the target.

Now, $= P(A \cap B \cap \overline{C}) + P(A \cap \overline{B} \cap C)$

$$= \left(1 - \frac{3}{4}\right) \times \left(1 - \frac{2}{3}\right) = \frac{1}{12}$$

Hence, the required probability $= 1 - \dfrac{1}{12} = \dfrac{11}{12}$

101. Let E_1 be the of throwing 6 with a pair of dice and E_2 be the event of throwing 7 with a pair of dice.

Thus, $E_1 = \{(1, 5), (5, 1), (2, 4), (4, 2), (3, 3)\}$

and

$E_2 = \{(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)\}$

The probability of A can win

$$P(A) = P(E_1 \text{ or } \overline{E_1}\cdot\overline{E_2}\cdot E_1 \text{ or } \overline{E_1}\cdot\overline{E_2}\cdot\overline{E_1}\cdot\overline{E_2}\cdot E_1 \text{ or } ...)$$

$$= \frac{5}{36} + \frac{31}{36}.\frac{5}{6}.\frac{5}{36} + \left(\frac{31}{36}\right)^2\left(\frac{5}{6}\right)^2\left(\frac{5}{36}\right) + ..\text{to } \infty$$

$$= \frac{5}{36} + \left(\frac{155}{216}\right).\frac{5}{36} + \left(\frac{155}{216}\right)^2\left(\frac{5}{36}\right) + ..\text{to } \infty$$

$$= \frac{\frac{5}{36}}{1-\frac{155}{216}} = \frac{30}{61}$$

The probability that B can win $= 1 - \frac{30}{61} = \frac{31}{61}$.

102. Let A be the event that a randomly selected ball is a red. E_1 and E_2 be denote two bags respectively.

Then $P(A) = P(E_1)\, P\{A/E_1) + P(E_2)\, P(A/E_2)$

$$= \frac{1}{2}\times\frac{4}{7} + \frac{1}{2}\times\frac{2}{6} = \frac{2}{7} + \frac{1}{6} = \frac{19}{42}$$

103. **Case I:** When a white ball is transferred from the first bag to a second bag.

Then $P(W) = \frac{4}{9}\times\frac{7}{14} = \frac{2}{9}$

Case II: When a black ball is transferred from the first bag to a 2nd bag.

Then $P(W) = \frac{4}{9}\times\frac{6}{14} = \frac{2}{3}\times\frac{2}{7} = \frac{4}{21}$

Hence, the required probability $= \frac{2}{9} + \frac{4}{21}$

$$= \frac{14+12}{63} = \frac{26}{63}.$$

104. Let E_1, E_2 and A denote the events, the lot contains 2 defective, 3 defective and all the defective articles are found by 12th test, i.e. the event A means that all but one defective article must be found in the first eleven testings.

Now, $P(E_1) = 0.4$, $P(E_2) = 0.6$

$$P(A/E_1) = \frac{{}^2C_1 \times {}^{18}C_{10}}{{}^{20}C_{11}} \times \frac{1}{9} = \frac{11}{190}$$

and $P(A/E_2) = \frac{{}^3C_2 \times {}^{17}C_9}{{}^{20}C_{11}} \times \frac{1}{9} = \frac{11}{228}$

Hence the required probability

$= P(A)$

$= P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2)$

$$= (0.4)\left(\frac{11}{190}\right) + (0.6)\left(\frac{11}{228}\right) = \frac{99}{1900}$$

105. Let E_{xy} denote the event that the first ball drawn has color x and second ball drawn has color y and A be the event that the third ball drawn is black.

$$P(E_{ww}) = \frac{2}{4}\times\frac{1}{3} = \frac{1}{6},\ P(E_{wb}) = \frac{2}{4}\times\frac{2}{3} = \frac{1}{3},$$

$$P(E_{bw}) = \frac{2}{4}\times\frac{2}{5} = \frac{1}{5},\ P(E_{bb}) = \frac{2}{4}\times\frac{3}{5} = \frac{3}{10}$$

Also, $P(A/E_{ww}) = \frac{2}{2} = 1,\ P(A/E_{wb}) = \frac{3}{4},$

$$P(A/E_{bw}) = \frac{3}{4},\ P(A/E_{bb}) = \frac{4}{6} = \frac{2}{3}$$

Hence the required probability

$= P(A)$

$= P(E_{ww}) \cdot P(A/E_{ww}) + P(E_{wb}) \cdot P(A/E_{wb})$
$+ P(E_{bw}) \cdot P(A/E_{bw}) + P(E_{bb}) \times P(A/E_{bb})$

$$= \frac{1}{6}\times 1 + \frac{1}{3}\times\frac{3}{4} + \frac{1}{5}\times\frac{3}{4} + \frac{3}{10}\times\frac{2}{3} = \frac{23}{30}$$

106. Let E_1, E_2 denote the events that the coin shows a head, tail and A be the event that the noted number is either 7 or 8.

We have $P(E_1) = \frac{1}{2}$ and $P(E_2) = \frac{1}{2}$

Now, $7 \rightarrow \{(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)\}$
and $8 \rightarrow \{(2, 6), (6, 2), (3, 5), (5, 3), (4, 4)\}$

Thus, $P(A/E_1) = \frac{11}{36},\ P(A/E_2) = \frac{2}{11}$

Hence, the required probability

$$= P(A) = P(E_1)P(A/E_1) + P(E_2)P(A/E_2)$$

$$= \left(\frac{1}{2}\right)\left(\frac{11}{36}\right) + \left(\frac{1}{2}\right)\left(\frac{2}{11}\right) = \frac{193}{792}$$

107. Let W be the event that a white ball is drawn at the first draw B be the event that a black ball is drawn at the first draw and A be the event that a white ball is drawn at the second draw.

We have, $P(W) = \frac{m}{m+n},\ P(B) = \frac{n}{m+n}$

and $P(A/W) = \frac{m+k}{m+n+k},\ P(A/B) = \frac{m}{m+n+k}$

Hence the required probability,

$P(A) = P(W) \times P(A/W) + P(B) \cdot P(\underline{A}/B)$

$$= \frac{m}{m+n}\cdot\frac{m+k}{m+n+k} + \frac{m}{m+n}\cdot\frac{m}{m+n+k}$$

$$= \frac{m}{m+n}.$$

108. Hence the required probability,

$$P(E_2/B) = \frac{P(E_2)\, P(A/E_2)}{P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2)}$$

$$= \frac{\frac{1}{2}\times\frac{5}{11}}{\frac{1}{2}\times\frac{3}{7} + \frac{1}{2}\times\frac{5}{11}} = \frac{35}{68}$$

109. It is given that

$$P(A)=\frac{4}{5}, P(B)=\frac{3}{4}, P(C)=\frac{2}{3}$$

(i) Hence, the required probability,

$$P(ABC) = P(A)P(B)P(C)$$
$$=\frac{4}{5}\times\frac{3}{4}\times\frac{2}{3}$$
$$=\frac{2}{5}$$

(ii) Hence, the required probability,

$$P(\overline{A}BC)=P(\overline{A})P(B)P(C)$$
$$=\left(1-\frac{4}{5}\right)\times\frac{3}{4}\times\frac{2}{3}$$
$$=\frac{1}{5}\times\frac{3}{4}\times\frac{2}{3}$$
$$=\frac{1}{10}$$

110. Let A be the event that the man reports that six occurs on throwing a die. E_1 is the event that six occurs and E_2 is the event that six does not occur.

$$P(E_1)=\frac{1}{6} \text{ and } P(E_2)=\frac{5}{6}$$
$$P(A/E_1)=\frac{3}{5} \text{ and } P(A/E_2)=\frac{2}{5}$$

Hence the required probability

$$P(E_1/A)=\frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)}$$
$$=\frac{\frac{1}{6}\times\frac{3}{5}}{\frac{1}{6}\times\frac{3}{5}+\frac{5}{6}\times\frac{2}{5}}=\frac{3}{3+10}=\frac{3}{13}$$

111. Hence, the required probability

$$=\frac{{}^{2}C_1\times{}^{2}C_1}{{}^{52}C_2}$$
$$=\frac{2}{663}$$

112. Hence, the required probability

$$=\frac{{}^{13}C_1\times{}^{2}C_1}{{}^{52}C_2}$$
$$=\frac{13\times2\times2}{52\times51}$$
$$=\frac{1}{51}$$

113. Hence, the required probability

$$=\frac{{}^{3}C_2\times{}^{2}C_2}{{}^{25}C_2}$$
$$=\frac{3\times2\times2}{25\times24}$$
$$=\frac{1}{50}$$

114. Hence, the required probability

= 1 – None of them solve the problem

$$=1-P(\overline{A}\,\overline{B}\,\overline{C})$$
$$=1-\left\{P(\overline{A})\cdot P(\overline{B})\cdot P(\overline{C})\right\}$$
$$=1-\left\{\frac{1}{3}\times\frac{2}{7}\times\frac{3}{8}\right\}$$
$$=1-\frac{1}{28}$$
$$=\frac{27}{28}.$$

115. (i) Hence, the required probability,

$$P(A\overline{B})=P(A)P(\overline{B})$$
$$=\frac{3}{7}\times\left(1-\frac{5}{7}\right)$$
$$=\frac{3}{7}\times\frac{2}{7}$$
$$=\frac{6}{49}$$

(ii) Hence, the required probability,

$$P(A\overline{B} \text{ or } \overline{A}B)=P(A\overline{B})+P(\overline{A}B)$$
$$=P(A)P(\overline{B})+P(\overline{A})P(B)$$
$$=\frac{3}{7}\times\frac{2}{7}+\frac{4}{7}\times\frac{5}{7}$$
$$=\frac{26}{49}$$

116. $P(W)=\frac{1}{6}, P(F)=\frac{5}{6}$

Hence, the required probability of A,

$$P(A)=P(W \text{ or } FFW \text{ or } FFFFW \text{ or ... to } \infty)$$
$$=P(W)+P(FFW)+P(FFFFW)+\ldots$$
$$=\frac{1}{6}+\left(\frac{5}{6}\right)^2\left(\frac{1}{6}\right)+\left(\frac{5}{6}\right)^4\left(\frac{1}{6}\right)+\ldots$$
$$=\frac{1}{6}\left(1+\left(\frac{5}{6}\right)^2+\left(\frac{5}{6}\right)^4+\ldots\right)$$
$$=\frac{1}{6}\times\frac{1}{1-\left(\frac{5}{6}\right)^2}$$

$$= \frac{1}{6} \times \frac{36}{36-25}$$
$$= \frac{6}{11}$$

Thus, $P(B) = 1 - P(A) = 1 - \frac{6}{11} = \frac{5}{11}$

117. Here, $6 \rightarrow (1, 5), (5, 1), (2, 4), (4, 2), (3, 3)$
and $7 \rightarrow (1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)$

$\therefore$ The probability of getting 6 is $\frac{5}{36}$

and the probability of getting 7 is $\frac{1}{6}$.

Thus, the probability of A

$$= \frac{5}{36} + \left(\frac{31}{36} \times \frac{5}{6}\right)\left(\frac{5}{36}\right) + \left(\frac{31}{36} \times \frac{5}{6}\right)^2 \left(\frac{5}{36}\right) + ...$$
$$= \frac{5}{36}\left(1 + \left(\frac{31 \times 5}{36 \times 6}\right) + \left(\frac{31 \times 5}{36 \times 6}\right)^2 + ...\right)$$
$$= \frac{5}{36} \times \frac{1}{1 - \left(\frac{31 \times 5}{36 \times 6}\right)}$$
$$= \frac{5}{36} \times \frac{36 \times 6}{216 - 155}$$
$$= \frac{30}{61}$$

Thus, the probability of B,

$$P(B) = 1 - \frac{30}{61} = \frac{31}{61}$$

118. Here, $P(W) = \frac{1}{6}, P(F) = \frac{5}{6}$

Thus, the probability of A winning,

$$P(A) = P(W \text{ or } FFFW \text{ or } FFFFFFW \text{ or } ....)$$
$$= P(W) + P(FFFW) + P(FFFFFFW) + ...$$
$$= \frac{1}{6} + \left(\frac{5}{6}\right)^3 \frac{1}{6} + \left(\frac{5}{6}\right)^6 \frac{1}{6} + ...$$
$$= \frac{1}{6}\left(1 + \left(\frac{5}{6}\right)^3 + \left(\frac{5}{6}\right)^6 + ...\right)$$
$$= \frac{1}{6} \times \frac{1}{1 - \left(\frac{5}{6}\right)^3}$$
$$= \frac{1}{6} \times \frac{216}{216 - 125}$$
$$= \frac{36}{91}$$

Now, the probability of B winning

$$P(B) = P(FW \text{ or } FFFFW \text{ or } FFFFFFFW \text{ or } .....)$$
$$= P(FW) + P(FFFFW) + P(FFFFFFFW) + ...$$
$$= \frac{5}{6} \times \frac{1}{6} + \left(\frac{5}{6}\right)^4 \times \frac{1}{6} + \left(\frac{5}{6}\right)^7 \times \frac{1}{6} + ...$$
$$= \frac{5}{6} \times \frac{1}{6}\left(1 + \left(\frac{5}{6}\right)^3 + \left(\frac{5}{6}\right)^6 + ...\right)$$
$$= \frac{5}{6} \times \frac{1}{6} \times \frac{1}{1 - \left(\frac{5}{6}\right)^3}$$
$$= \frac{5}{6} \times \frac{1}{6} \times \frac{216}{216 - 125}$$
$$= \frac{30}{91}$$

Hence, the probability of C winning

$$P(B) = 1 - \{P(A) + P(B)\}$$
$$= 1 - \left(\frac{36}{91} + \frac{30}{91}\right)$$
$$= 1 - \frac{66}{91}$$
$$= \frac{25}{91}$$

119. Hence, the required probability

$$= \frac{1}{2} \times \frac{4}{7} + \frac{1}{2} \times \frac{2}{6}$$
$$= \frac{1}{2} \times \left(\frac{4}{7} + \frac{1}{3}\right)$$
$$= \frac{1}{2} \times \frac{12 + 7}{21}$$
$$= \frac{19}{42}$$

120. Hence, the required probability $= \frac{4}{9} \times \frac{7}{14} + \frac{5}{9} \times \frac{6}{14}$

$$= \frac{28 + 30}{126}$$
$$= \frac{58}{126}$$
$$= \frac{29}{63}$$

121. Let E_1, E_2 and D denote the events, a lot contains 2 defective articles, second lot contains 3 defective articles and all the defective article are found by the 12th test, i.e. the event D means one defective articles must be found in the first eleven testings.

Now, $P(E_1) = 0.4$, $P(E_2) = 0.6$

$$P(D/E_1) = \frac{{}^{2}C_1 \times {}^{18}C_{10}}{{}^{20}C_{11}} \times \frac{1}{9} = \frac{11}{190}$$
$$P(D/E_2) = \frac{{}^{3}C_2 \times {}^{17}C_9}{{}^{20}C_{11}} \times \frac{1}{9} = \frac{11}{228}$$

Hence, the required probability,

$$P(D) = P(E_1).P(D/E_1) + P(E_2).P(D/E_2)$$
$$= 0.4\times\frac{11}{190}+0.6\times\frac{11}{228}$$
$$= \frac{99}{1900}$$

122. Hence, the required probability

$$= \frac{25}{100}\times\frac{5}{100}+\frac{35}{100}\times\frac{4}{100}+\frac{40}{100}\times\frac{2}{100}$$
$$= \frac{125+140+80}{10000}$$
$$= \frac{345}{10000}$$
$$= \frac{69}{2000}$$

123. Hence, the required probability

$$= \frac{1}{3}\times\frac{25}{100}+\frac{2}{3}\times\frac{28}{100}$$
$$= \frac{25+56}{300}$$
$$= \frac{81}{300}$$
$$= \frac{27}{100}$$
$$= 27\%$$

124. For the first two draw, Let the balls taken out may be

E_1 = white and white
E_2 = white and black
E_3 = black and white
E_4 = black and black

Now, $P(E_1) = \frac{2}{4}.\frac{1}{3} = \frac{1}{6}$

$$P(E_2) = \frac{2}{4}.\frac{2}{3} = \frac{1}{3}$$
$$P(E_3) = \frac{2}{4}.\frac{2}{5} = \frac{1}{5}$$
$$P(E_4) = \frac{2}{4}.\frac{3}{5} = \frac{3}{10}$$

Now,

$$P(E_1) + P(E_2) + P(E_3) + P(E_4)$$
$$= \frac{1}{6}+\frac{1}{3}+\frac{1}{5}+\frac{3}{10}$$
$$= \frac{10+20+12+18}{60}$$
$$= \frac{60}{60} = 1$$

Thus, the events E_1, E_2, E_3 and E_4 are exhaustive and hence they are mutually exclusive.

Hence, the required probability,

$$P(E) = P(E_1)P\left(\frac{E}{E_1}\right)+P(E_2)P\left(\frac{E}{E_2}\right)$$
$$+P(E_3)P\left(\frac{E}{E_3}\right)+P(E_4)P\left(\frac{E}{E_4}\right)$$
$$= \frac{1}{6}\times 1+\frac{1}{3}\times\frac{3}{4}+\frac{1}{5}\times\frac{3}{4}+\frac{3}{10}\times\frac{2}{3}$$
$$= \frac{1}{6}+\frac{1}{4}+\frac{3}{20}+\frac{1}{5}$$
$$= \frac{10+15+9+12}{60}$$
$$= \frac{46}{60} = \frac{23}{30}$$

125. Let E_1, E_2 denote the events that the coin shows a head, tail and A be the event that the noted number is either 7 or 8.

We have,

$$P(E_1) = \frac{1}{2} \text{ and } P(E_2) = \frac{1}{2}$$

Now, $7 \rightarrow \{(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)\}$
and $8 \rightarrow \{(2, 6), (6, 2), (3, 5), (5, 3), (4, 4)\}$

Thus, $P(A/E_1) = \frac{11}{36}$, $P(A/E_2) = \frac{2}{11}$

Hence, the required probability

$$P(A) = P(E_1)P(A/E_1) + P(E_2)P(A/E_2)$$
$$= \left(\frac{1}{2}\right)\left(\frac{11}{36}\right)+\left(\frac{1}{2}\right)\left(\frac{2}{11}\right)$$
$$= \frac{193}{792}$$

126. An urn contains m white and n black balls. A ball is drawn at random and is put into the urn along with k additional balls of the same color as that of the ball drawn. A ball is again drawn at random. What is the probability that the ball drawn now is white?
Baye's Theorem.

127. Hence, the required probability

$$= \frac{\frac{1}{2}\times\frac{5}{11}}{\frac{1}{2}\times\frac{3}{7}+\frac{1}{2}\times\frac{5}{11}}$$
$$= \frac{\frac{5}{11}}{\frac{3}{7}+\frac{5}{11}}$$
$$= \frac{35}{33+35} = \frac{35}{68}$$

128. Hence, the required probability

$$= \frac{\frac{1}{6}\times\frac{1}{100}}{\frac{1}{6}\times\frac{1}{100}+\frac{1}{3}\times\frac{3}{100}+\frac{1}{2}\times\frac{15}{100}}$$

$$= \frac{1}{1+6+45} = \frac{1}{52}$$

129. Hence, the required probability

$$= \frac{\frac{3}{5}\times\frac{1}{6}}{\frac{3}{5}\times\frac{1}{6}+\frac{2}{5}\times\frac{5}{6}}$$

$$= \frac{3}{3+5} = \frac{3}{8}$$

130. Hence, the required probability,

$$P(A\bar{B} \text{ or } \bar{A}B) = P(A\bar{B}) + P(\bar{A}B)$$

$$= P(A)P(\bar{B}) + P(\bar{A})P(B)$$

$$= \frac{3}{4}\times\frac{1}{6}+\frac{1}{4}\times\frac{5}{6}$$

$$= \frac{8}{24} = \frac{1}{3}$$

131. The probability of drawing a red ball = 1/6
The probability of A speaks truth = 2/3
The probability of B speaks truth = 4/5
Thus, the probability that A and B both speak truth

$$= \frac{1}{6}\times\frac{2}{3}\times\frac{4}{5}$$

$$= \frac{4}{45}$$

Now, the probability that A falsely asserts a ball as red.

$$= \frac{1}{3}\times\frac{1}{5} = \frac{1}{15}$$

Similarly, the probability that B falsely asserts a ball as red.

$$= \frac{1}{5}\times\frac{1}{5} = \frac{1}{25}$$

The probability that both A and B speak false

$$= \frac{5}{6}\times\frac{1}{15}\times\frac{1}{25}$$

$$= \frac{1}{450}$$

Hence, the required probability

$$= \frac{\frac{4}{45}}{\frac{4}{45}+\frac{1}{450}}$$

$$= \frac{\frac{4}{45}}{\frac{41}{450}} = \frac{40}{41}$$

132. Hence, the required probability

$$= \frac{\frac{1}{2}\times\frac{40}{100}\times\frac{90}{100}}{\frac{1}{2}\times\frac{60}{100}\times\frac{80}{100}+\frac{1}{2}\times\frac{40}{100}\times\frac{90}{100}}$$

$$= \frac{40\times 90}{60\times 80+40\times 90}$$

$$= \frac{3600}{4800+3600}$$

$$= \frac{36}{48+36}$$

$$= \frac{36}{84} = \frac{3}{7}$$

133. Let E_1, E_2 and E_3 be the events that the answer is guessed, copied and knows the answer, respectively and E be the event that the examinee answers correctly.

Given $P(E_1) = \frac{1}{3}, P(E_2) = \frac{1}{6}$

Here, E_1, E_2 and E_3 are exhaustive events.

Thus,

$$P(E_1) + P(E_2) + P(E_3) = 1$$

$$\Rightarrow \quad P(E_3) = 1 - P(E_1) - P(E_2)$$

$$= 1 - \frac{1}{3} - \frac{1}{6} = \frac{6-2-1}{6} = \frac{3}{6} = \frac{1}{2}$$

Now,

$P(E/E_1)$ = Probability of getting correct answer by guessing = 1/4

$P(E/E_2)$ = Probability of getting correct answer by copying = 1/8.

$P(E/E_3)$ = Probability of getting correct answer by knowing = 1.

Hence, the required probability, $P(E_3/E)$

$$= \frac{P(E_3)\cdot P(E/E_3)}{P(E_1)\cdot P(E/E_1)+P(E_2)\cdot P(E/E_2)+P(E_3)\cdot P(E/E_3)}$$

$$= \frac{\frac{1}{2}\times 1}{\frac{1}{3}\times\frac{1}{4}+\frac{1}{6}\times\frac{1}{8}+\frac{1}{2}\times 1} = \frac{24}{29}$$

134. Let E_1/E_2, denote the events that the black and red cards are lost and A denotes the event that the occurrence of 13 cards which are examined and found all red.

Obviously, $P(E_1) = \frac{1}{2} = P(E_2)$

Also $P(A/E_1) = \frac{^{26}C_{13}}{^{51}C_{13}}$ and $P(A/E_2) = \frac{^{25}C_{13}}{^{51}C_{13}}$

Hence the required probability,

$$P(E_1/A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2)}$$

$$= \frac{\frac{1}{2}\cdot\frac{^{26}C_{13}}{^{51}C_{13}}}{\frac{1}{2}\cdot\frac{^{26}C_{13}}{^{51}C_{13}} + \frac{1}{2}\cdot\frac{^{25}C_{13}}{^{51}C_{13}}}$$

$$= \frac{^{26}C_{13}}{^{26}C_{13} + {}^{25}C_{13}} = \frac{2}{2+1} = \frac{2}{3}$$

135. Let E be the event that an employee receive the letter and A be the event that employer received the reply.

$$P(E) = \frac{n-1}{n} \text{ and } P(\overline{E}) = \frac{1}{n}$$

$$P(A/E) = \frac{n-1}{n} \text{ and } P(A/\overline{E}) = 0$$

Now, $P(A) = P(E \cap A) + P(\overline{E} \cap A)$

$$= P(E)\cdot P(A/E) + P(\overline{E})\cdot P(A/\overline{E})$$

$$= \left(\frac{n-1}{n}\right)\cdot\left(\frac{n-1}{n}\right) + \frac{1}{n}\cdot 0$$

Hence the required probability,

$$P(E/\overline{A}) = \frac{P(E \cap \overline{A})}{P(\overline{A})}$$

$$= \frac{P(E) - P(E \cap A)}{P(\overline{A})}$$

$$= \frac{P(E) - P(E)\cdot P(A/E)}{P(\overline{A})}$$

$$= \frac{\left(\frac{n-1}{n}\right) - \left(\frac{n-1}{n}\right)\cdot\left(\frac{n-1}{n}\right)}{\frac{2n-1}{n^2}}$$

$$= \frac{n-1}{2n-1}$$

136. We have to find the probability of A being hit by B but not by C.
Thus,

$$P(BC'/A) = \frac{P(BC')\cdot P(A/BC')}{P(BC')\cdot P(A/BC') + P(B'C)\cdot P(A/B'C) + P(BC)\cdot P(A/BC) + P(B'C')\cdot P(A/B'C')}$$

$$= \frac{1\cdot\frac{1}{2}\cdot\frac{2}{3}}{1\cdot\frac{1}{2}\cdot\frac{2}{3} + 1\cdot\frac{1}{2}\cdot\frac{1}{3} + 1\cdot\frac{1}{2}\cdot\frac{1}{3} + 0\cdot\frac{1}{2}\cdot\frac{2}{3}}$$

$$= \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{6} + \frac{1}{6}} = \frac{1}{2}$$

137. Let E_1 be the event that the coin drawn is fair and E_2 be the event that the coin drawn is biased.

$$P(E_1) = \frac{m}{N} \text{ and } P(E_2) = \frac{N-m}{N}$$

Let A is the event that on tossing a coin the head appears first and then appears tail.

Now, $P(A) = P(E_1 \cap A) + P(E_2 \cap A)$

$$= P(E_1)\cdot P(A/E_1) + P(E_2)\cdot P(A/E_2)$$

$$= \frac{m}{N}\left(\frac{1}{2}\right)^2 + \left(\frac{N-m}{N}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)$$

Hence the required probability

$$P\left(\frac{E_1}{A}\right) = \frac{P(E_1 \cap A)}{P(A)} = \frac{P(E_1)\cdot P(A/E_1)}{P(A)}$$

$$= \frac{\frac{m}{N}\left(\frac{1}{2}\right)^2}{\frac{m}{N}\left(\frac{1}{2}\right)^2 + \frac{N-m}{N}\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)}$$

$$= \frac{9m}{m+8N}$$

138. Let E_i denotes the event that exactly i white balls are drawn.

We have $P(E_i) = \dfrac{^6C_i \times {}^{12}C_{6-i}}{^{18}C_6}$

Let E denotes the event that at least 4 white balls are drawn, then

$$E = E_4 \cup E_5 \cup E_6$$

Let A denotes the event that exactly one of the two balls next drawn is white.

Now, $P(A/E_4) = \dfrac{^2C_1 \times {}^{11}C_1}{^{12}C_2} = \dfrac{20}{66}$,

and $P(A/E_5) = \dfrac{^1C_1 \times {}^{11}C_1}{^{12}C_2} = \dfrac{1}{66}$, $P(A/E_6) = 0$

Thus, the required probability,

$$P(A/E) = \frac{P(A \cap E)}{P(E)}$$

$$= \frac{P(A \cap E_4) + P(A \cap E_5) + P(A \cap E_6)}{P(E)}$$

$$= \frac{P(E_4)P(A/E_4) + P(E_5)P(A/E_5) + P(E_6)P(A/E_6)}{P(E_4) + P(E_4) + P(E_4)}$$

$$= \frac{20 \times {}^6C_4 \times {}^{12}C_4 + 11 \times {}^6C_5 \times {}^{12}C_1}{66\left[{}^6C_4 \times {}^{12}C_2 + {}^6C_5 \times {}^{12}C_1 + 1\right]}$$

139. Let C, S, B and T denote the events that the person goes to the office by a car, a scooter, a bus and a train respec-

tively and let L denote the event that he reaches late in the office.

We have,

$$P(C)=\frac{1}{7},\ P(S)=\frac{2}{7},\ P(B)=\frac{3}{7},\ P(T)=\frac{1}{7}$$

$$P(L/C)=\frac{2}{9},\ P(L/S)=\frac{4}{9},\ P(L/B)=\frac{1}{9},\ P(L/T)=\frac{1}{9}$$

Thus, $P(L'/C)=\frac{7}{9},\ P(L'/S)=\frac{5}{9},$

$$P(L'/B)=\frac{8}{9}, P(L'/T)=\frac{8}{9}$$

Hence the required probability,

$$P(C/L')=\frac{P(C)P(L'/C)}{P(C)P(L'/C)+P(S)P(L'/S)+P(B)P(L'/B)+P(T)P(L'/T)}$$

$$=\frac{\frac{1}{7}\cdot\frac{2}{9}}{\frac{1}{7}\cdot\frac{7}{9}+\frac{2}{7}\cdot\frac{5}{9}+\frac{3}{7}\cdot\frac{8}{9}+\frac{1}{7}\cdot\frac{8}{9}}$$

$$=\frac{1}{7}$$

140. Let E_1, E_2, E_3, E_4 be the events of losing a card from spades, hearts, diamonds and clubs and E be the event of drawing 2 spades from the remaining cards.
Hence the required probability,

$$P(E_1/E)=\frac{P(E_1)P(E/E_1)}{P(E_1)P(E/E_1)+P(E_2)P(E/E_2)+P(E_3)P(E/E_3)+P(E_4)P(E/E_4)}$$

$$=\frac{\frac{^{13}C_1}{^{52}C_1}\times\frac{^{12}C_2}{^{51}C_2}}{\frac{^{13}C_1}{^{52}C_1}\times\frac{^{12}C_2}{^{51}C_2}+3\times\frac{^{13}C_1}{^{52}C_1}\times\frac{^{13}C_2}{^{51}C_2}}$$

$$=\frac{\frac{1}{4}\times\frac{66}{1275}}{\frac{1}{4}\times\frac{66}{1275}+3\times\frac{1}{4}\times\frac{78}{1275}}$$

$$=\frac{66}{66+78+78+78}$$

$$=\frac{66}{300}=\frac{11}{50}$$

141. Let T and L be the events that both A and B speak truth and lies and B the event that a ball is drawn from the bag.

Clearly $P(B)=\frac{1}{6}$ and $P(\overline{B})=\frac{5}{6}$

$$P(T/B)=\frac{3}{5}\times\frac{7}{10},\ P(L/\overline{B})=\frac{2}{5}\times\frac{3}{10}$$

Hence, the required probability

$$=\frac{\frac{1}{6}\times\frac{3}{5}\times\frac{7}{10}}{\frac{1}{6}\times\frac{3}{5}\times\frac{7}{10}+\frac{1}{6}\times\frac{2}{5}\times\frac{3}{10}}$$

$$=\frac{21}{21+6}=\frac{21}{27}=\frac{7}{9}$$

142. Let E_1, E_2 be the event that the letter came from CALCUTTA and TATANAGAR and A the event that two consecutive alphabets visible on the envelope are TA.

Clearly, $P(E_1)=\frac{1}{2}=P(E_2)$

and $P(A/E_1)=\frac{1}{7},\ P(A/E_2)=\frac{2}{8}=\frac{1}{4}$

Hence the required probability,

$$P(E_2/A)=\frac{P(E_2)P(A/E_2)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)}$$

$$=\frac{\frac{1}{2}\cdot\frac{1}{7}}{\frac{1}{2}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{7}}=\frac{4}{11}$$

143. Let E_1, E_2 be the event that the letter came from MAHARASTRA and MADRAS and A the event that two consecutive alphabets visible on the envelope are RA

Clearly, $P(E_1)=\frac{1}{2}=P(E_2)$

and $P(A/E_1)=\frac{2}{9},\ P(A/E_2)=\frac{1}{5}$

Hence the required probability,

$$P(E_2/A)=\frac{P(E_2)P(A/E_2)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)}$$

$$=\frac{\frac{1}{2}\times\frac{2}{9}}{\frac{1}{2}\times\frac{2}{9}+\frac{1}{2}\times\frac{2}{5}}$$

$$=\frac{\frac{2}{9}}{\frac{2}{9}+\frac{2}{5}}$$

$$=\frac{10}{10+9}$$

$$=\frac{10}{19}$$

144. A can hit a target 4 times in 5 shots, B can hit 3 times in 4 shots and C twice in 3 shots. They fire once each. If two of them hit. What is the chance that C has missed it?

144.

145. Let E_1 be the event that both A and B speak truth, E_2 be the event that both A and B tell a lie, E be the event that A and B agree in a certain statement, C be the event that A speaks the truth and D be the event that B speaks truth.

Thus, $E_1 = C \cap D$, $E_2 = C' \cap D'$

Therefore, $P(E_1) = P(C \cap D)$
$= P(C)P(D) = xy$

and $P(E_2) = P(C' \cap D')$
$= P(C')P(D')$
$= (1 - x)(1 - y)$
$= 1 - x - y + xy$

Now, $P\left(\frac{E}{E_1}\right)$ = Probability that A and B will agree when both of them speak the truth
$= 1$

and $P\left(\frac{E}{E_2}\right)$ = Probability that A and B will agree when both of them tell a lie
$= 1$

Hence, the required probability,

$$P\left(\frac{E_1}{E}\right) = \frac{P(E_1)\cdot P\left(\frac{E}{E_1}\right)}{P(E_1)\cdot P\left(\frac{E}{E_1}\right) + P(E_2)\cdot P\left(\frac{E}{E_2}\right)}$$

$$= \frac{xy\cdot 1}{xy\cdot 1 + (1 - x - y + xy)\cdot 1}$$

$$= \frac{xy}{1 - x - y + 2xy}$$

146. $P(E_1) = P(1 \text{ or } 3 \text{ on the die}) = \frac{2}{6} = \frac{1}{3}$

$P(E_2) = P(1 \text{ or } 3 \text{ not on the die}) = \frac{4}{6} = \frac{2}{3}$

Let B be = the ball selected is black.

Thus, $P\left(\frac{B}{E_1}\right) = \frac{3}{7}$, $P\left(\frac{B}{E_2}\right) = \frac{4}{7}$

Therefore, the probability of a black ball

$$= \frac{1}{3}\times\frac{3}{7} + \frac{2}{3}\times\frac{4}{7}$$

$$= \frac{11}{21}$$

Hence, the required probability

$$= \frac{\frac{2}{3}\times\frac{4}{7}}{\frac{1}{3}\times\frac{3}{7} + \frac{2}{3}\times\frac{4}{7}}$$

$$= \frac{8}{11}$$

147. Given S = {HH, HT, TH, TT}
and $R_X = \{0, 1, 2\}$

Now, $P(X = 0) = \frac{1}{4}$

$P(X = 1) = \frac{2}{4}$

and $P(X = 2) = \frac{1}{4}$

Hence, the probability distribution is

X:	0	1	2
$P(X)$:	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

148. Here, $R_X = \{0, 1, 2, 3, 4\}$

Now, $P(X = 0) = \frac{^6C_4}{^{10}C_4}$

$P(X = 1) = \frac{^4C_1 \times {}^6C_3}{^{10}C_4}$

$P(X = 2) = \frac{^4C_2 \times {}^6C_2}{^{10}C_4}$

$P(X = 3) = \frac{^4C_3 \times {}^6C_1}{^{10}C_4}$

and $P(X = 4) = \frac{^4C_4}{^{10}C_4}$

Hence, the probability distribution is

X:	0	1	2	3	4
$P(X)$:	$\frac{^6C_4}{^{10}C_4}$	$\frac{^4C_1 \times {}^6C_3}{^{10}C_4}$	$\frac{^4C_2 \times {}^6C_2}{^{10}C_4}$	$\frac{^4C_3 \times {}^6C_1}{^{10}C_4}$	$\frac{^4C_4}{^{10}C_4}$

149. Here, $R_X = \{0, 1, 2\}$

Now, $P(X = 0) = \frac{5}{6}\times\frac{5}{6} = \frac{25}{36}$

$P(X = 1) = \frac{1}{6}\times\frac{5}{6} + \frac{5}{6}\times\frac{1}{6} = \frac{10}{36}$

and $P(X = 2) = \frac{1}{6}\times\frac{1}{6} = \frac{1}{36}$

Hence, the required probability distribution

X:	0	1	2
$P(X)$:	$\frac{25}{36}$	$\frac{10}{36}$	$\frac{1}{36}$

150. Here, $R_X = \{0, 1, 2\}$

Now, $P(X = 0) = \frac{12}{13}\times\frac{12}{13} = \frac{144}{169}$

$P(X = 1) = \frac{1}{13}\times\frac{12}{13} + \frac{12}{13}\times\frac{1}{13} = \frac{24}{169}$

and $P(X = 2) = \frac{1}{13}\times\frac{1}{13} = \frac{1}{169}$

Hence, the probability distribution is

X:	0	1	2
$P(X)$:	$\frac{144}{169}$	$\frac{24}{169}$	$\frac{1}{169}$

151. Here, $R_X = \{0, 1, 2, 3\}$

Now, $P(X=0) = \frac{^{48}C_3}{^{52}C_3}$

$P(X=1) = 3 \times \frac{^{4}C_1 \times {}^{48}C_2}{^{52}C_3}$

$P(X=2) = 3 \times \frac{^{4}C_2 \times {}^{48}C_1}{^{52}C_3}$

and $P(X=3) = \frac{^{4}C_3}{^{52}C_3}$

Hence, the required probability distribution is

X:	0	1	2	3
$P(X)$:	$\frac{^{48}C_3}{^{52}C_3}$	$\frac{3 \times {}^{4}C_1 \times {}^{48}C_2}{^{52}C_3}$	$\frac{3 \times {}^{4}C_2 \times {}^{48}C_1}{^{52}C_3}$	$\frac{^{4}C_3}{^{52}C_3}$

152. Here, $R_X = \{0, 1, 2, 3, 4\}$

Now, $P(X=0) = \left(\frac{4}{5}\right)^4$

$P(X=1) = 4 \times \frac{1}{5} \times \left(\frac{4}{5}\right)^3$

$P(X=2) = 6 \times \left(\frac{1}{5}\right)^2 \times \left(\frac{4}{5}\right)^2$

$P(X=3) = 4 \times \left(\frac{1}{5}\right)^3 \times \left(\frac{4}{5}\right)$

and $P(X=4) = \left(\frac{1}{5}\right)^4$

Hence, the required probability distribution is

X:	0	1	2	3	4
$P(X)$:	$\frac{256}{625}$	$\frac{256}{625}$	$\frac{96}{625}$	$\frac{16}{625}$	$\frac{1}{625}$

154. Let

$2P(X = x_1) = 3P(X = x_2) = P(X = x_3) = 5P(X = x_4) = k$

As we know that,

$p_1 + p_2 + p_3 + p_4 = 1$

$\Rightarrow \quad \frac{k}{2} + \frac{k}{3} + k + \frac{k}{5} = 1$

$\Rightarrow \quad 15k + 10k + 30k + 12k = 30$

$\Rightarrow \quad 67k = 30$

$\Rightarrow \quad k = \frac{30}{67}$

Hence, the required probability distribution is

X:	x_1	x_2	x_3	x_4
$P(X)$:	$\frac{15}{67}$	$\frac{10}{67}$	$\frac{30}{67}$	$\frac{6}{67}$

155. Here $n = 6$, $p = \frac{1}{6}, q = 1 - \frac{1}{6} = \frac{5}{6}$

(i) $P(X \geq 5) = P(X = 5) + P(X = 6)$

$= {}^6C_5 p^5 q^1 + {}^6C_6 p^6 q^0$

$= 6p^5 q + p^6 = p^5(6p + q) = \left(\frac{1}{6}\right)^5 \times \left(\frac{11}{6}\right)$

(ii) $P(X \leq 5) = 1 - P(X) > 5)$

$= 1 - P(X = 6)$

$= 1 - {}^6C_6 p^6 q^0 = 1 - \left(\frac{1}{6}\right)^5$

(iii) $P(X = 5) = {}^6C_5 p^5 q^1 = 6 \times \left(\frac{1}{6}\right)^5 \times \left(\frac{5}{6}\right)$

(iv) $P(X = 0) = {}^6C_0 p^0 q^6 = 6 \times \left(\frac{5}{6}\right)^6$

156. Here, $n = 6$, $p = \frac{1}{2}, q = 1 - \frac{1}{2} = \frac{1}{2}$

(i) $P(X = 5) = {}^6C_5 p^5 q^1 = 6 \times \left(\frac{1}{2}\right)^6$

(ii) $P(X \geq 5) = P(X = 5) + P(X = 6)$

$= {}^6C_5 p^5 q^1 + {}^6C_6 p^6$

$= 6 \times \left(\frac{1}{2}\right)^6 + \left(\frac{1}{2}\right)^6$

$= 7 \times \left(\frac{1}{2}\right)^6$

(iii) $P(X \leq 5)$

$= 1 - P(X > 5)$

$= 1 - P(X = 6)$

$= 1 - \left(\frac{1}{2}\right)^6$

(iv) $P(X \geq 1) = 1 - P(X < 1)$

$= 1 - P(X = 0)$

$= 1 - \left(\frac{1}{2}\right)^6$

(v) $P(X = 0) = \left(\frac{1}{2}\right)^6$

157. Here, $n = 7$, $p = \frac{1}{6}, q = \frac{5}{6}$

(i) $P(X = 0) = \left(\frac{5}{6}\right)^7$

(ii) $P(X=6) = {}^7C_6 \times \left(\frac{1}{6}\right)^6 \times \left(\frac{5}{6}\right)$

(iii) $P(X \geq 6) = P(X=6) + P(X=7)$

$= {}^7C_6 \times \left(\frac{1}{6}\right)^6 \times \left(\frac{5}{6}\right) + \left(\frac{1}{6}\right)^7$

(iv) $P(X \leq 6) = 1 - P(X > 6)$

$= 1 - P(X=7)$

$= 1 - \left(\frac{1}{6}\right)^7$

158. Here $S = \{(1, 1), (1, 2), ..., (6, 6)\}$ and $n = 5$.
Now, $7 \to \{(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)\}$

Thus, $p = \frac{6}{36} = \frac{1}{6}, q = 1 - \frac{1}{6} = \frac{5}{6}$

Hence, the required probability,

$P(X=4) = {}^7C_4\ p^4\ q^3 = 35 \times \left(\frac{1}{6}\right)^4 \times \left(\frac{5}{6}\right)^3$

159. Here, $n = 5$, $p = \frac{1}{5}, q = 1 - \frac{1}{5} = \frac{4}{5}$.

Hence, the required probability,

$P(X=4) = {}^5C_4\, P^4 q^1$

$= 5 \times \left(\frac{1}{5}\right)^4 \times \left(\frac{4}{5}\right) = \frac{4}{625}$

160. Here, $n = 4$.
The possible outcome of doublet is $\{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$

Thus, $p = \frac{6}{36} = \frac{1}{6}, q = 1 - \frac{1}{6} = \frac{5}{6}$

Hence, the required probability,

$P(X \geq 2) = 1 - P(X < 2)$

$= 1 - \{P(X = 1) + P(X = 0)\}$

$= 1 - ({}^4C_1\, p^1\, q^3 + C_0\, p^0\, q^4\}$

$= 1 - (4pq^3 + q^3)$

$= 1 - q^3(4p + q)$

$= 1 - \frac{3}{2}\left(\frac{5}{6}\right)^3$

$= 1 - \frac{125}{144} = \frac{19}{144}$

161. Here, $p = \frac{1}{2}, q = 1 - \frac{1}{2} = \frac{1}{2}$

Hence, the required probability,

$P(X = 1 \text{ or } X = 3 \text{ or } X = 5 \text{ or } \ldots \text{ to } \infty)$

$= P(X = 1) + P(X = 3) + P(X = 5) + \ldots \text{ to } \infty)$

$= {}^nC_1\, p^1\, q^{n-1} + {}^nC_3\, p_3\, q^{n-3} + {}^nC_5\, p^5\, q^{n-5} + \ldots$

$= {}^nC_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^{n-1} + {}^nC_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{n-3}$

$+ {}^nC_1 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^{n-5} + {}^nC_1 \left(\frac{1}{2}\right)^7 \left(\frac{1}{2}\right)^{n-7} + \ldots$

$= \left(\frac{1}{2}\right)^n ({}^nC_1 + {}^nC_3 + {}^nC_5 + {}^nC_7 + \ldots \text{to } \infty)$

$= \left(\frac{1}{2}\right)^n \times 2^{n-1} = \frac{1}{2}$

162. Here, $n = 10$, $p = 1 - \frac{1}{10} = \frac{9}{10}, q = \frac{1}{10}$

Hence, the required probability,

$P(X = 7) = {}^{10}C_7 p^7 q^3$

$= {}^{10}C_7 \times \left(\frac{9}{10}\right)^7 \times \left(\frac{1}{7}\right)^3$

163. Here, $n = 5$, $p = \frac{3}{4}, q = 1 - \frac{3}{4} = \frac{1}{4}$.

Hence, the required probability,

$P(X \geq 3) = P(X = 3) + P(X = 4) + P(X = 5)$

$= {}^5C_3\, p^3\, q^2 + {}^5C_4\, p^4\, q + {}^5C_5\, \mathrm{p}^5\, q^0$

$= 10p^3\, q^2 + 5p^4\, q + p^5\, q^0$

$= p^3(10q^2 + 5pq + p^2)$

$= \frac{27}{64}\left(\frac{10}{16} + \frac{15}{16} + \frac{3}{16}\right)$

$= \frac{27}{64} \times \frac{28}{64}$

$= \frac{27}{64} \times \frac{7}{16} = \frac{189}{1024}$

164. Here, $n = 10$, $p = \frac{2}{7}, q = 1 - \frac{2}{7} = \frac{5}{7}$

Hence, the required probability,

$P(X \geq 8) = P(X = 8) + P(X = 9) + P(X = 10)$

$= {}^{10}C_8\, p^8\, q^2 + {}^{10}C_9\, p^9\, q^1 + {}^{10}C_{10}\, p^{10}\, q^9$

$= p^8(45q^2 + 10pq + p^2)$

$= \left(\frac{2}{7}\right)^9 \times \left(\frac{45 \times 25}{49} + \frac{100}{49} + \frac{4}{49}\right)$

$= \left(\frac{2}{7}\right)^9 \times \left(\frac{1129}{49}\right)$

165. Here, $n = 5$, $p = \frac{1}{2}, q = 1 - \frac{1}{2} = \frac{1}{2}$

Hence, the required probability,

$P(X \geq 2) = 1 - P(X < 2)$

$= 1 - \{P(X = 1) + P(X = 0)\}$

$= 1 - \{({}^5C_1\, p^1\, q^4 + {}^5C_0\, p^0\, q^5)\}$

$$=1-\left(\frac{5}{32}+\frac{1}{32}\right)$$

$$=1-\frac{6}{32}=\frac{26}{32}=\frac{13}{32}$$

166. Hence, the required probability,

$$P(X=r)={}^nC_r\,p^r\,q^{n-r}$$

$$={}^nC_r\times\left(\frac{1}{m}\right)^r\times\left(\frac{m-1}{m}\right)^{n-r}$$

$$={}^nC_r\times\frac{(m-1)^{n-r}}{m^n}$$

167. Hence, the required probability,

$$P(X=5 \text{ or } X=6)=P(X=5)+\text{ or } P(X=6)$$

$$={}^{11}C_5p^5q^6+{}^{11}C_6p^6q^5$$

$$={}^{11}C_5p^5q^6+{}^{11}C_6p^6q^5$$

$$={}^{11}C_5p^5q^5\,(p+q)$$

$$={}^{11}C_5p^5q^5$$

$$={}^{11}C_5\times\left(\frac{2}{5}\right)^5\times\left(\frac{3}{5}\right)^5$$

168. It is given that

$$P(X\geq 1)>\frac{2}{3}$$

$$\Rightarrow\quad 1-P(X<1)>\frac{2}{3}$$

$$\Rightarrow\quad 1-P(X=0)>\frac{2}{3}$$

$$\Rightarrow\quad 1-{}^nC_0p^0q^n>\frac{2}{3}$$

$$\Rightarrow\quad 1-q^n>\frac{2}{3}$$

$$\Rightarrow\quad q^n<1-\frac{2}{3}$$

$$\Rightarrow\quad \left(\frac{3}{4}\right)^n<\frac{1}{3}$$

Clearly, $n=4$.

170. Probability that any one one thing is received by a man

$$=\frac{a}{a+b}=p\,(\text{say})$$

Probability that any one thing is received by a woman

$$=\frac{b}{a+b}=q\,(\text{say})$$

Clearly, $p+q=1$

Out of m things, if r received by men, the rest $(n-r)$ will be received by women.

Hence, the required probability,

$$P(1)+P(3)+P(5)+\ldots$$

$$={}^mC_1\,pq^{m-1}+{}^mC_3\,p^3\,q^{m-3}+{}^mC_5p^5\,q^{m-5}+\ldots$$

$$=\frac{1}{2}\left[(p+q)^m-(p-q)^m\right]$$

$$=\frac{1}{2}\left[1-(p-q)^m\right]$$

$$=\frac{1}{2}\left[1-\left(\frac{b-a}{a+b}\right)^m\right]$$

$$=\frac{1}{2}\left[\frac{(b+a)^m-(b-a)^m}{(a+b)^m}\right]$$

171. Let X be the number of games A wins against B and p be the probability A wins the game against B.

Now,

$$P(A \text{ wins the match})=P(X\geq 2)$$

$$=P(X=2)+P(X=3)$$

$$=3C^2p^2q+3C_3p^3$$

$$=3p^2q+p^3$$

$$=3\cdot(0.4)^2\cdot(0.6)+(0.4)^3$$

$$=(0.4)^2\,(3\cdot(0.6)+(0.4))$$

$$=(0.16)\cdot(2.2)=0.352$$

Also,

$$P(A \text{ wins the match})=P(X\geq 3)$$

$$=P(X=3)+P(X=4)+P(X=5)$$

$$=5C_3p^3q^2+5C_4p^4q+{}^5C_5p^5$$

$$=10p^3q^2+5p^4q+p^5$$

$$=p^3(10q^2+5pq+p^2)$$

$$=(0.4)^3(10(0.6)^2+5(0.24)+(0.4)^2)$$

$$=(0.4)^3(3.6+1.2+.16)$$

$$=(0.064)(4.96)$$

$$=0.31744$$

It shows that, best of 3 games 'option is better'.

172. Here $7\to(1,6),(6,1),(2,5),(5,2),(3,4),(4,3)$

Now, the probability of getting $7=p=\frac{1}{6}$

and the probability of not getting $7=q=\frac{5}{6}$

It is given that,

$$P(X\geq 1)>\frac{95}{100}$$

$$\Rightarrow\quad 1-P(X<1)>\frac{19}{20}$$

$$\Rightarrow\quad 1-P(X=0)>\frac{19}{20}$$

$$\Rightarrow\quad 1-{}^nC_0p^0q^n>\frac{19}{20}$$

$$\Rightarrow\quad 1-q^n>\frac{19}{20}$$

$$\Rightarrow\quad 1-\left(\frac{5}{6}\right)^n>\frac{19}{20}$$

$$\Rightarrow\quad \left(\frac{5}{6}\right)^n<1-\frac{19}{20}$$

$$\Rightarrow \quad \left(\frac{5}{6}\right)^n < \frac{1}{20}$$

Clearly, n is 4.

173. Let X be the event of getting 1, 3 or 5 in a single throw of a die.

$$P(X) = \frac{3}{6} = \frac{1}{2}$$

Hence, the required probability,

$$P(X = 0) + P(X = 1) + P(X = 3) + \ldots$$

$$= {}^{2n+1}C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^{2n+1} + {}^{2n+1}C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^{2n} + \ldots + {}^{2n+1}C_n\left(\frac{1}{2}\right)^n\left(\frac{1}{2}\right)^{n+1}$$

$$= \frac{1}{2^{2n+1}}({}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + \ldots + {}^{2n+1}C_n)$$

$$= \frac{2^{2n}}{2^{2n+1}}$$

$$= \frac{1}{2}$$

174. Here, $n = 10$, $p = 5/6$ and $q = 1/6$

Hence, the required probability,

$P(X < 2) = P(X = 0) + P(X = 1)$

$$= {}^{10}C_0 p^0 q^{10} + {}^{10}C_1 p^1 q^9$$

$$= q^{10} + 10p^1 q^9$$

$$= q^9(q + 10p)$$

$$= \left(\frac{1}{6}\right)^9\left(\frac{1}{6} + \frac{50}{6}\right)$$

$$= \left(\frac{1}{6}\right)^9 \times \frac{51}{6}$$

$$= \left(\frac{1}{6}\right)^9 \times \frac{17}{2}$$

$$= \frac{17}{2 \times 6^9}$$

175. Here $n = 4$, $p = 1/3$ and $q = 2/3$

Now, mean = $np = 4/3$

Variance = $npq = 8/9$

Standard Deviation = $+\sqrt{\text{Var}(X)}$

$$= +\sqrt{\frac{8}{9}}$$

$$= \frac{2\sqrt{2}}{3}$$

176. Here, $np = 4$, $npq = 4/3$

Thus, $p = 2/3$, $q = 1/3$, $n = 6$

Hence, the required probability = $P(X \geq 1)$

$$= 1 - P(X < 1)$$

$$= 1 - P(X = 0)$$

$$= 1 - {}^{10}C_0 p^0 q^6$$

$$= 1 - \left(\frac{1}{3}\right)^6$$

$$= \left(\frac{3^6 - 1}{3^6}\right)$$

177. Here, $n = 5$

It is given that,

$$np + npq = 1.8$$

$$\Rightarrow \quad np(1 + p) = 1.8$$

$$\Rightarrow \quad 5p(1 + 1 - p) = \frac{9}{5}$$

$$\Rightarrow \quad 25p(2 - p) = 9$$

$$\Rightarrow \quad 50p - 25p^2 - 9 = 0$$

$$\Rightarrow \quad -50p + 25p^2 + 9 = 0$$

$$\Rightarrow \quad 25p^2 - 50p + 9 = 0$$

$$\Rightarrow \quad 25p^2 - 5p - 45p + 9 = 0$$

$$\Rightarrow \quad 5p(5p - 1) - 9(5p - 1) = 0$$

$$\Rightarrow \quad (5p - 9)(5p - 1) = 0$$

$$\Rightarrow \quad p = \frac{1}{5}, \frac{9}{5}$$

$$\Rightarrow \quad p = \frac{1}{5}$$

Thus, $q = \dfrac{4}{5}$

Hence, the binomial distribution,

$$P(X = r) = {}^nC_r\, p^r\, q^{n-r}$$

$$= {}^6C_r\left(\frac{1}{5}\right)^r\left(\frac{4}{5}\right)^{6-r},$$

where $r = 0, 1, 2, 3, 4, 5, 6$.

178. Let A be the event in the call taking place in the last 20 minutes of the hour.

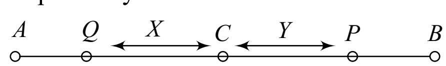
Let us represent the sample space as a line segment of length 20.

Hence the required probability

$$= \frac{\text{Measure of specified part of the region}}{\text{Measure of the whole region}}$$

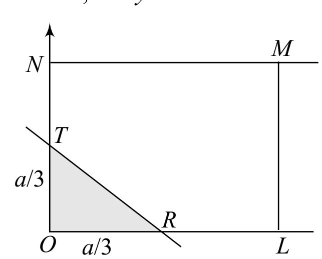
$$= \frac{20}{60} = \frac{1}{3}$$

179. Consider two points P and Q on the segments CB and CA respectively.



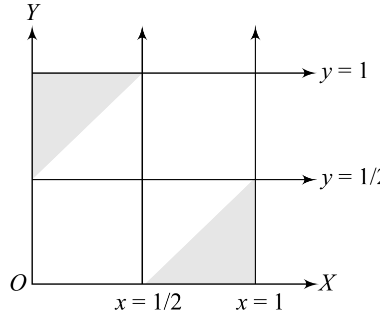
Let $CP = y$ and $CQ = x$ so that

$$0 \leq x \leq a/2,\ 0 \leq y \leq a/2$$



We are interested in the event $x + y < a/3$.

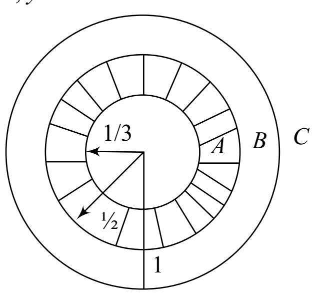
Here, first we make a square of side $a/2$ and its sub-region triangle with base $a/3$ and height is also $a/3$.



Thus, the required probability $= \dfrac{\frac{1}{2}\left(\frac{a}{3}\right)\left(\frac{a}{3}\right)}{\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)} = \dfrac{2}{9}$

180. Let the two numbers be x and y such that

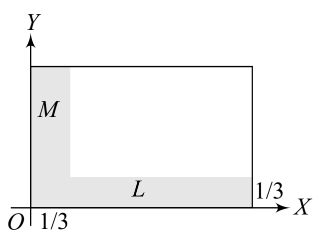
$$0 \le x, y \le 1.$$



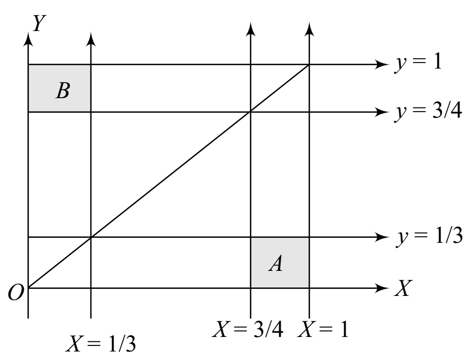
Now, the probability of

$$P\left(|x - y| > \frac{1}{2}\right) = \frac{\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}}{1\times 1} = \frac{1}{4}$$

181. Let us consider two numbers be x and y where $0 \le x, y \le 1$, so that the geometric configuration is a square.



Let $M = \max. \{x, y\}$ and $m = \min. \{x, y\}$



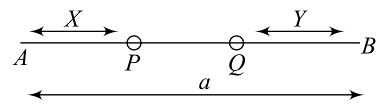
Hence the required probability,

$$P\left(\frac{M \ge 3/4}{m \le 1/3}\right) = \frac{P(M \ge 3/4, m \le 1/3)}{P(m \le 1/3)}$$

$$= \frac{2\cdot\frac{1}{3}\cdot\frac{1}{4}}{\frac{1}{3}+\frac{1}{3}-\frac{1}{9}} = \frac{\frac{1}{6}}{\frac{5}{9}} = \frac{3}{10}$$

182. Let AB be the straight line divided at P and Q such that $AB = a$.

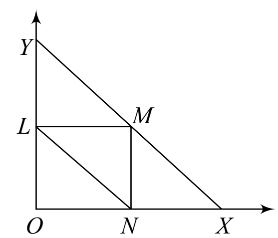
Let $AP = x$ and $BQ = y$



Now in favourable cases, we must have,

$$x < a/2, y < a/2$$

and $\quad x + y > a/2$



Hence, the required probability

$$= \frac{\frac{1}{2}\cdot\frac{a}{2}\cdot\frac{a}{2}}{\frac{1}{2}\cdot a\cdot a} = \frac{1}{4}$$

183 Let two numbers be x and y such that

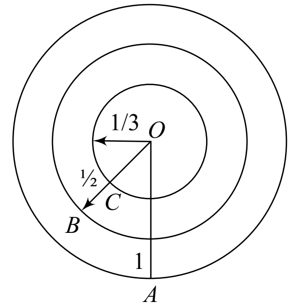
$$x + y = 24.$$

For extremum, it is clear that, the greatest value of xy is $(12)^2$

Hence the required probability,

$$P(xy \ge 108) = P(x(24 - x) \ge 108)$$
$$= \mathrm{P}((x - 18)(x - 6) \le 0)$$
$$= P(6 \le x \le 18)$$
$$= \frac{12}{24} = \frac{1}{2}$$

184. Let the probability of a shot falling in the region C of radius 1/3 is



$$p_1 = \frac{\pi\cdot(1/3)^2}{\pi\cdot(1)^2} = \frac{1}{9}$$

the probability of a shot falling in the region B of radius 1/2 is

$$p_2 = \frac{\pi\cdot(1/2)^2 - \pi(1/3)^2}{\pi\cdot(1)^2} = \frac{5}{36}$$

and the probability of a shot falling in the region A of radius 1 is

$$p_3 = \frac{\pi\cdot(1)^2 - \pi(1/2)^2}{\pi\cdot(1)^2} = \frac{3}{4}$$

Hence the required probability is

$$= P(p_1 \cdot p_2 \cdot p_3 \times 3!) = \frac{6\times 5}{432} = \frac{5}{72}.$$

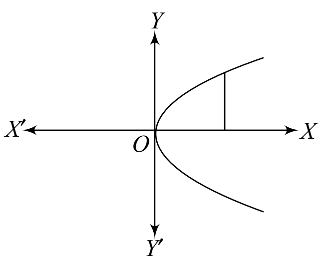
185. Let A be the event that the length of the interval in which the given equation has real roots and S be the sample space, i.e. the length of the interval [0, 5].
Since roots are real,

$$D \ge 0 \quad \Rightarrow \quad p^2 - 4.1.\frac{(p+2)}{4} \ge 0$$

$\Rightarrow \quad p^2 - p - 2 \ge 0$
$\Rightarrow \quad (p-2)(p+1) \ge 0$
$\Rightarrow \quad p \le -1, p \ge 2$
Thus, $A = [2, 5]$, since $p \in [0, 5]$.

Hence the required probability $= \dfrac{n(A)}{n(S)} = \dfrac{3}{5}$.

186. Consider a point with co-ordinates (x, y) corresponds to the pair of numbers x and y.
The sample space is a square whose sides are unit segments to the co-ordinate axes.
The region, whose set of points corresponds to the outcome favourable to the event $y^2 \le x$ is bounded by the graphs of the function $y = 0$, $x = 1$ and $y^2 = x$.



Hence the required probability $= \displaystyle\int_0^1 \sqrt{x}\, dx = \left(\frac{2}{3}x^{3/2}\right)_0^1 = \frac{2}{3}.$

187. Let A and B arrive at the place of their meeting x and y mins. after 11 am.
As per the condition, their meeting is possible only when $|x - y| \le 20$.
Hence the required probability

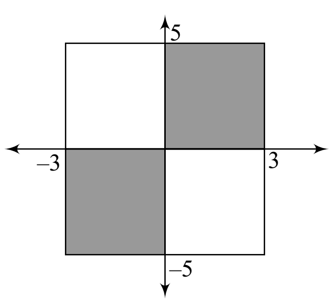
$$= \frac{60\times 60 - \frac{1}{2}\times 40\times 40 - \frac{1}{2}\times 40\times 40}{60\times 60}$$

$$= \frac{2000}{3600} = \frac{5}{9}$$

188. Hence, the required probability

$$= \frac{15+15}{6\times 10}$$

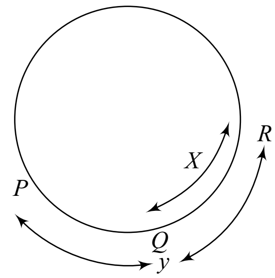
$$= \frac{30}{60} = \frac{1}{2}$$



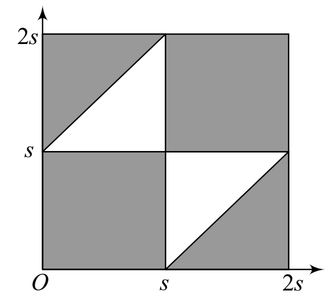
189. Clearly, $x \in [-2, 3]$
Now, the solution set of $\dfrac{(x+1)(x-2)}{(x-4)} \ge 0$ is $x \in [-1, 2]$.

Hence, the required probability $= \dfrac{3}{5}$.

190. Let the length of the circumference is $2s$.
Let x denotes the clockwise arc length of PQ and let y denotes the clockwise arc length of PR.
Thus, $0 < x < 2s$, $0 < y < 2s$ …(i)



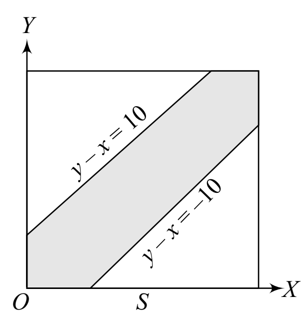
Let A denotes the subset of s for which any of the following conditions hold.
(i) $x, y < s$
(ii) $x < s$ and $y - x > s$
(iii) $x, y > s$
(iv) $y < s$ and $x - y > s$



Thus A consists of these points for which P, Q, and R lie on a semi-circle.

Thus, $p = \dfrac{\text{Area of } A}{\text{Area of } S} = \dfrac{3s^2}{4s^2} = \dfrac{3}{4}$.

191. The sample space is
$S = \{(x, y) : 0 \le x, y \le 30\}$
Let $A = \{(x, y) : |x - y| \le 10\}$



Hence, the required probability

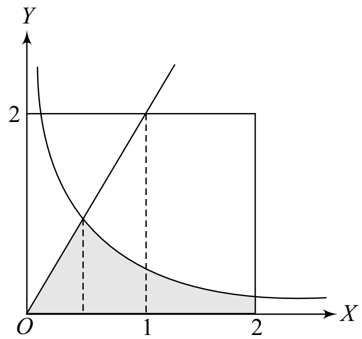
$$= \frac{30\times 30 - \frac{1}{2}\times 20\times 20 - \frac{1}{2}\times 20\times 20}{30\times 30}$$

$$= \frac{900 - 400}{900}$$

$$= \frac{500}{900}$$

$$= \frac{5}{9}$$

193. Clearly, $0 < x, y \le 2$.



Let A be the area of the shaded part

Now, $A = \int_0^{\frac{1}{\sqrt{2}}} 2x\,dx + \int_{\frac{1}{\sqrt{2}}}^{2} \frac{dx}{x}$

$= (x^2)_0^{\frac{1}{\sqrt{2}}} + (\log|x|)_{\frac{1}{\sqrt{2}}}^{2}$

$= \frac{1}{2} + \log 2 - \log\left(\frac{1}{\sqrt{2}}\right)$

$= \frac{1}{2} + \log 2 + \log(\sqrt{2})$

$= \frac{1}{2} + \log 2 + \frac{1}{2}\log(2)$

$= \frac{1}{2} + \frac{3}{2}\log 2$

Hence, the required probability,

$$P(A) = \frac{\frac{1}{2} + \frac{3}{2}\log 2}{4}$$

$$= \left(\frac{1 + 3\log 2}{8}\right)$$

194. Let two friends A and B arrive at the place of their meeting x and y mins. after 2 p.m.
As per the condition, their meeting is possible only when $|x - y| \le 15$.
Hence the required probability

$$= \frac{60 \times 60 - \frac{1}{2} \times 45 \times 45 - \frac{1}{2} \times 45 \times 45}{60 \times 60}$$

$$= \frac{3600 - 2025}{3600}$$

$$= \frac{1575}{3600}$$

$$= \frac{63}{144}$$

$$= \frac{7}{16}$$

Level III

(Problems For JEE-Advanced)

1. $15 \to (6, 6, 3) = \frac{3!}{2!} = 3$
$\to (6, 4, 5) = 3! = 6$
$\to (5,\ 5, 5) = \frac{3!}{3!} = 1$
Total possible ways to get sum 15 = 10
Hence, the required probability $= \frac{10}{216} = \frac{5}{108}$

2. Total number of aces = 4
and total number of face cards = 12
Hence, the required probability $= \frac{4}{52} \times \frac{12}{51}$
$= \frac{4}{13 \times 17} = \frac{4}{221}$

3. Required probability
$= P(GBB \text{ or } BGB \text{ or } BBG)$
$= P(GBB) + P(BGB) + P(BBG)$
$= P(G)P(B)P(B)$ or $P(B)P(G)P(B)$
$+ P(B)P(B)P(G)$
$= \frac{3}{4} \times \frac{2}{4} \times \frac{3}{4} + \frac{1}{4} \times \frac{2}{4} \times \frac{3}{4} + \frac{1}{4} \times \frac{2}{4} \times \frac{1}{4}$
$= \frac{18}{64} + \frac{6}{64} + \frac{2}{64}$
$= \frac{26}{64}$
$= \frac{13}{32}$

4. Let F, G, B denote respectively the events that filament, glass casings, bases are defective
Then $P(G) = 0.02$, $P(G) = 0.03$, $P(B) = 0.01$
Hence, the required probability, that selected bulb, at random, is defective,
$1 - P(\bar{F} \cap \bar{G} \cap \bar{B}) = 1 - P(\bar{F})P(\bar{G})P(\bar{B})$
$= 1 - (0.98)(0.97)(0.99)$
$= 1 - 0.94 = 0.6.$

5 Let a, b, c are in AP, where $a < b < c$
Then $2b = a + c$
Since b is an integer, then a and c are also integers.
Thus, a and c must be both odd or even.
Here out of 21 tickets numbers, 11 are odd no and 10 are even numbers tickets.
Hence, the required probability,

$$\frac{{}^{11}C_2 + {}^{10}C_2}{{}^{21}C_3} = \frac{55 + 45}{7 \times 10 \times 19}$$

$$= \frac{100}{7 \times 10 \times 19}$$

$$= \frac{10}{7 \times 19} = \frac{10}{133}$$

6. Total numbers of five-digit numbers that can be formed with the help of given digits = (5)! = 120.
   The five-digit numbers which is to be divisible by 4 must have 2 at the ten's position and 4 is in the unit position.
   Thus, total number of ways = (3)! × (1)! = 6
   Hence, the required probability $= \frac{6}{120} = \frac{1}{20}$.
7. The number of ways in which 5 persons can leave at different floors = $^7P_5$.
   The number of ways in which each of the five persons may leave the lift cabin at any of the seven floors except the ground floor = 7^5
   Hence, the required probability,
   $$\frac{^7P_5}{7^5} = \frac{(7)!}{(2)! \times 7^5} = \frac{30 \times 12}{49 \times 49}$$
   $$= \frac{40}{539}$$
8. Hence, the required probability
   = P(he shoots 10 – times)
   = $(0.3)^{10}$
9. Total 10-digit numbers, which are divisible by 4, if the last two digits are divisible by 4.
   = 7 × (7)! × 5 × 1 + (8)! × 4 × 1 + 7 × (7)! × 5 × 1 + 7 × (7)! × 3 × 1 + (8)! × 1 × 1
   = 176400+349440 + 176400 + 105840 + 40320
   = 848400
   Hence, the required probability
   $$= \frac{848400}{9 \cdot (9)!} = 0.2597 = 0.26$$
10. Hence, the required probability,
   $$P(\bar{M} \cap \bar{W}) = P(\bar{M}) \cdot P(\bar{W})$$
   $$= \frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$$
11. $P(\text{spade card}) = \frac{13}{52} = \frac{1}{4}$
   and $P(\text{not a spade card}) = \frac{3}{4}$.
   Chance of A winning a prize
   $P(A) = P$(W or FFFW or FFFFFFW or …)
   $= P(\text{W}) + P(\text{FFFW}) + P(\text{FFFFFFW}) + \ldots$
   $$= \frac{1}{4} + \left(\frac{3}{4}\right)^3 \cdot \frac{1}{4} + \left(\frac{3}{4}\right)^6 \cdot \frac{1}{4} + \ldots$$
   $$= \frac{1}{4}\left(1 + \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^6 + \ldots\right)$$
   $$= \frac{1}{4} \times \frac{1}{1-(3/4)^3}$$
   $$= \frac{1}{4} \times \frac{64}{64-27} = \frac{16}{37}$$
   Chance of B winning a prize
   $P(B) = P$(W or FFFFW or FFFFFFFW or …)
   $= P(\text{FW}) + P(\text{FFFFW}) + P(\text{FFFFFFFW}) + \ldots$
   $$= \frac{3}{4} \cdot \frac{1}{4} + \left(\frac{3}{4}\right)^4 \cdot \frac{1}{4} + \left(\frac{3}{4}\right)^7 \cdot \frac{1}{4} + \ldots$$
   $$= \frac{3}{4} \cdot \frac{1}{4}\left(1 + \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^6 + \ldots\right)$$
   $$= \frac{3}{4} \cdot \frac{1}{4} \times \frac{1}{1-(3/4)^3}$$
   $$= \frac{3}{4} \cdot \frac{1}{4} \times \frac{64}{64-27}$$
   $$= \frac{12}{37}$$
   Chance of C winning a prize
   $$P(C) = 1 - \left(\frac{16}{37} + \frac{12}{37}\right) = 1 - \frac{28}{37} = \frac{9}{37}.$$
12. The probability that none of three persons have same date and same month as their date of birth.
   $$= \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} = \frac{364 \times 353}{(365)^2}$$
   Hence, the required probability $= 1 - \frac{364 \times 353}{(365)^2}$
13. $P(\text{getting the sum is 7}) = \frac{6}{36} = \frac{1}{6}$
   $P(\text{not getting the sum is 7}) = \frac{5}{6}$
   Given $P(X \geq 1) > 0.95$
   $\Rightarrow \quad 1 - P(X = 0) > 0.95$
   $\Rightarrow \quad 1 - {^nC_0} \times p^0 \times q^n > 0.95$
   $\Rightarrow \quad 1 - q^n > 0.95$
   $\Rightarrow \quad 1 - \left(\frac{5}{6}\right)^n > 0.95$
   $\Rightarrow \quad \left(\frac{5}{6}\right)^n < 1 - 0.95 = 0.05$
   $\Rightarrow \quad n \log_{10}\left(\frac{5}{6}\right) < \log_{10}\left(\frac{5}{100}\right)$
   $\Rightarrow \quad n(\log_{10} 5 - \log_{10} 6) < \log_{10} 5 - \log_{10} 10$
   $\Rightarrow \quad n(0.6989 - 0.7781) < (0.6980 - 2)$
   $\Rightarrow \quad n(0.07918) < -(1.3010)$
   $\Rightarrow \quad n(0.07918) > (1.3010)$
   $\Rightarrow \quad n > \frac{(1.3010)}{0.07918} = \frac{13010}{7918} = 16.43$
   $\Rightarrow \quad n = 17$
14. Hence, the required probability $= \frac{^{33}C_1 \times {^{99}C_1}}{^{100}C_2}$
   $$= \frac{33 \times 99 \times 2}{100 \times 99} = \frac{33}{50}$$

15. Let E be the event that the new product is introduced.
Given $P(A) = 0.5$, $P(B) = 0.3$ and $P(C) = 0.2$
$P(E/A) = 0.7$, $P(E/B) = 0.6$ and $P(E/C) = 0.5$
Hence, the required probability,
$$P(E) = P(A).P(E/A) + P(B) \cdot P(E/B) + P(C) \cdot P(E/C)$$
$$= 0.5 \times 0.7 + 0.3 \times 0.6 + 0.2 \times 0.5$$
$$= 0.35 + 0.18 + 0.10$$
$$= 0.63$$

16. Probability of none of them is defective
= P(not defective from A)
+ P(not defective from B)
+ P(not defective from A and B)
$$= \frac{^4C_2}{^9C_2} \times (0.1)^2 + \frac{^5C_2}{^9C_2} \times (0.2)^2 + \frac{^4C_1 \times {^5C_1}}{^9C_2} \times (0.1) \times (0.2)$$
$$= \frac{6}{36} \times (0.01) + \frac{10}{36} \times (0.04) + \frac{20}{36} \times (0.02)$$
$$= \frac{0.06 + 0.40 + 0.40}{36}$$
$$= \frac{0.86}{36}$$
Hence, the required probability
$$= 1 - \frac{0.86}{36}$$
$$= 1 - 0.023$$
$$= 0.977$$
$$= 0.98.$$

17. We have,
$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$
$$= 0.6 + 0.4 - 0.8 = 0.2$$
Given $P(A \cup B \cup C) \geq 0.85$
Now, $P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C)$
$-P(A \cap B) + P(A \cap B \cap C) \geq 0.85$
$\Rightarrow$ $0.6 + 0.4 + 0.5 - 0.2 - 0.3 + 0.2 - 0.85 \geq P(B \cap C)$
$\Rightarrow$ $1.5 - 1.15 \geq P(B \cap C)$
$\Rightarrow$ $P(B \cap C) \leq 0.35.$

18. Solution

19. Probability of getting 6 = $\frac{1}{6}$
Probability of getting 6 in n trials = $n \times \left(\frac{1}{6}\right) = \frac{n}{6}$

Given,
$$P(X \geq 1) > \frac{1}{2}$$
$$\Rightarrow \quad 1 - P(X < 1) > \frac{1}{2}$$
$$\Rightarrow \quad 1 - P(X = 0) > \frac{1}{2}$$
$$\Rightarrow \quad P(X = 0) < \frac{1}{2}$$
$$\Rightarrow \quad {^nC_0} \cdot p^0 \cdot q^n < \frac{1}{2}$$
$$\Rightarrow \quad \left(\frac{5}{6}\right)^n < \frac{1}{2}$$
It is possible only when $n > 3$.

20. P(at least one rainy day) = $P(X \geq 1)$
$$= 1 - P(X < 1)$$
$$= 1 - P(X = 0)$$
$$= 1 - {^7C_0} \cdot p^0 q^7$$
$$= 1 - q^7 = 1 - \left(\frac{7}{10}\right)^7.$$
Now, P(at least two rainy days) = $P(X \geq 2)$
$$= 1 - P(X < 2)$$
$$= 1 - \{P(X = 0) + P(X = 1)\}$$
$$= 1 - ({^7C_0}p^0q^7 + {^7C_1}p^1q^6)$$
$$= 1 - (q^7 + 7pq^6)$$
$$= 1 - (q + 7p)q^6$$
$$= 1 - 0.3294$$
$$= 0.6706$$
$$= 0.67$$
Hence the required probability
$$= \frac{P(\text{atleast two rainy day})}{P(\text{atleast one rainy day})}$$
$$= \frac{0.67}{1 - (0.7)^7}$$
$$= \frac{0.67}{1 - 0.0823}$$
$$= \frac{0.670}{0.917}$$
$$= 0.732$$

21. For equal roots, $b^2 = 4ac$.
Now, we shall find the total favorable cases from the following table.

b	(a, c)	Possible cases
2	(1, 1)	1
4	(2, 2), (1, 4), (4, 1)	3
6	(3, 3)	1

Total possible ways, we can get the equal roots = 5
Hence, the required probability = $\frac{5}{216}$.

22. This can be done in four different ways a pair of cards is drawn.
Case I: Let second card makes pair with first and third and fourth are different.
Thus, the probability = $1 \times \frac{3}{51} \times \frac{48}{50} \times \frac{47}{49}$
$= 0.0542$
Case II: Let first and third cards make a pair and other two are different.

Then, the probability $= 1\times\frac{48}{51}\times\frac{3}{50}\times\frac{47}{49}$

$= 0.0542$

Case III: Let first and fourth are pair cards and rest are different.

Then, the probability $= 1\times\frac{48}{51}\times\frac{46}{50}\times\frac{3}{49}$

$= 0.0530$

Case IV: Let first, second and third are different and fourth are matched with either first, second or third.

Then, the probability $= 1\times\frac{48}{51}\times\frac{46}{50}\times\frac{9}{49}$

$= 0.1590$

Hence, the required probability

$= 0.0542 + 0.0542 + 0.0530 + 0.1590$

$= 0.3204.$

23. Given $P(A\cap B)=\frac{1}{6}$ and $P(\bar{A}\cap\bar{B})=\frac{1}{3}$

$P(A)\cdot P(B)=\frac{1}{6}$ and $P(\bar{A})P(\bar{B})=\frac{1}{3}$

Now, $P(\bar{A})P(\bar{B})=\frac{1}{3}$

$\Rightarrow \quad (1-P(A))(1-P(B))=\frac{1}{3}$

$\Rightarrow \quad 1-P(A)-P(\text{B})+P(\text{A})\cdot P(B)=\frac{1}{3}$

$\Rightarrow \quad P(A)+P(B)=1+\frac{1}{6}-\frac{1}{3}$

$\Rightarrow \quad P(A)+P(B)=\frac{5}{6}$

Thus, $P(A)$ and $P(B)$ are the roots of

$x^2-\left(\frac{5}{6}\right)x+\frac{1}{6}=0.$

$\Rightarrow \quad 6x^2 - 5x + 1 = 0$

$\Rightarrow \quad (2x - 1)(3x - 1) = 0$

$\Rightarrow \quad x=\frac{1}{2},\frac{1}{3}$

Thus, $P(A)=\frac{1}{2}$ and $P(B)=\frac{1}{3}$

24. Hence, the required probability

$= P(\text{first heart})\times P(\text{second ace/first heart})$

$+ P(\text{first ace})\times P(\text{second heart/first ace})$

$=\frac{13}{52}\times P(\text{second ace/first heart})$

$+\frac{4}{52}\times P(\text{second heart/first ace})$

$=\frac{1}{4}[P(\text{second ace/first ace of hearts})$

$+P(\text{second ace/first not a ace of hearts})]$

$+\frac{1}{13}[P(\text{second heart/first ace of hearts})$

$+ P(\text{second heart/first not an ace of hearts})]$

$=\frac{1}{4}\left[\frac{3}{51}+\frac{4}{51}\right]+\frac{1}{13}\left[\frac{12}{51}+\frac{13}{51}\right]$

$=\frac{1}{51}\left(\frac{3}{4}+\frac{4}{4}+\frac{12}{13}+\frac{13}{13}\right)$

$=\frac{1}{51}\left(\frac{39+52+48+52}{52}\right)$

$=\frac{1}{51}\times\frac{191}{52}=\frac{191}{2652}$

Level IV

1. Hence, the required probability

$=\frac{{}^4C_4\times{}^{48}C_9}{{}^{52}C_{13}}=\frac{{}^{48}C_9}{{}^{52}C_{13}}=\frac{11}{4165}$

2. Hence, the probability of occurrence of odd number of times $=\frac{{}^nC_1+{}^nC_3+{}^nC_5+\ldots}{2^n}=\frac{2^{n-1}}{2^n}=\frac{1}{2}$

3. Let n be the number of trials.

It is given that

$P(X\geq 1)\geq 0.9$

$\Rightarrow \quad 1-P(X<1)\geq 0.9$

$\Rightarrow \quad 1-P(X=0)\geq 0.9$

$\Rightarrow \quad 1-{}^nC_0p^0q^n\geq 0.9$

$\Rightarrow \quad 1-q^n\geq 0.9$

$\Rightarrow \quad 1-\left(\frac{1}{2}\right)^n\geq 0.9$

$\Rightarrow \quad 1-\frac{1}{2^n}\geq 0.9$

$\Rightarrow \quad \frac{1}{2^n}\leq 0.1=\frac{1}{10}$

$\Rightarrow \quad 2n\geq 10$

It is true for $n = 4$.

4. Let A = the event whose sum is 5

B = the event whose sum is 7

and C = the event whose sum is neither 5 nor 7

Thus, $P(A)=\frac{4}{36}=\frac{1}{9}$,

$P(B)=\frac{6}{36}=\frac{1}{6}$

and $P(C)=\frac{26}{36}=\frac{13}{18}$

Hence, the required probability

= Probability of A occurs before B

$= P(A \text{ or } CA \text{ or } CCA \text{ or } CCCCA \text{ or } ...)$

$= P(A) + P(CA) + P(CCA) + P(CCCCA) + ...$

$= P(A) + P(C)P(A) + P(C)P(C)P(A) + ...$

$$= \frac{P(A)}{1 - P(C)}$$

$$= \frac{1/9}{1 - \frac{13}{18}} = \frac{18/9}{18 - 13} = \frac{2}{5}$$

5. Here, $p = \frac{3}{6} = \frac{1}{2}, q = \frac{1}{2}$

Hence, the required probability

$$= \frac{P(X \le n)}{2^{n+1}}$$

$$= \frac{P(X = 0) + P(X = 1) + P(X = 2) + ... + P(X = n)}{2^{n+1}}$$

$$= \frac{{}^{2n+1}C_0 + {}^{2n+1}C_1 + {}^{2n+1}C_2 + ... + {}^{2n+1}C_n}{2^{n+1}}$$

$$= \frac{2^{2n}}{2^{n+1}} = \frac{1}{2}$$

6. Clearly, n(S) = $2 \times 2 \times 2 \times 2 = 2^4 = 16$

The zero determinants are

$$\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}, \begin{vmatrix} -1 & -1 \\ -1 & -1 \end{vmatrix}, \begin{vmatrix} -1 & 1 \\ 1 & -1 \end{vmatrix}, \begin{vmatrix} -1 & -1 \\ 1 & 1 \end{vmatrix}$$

$$\begin{vmatrix} -1 & 1 \\ -1 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 1 \\ -1 & -1 \end{vmatrix}, \begin{vmatrix} 1 & -1 \\ 1 & -1 \end{vmatrix}, \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}$$

Let E be the event, where the value of the determinant is non-zero.

Hence, the required probability,

$$P(E) = \frac{n(E)}{n(S)} = \frac{8}{16} = \frac{1}{2}$$

7. When A gets 1, then B gets any of the numbers 1, 2, 3, 4, 5 and 6

When A gets 2, then B gets 2, 3, 4, 5, 6

... A gets 3, ... B gets 3, 4, 5, 6

⋮

when A gets 6, then B gets also 6.

Thus, the number of favourable ways

$= 6 + 5 + 4 + 3 + 2 + 1 = 21$

Hence, the required probability $= \frac{21}{36} = \frac{7}{12}$

8. Here, $P(\text{min. of } 4) = \frac{{}^1C_1 \times {}^6C_2}{{}^{10}C_3}$,

since greater than 4 can be chosen from {5, 6, 7, 8, 9, 10}

$$P(\text{max. of } 8) = \frac{{}^1C_1 \times {}^7C_2}{{}^{10}C_3},$$

since greater than 8 can be chosen from {1, 2, 3, 4, 5, 6, 7}

$$P(\text{min. 4 and max. 8}) = \frac{{}^1C_1 \times {}^3C_1 \times {}^1C_1}{{}^{10}C_3},$$

since 4 and 8 plus any one can be chosen from {5, 6, 7}

Hence, the required probability

$= P(\text{min } 4) + P(\text{max } 8) - P(\text{min 4 and max } 8)$

$$= \frac{{}^6C_2 + {}^7C_2 - {}^3C_1}{{}^{10}C_3}$$

$$= \frac{15 + 21 - 3}{120} = \frac{33}{120} = \frac{11}{40}$$

9. We have

$f(x) = x^3 + ax^3 + bx + c$

$\Rightarrow \quad f'(x) = 3x^2 + 2ax^2 + b$

$\Rightarrow \quad f'(x) = 3x^2 + 2ax^2 + b > 0,$

since $f(x)$ is strictly increasing function.

So, $\quad D < 0$

$\Rightarrow \quad 4a^2 - 12b < 0$

$\Rightarrow \quad a^2 - 3b < 0$

which is true for the ordered pairs (a, b) where $1 < a, b < 6$

i.e. (1, 2), …, (1, 6), (2, 3), …, (2, 6), (3, 4), …, (3, 6), (4, 5), (4, 6).

Hence, the required probability

$$= \frac{5 + 4 + 3 + 2}{36} = \frac{14}{36} = \frac{7}{18}$$

10. Here, $n(S) = 6 \times 6 \times 6 \times 6 = 6^4$

Let A be the event of getting a sum 12 of throwing a die 4 times

$n(\text{A})$ = Co-efficients of x^{12} in $(x + x^2 + x^3 + \ldots + x^6)^4$

= Co-efficients of x^8 in $(1 + x + x^2 + \ldots + x^5)^4$

= Co-efficients of x^8 in $\left(\frac{1 - x^6}{1 - x}\right)^4$

= Co-efficients of x^8 in $(1 - 4x^6) \times (1 - x)^{-4}$

= Co-efficients of x^8 in $(1 - 4x^6) \times (1 + {}^4C_1x + {}^5C_2x^2 + \ldots + {}^{11}C_8x^8 + \ldots)$

$= ({}^{11}C_8 - 4 \cdot {}^5C_2)$

$$= \frac{11.10.9}{6} - 40 = 165 - 40 = 125$$

Hence, the required probability,

$$\frac{n(A)}{n(S)} = \frac{125}{1296}.$$

11. Hence, the required probability

$$= \frac{\text{Co-efficients of } x^k \text{ in } (x + x^2 + ... + x^6)^3}{216}$$

$$= \frac{{}^{k-1}C_2}{216} = \frac{(k-1)(k-2)}{432}$$

12. Clearly, $n(S) = 6 \times 6 = 6^2$

Let E be the event in which the maximum of two is greater than 4.

Thus, $E = \{(1, 5), (2, 5), (3, 5), (4, 5), (5, 5), (6, 5), (5, 1), (5, 2), (5, 3), (5, 4), (5, 6), (6, 6), (1, 6), (2, 6), (3, 6), (4, 6), (6, 1), (6, 2), (6, 3), (6, 4)\}$

Hence, the required probability,

$$P(E) = \frac{n(E)}{n(S)} = \frac{20}{36} = \frac{5}{9}$$

13. Let defective bulbs is denoted as p and good bulbs is denoted as q.

$$P(p) = \frac{4}{10} = \frac{2}{5}, P(q) = \frac{3}{5}$$

Hence, the required probability

= Probability that the room is lighted

$= P(X = 1 \text{ or } 2 \text{ or } 3)$

$= P(X = 1) + P(X = 2) + P(X = 3)$

$$= \frac{3}{5}\times\frac{2}{5}\times\frac{2}{5}\times 3 + \frac{3}{5}\times\frac{3}{5}\times\frac{2}{5}\times 3 + \frac{3}{5}\times\frac{3}{5}\times\frac{3}{5}$$

$$= \frac{36}{125} + \frac{54}{125} + \frac{27}{125}$$

$$= \frac{36+54+27}{125}$$

$$= \frac{117}{125}$$

14. Hence, the required probability

$= P(X = 2) + P(X = 3) + P(X = 4)$

$= {}^4C_2 p^2 q^2 + {}^4C_3 p^3 q^1 + {}^4C_4 p^4 q^0$

$$= 6\times\left(\frac{2}{5}\right)^2 \times\left(\frac{3}{5}\right)^2 + 4\left(\frac{2}{5}\right)^3\left(\frac{3}{5}\right)^1 + \left(\frac{2}{5}\right)^4$$

$$= \left(\frac{216+96+16}{625}\right) = \frac{328}{625}$$

15. Hence, the required probability,

$P(X \geq 2) = P(X = 2) + P(X = 3)$

$= {}^3C_2 p^2 q^1 + {}^3C_3 p^3 q^0$

$$= 3\cdot\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^1 + \left(\frac{1}{2}\right)^3$$

$$= \frac{3}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$$

16. Clearly,

$$n(S) = {}^{10}C_3 = \frac{10\cdot 9\cdot 8}{6} = 120$$

Let p_1 = 3 numbers having c.d is 1

$= \{(1, 2, 3), (2, 3, 4), \ldots, (8, 9, 10)\}$

Then $n(p_1) = 8$

Let p_2 = 3 numbers having c.d 2

$= \{(1, 3, 5), (2, 4, 6), (3, 5, 7), \ldots, (6, 8, 10)\}$

Then $n(p_2) = 8$.

Let p_3 = 3 numbers having c.d 3

$= \{(1, 4, 7), (2, 5, 8), (3, 6, 9), (4, 7, 10)\}$

Then $n(p_3) = 4$

Let p_4 = 3 numbers having c.d 4

$= \{(1, 5, 9), (2, 6, 10)\}$

Then $n(p_4) = 2$

Hence, the required probability

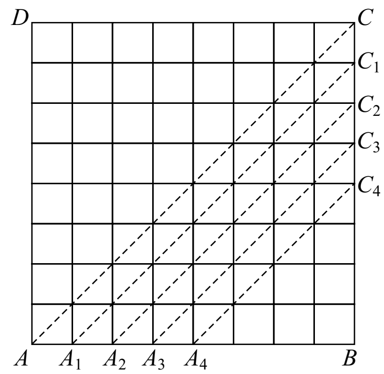
$$= \frac{8+6+4+2}{{}^{10}C_3} = \frac{20}{120} = \frac{1}{6}$$

17. Hence, the required probability

$$= \frac{{}^{13}C_{13} + {}^{13}C_{13} + {}^{13}C_{13} + {}^{13}C_{13}}{{}^{52}C_{13}}$$

$$= \frac{4}{{}^{52}C_{13}}$$

18. There are 64 squares in a chess board.



In ΔACB, the number of ways in which 4 selected squares lie along the lines

A_4C_4, A_3C_3, A_2C_2, A_1C_1 and AC are

${}^4C_4$, ${}^5C_4$, ${}^6C_4$, ${}^7C_4$ and ${}^8C_4$, respectively

Similarly, in ΔACD, there are equal number of ways of selecting 4 squares in a diagonal parallel to AC.

Thus, the total number of ways 4 squares can be chosen in $(2({}^4C_4 + {}^5C_4 + {}^6C_4 + {}^7C_4) + {}^8C_4)$ ways.

Since there is an equal number of ways in which 4 selected squares are in a diagonal line parallel to BD.

So, the number of possible ways

$= 2[2({}^4C_4 + {}^5C_4 + {}^6C_4 + {}^7C_4) + {}^8C_4]$

Hence, the required probability

$$= \frac{2[2({}^4C_4 + {}^5C_4 + {}^6C_4 + {}^7C_4) + {}^8C_4]}{{}^{64}C_4}$$

$$= \frac{[4(1+5+15+35)+140]\times 4\times 3\times 2}{64\times 63\times 62\times 61}$$

$$= \frac{91}{158844}$$

19. Hence, the required probability

$= P(\text{at least one do not get anything})$

$= 1 - P(\text{None of them is empty})$

$$= \left(1 - \frac{10!}{(10)^{10}}\right)$$

20. Out of 2 cards, one is an ace of spade and other card can take 51 ways.
or 1 ace of any other 3 aces and 1 spade from any other 12 spades in $3 \times 12 = 36$ ways.
Hence, the required probability,
$$\frac{(51+36)}{^{52}C_2} = \frac{87}{26 \times 51} = \frac{29}{26 \times 17} = \frac{29}{442}$$

21. For $7^m + 7^n$ is be divisible by 5.
Case I: When m ends with 9 and n ends with 1.
Thus, $m = 2, 6, 10, \ldots, 98$ (25 cases)
and $n = 4, 8, 12, \ldots, 100$ (25 cases)
Therefore, it can be possible in 25×25 ways.
Case II: When m ends with 7 and n ends with 3
Thus $m = 1, 5, 9, \ldots, 97$ (25 cases)
and $n = 1, 5, 9, \ldots, 97$ (25 cases)
Therefore, it can also be possible in 25×25 ways.
Hence, the required probability
$$= \frac{2 \times 2 \times 25 \times 25}{100 \times 100}$$
$$= \frac{1}{4}$$

22. Clearly, the last digit of the product of n integers is 1, 2, 3, 4, 6, 7, 8 or 9.
Hence, the required probability
$$= \left(\frac{8}{10}\right)^n = \left(\frac{4}{5}\right)^n$$

23. Hence, the required probability,
$$\frac{^{365}C_1}{(365)^3} = \frac{365}{(365)^3} = \frac{1}{(365)^2}$$

24. Given equation is $3x = 2x^2 + 1$
Clearly, it has three solution at $x = 0$, 1 and 2.
Hence, the required probability $= \frac{3}{10}$.

25. Clearly, 3 integers can be chosen in $^{20}C_3$ ways.
The product of three integers will be even if at least one of the integers is even.
Hence, the required probability
= 1 – probability that none of three integers is even.
$$= \left(1 - \frac{^{10}C_3}{^{20}C_3}\right)$$
$$= 1 - \frac{10 \cdot 9 \cdot 8}{20 \cdot 19 \cdot 18} = 1 - \frac{2}{19} = \frac{17}{19}$$

Integer Type Questions

1. For each toss, there are 4 cases will be arise
Case I: A gets head and B gets head
Case II: A gets head and B gets tail
Case III: A gets tail and B gets head
Case IV: Both A and B get tail
So, out of 4 choices, only one choice is there, where A and B both get tail
Hence, the required probability $= \left(\frac{3}{4}\right)^{50}$
Clearly, $p = 3$ and $q = 4$
Hence, the value of $(p + q + 2)$ is 9.

2. We have,
$x^2 - 13x \le 30$
$\Rightarrow x^2 - 13x - 30 \le 0$
$\Rightarrow (x - 3)(x - 10) \le 0$
$\Rightarrow 3 \le x \le 10$
Thus, $x = 3, 4, 5, 6, 7, 8, 9, 10$
Hence, the required probability of $x = \frac{8}{10} = \frac{4}{5}$
Clearly, $a = 4$ and $b = 5$.
Hence, the value of $(b - a + 2) = 1 + 2 = 3$.

3. Hence, the required probability
$$= \frac{\frac{1}{4} \times \frac{1}{6}}{\frac{3}{4} \times \frac{5}{6} + \frac{1}{4} \times \frac{1}{6}} = \frac{1}{15+1} = \frac{1}{16}$$
Clearly $m = 1$ and $n = 16$
Hence, the value of $\left(\frac{n}{m+1}\right)$ is 8.

4. Hence, the required probability,
$$\frac{^3C_1 \times {^3C_1}}{^6C_2} = \frac{3 \times 3 \times 2}{6 \times 5} = \frac{3}{5}$$
Clearly $a = 3$ and $b = 5$
Hence, the value of $(a + b)$ is 8.

5. Clearly $P(H) = \frac{1}{2}, P(6) = \frac{1}{6}$
Hence, the required probability of getting a head before getting 6
$$= \frac{1}{2} + \frac{5}{6}\left(\frac{1}{2}\right)^2 + \left(\frac{5}{6}\right)^2\left(\frac{1}{2}\right)^3 + \ldots$$
$$= \frac{\frac{1}{2}}{1 - \frac{5}{12}} = \frac{\frac{1}{2}}{\frac{7}{12}} = \frac{6}{7}$$
Thus, $p = 6/7$
Hence, the value of $(7p + 1) = 6 + 1 = 7$.

6. Here, $n(S) = 6 \times 6 \times 6 \times 6 = 1296$
Let E be the event of getting a sum 12.
Now, $n(E)$
$n(A)$ = Co-efficients of x^{12} in $(x + x^2 + x^3 + \ldots + x^6)^4$
= Co-efficients of x^8 in $(1 + x + x^2 + \ldots + x^5)^4$
= Co-efficients of x^8 in $\left(\frac{1 - x^6}{1 - x}\right)^4$

= Co-efficients of x^8 in $(1-4x^6)\times(1-x)^{-4}$
= Co-efficients of x^8 in $(1-4x^6)\times(1+{}^4C_1x + {}^5C_2x^2 + \ldots + {}^{11}C_8x^8 + \ldots)$
$= ({}^{11}C_8 - 4\cdot{}^5C_2)$
= 125

Hence, the required probability

$$=\frac{125}{1296}=\left(\frac{5}{6}\right)^3\times\frac{1}{6}$$

Thus, $m = 5$ and $n = 6$.
Hence, the value of $(n - m + 2) = 6 - 5 + 2 = 3$.

7. Two numbers a and b can be chosen as
(3, 2), (6, 4), (9, 16), (12, 8), (15, 10).
Hence, the required probability,

$$\frac{5}{{}^{15}C_2}=\frac{5\times 2}{15\times 14}=\frac{1}{21}$$

Thus, $p=\frac{1}{21}$

Hence, the value of $(84p + 2) = 4 + 2 = 6$.

8. First we arrange the given set of numbers as
1, 4, 7, ..., 97
2, 5, 8, ..., 98
3, 6, 9, ..., 99

Now, $(x^3 + y^3)$ is divisible by 3 only when one number from the first row and 1 number from the second row or 2 numbers from the third row.
Thus, the number of favourable ways,

$$^{33}C_1\times{}^{33}C_1+{}^{33}C_2 = 33.33+\frac{33.32}{2}$$
$$= 33(33 + 16) = 33 \times 49$$

Hence, the required probability,

$$\frac{33\times 49}{{}^{99}C_2}=\frac{33\times 49}{99\times 49}$$
$$=\frac{1}{3}$$

Thus, $p = 1$ and $q = 3$
Hence, the value of $p + q + pq = 1 + 3 + 7 = 7$.

9. Probability of drawing a white ball 4th time at 7th draw.
= Probability of drawing 3 white balls in first 6 draws × probability of drawing a white ball at the 7th draw.

$$=({}^6C_3p^3q^3)\times\frac{16}{24}$$
$$=\left(20\times\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^3\right)\times\frac{2}{3}$$
$$=\left(\frac{2}{3}\right)^3\times\frac{40}{81}$$

Clearly $a = 2$, $b = 3$.
Hence, the value of $a + b + 1$ = is $2 + 3 + 1 = 6$.

10. Let the number of elements in set X is m.
∴ Number of selections of two subsets is ${}^{2m}C_2$ ways.
Let E be the event for which two subsets A and B are selected in such a way that
$A\cup B=\varphi$, $A\cup B = X$

Thus, $n(E)=\frac{1}{2}({}^mC_0+{}^mC_1+{}^mC_2+\ldots+{}^mC_m)$

$$=\frac{2^m}{2}=2^{m-1}$$

Hence, the required probability,

$$\frac{2^{m-1}}{{}^{2^m}C_2}=\frac{2^{m-1}\times 2}{2^m(2^m-1)}$$
$$=\frac{2^m}{2^m(2^m-1)}=\frac{1}{(2^m-1)}$$

Thus, $\frac{1}{(2^m-1)}=\frac{1}{127}$

⇒ $(2^m - 1) = 127$
⇒ $2^m = 127 + 1 = 128 = 2^7$
⇒ $m = 7$

Hence, the number of elements in set X is 7.

11. Here, $P(\text{white ball})=\frac{3}{9}=\frac{1}{3}$

and $P(\text{non-white ball})=\frac{6}{9}=\frac{2}{3}$

Let X be the number of white balls drawn.
Clearly, $n = 4$
Thus, the required probability,

$$P(X=1)={}^4C_1\times p^1\times q^3$$
$$=4\times\left(\frac{1}{3}\right)\times\left(\frac{2}{3}\right)^3=2\times\left(\frac{2}{3}\right)^4$$

Clearly, $a = 2$ and $b = 3$
Hence, the value of $a + b + 1 = 2 + 3 + 1 = 6$.

12. Here, $P(\text{leap year})=\frac{1}{4}$ and

$P(\text{non-leap year})=\frac{3}{4}$

Let A and B be the events that a non-leap year and a leap year, respectively have 53 Sundays.

Thus, $P(A)=\frac{1}{4}\times\frac{2}{7}$ and $P(B)=\frac{3}{4}\times\frac{1}{7}$

Hence, the required probability,

$$P(A \text{ or } B) = P(A\cup B)$$
$$= P(A) + P(B)$$
$$=\frac{1}{4}\times\frac{2}{7}+\frac{3}{4}\times\frac{1}{7}$$
$$=\frac{5}{28}=\frac{5}{4}\times\frac{1}{7}$$

Clearly, $a = 5$ and $b = 4$

Hence, the value of $a + b - 3 = 5 + 4 - 3 = 6$.

13. Probability of getting 5 is $\frac{4}{36} = \frac{1}{9}$

Probability of getting 7 is $\frac{6}{36} = \frac{1}{6}$

Probability that neither 5 nor 7 is $= 1 - \frac{10}{36} = \frac{13}{18}$

Hence, the required probability

$$= \frac{1}{9} + \left(\frac{13}{18}\right)\left(\frac{1}{9}\right) + \left(\frac{13}{18}\right)^2\left(\frac{1}{9}\right) + \left(\frac{13}{18}\right)^3\left(\frac{1}{9}\right) + \ldots$$

$$= \frac{\frac{1}{9}}{1 - \left(\frac{13}{18}\right)} = \frac{\frac{1}{9}}{\frac{5}{18}} = \frac{2}{5}$$

Clearly, $m = 2$ and $n = 5$

Hence, the value of $(m + n) = 2 + 5 = 7$.

Previous Years' JEE-Advanced Examinations

1. (i) Required probability $= \frac{7! \times 6!}{12!} = \frac{1}{132}$

(ii) Required probability $= \frac{2 \times 6! \times 6!}{12!} = \frac{1}{462}$

2. Ans. 1/1260

Required probability $= \frac{{}^2P_2 \times {}^4P_4 \times {}^3P_3}{9!} = \frac{1}{1260}$

3. Required probability,

$$P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B})$$
$$= 1 - P(A \cup B)$$
$$= 1 - \{P(A) + P(B) - P(A \cap B)\}$$
$$= 1 - \{0.25 + 0.50 - 0.14\}$$
$$= 1 - \{0.75 - 0.14\}$$
$$= 1 - 0.61$$
$$= 0.39$$

4. Required probability,

$$P(X \geq 1) = 1 - P(X < 1)$$
$$= 1 - P(X = 0)$$
$$= 1 - {}^3C_0 p^0 q^3$$
$$= 1 - q^3$$
$$= 1 - (0.6)^3$$
$$= 1 - 0.216$$
$$= 0.784$$

5. We have,

$$P\left(\frac{\bar{A}}{\bar{B}}\right) = \frac{P(\bar{A} \cap \bar{B})}{P(\bar{B})}$$
$$= \frac{P(\overline{A \cup B})}{P(\bar{B})}$$
$$= \frac{1 - P(A \cup B)}{P(\bar{B})}$$
$$= \frac{1 - \{P(A) + P(B) - P(A \cap B)\}}{P(\bar{B})}$$
$$= \frac{\{1 - P(B)\} - \{P(A) - P(A \cap B)\}}{P(\bar{B})}$$
$$= \frac{P(\bar{B}) - P(A \cap \bar{B})}{P(\bar{B})}$$
$$= 1 - \frac{P(A \cap \bar{B})}{P(\bar{B})}$$
$$= 1 - P\left(\frac{A}{\bar{B}}\right)$$

6. Let S_i be the shots of the anticraft gun

Here, $P(S_1) = 0.4$, $P(S_2) = 0.3$

and $P(S_3) = 0.2$, $P(S_4) = 0.1$

Now,

$$P(\bar{S_1} \cap \bar{S_2} \cap \bar{S_3} \cap \bar{S_4}) = P(\bar{S_1}) \cdot P(\bar{S_2}) \cdot P(\bar{S_3}) \cdot P(\bar{S_4})$$
$$= 0.4 \times 0.3 \times 0.2 \times 0.1$$
$$= 0.0024$$

Hence, the required probability,

$$= P(S_1 \cup S_2 \cup S_3 \cup S_4)$$
$$= 1 - P(\bar{S_1} \cap \bar{S_2} \cap \bar{S_3} \cap \bar{S_4})$$
$$= 1 - 0.0024$$
$$= 0.6976.$$

7. Let E_i denote the event that face i turns up.

$P(E_1) = 0.1$ and $P(E_2) = 0.32$

Also, $P(E_1 \cup E_2) = P(E_1) + (E_2)$

$= 0.1 + 0.32$

$= 0.42$

Thus, required probability,

$$P\left(\frac{E_1}{E_1 \cup E_2}\right) = \frac{P(E_1)}{P(E_1 \cup E_2)}$$
$$= \frac{0.1}{0.42}$$
$$= \frac{10}{42} = \frac{5}{21}$$

8. Given $P(A) = 0.5$, $P(A \cap B) \leq 0.3$

We have $P(A \cup B) \leq 1$

$\Rightarrow \quad P(A) + P(B) - P(A \cap B) \leq 1$

$\Rightarrow \quad P(A) + P(B) \leq 1 + P(A \cap B)$

$\Rightarrow \quad P(A) + P(B) \leq 1 + 0.3 = 1.3$

$\Rightarrow \quad 0.5 + P(B) \leq 1.3$

$\Rightarrow \quad P(B) \leq 1.3 - 0.5 = 0.8$

9. Hence the required probability

$$= \frac{9^7 - 8^7}{15^7}$$

10. Hence, the required probability,

$P(N = n) = P$(drawing one ace in first $(n - 1)$ draws)

$\times P$(drawing an ace at the nth draws)

$$= \frac{^4C_1 \times {}^{48}C_{n-2}}{^{52}C_{n-1}} \times \frac{3}{52-(n-1)}$$

$$= \frac{4 \times (48)!}{(n-2)! \times (50-n)!} \times \frac{(n-1)! \times (53-n)!}{52!} \times \frac{3}{53-n}$$

$$= \frac{4 \times (n-1) \times (52-n) \times (51-n) \times 3}{52 \times 51 \times 50 \times 49}$$

$$= \frac{(n-1) \times (52-n) \times (51-n)}{50 \times 49 \times 17 \times 13}$$

11. We have,

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(BC) - P(AC) + P(ABC)$

$= 0.3 + 0.4 + 0.8 - 0.08 - P(BC) - 0.28 + 0.09$

$= 1.23 - P(BC)$

Now, $0.75 \leq P(A \cup B \cup C) \leq 1$

$\Rightarrow \quad 0.75 \leq 1.23 - P(BC) \leq 1$

$\Rightarrow \quad 0.23 \leq P(BC) \leq 0.48$

12. Hence, the required probability

$$= \frac{\frac{4!}{2!} \times \frac{^5P_4}{4!}}{\frac{8!}{4! \times 2!}}$$

$$= \frac{4! \times 5!}{8!}$$

$$= \frac{24}{8 \times 7 \times 6} = \frac{1}{14}.$$

13. Hence, the required probability

$$= \frac{6}{6^3} = \frac{1}{36}$$

14. Hence, the required probability

= P(drawing a white ball 4th time on the 7th draw)

= P(drawing 3 white balls in the first 6 draws) × P(drawing a white ball at the 7th draw)

$$= \left({}^6C_3 \left(\frac{12}{24}\right)^3 \left(\frac{12}{24}\right)^3 \right) \times \left(\frac{12}{24}\right)$$

$$= \left(\frac{6 \cdot 5 \cdot 4}{6} \times \frac{12^3 \times 12^3 \times 12}{24^3 \times 24^3 \times 24} \right)$$

$$= \frac{20}{2^7} = \frac{5}{2^5} = \frac{5}{32}$$

15. Hence, the required probability,

$P(M \cap \bar{N} \text{ or } \bar{M} \cap N)$

$= P(M \cap \bar{N}) + P(\bar{M} \cap N)$

$= P(M) - P(M \cap N) + P(N) - P(M \cap N)$

$= P(M) + P(N) - 2P(M \cap N)$

$= \left[1 - P(\bar{M})\right] + \left[1 - P(\bar{N})\right] - 2\left[1 - P(\bar{M} \cup \bar{N})\right]$

$= 2P(\bar{M} \cup \bar{N}) - P(\bar{M}) - P(\bar{N})$

$= 2\left[P(\bar{M}) + P(\bar{N}) - P(\bar{M} \cap \bar{N})\right] - P(\bar{M}) - P(\bar{N})$

$= \left[P(\bar{M}) + P(\bar{N}) - 2P(\bar{M} \cap \bar{N})\right]$

16. Given $P(A \cap B) = \frac{1}{6}$ and $P(\bar{A} \cap \bar{B}) = \frac{1}{3}$

$P(A)P(B) = \frac{1}{6}$ and $P(\bar{A})P(\bar{B}) = \frac{1}{3}$

Now, $P(\bar{A})P(\bar{B}) = \frac{1}{3}$

$\Rightarrow \quad (1 - P(A))(1 - P(B)) = \frac{1}{3}$

$\Rightarrow \quad 1 - P(A) - P(B) + P(A)P(B) = \frac{1}{3}$

$\Rightarrow \quad P(A) + P(B) = 1 + \frac{1}{6} - \frac{1}{3} = \frac{5}{6}$

Thus, $P(A) \cdot P(B) = \frac{1}{6}, P(A) + P(B) = \frac{5}{6}$

Therefore, $P(A)$ and $P(B)$ are the roots of

$\Rightarrow \quad x^2 - \left(\frac{5}{6}\right)x + \frac{1}{6} = 0$

$\Rightarrow \quad 6x^2 - 5x + 1 = 0$

$\Rightarrow \quad (3x - 1)(2x - 1) = 0$

$\Rightarrow \quad x = \frac{1}{2}$ or $\frac{1}{3}$

Thus, $P(A) = \frac{1}{2}$ or $\frac{1}{3}$

17. Given $P(A) = 0.25$, $P(B) = 0.20$ and $P(A \cap B) = 0.08$

Now, $P(A \cap B') = P(A) - P(A \cap B)$

$= 0.25 - 0.08 = 0.17$

and $\quad P(B \cap A') = P(B) - P(A \cap B)$

$= 0.20 - 0.08 = 0.12$

Let E denote the events that a person looks into advertisement.

Thus, $P\left(\frac{E}{A \cap B'}\right) = 0.30$

$P\left(\frac{E}{A' \cap B}\right) = 0.40$

and $\quad P\left(\frac{E}{A \cap B}\right) = 0.50$

We have,

$$P(E) = P(A \cap B')P\left(\frac{E}{A \cap B'}\right) + P(A' \cap B)P\left(\frac{E}{A' \cap B}\right) + P(A \cap B)P\left(\frac{E}{A \cap B}\right)$$

$= (0.17)\times(0.30)+(0.12)\times(0.40)+(0.08)\times(0.50)$
$= 0.1390$
$= \frac{1390}{1000}$
$= \frac{13.9}{100}$
$= 13.9\%$

18. The number of ways in which a candidate can tick one or more of the alternatives is
$= {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4$
$= 2^4 - 1 = 15$
Hence, the required probability $= \frac{3}{15} = \frac{1}{5}$

19. Given $P(A \cup B) = P(A \cap B)$
$\Rightarrow P(A) + P(B) - P(A \cap B) = P(A \cap B)$
$\Rightarrow P(A) - P(A \cap B) + P(B) - P(A \cap B) = 0$ …(i)
Since $P(A) \cdot P(B) \geq P(A \cap B)$,
Equation (i) is possible only when
$P(A) - P(A \cap B) = 0, P(B) - P(A \cap B) = 0$
Thus, $P(A) = P(B) = P(A \cap B)$.

20. Let B denote the event that max number on the chosen ticket is not more than 10 and A be the event that min number of them is 5.
We have, required probability $= P\left(\frac{A}{B}\right)$
$= P\left(\frac{A \cap B}{B}\right)$
$= \frac{{}^5C_1}{{}^{10}C_2} = \frac{5.2}{10.9} = \frac{1}{9}$

21. Let E_1, E_2 and D denote the events, a lot contains 2 defective articles a lot contains 3 defective articles and all the defective article are found by the 12th test, i.e. the event D means one defective articles must be found in the first eleven testings.
Now, $P(E_1) = 04$, $P(E_2) = 0.6$

$$P(D/E_1) = \frac{{}^2C_1 \times {}^{18}C_{10}}{{}^{20}C_{11}} \times \frac{1}{9} = \frac{11}{190}$$

$$P(D/E_2) = \frac{{}^3C_2 \times {}^{17}C_9}{{}^{20}C_{11}} \times \frac{1}{9} = \frac{11}{228}$$

Hence, the required probability,

$$P(D) = P(E_1) \cdot P(D/E_1) + P(E_2) \cdot P(D/E_2)$$
$$= 0.4 \times \frac{11}{190} + 0.6 \times \frac{11}{228}$$
$$= \frac{99}{1900}$$

22. Let E_i denote the events that the student passes in the ith test.
So, $P(E_1) = p$, $P(E_2) = q$ and $P(E_3) = \frac{1}{2}$
Given
$\Rightarrow P((E_1 \cap E_2) \cup (E_1 \cap E_3)) = \frac{1}{2}$
$\Rightarrow P(E_1 \cap E_2) + P(E_1 \cap E_3) - P(E_1 \cap E_2 \cap E_3) = \frac{1}{2}$
$\Rightarrow P(E_1)P(E_2) + P(E_1)P(E_3) - P(E_1)P(E_2)P(E_3) = \frac{1}{2}$
$\Rightarrow pq + p \cdot \frac{1}{2} - pq \cdot \frac{1}{2} = \frac{1}{2}$
$\Rightarrow \frac{pq}{2} + \frac{p}{2} = \frac{1}{2}$
$\Rightarrow pq + p = 1$
$\Rightarrow p(q + 1) = 1$
It is possible only when $q = 0$ and $p = 1$.

23. Given $\frac{1+3p}{3} \geq 0, \frac{1-p}{4} \geq 0, \frac{1-2p}{2} \geq 0$
and $\frac{1+3p}{3} + \frac{1-p}{4} + \frac{1-2p}{2} \leq 1$
$\Rightarrow p \geq -\frac{1}{3}, p \leq 1, p \leq \frac{1}{2}$
and $4 + 12p + 3 - 3p + 6 - 12p \leq 12$
$\Rightarrow -\frac{1}{3} \leq p \leq \frac{1}{2}$ and $p \geq \frac{1}{3}$
$\Rightarrow \frac{1}{3} \leq p \leq \frac{1}{2}$

24. Given $P(A \cup B) = 0.6$ and $P(A \cap B) = 0.2$
Now, $P(A \cup B) = 0.6$
$\Rightarrow P(A) + P(B) - P(A \cap B) = 0.6$
$\Rightarrow P(A) + P(B) - 0.2 = 0.6$
$\Rightarrow P(A) + P(B) = 0.2 + 0.6 = 0.8$
We have,
$P(\bar{A}) + P(\bar{B}) = 1 - P(A) + 1 - P(B)$
$= 2 - (P(A) + P(B))$
$= 2 - 0.8 = 1.2$

25. Let X be the number of steps taken in the forward direction. Here $n = 11$, $p = 0.4$ and $q = 0.6$
Hence, the required probability,
$P(X = 5) + P(X = 6) = {}^{11}C_5 p^5 q^6 + {}^{11}C_6 p^6 q^5$
$= {}^{11}C_5 p^5 q^6 + {}^{11}C_5 p^6 q^5$
$(\because {}^{11}C_6 = {}^{11}C_5)$
$= {}^{11}C_5 p^5 q^5 (p + q)$
$= {}^{11}C_5 p^5 q^5 \cdot 1$
$= {}^{11}C_5 (pq)^5$
$= 462 \times (0.24)^5$

26. Let E_{xy} denote the event that the first ball drawn has color x and second ball drawn has color y and A denotes the event that the third ball drawn is black.

$$P(E_{ww}) = \frac{2}{4} \times \frac{1}{3} = \frac{1}{6},\ P(E_{wb}) = \frac{2}{4} \times \frac{2}{3} = \frac{1}{3}$$

$$P(E_{bw}) = \frac{2}{4} \times \frac{2}{5} = \frac{1}{5},\ P(E_{bb}) = \frac{2}{4} \times \frac{3}{5} = \frac{3}{10}$$

Also, $P(A/E_{ww}) = \frac{2}{2} = 1, P(A/E_{wb}) = \frac{3}{4}$

$P(A/E_{bw}) = \frac{3}{4}, P(A/E_{bb}) = \frac{4}{6} = \frac{2}{3}$

Hence, the required probability

$$P(E_{ww})P(A/E_{ww}) + P(E_{wb})P(A/E_{wb}) + P(E_{bb})P(A/E_{bb}) + P(E_{bw})P(A/E_{bw})$$

$$= \frac{1}{6} \times 1 + \frac{1}{3} \times \frac{3}{4} + \frac{1}{5} \times \frac{3}{4} + \frac{3}{10} \times \frac{2}{3}$$

$$= \frac{23}{30}$$

27. Let X denote the number of coins showing heads.

Given,

$P(X = 50) = P(X = 51)$

$\Rightarrow \quad {}^{100}C_{50} p^{50}(1 - p)^{50} = {}^{100}C_{51} p^{51}(1 - p)^{49}$

$\Rightarrow \quad {}^{100}C_{50}(1 - p) = {}^{100}C_{51} p$

$\Rightarrow \quad \frac{{}^{100}C_{50}}{{}^{100}C_{51}} = \frac{p}{(1-p)}$

$\Rightarrow \quad \frac{p}{(1-p)} = \frac{51}{50}$

$\Rightarrow \quad 50p = 51 - 51p$

$\Rightarrow \quad 101p = 51$

$\Rightarrow \quad p = \frac{51}{101}$

28. We have $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow \quad P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$\Rightarrow \quad P(A \cap B) \geq P(A) + P(B) - 1$

$\because \quad P(A \cap B) \leq 1$

Also, $P(A \cap B) = P(A) + P(B) - P(A \cup B)$

$\Rightarrow \quad P(A \cap B) \leq P(A) + P(B)$

29. Clearly, the total number of coins is $N + 7$.

So, the number of ways in which 5 coins can be taken out of these is ${}^{N+7}C_5$.

Hence, the required probability

= 1 – Probability that the total value of 5 coins is greater than or equal to ₹1.50

$$= 1 - \frac{{}^2C_2 \times {}^5C_3 \times {}^8C_0 + {}^2C_2 \times {}^5C_2 \times {}^8C_1 + {}^2C_2 \times {}^5C_1 \times {}^8C_2}{{}^{N+7}C_5}$$

$$= 1 - \frac{10 + 10N + 10}{{}^{N+7}C_5}$$

$$= 1 - \frac{10(N+2)}{{}^{N+7}C_5}$$

30. Ans. 32/55.

31. As E and F are independent, so

$P(E \cap F) = P(E) \cdot P(F)$

Now, $P(E \cap F^c) = P(E) - P(E \cap F)$

$= P(E) - P(E) \cdot P(F)$

$= P(E)(1 - P(F))$

$= P(E)P(F^c)$

Thus, E and F^c are independent.

Also,

$P(E^c \cap F^c) = P(\overline{E \cup F})$

$= 1 - P(E \cup F)$

$= 1 - \{P(E) + P(F) - P(E \cap F)\}$

$= 1 - P(E) - P(F) + P(E \cap F)$

$= 1 - P(E) - P(F) + P(E) \cdot P(F)$

$= (1 - P(E))(1 - P(F))$

$= P(E^c)P(F^c)$.

Thus, E^c and F^c are independent.

32. Let X be the number of games A wins against B and p be the probability A wins the game against B.

Now,

$P(A$ wins the match$)$

$= P(X \geq 2)$

$= P(X = 2) + P(X = 3)$

$= {}^3C_2 p^2 q + {}^3C_3 p^3$

$= 3p^2 q + p^3$

$= 3 \cdot (0.4)^2 \cdot (0.6) + (0.4)^3$

$= (0.4)^2(3 \cdot (0.6) + (0.4))$

$= (0.16) \cdot (2.2) = 0.352$

Also,

$P(A$ wins the match$)$

$= P(X \geq 3)$

$= P(X = 3) + P(X = 4) + P(X = 5)$

$= {}^5C_3 p^3 q^2 + {}^5C_4 p^4 q + {}^5C_5 p^5$

$= 10p^3 q^2 + 5p^4 q + p^5$

$= p^3(10q^2 + 5pq + p^2)$

$= (0.4)^3(10(0.6)^2 + 5(0.24) + (0.4)^2)$

$= (0.4)^3(3.6 + 1.2 + .16)$

$= (0.064)(4.96)$

$= 0.31744$

It shows that, best of 3 games 'option is better'.

33. $P(A \cup B) = 1 - P(\overline{A \cup B})$

$= 1 - P(\bar{A} \cap \bar{B})$

$= 1 - P(\bar{A}) \cdot P(\bar{B})$

$= 1 - (0.8)(0.7)$

$= 1 - 0.56 = 0.44$

34. $5 \rightarrow \{(1, 4), (4, 1), (2, 3), (3, 2)\}$

$7 \rightarrow \{(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)\}$

Let F be the event, that neither a sum of 5 or 7 occurs.

$P(F) = 1 - \left(\frac{4+6}{36}\right) = \frac{26}{36} = \frac{12}{13}$

Hence, the required probability

$P(W$ or WF or WWF or $\ldots)$

$= P(W) + P(FW) + P(FFW) + \ldots$

$$= \frac{1}{9} + \left(\frac{13}{18}\right)\left(\frac{1}{9}\right) + \left(\frac{13}{18}\right)^2\left(\frac{1}{9}\right) + \ldots$$

$$= \frac{1}{9}\left(1 + \left(\frac{13}{18}\right) + \left(\frac{13}{18}\right)^2 + \ldots\right)$$

$$= \frac{1}{9}\left(\frac{1}{1 - \left(\frac{13}{18}\right)}\right)$$

$$= \frac{1}{9}\left(\frac{18}{18 - 13}\right)$$

$$= \frac{2}{5}$$

35. Let A = $\{a_1, a_2, \ldots, a_n\}$
For each $a_i \in A)1 \le i \le n)$
We have the following four choices:
(i) $a_i \in P, a_i \in Q$ (ii) $a_i \notin P, a_i \in Q$
(iii) $a_i \in P, a_i \notin Q$ (iv) $a_i \notin P, a_i \notin Q$
Thus, the total number of ways of choosing P and Q is 4^n.
Out of these four choices (i) is not favourable for
$P \cap Q = \varphi$
Hence, the required probability is = $\left(\frac{3}{4}\right)^n$.

36. Let $P(B) = x$
Given $P(A) = 0.3$ and $P(A \cup B) = 0.8$
Now, $P(A \cup B) = 0.8$
$\Rightarrow$ $P(A) + P(B) - P(A \cap B) = 0.8$
$\Rightarrow$ $P(A) + P(B) - P(A) \cdot P(B) = 0.8$
$\Rightarrow$ $0.3 + x - (0.3)x = 0.8$
$\Rightarrow$ $(0.7)x = 0.8 - 0.3$
$\Rightarrow$ $(0.7)x = 0.5$
$\Rightarrow$ $x = \frac{0.5}{0.7}$
$\Rightarrow$ $x = \frac{5}{7}$
$\Rightarrow$ $P(B) = \frac{5}{7}$

37. $P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$

$$\frac{P(A) + P(B) - P(A \cup B)}{P(B)}$$

$$\ge \frac{P(A) + P(B) - 1}{P(B)}$$

Thus, (a) is correct.
Also, $P(A \cap B') = P(A) - P(A \cap B)$ holds.
So (b) is incorrect.
Also, $P(A \cup B) = 1 - P(A' \cap B')$
$= 1 - P(A')P(B')$, since A and B are independent events.
Hence (c) is correct.
Again, $P(A \cup B) = P(A) + P(B)$
$\neq P(A')\, P(B')$
So (d) is incorrect.

38. Let E_1, E_2 and E_3 be the events that the answer is guessed, copied and knows the answer and E be the event that the examinee answers correctly.
Given $P(E_1) = \frac{1}{3}, P(E_2) = \frac{1}{6}$
Here, E_1, E_2 and E_3 are exhaustive events.
Thus $P(E_1) + P(E_2) + P(E_3) = 1$
$\Rightarrow$ $P(E_3) = 1 - P(E_1) - P(E_2)$

$$= 1 - \frac{1}{3} - \frac{1}{6} = \frac{6-2-1}{6} = \frac{3}{6} = \frac{1}{2}$$

Now,
$P(E/E_1)$ = Probability of getting correct answer by guessing = 1/4
$P(E/E_2)$ = Probability of getting correct answer by copying = 1/8.
$P(E/E_3)$ = Probability of getting correct answer by knowing = 1.
Hence, the required probability,

$$= P(E_3/E)$$

$$= \frac{P(E_3) \cdot P(E/E_3)}{P(E_1) \cdot P(E/E_1) + P(E_2) \cdot P(E/E_2) + P(E_3) \cdot P(E/E_3)}$$

$$= \frac{\frac{1}{2} \times 1}{\frac{1}{3} \times \frac{1}{4} + \frac{1}{6} \times \frac{1}{8} + \frac{1}{2} \times 1} = \frac{24}{29}$$

39. Given $np = 2$ and $npq = 1$
Thus, $q = \frac{1}{2}, p = 1 - \frac{1}{2} = \frac{1}{2}$ and $n = 4$
Now,
$P(X > 1) = 1 - P(X \le 1)$
$= 1 - \{P(X = 0) + P(X = 1)\}$
$= 1 - \{{}^4C_0 \cdot (p)^0 q^4 + {}^4C_1 \cdot (p)^1 q^3\}$
$= 1 - \{q^4 + 4 \cdot pq^3\}$

$$= 1 - \left\{\left(\frac{1}{2}\right)^4 + 4 \cdot \left(\frac{1}{2}\right)^4\right\}$$

$$= 1 - \left\{\frac{1}{16} + \frac{4}{16}\right\}$$

$$= 1 - \frac{5}{16} = \frac{11}{16}$$

40. Required probability
$P(X \ge 7) = P(X = 7 \text{ or } X = 8)$
$= P(X = 7) + P(X = 8)$
$= P$(India wins 3 matches and draws one) $+ P$(India wins all 4 matches)
$= {}^4C_3(0.5)^3 \cdot (0.05) + {}^4C_4(0.5)^4$
$= (0.5)^3(0.2 + 0.5)$
$= (0.125) \cdot (0.7) = 0.0875$

41. (i) Let D_i and N_i respectively denote the occurrence of a defective and non-defective bulb at the ith draw where $i = 1, 2$

We have, $P(D_i) = \frac{50}{100} = \frac{1}{2}, i = 1,2$

$$P(N_i) = \frac{50}{100} = \frac{1}{2}, i = 1,2$$

Here, D_1N_2, D_2N_2, D_1D_2 and N_1N_2 all are independents.

Given A = {the first bulb is defective}
= $\{D_1D_2, D_1N_2\}$
B = {the 2nd bulb is non-defective)
= $\{D_1N_2, N_1N_2\}$
and C = {both bulbs are defective or non defective}
= $\{D_1D_2, N_1N_2\}$

(i) $P(A) = P(D_1D_2) + P(N_1N_2)$
$= P(D_1)P(D_2) + P(N_1)P(N_2)$
$= \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2}$

Similarly, $P(B) = \frac{1}{2} = P(C)$

Also, $P(A \cap B) = P(D_1N_2) = P(D_1)P(N_2)$
$= \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$

Similarly, $P(B \cap C) = \frac{1}{4} = P(C \cap A)$

Thus, $P(A \cap B) = P(A)P(B)$
$P(B \cap B) = P(B)P(C)$
and $P(C \cap A) = P(C)P(A)$

Hence, A, B and C are pairwise independents.

(ii) Now, $(A \cap B \cap C) = \varphi$
Thus, $P(A \cap B \cap C) \neq P(A)P(B)P(C)$
Hence, A, B and C are not independent.

42. $P(\text{yellow at the first toss}) = \frac{3}{6} = \frac{1}{2}$

$P(\text{red at the second toss}) = \frac{2}{6} = \frac{1}{3}$

$P(\text{blue at the third toss}) = \frac{1}{6}$

Hence, the required probability $= \frac{1}{2}\times\frac{1}{3}\times\frac{1}{6}$
$= \frac{1}{36}$

43. Let E be the event whose face value not less than 2 and not more than 5 in a single throw of a die, i.e.
$E = \{2, 3, 4, 5\}$

So, $P(E) = \frac{4}{6} = \frac{2}{3}$

Required probability,
$P(EEEE) = P(E)\cdot P(E)\cdot P(E)\cdot P(E)$
$= \left(\frac{2}{3}\right)^4 = \frac{16}{81}$

44. Given $P(E \cap F) = \frac{1}{12}$ and $P(\bar{E} \cap \bar{F}) = \frac{1}{2}$

Thus, $P(E)\cdot P(F) = \frac{1}{12}$ …(i)

Also, $P(\bar{E} \cap \bar{F}) = \frac{1}{2}$

$\Rightarrow$ $P(\overline{E \cup F}) = \frac{1}{2}$

$\Rightarrow$ $1 - P(E \cup F) = \frac{1}{2}$

$\Rightarrow$ $P(E \cup F) = 1 - \frac{1}{2} = \frac{1}{2}$

$\Rightarrow$ $P(E) + P(F) - P(E \cap F) = \frac{1}{2}$

$\Rightarrow$ $P(E) + P(F) - \frac{1}{12} = \frac{1}{2}$

$\Rightarrow$ $P(E) + P(F) = \frac{1}{2} + \frac{1}{12} = \frac{7}{12}$ …(ii)

From Eqs (i) and (ii), we can say that, $P(E)$ and $P(F)$ are the roots of $x^2 - \frac{7}{12}x + \frac{1}{12} = 0$

$\Rightarrow$ $12x^2 - 7x + 1 = 0$
$\Rightarrow$ $(4x - 1)(3x - 1) = 0$
$\Rightarrow$ $x = \frac{1}{4}, \frac{1}{3}$

Thus,
$P(E) = \frac{1}{4}, P(F) = \frac{1}{3}$ or $P(E) = \frac{1}{3}, P(F) = \frac{1}{4}$

45. Let E be the event, where the product of two-digit selected number is 18, i.e.
$E = \{(2, 9), (9, 2), (3, 6), (6, 3)\}$

Thus, $P(E) = \frac{4}{100} = \frac{1}{25}$ and $P(\bar{E}) = \frac{24}{25}$

Required probability,
$P(X \geq 3) = P(X = 3) + P(X = 4)$
$= {}^4C_3p^3q + {}^4C_4p^4q^0$
$= 4p^3q + p^4$
$= p^3(4q + p)$
$= p^3(1 + 3q)$
$= \left(\frac{1}{25}\right)^3\left(1 + 3\cdot\frac{24}{25}\right)$
$= \left(\frac{1}{25}\right)^3\left(.\frac{25+72}{25}\right)$
$= \left(.\frac{97}{25^4}\right)$

46. We have,

$P(A \cap (B \cap C) = P(A)P(B \cap C)$
$= P(A)P(B)P(C)$

Thus, the statement S_2 is true.

Also,

$P(A \cap (B \cup C))$
$= P((A \cap B) \cup (A \cap C))$
$= P(A \cap B) + P(A \cap C) - P((A \cap B) \cap (A \cap C))$
$= P(A \cap B) + P(A \cap C) - P((A \cap B \cap C)$
$= P(A)P(B) + P(A)P(C) - P(A)P(B)P(C)$
$= P(A)(P(B) + P(C) - P(B)P(C))$
$= P(A)(P(B \cup C))$
$= P(A)P(B \cup C)$

Thus, the statement S_1 is true.

47. Let E_1, E_2 denote the events that the coin shows a head, tail and A be the event that the noted number is either 7 or 8.

We have $P(E_1) = \frac{1}{2}$ and $P(E_2) = \frac{1}{2}$

Now, $7 \to \{(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)\}$
and $8 \to \{(2, 6), (6, 2), (3, 5), (5, 3), (4, 4)\}$

Thus, $P(A/E_1) = \frac{11}{36}$, $P(A/E_2) = \frac{2}{11}$

Hence, the required probability,

$$P(A) = P(E_1)P(A/E_1) + P(E_2)P(A/E_2)$$
$$= \left(\frac{1}{2}\right)\left(\frac{11}{36}\right) + \left(\frac{1}{2}\right)\left(\frac{2}{11}\right) = \frac{193}{792}.$$

48. Given $P(\overline{A}) = 0.3$

$P(A) = 1 - P(\overline{A}) = 1 - 0.3 = 0.7$

Also, $P(A \cap \overline{B}) = 0.5$

Now, $P\{B \cap (A \cup B')\} = P((B \cap A) \cup (B \cap B))$
$= P((B \cap A) \cup \varphi)$
$= P(B \cap A)$
$= P(A) - P(A \cap B)$
$= 0.7 - 0.5$
$= 0.2$

$P(A \cup \overline{B}) = P(A) + P(\overline{B}) - P(A \cap \overline{B})$
$= P(A) + P(\overline{B}) - P(A \cap \overline{B})$
$= 0.7 + 0.6 - 0.5$
$= 0.8$

We have,

$$P(B/(A \cup B^c)) = P\left(\frac{B}{A \cup \overline{B}}\right)$$
$$= \frac{P(B \cap (A \cup \overline{B})}{P(A \cup \overline{B})}$$
$$= \frac{0.2}{0.8}$$
$$= \frac{1}{4}$$

49. Required probability

$= P(\text{2nd win at the third test})$
$= P(\text{Exactly one win in first two matches} \times P(\text{winning the third test})$
$= P(WL \text{ or } LW) \times P(W)$
$= (P(WL) + P(LW)) \times P(W)$
$= (P(W)P(L) + P(L)P(W)) \times P(W)$

$$= \left(\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}\right) \times \frac{1}{2}$$
$$= \left(\frac{1}{4} + \frac{1}{4}\right) \times \frac{1}{2}$$
$$= \frac{1}{2} \times \frac{1}{2}$$
$$= \frac{1}{4}$$

50. Given $P(A \cup B) = P(A) + P(B) - P(A)P(B)$

$\Rightarrow P(A) + P(B) - P(A \cap B)$
$= P(A) + P(B) - P(A)P(B)$

$\Rightarrow P(A \cap B) = P(A)P(B)$

A and B are independent events.

Now,

$P(A \cup B)' = P(A' \cap B')$
$= P(A') \cap P(B')$

$$\text{Also, } P\left(\frac{A}{B}\right) = \frac{P(A \cap B)}{P(B)}$$
$$= \frac{P(A) \cdot P(B)}{P(B)}$$
$$= P(A)$$

51. Required probability,

$$\frac{2}{^6C_3} = \frac{2}{(6 \cdot 5 \cdot 4)/6}$$
$$= \frac{2}{20}$$
$$= \frac{1}{10}$$

52. $P(\text{Exactly one of } A \text{ or } B \text{ occurs})$,

$P(A \cap \overline{B} \text{ or } \overline{A} \cap B)$
$= P(A \cap \overline{B}) + P(\overline{A} \cap B)$
$= P(B) - P(A \cap B) + P(A) - P(A \cap B)$
$= P(A) + P(B) - 2P(A \cap B) = p$ …(i)

Similarly, $P(\text{Exactly one of } B \text{ or } C \text{ occurs})$,

$P(B) + P(C) - 2P(B \cap C) = p$ …(ii)

and $P(\text{Exactly one of } C \text{ or A occurs})$,

$P(C) + P(A) - 2P(A \cap C) .= p$ …(iii)

Adding all these results, we get,

$2[(A) + P(B) + P(C) - P(A \cap B)$
$-P(B \cap C) - P(C \cap A)] = 3p$

$[P(A) + P(B) + P(C) - P(A \cap B)$

$$-P(B \cap C) - P(C \cap A)] = \frac{3p}{2} \quad \ldots\text{(iv)}$$

Also, $P(A \cap B \cap C)$ ...(v)

Adding Eqs (iii) and (iv), we get

$[P(A) + P(B) + P(C) - P(A \cap B)$

$- P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)]$

$$= \frac{3p}{2} + p^2$$

Thus, $P(A \cup B \cup C) = \frac{3p}{2} + p^2$

53. The roots of $x^2 + px + q = 0$ are imaginary if and only if $p^2 - 4q < 0$, i.e. $p^2 < 4q$

The possible pairs of p and q can happen is as follows,

q	p	No. of pairs of p and q
1	1	1
2	1,2	2
3	1,2,3	3
4	1,2,3	3
5	1,2,3,4	4
6	1,2,3,4	4

q	p	No. of pairs of p and q
7	1,2,3,4,5	5
8	1,2,3,4,5	5
9	1,2,3,4,5	5
10	1,2,3,4,5,6	6

Thus, the total possible pairs of p and q = 38

Required probability,

P(roots are real)

$= 1 - P$(roots are imaginary)

$$= \left(1 - \frac{38}{100}\right)$$

$$= \frac{62}{100} = 0.62$$

54. (a) The number of ways of choosing 8 winners out of 16 is ${}^{16}C_8$.

The number of ways of choosing S_1 and 7 other winners out of 15 is ${}^{15}C_7$.

Required probability,

$$P(S_1 \text{ will win}) = \frac{{}^{15}C_7}{{}^{16}C_8}$$

$$= \frac{(15)!}{(7)! \times (8)!} \times \frac{(8)! \times (8)!}{(16)!} = \frac{8}{16} = \frac{1}{2}$$

(b) Let E_1 denotes the event S_1 and S_2 are paired and E_2 denotes the event S_1 and S_2 are not paired and A denotes the event that one of the two players S_1 and S_2 is among the winners.

So, $P(E_1) = \frac{1}{15}$ and $P(E_2) = \frac{14}{15}$

In case E_2 occurs, the probability that exactly one of S_1 and S_2 is among the winners., i.e.

$$P(S_1 \cap \bar{S}_2 \text{ or } \bar{S}_1 \cap S_2)$$

$$= P(S_1 \cap \bar{S}_2) + P(\bar{S}_1 \cap S_2)$$

$$= P(S_1) \cdot P(\bar{S}_2) + P(\bar{S}_1) \cdot P(S_2)$$

$$= \frac{1}{2}\left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{2}\right)\frac{1}{2}$$

$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Now, $P(A/E_1) = 1$ and $P(A/E_2) = \frac{1}{2}$

Hence, the required probability,

$$P(A) = P(E_1)P\left(\frac{A}{E_1}\right) + P(E_2)P\left(\frac{A}{E_2}\right)$$

$$= \left(\frac{1}{15}\right) \cdot 1 + \left(\frac{14}{15}\right) \cdot \frac{1}{2}$$

$$= \frac{16}{30} = \frac{8}{15}$$

55. Let A the event, minimum of the chosen nos is 3 and B the event, maximum of the chosen nos is 7.

Now,

$P(A) = P$(Choosing 3 and two other numbers from 4 to 10)

$$= \frac{{}^7C_2}{{}^{10}C_3} = \frac{7.6}{2} \times \frac{6}{10.9.8} = \frac{7}{40}$$

$P(B) = P$(Choosing 7 and two others numbers from 1 to 6)

$$= \frac{{}^6C_2}{{}^{10}C_3} = \frac{6.5}{2} \times \frac{6}{10.9.8} = \frac{1}{8}$$

and $P(A \cap B) = P$(Choosing 3 and 7 and one number from 4 to 6)

$$= \frac{3}{{}^{10}C_3} = \frac{3.6}{10.9.8} = \frac{1}{40}$$

Hence, the required probability,

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{7}{40} + \frac{1}{8} - \frac{1}{40}$$

$$= \frac{11}{40}$$

56. Here, we fill 3 black balls in between 7 white balls.

$$\text{Required probability} = \frac{{}^8C_3}{\frac{(10)!}{(7)! \times (3)!}} = \frac{8.7.6}{10.9.8} = \frac{7}{15}$$

57. Required probability

P(2 white ball and 1 black ball)

$= P(W_1W_2B_3 \text{ or } W_1B_2W_3 \text{ or } B_1W_2W_3)$

$= P(W_1W_2B_3) + P(W_1B_2W_3) + P(B_1W_2W_3)$

$$= \frac{3}{4}.\frac{2}{4}.\frac{3}{4} + \frac{3}{4}.\frac{2}{4}.\frac{1}{4} + \frac{1}{4}.\frac{2}{4}.\frac{1}{4}$$
$$= \frac{18+6+2}{64}$$
$$= \frac{26}{64} = \frac{13}{32}$$

58. Now,

$$P(E/F) + P(\bar{E}/F) = \frac{P(E \cap F)}{P(F)} + \frac{P(\bar{E} \cap F)}{P(F)}$$
$$= \frac{P(E \cap F) + P(\bar{E} \cap F)}{P(F)}$$
$$= \frac{P(E \cap F) + P(F) - P(E \cap F)}{P(F)}$$
$$= \frac{P(F)}{P(F)} = 1$$

Also,

$$P(E/\bar{F}) + P(\bar{E}/\bar{F}) = \frac{P(E \cap \bar{F})}{P(\bar{F})} + \frac{P(\bar{E} \cap \bar{F})}{P(\bar{F})}$$
$$= \frac{P(E \cap \bar{F}) + P(\bar{E} \cap \bar{F})}{P(\bar{F})}$$
$$= \frac{P(E \cap \bar{F}) + P(\bar{F}) - P(E \cap \bar{F})}{P(\bar{F})}$$
$$= \frac{P(\bar{F})}{P(\bar{F})} = 1$$

59. Here, (a), (b) and (c) cannot hold.

60. The occurrence of head and tail are independent.
Required probability
= P(head comes up on the fifth toss)
$$= \frac{1}{2}$$

61. Required probability,
= P(only two tests are needed)
= P(the first machine tested is faulty)
$$\times P\left(\begin{array}{r}\text{2nd mechine tested is faulty given that}\\ \text{the first mechine tested is faulty}\end{array}\right)$$
$$= \frac{2}{4} \times \frac{1}{3}$$
$$= \frac{1}{6}$$

62. We have,
$a = P(A$ is first to get head)
$= P(H$ or $TTTH$ or $TTTTTTH$ or …)
= P(H) + P(TTTH) + P(TTTTTTH) + …
$= P(H) + P(T)^3 P(H) + P(T)^6 P(H) + \ldots$
$= P(H)(1 + P(T)^3 + P(T)^6 + \ldots$

$$= P(H) \cdot \left(\frac{1}{1-P(T)^3}\right)$$
$$= .\left(\frac{P(H)}{1-P(T)^3}\right)$$
$$= .\left(\frac{p}{1-(1-p)^3}\right)$$

Also,
$\beta = P(B$ is the first to get the head)
$= P(TH$ or $TTTTH$ or $TTTTTTTH$ or …)
$= P(TH) + P(GGGGH) + P(TTTTTTTT) + \ldots$
$= P(T)P(H) + P(T)^4 P(H) + P(T)^7 P(H) + \ldots$
$= P(T)P(H)(1 + P(T)^3 + P(T)^6 + \ldots$
$$= \frac{P(T)P(H)}{1-P(T)^3}.$$
$$= \frac{(1-p)p}{1-(1-p)^3} = (1-p)\alpha$$

Finally,
$\gamma = 1 - (\alpha + \beta)$
$= 1 - (\alpha + (1-p)\alpha)$
$= 1 - (1 + (1-p))\alpha$
$= 1 - (2-p)\alpha$
$$= 1 - \frac{(2-p)p}{1-(1-p)^3}$$
$$= \frac{1-(1-p)^3 - (2-p)p}{1-(1-p)^3}$$
$$= \frac{1-(1-3p+3p^2-p^3) - 2p + p^2}{1-(1-p)^3}$$
$$= \frac{p^3 - 2p^2 + p}{1-(1-p)^3}$$
$$= \frac{(p^2-2p+1)p}{1-(1-p)^3}$$
$$= \frac{(p-1)^2 p}{1-(1-p)^3}$$

63 $7^m + 7^n$ is divisible by 5, only when its unit's digit becomes zero

Sl. no.	m	n
1	$4r$	$4s + 2$
2	$4r + 1$	$4s + 3$
3	$4r + 2$	$4s$
4	$4r + 3$	$4s + 1$

Thus, for a given value of m, there are just 25 values of n for which $7^m + 7^n$ end in 0.
For example, when $m = 4r$, then n can be taken as 2, 6, 10, 14, …, 98.
Hence, the required probability

$$= \frac{100 \times 25}{100 \times 100}$$

$$= \frac{1}{4}$$

64. Given $P(M \cup P \cup C) = 0.75$

$\Rightarrow \quad 1 - P(\overline{M \cup P \cup C}) = 0.75$

$\Rightarrow \quad 1 - P(\bar{M} \cap \bar{P} \cap \bar{C}) = 0.75$

$\Rightarrow \quad 1 - P(\bar{M})P(\bar{P})P(\bar{C}) = 0.75$

$\Rightarrow \quad 1 - (1 - m)(1 - p)(1 - c) = 0.75$

$\Rightarrow \quad (1 - m)(1 - p)(1 - c) = 0.25 \quad \ldots$(i)

Also, $P(M \cap P \cap \bar{C}) + P(M \cap \bar{P} \cap C)$

$+ P(\bar{M} \cap P \cap C) + P(M \cap P \cap C) = 0.50$

$\Rightarrow \quad pm(1 - c) + pc(1 - m) + mc(1 - p) + pcm = 0.50$ …(ii)

Again $P(M \cap P \cap \bar{C}) + P(M \cap \bar{P} \cap C)$

$+ P(\bar{M} \cap P \cap C) = 0.40$

$\Rightarrow \quad pm(1 - c) + pc(1 - m) + mc(1 - p) = 0.40$ …(iii)

From Eqs (ii) and (iii), we get

$pcm = 0.10$ …(iv)

and $pm + pc + cm = 0.40 + 3\,pcm$

$= 0.40 + 0.30$

$= 0.70$ …(v)

From Eqs (i), (iv) and (v), we get

$1 - (p + c + m) + (pm + pc + cm) - pmc = 0.25$

$(p + c + m) = 1 + 0.70 - 0.10 - 0.25 = 1.35$

$= \frac{27}{20}$

65. Total number of ways in which $P_1, P_2, \ldots, P_8$ can be paired in four pairs

$$= \frac{{}^8C_2 \times {}^6C_2 \times {}^4C_2 \times {}^2C_2}{(4)!}$$

$= 105.$

Note that at least two of P_1, P_2, P_3 will certainly reach the second round. P_4 can reach the final if exactly two of P_1, P_2, P_3 play against each other and remaining out of P_1, P_2, P_3 plays against one of P_5, P_6, P_7, P_8 and P_4 plays against one of the remaining three from P_5, P_6, P_7, P_8.

This can be happen in $= {}^3C_2 \times {}^4C_1 \times {}^3C_1 = 36$ ways

Thus, the probability that P_4 and exactly one of P_5, P_6, P_7, P_8 reach the second round

$= 36/105 = 12/35$

If P_1, P_i, P_2 and P_j where $i = 2$ or 3 and $j = 5$, 6 or 7 reach the 2nd round they can paired in

$$= \frac{{}^4C_2 \times {}^2C_2}{(2)!} = \frac{6}{2} = 3 \text{ ways}$$

But P_4 will reach the final if P_1 plays against P_i and P_4 plays against P_j.

Therefore, the probability that P_4 reaches the final round from the end round $= \frac{1}{3}$

Hence, the required probability,

$$P(P_4 \text{ reaches the final}) = \frac{12}{35} \times \frac{1}{3} = \frac{4}{35}$$

66. When $n = 1$, the two possible outcomes H and T satisfy the condition that no two or more consecutive heads occur

Thus, $p_1 = 1$

When $n = 2$, the possible outcomes are HH, HT, TH and TT

Thus, $p_2 = 1 - P(HH) = 1 - p \cdot p = 1 - p^2$

When $n \geq 3$, if the last outcome is T, the probability that the first $(n - 1)$ tosses do not contain two consecutive or more heads is p_{n-1} and if the last outcome is H, then $(n - 1)$th outcome must be T and the probability that the first $(n - 2)$ tosses do not contain two or more consecutive heads is p_{n-2}.

Hence, $p_n = p_{n-1} \times P(n\text{th results toss in tail})$

$p_{n-2} \times P[(n - 1)\text{th results toss in tail and } n\text{th results toss in head}]$

$= p_{n-1}(1 - p) + p_{n-2}(1 - p)p$

$= (1 - p)p_{n-1} + p(1 - p)p_{n-2}$.

67. Let W, B and A denote the following events:

W : a white ball is drawn at the first draw

B : a black ball is drawn at the first draw

A : a white ball is drawn at the second draw

We have

$$P(W) = \frac{m}{m+n}, \; P(B) = \frac{m}{m+n}$$

$$P\left(\frac{A}{W}\right) = \frac{m+k}{m+n+k}, \; P\left(\frac{A}{B}\right) = \frac{m}{m+n+k}$$

By the total probability rule

$$P(A) = P(W)P\left(\frac{A}{W}\right) + P(B)P\left(\frac{A}{B}\right)$$

$$= \frac{m}{m+n} \cdot \frac{m+k}{m+n+k} + \frac{n}{m+n} \cdot \frac{m}{m+n+k}$$

$$= \frac{m(m+n+k)}{(m+n)(m+n+k)}$$

$$= \frac{m}{(m+n)}$$

68. The total number of outcomes $= 6^n$.

We can choose three numbers out of 6 in ${}^6C_3$ ways.

By using these three numbers, we can get $3n$ sequences of length n.

But these include sequences which use exactly two numbers or exactly one number.

The number of n sequences which use exactly two numbers is ${}^3C_2(2^n - 2) = 3(2^n - 2)$

and the number of n sequences which has exactly one number $= {}^3C_1 \times (1^n) = 3$

Thus, the number of favourable cases

$= {}^6C_3(3^n - 3(2^n - 2)) = 20(3^n - 3.2^n + 3)$

Hence, the required probability of an event

$$= \frac{20(3^n - 3.2^n + 3)}{6^n}$$

69. Let E_1: Coin selected is fair.

E_2: Coin selected is biased

and A: the first toss results in a head and the second toss results in atrial

$$P(E_1) = \frac{m}{N}, P(E_2) = \frac{N-m}{N}$$

$$P(A/E_1) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

and $P(A/E_2) = \frac{2}{3} \times \frac{1}{3} = \frac{2}{9}$

By the Bayes theorem

$$P(E_1/A) = \frac{P(A/E_1)P(E_1)}{P(A/E_1)P(E_1) + P(A/E_2)P(E_2)}$$

$$= \frac{9\,m}{8\,N + m}$$

70. Total possible cases $= 2! \times {}^6C_2 = 30$

Total favourable cases $= 30 - 6 = 24$

Hence, the required probability $= \frac{24}{30} = \frac{4}{5}0$

71. $P(B \cap C') = P[(A \cup A') \cap (B \cap C)]$

$= P(A \cap B \cap C') + P(A' \cap B \cap C')$

$= \frac{1}{3} + \frac{1}{3} = \frac{2}{3}$

Now, $P(B \cap C) = P(B) - P(B \cap C')$

$= \frac{3}{4} - \frac{2}{3} = \frac{1}{12}$

72. Hence, the required probability,

$$P\left(\frac{BC'}{A}\right) = \frac{P\left(\frac{A}{BC'}\right)P(BC')}{P\left(\frac{A}{BC'}\right)P(BC') + P\left(\frac{A}{B'C}\right)P(B'C) + P\left(\frac{A}{B'C'}\right)P(B'C') + P\left(\frac{A}{BC}\right)P(BC)}$$

$$= \frac{1 \times \frac{1}{2} \times \frac{2}{3}}{1 \times \frac{1}{2} \times \frac{2}{3} + 1 \times \frac{1}{2} \times \frac{2}{3} + 1 \times \frac{1}{2} \times \frac{2}{3} + 0 \times \frac{1}{2} \times \frac{2}{3}}$$

$$= \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{6} + \frac{1}{6}} = \frac{1}{2}$$

74. Numbers between 1 to 100, which are divisible by both 2 and 3 are 16, i.e. multiple of 6.

Hence, the required probability $= \frac{{}^{16}C_3}{{}^{100}C_3} = \frac{4}{1155}$

75. Let $P(A)$ be the probability that at least 4 white balls have been drawn.

$P(A_1)$ be the probability that exactly 4 white balls have been drawn.

$P(A_2)$ be the probability that exactly 5 white balls have been drawn.

$P(A_3)$ be the probability that exactly 6 white balls have been drawn.

$P(B)$ be the probability that exactly 1 white ball is drawn from two draws.

$$P(B/A) = \frac{\sum_{i=1}^{3} P(A_i)\,P(B/A_i)}{\sum_{i=1}^{3} P(A_i)}$$

$$= \frac{\frac{{}^{12}C_2 \cdot {}^6C_4}{{}^{18}C_6} \cdot \frac{{}^{10}C_1 \cdot {}^2C_1}{{}^{12}C_2} + \frac{{}^{12}C_1 \cdot {}^6C_5}{{}^{18}C_6} \cdot \frac{{}^{11}C_1 \cdot {}^1C_1}{{}^{12}C_2}}{\frac{{}^{12}C_2 \cdot {}^6C_4}{{}^{18}C_6} + \frac{{}^{12}C_1 \cdot {}^6C_5}{{}^{18}C_6} + \frac{{}^{12}C_0 \cdot {}^6C_6}{{}^{18}C_6}}$$

$$= \frac{{}^{12}C_2 \cdot {}^6C_4\, {}^{10}C_1 \cdot {}^2C_1 + {}^{12}C_1 \cdot {}^6C_5\, {}^{11}C_1 \cdot {}^1C_1}{{}^{12}C_2({}^{12}C_2 \cdot {}^6C_4 + {}^{12}C_1 \cdot {}^6C_5 + {}^{12}C_0 \cdot {}^6C_6)}$$

76. $P(A \cup B) \cdot P(A')P(B')$

$\leq (P(A) + P(B))P(A')P(B')$

$= P(A) \cdot P(A')P(B') + P(B) \cdot P(A')P(B')$

$= P(A) \cdot P(B')(1 - P(A)) + P(B) \cdot P(A')(1 - P(B))$

$\leq P(A)P(B') + P(B)P(A') = P(C)$

77. Hence, the required probability

$$= \frac{1}{6} \cdot \frac{5}{6} + \frac{1}{6} \cdot \left(\frac{5}{6}\right)^3 + \ldots\ldots$$

$$= \frac{1.5}{6.6}\left(1 + \left(\frac{5}{6}\right)^2 + \left(\frac{5}{6}\right)^2 + \ldots\right)$$

$$= \frac{1.5}{6.6}\left(\frac{1}{1 - (5/6)^2}\right)$$

$$= \frac{5}{11}$$

78. Let C, S, B, T be the events of the person going by a car, a scooter, a bus or a train, respectively.

Given that

$$P(C) = \frac{1}{7}, P(S) = \frac{3}{7},$$

$$P(B) = \frac{2}{7}, P(T) = \frac{1}{7}$$

Let $\bar{L}$ be the event of the person reaching the office in time.

$$\therefore \quad P\left(\frac{\bar{L}}{C}\right)=\frac{7}{9}, P\left(\frac{\bar{L}}{S}\right)=\frac{8}{9}, P\left(\frac{\bar{L}}{B}\right)=\frac{5}{9}, P\left(\frac{\bar{L}}{T}\right)=\frac{8}{9}$$

$$\text{Now, } P\left(\frac{C}{\bar{L}}\right)=\frac{P\left(\frac{\bar{L}}{C}\right)\cdot P(C)}{P(\bar{L})}$$

$$=\frac{\frac{1}{7}\times\frac{7}{9}}{\frac{1}{7}\times\frac{7}{9}+\frac{3}{7}\times\frac{8}{9}+\frac{2}{7}\times\frac{5}{9}+\frac{1}{7}\times\frac{8}{9}}$$

$$=\frac{1}{7}$$

79. $P(u_i)=ki, \sum P(u_i)=1$

and $k=\dfrac{2}{n(n+1)}$

$$\text{Now } \lim_{n\to\infty} P(w)=\lim_{n\to\infty}\sum_{i=1}^{n}\left(\frac{2i^2}{n(n+1)^2}\right)$$

$$=\lim_{n\to\infty}\left(\frac{2n(n+1)(2n+1)}{n(n+1)^2 6}\right)=\frac{2}{3}$$

80. $$P\left(\frac{u_B}{w}\right)=\frac{c\left(\frac{n}{n+1}\right)}{c\left(\frac{\sum i}{(n+1)}\right)}=\frac{2}{(n+1)}$$

81. $$P\left(\frac{w}{E}\right)=\frac{2+4+6+....+n}{\frac{n(n+1)}{2}}=\frac{n+2}{2(n+1)}$$

82. Fixing four American couples and one Indian man in between any two couples; we have 5 different ways in which his wife can be seated, of which 2 cases are favorable.

Hence, the required probability $=\dfrac{2}{5}$.

83. We have,

$$P\left(\frac{E^C\cap F^C}{G}\right)=\frac{P(E^C\cap F^C\cap G)}{P(G)}$$

$$=\frac{P(G)-P(E\cap G)-P(G\cap F)}{P(G)}$$

$$=\frac{P(G)[1-P(E)-P(F)]}{P(G)}$$

$$=[1-P(E)-P(F)]$$

$$=P(E^C)-P(F)$$

84. We have,

$$P(A\cap B)=\frac{4}{10}\times\frac{p}{10}=\frac{2p/5}{10}$$

$$\Rightarrow \quad \frac{2p}{5} \text{ is an integer}$$

Thus, $p = 5$ or 10.

85. For unique solution, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \neq 0$

where $a, b, c, d \in \{0, 1\}$

Total possible cases = 16

Favourable cases = 6

(Either $ad = 1$, $bc = 0$ or $ad = 0$, $bc = 1$)

The probability that the system of equations has unique solution $=\dfrac{6}{16}=\dfrac{3}{8}$

and the system of equations has either unique solution or infinite solutions so that probability for system to have a solution is 1.

86. (i) $P(X=3)=\left(\frac{5}{6}\right)\cdot\left(\frac{5}{6}\right)\cdot\left(\frac{1}{6}\right)=\frac{25}{216}$

(ii) $P(X\ge 3) = 1 - P(X < 3) = 1 - P(X \le 2)$

$$=1-\left(\frac{1}{6}+\frac{5}{6}\times\frac{1}{6}\right)$$

$$=1-\frac{11}{36}=\frac{25}{36}$$

(iii) $$P(X\ge 6)=\frac{5^5}{6^6}+\frac{5^5}{6^7}+.....$$

$$=\frac{5^5}{6^6}\left(\frac{1}{(1-5/6)}\right)=\left(\frac{5}{6}\right)^5$$

Also,

$$P(X>3)=\frac{5^3}{6^4}+\frac{5^4}{6^5}+\frac{5^5}{6^6}+....$$

$$=\frac{5^3}{6^4}\left(1+\frac{5}{6}+\frac{5^2}{6^2}+...\right)$$

$$=\frac{5^3}{6^4}\left(\frac{1}{1-5/6}\right)=\left(\frac{5}{6}\right)^3$$

Hence, the required probability

$$=\frac{(5/6)^5}{(5/6)^3}=\frac{25}{36}$$

87. Here $r_1, r_2, r_3 \in \{1, 2, 3, 4, 5, 6\}$

Where r_1, r_2, r_3 are of the form $3k$, $3k + 1$, $3k + 2$

Hence, the required probability

$$=\frac{3!\times {}^2C_1\times {}^2C_1\times {}^2C_1}{6\times 6\times 6}$$

$$=\frac{6\times 8}{216}=\frac{2}{9}$$

88. Let G = Original signal is green
$E_1 = A$ receives the signal correct
$E_2 = B$ receives the signal correct
E_2 = Signal received by B is green
Now, P(signal received by B is green)
$$= P(GE_1E_2) + P(G\bar{E}_1\bar{E}_2) + P(\bar{G}E_1\bar{E}_2) + P(\bar{G}\,\bar{E}_1E_2)$$
$$P(E) = \frac{46}{5\times 16}$$
and $P(G/E) = \dfrac{\frac{40}{5}\times 16}{\frac{46}{5}\times 16} = \dfrac{20}{23}$

89. Let $P(E) = a$ and $P(F) = b$
We have,
$$P(E\cup F) - P(E\cap F) = \frac{11}{25}$$
$$\Rightarrow \quad a + b - 2ab = \frac{11}{25} \quad \ldots\text{(i)}$$
Also,
$$P(\bar{E}\cap\bar{F}) = \frac{2}{25}$$
$$\Rightarrow \quad (1-a)(1-b) = \frac{2}{25}$$
$$\Rightarrow \quad 1 - a - b - ab = \frac{2}{25} \quad \ldots\text{(ii)}$$
From Eqs (i) and (ii), we get,
$$ab = \frac{12}{25},\ a + b = \frac{7}{5}$$
$$\Rightarrow \quad a = \frac{4}{5},\ b = \frac{3}{5} \text{ or } a = \frac{3}{5},\ b = \frac{4}{5}$$

90. (i) Let $H \to 1$ ball from U_1 to U_2,
$T \to 2$ balls from U_1 to U_2
and $E \to 1$ ball from U_2
Hence, the required probability of a white ball from the urn U_2
$$= \frac{1}{2}\times\left(\frac{3}{5}\times 1\right) + \frac{1}{2}\times\left(\frac{2}{5}\times\frac{1}{2}\right) + \frac{1}{2}\times\left(\frac{{}^3C_2}{{}^5C_2}\times 1\right) + \frac{1}{2}\times\left(\frac{{}^2C_2}{{}^5C_2}\times\frac{1}{3}\right) + \frac{1}{2}\times\left(\frac{{}^3C_1\cdot{}^2C_1}{{}^5C_2}\times\frac{2}{3}\right)$$
$$= \frac{23}{30}$$
(ii) $$P\left(\frac{H}{W}\right) = \frac{P(W/H)\times P(H)}{P(W/T)\times P(T) + P(W/H)\times P(H)}$$
$$= \frac{\frac{1}{2}\left(\frac{3}{5}\times 1 + \frac{2}{5}\times\frac{1}{2}\right)}{\frac{23}{30}} = \frac{12}{23}$$

91. $P(X_1) = \dfrac{1}{2},\ P(X_2) = \dfrac{1}{4},\ P(X_3) = \dfrac{1}{4}$
We have
$$P(X) = P(X_1\cap X_2\cap X_3{}^C) + P(X_1\cap X_2{}^C\cap X_3) + P(X_1{}^C\cap X_2\cap X_3) + P(X_1\cap X_2\cap X_3) = \frac{1}{4}$$
$$P(X_1{}^C/X) = \frac{P(X\cap X_1{}^C)}{P(X)} = \frac{\frac{1}{32}}{\frac{1}{4}} = \frac{1}{8}$$
$$P\begin{bmatrix}\text{Exactly two engines}\\ \text{are functioning}/X\end{bmatrix} = \frac{\frac{7}{32}}{\frac{1}{4}} = \frac{7}{8}$$
$$P\left(\frac{X}{X_2}\right) = \frac{\frac{7}{32}}{\frac{1}{4}} = \frac{7}{8}$$
$$P\left(\frac{X}{X_1}\right) = \frac{\frac{7}{32}}{\frac{1}{2}} = \frac{7}{16}$$

92. Hence, the required probability
$$= 1 - \frac{6.5^3}{6^4} = 1 - \frac{125}{216} = \frac{91}{216}$$

93. We have $P\left(\dfrac{X}{Y}\right) = \dfrac{P(X\cap Y)}{P(Y)} = \dfrac{1}{2}$
and $P\left(\dfrac{Y}{X}\right) = \dfrac{P(X\cap Y)}{P(X)} = \dfrac{1}{3}$
Also, $P(X\cap Y) = \dfrac{1}{6}$
$$P(Y) = \frac{1}{3} \text{ and } P(X) = \frac{1}{2}$$
Clearly, X and Y are independent.
Thus, $P(X\cup Y) = \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{6} = \dfrac{2}{3}$

94. P(at least one of them solves correctly)
= 1 – P(none of them solves correctly)
$$= 1 - \left(\frac{1}{2}\times\frac{1}{4}\times\frac{3}{4}\times\frac{7}{8}\right)$$
$$= \frac{235}{256}$$

95. Let $P(E_1) = x$, $P(E_2) = y$ and $P(E_3) = z$
then $(1-x)(1-y)(1-z) = p$
$x(1-y)(1-z) = \alpha$, $(1-x)y(1-z) = \beta$
and $(1-x)(1-y)x = \gamma$
So, $\dfrac{p}{\alpha} = \dfrac{1-x}{x} \Rightarrow x = \dfrac{\alpha}{\alpha + p}$

Similarly, $z = \dfrac{\gamma}{\gamma + p}$

$$\frac{P(E_1)}{P(E_2)} = \frac{\dfrac{\alpha}{\alpha+p}}{\dfrac{\gamma}{\gamma+p}} = \frac{\dfrac{\gamma+p}{\gamma}}{\dfrac{\alpha+p}{p}} = \frac{1+\dfrac{p}{\gamma}}{1+\dfrac{p}{\alpha}}$$

Also, given $\dfrac{\alpha\beta}{\alpha - 2\beta} = p = \dfrac{2\beta\gamma}{\beta - 3\gamma}$

$\Rightarrow \quad \beta = \dfrac{5\alpha\gamma}{\alpha + 4\gamma}$

Thus, $\left(\alpha - 2\left(\dfrac{5\alpha\gamma}{\alpha+4\gamma}\right)\right)p = \dfrac{\alpha \cdot 5\alpha\gamma}{\alpha + 4\gamma}$

$\Rightarrow \quad \alpha p - 6p\gamma = 5\alpha\gamma$

$\Rightarrow \quad \left(\dfrac{p}{\gamma}+1\right) = 6\left(\dfrac{p}{\alpha}+1\right)$

$\Rightarrow \quad \dfrac{\left(\dfrac{p}{\gamma}+1\right)}{\left(\dfrac{p}{\alpha}+1\right)} = 6$

96. We have

$1 + 2 + 3 + \ldots + n - 2 \le 1224 \le 3 + 4 + \ldots + n$

$\Rightarrow \quad \dfrac{(n-1)(n-2)}{2} \le 1224 \le \dfrac{(n-2)(n+3)}{2}$

$\Rightarrow \quad n^2 - 3n - 2446 \le 0$ and $n^2 + n - 2454 \ge 0$

$\Rightarrow \quad 49 < n < 51$

$\Rightarrow \quad n = 50$

$\Rightarrow \quad \dfrac{n(n+1)}{2} - (2k+1) = 1224$

$\Rightarrow \quad k = 25$

Thus, $k - 20 = 25 - 20 = 5$

97. (i) $P(WWW) + P(RRR) + P(BBB)$

$$= \frac{1}{6}\times\frac{2}{9}\times\frac{3}{12} + \frac{3}{6}\times\frac{3}{9}\times\frac{4}{12} + \frac{2}{6}\times\frac{4}{9}\times\frac{5}{12}$$

$$= \frac{6+36+40}{6\times9\times12}$$

$$= \frac{82}{648}$$

(ii) P(Ball drawn from box 2/one is W one is R)

$$= \frac{P(A \cap B)}{P(B)}$$

$$= \frac{\dfrac{1}{3}\times\dfrac{2.3}{{}^9C_2}}{\dfrac{1}{3}\left(\dfrac{1.3}{{}^6C_2} + \dfrac{2.3}{{}^9C_2} + \dfrac{3.4}{{}^{12}C_2}\right)}$$

$$= \frac{\dfrac{1}{6}}{\dfrac{1}{5}+\dfrac{1}{6}+\dfrac{2}{12}} = \frac{55}{181}$$

98. Either a girl will start the sequence or will be at second position and will not acquire the last position as well.

Required probability = $\dfrac{{}^3C_1 + {}^3C_1}{{}^5C_2} = \dfrac{1}{2}$

99. (i) **Case I:** One odd, 2 even

Total number of ways = 2.2.3 + 1.3.3 + 1.2.4 = 29

Case II: All 3 odd

Number of ways = 2.3.4 = 24

Favourable ways = 29 + 24 = 53

Required probability = $\dfrac{53}{3\times5\times7} = \dfrac{53}{105}$

(ii) Here, $2x_2 = x_1 + x_3$

$\Rightarrow \quad x_1 + x_3 =$ Even

Hence number of favourable ways

$= {}^2C_1 \times {}^4C_2 + {}^1C_1 \times {}^3C_1$

$= 8 + 3$

$= 11$